```
> restart:
  with(plots):
  with(plottools):
  with(Statistics):
```

```
> Digits:=40;
  digits:=Digits:
```

$$Digits := 40 \tag{1}$$

```
> cv := 299792458:
  MEv:= 597237*10^19:
  Gv:=667430*10^(-16):
  alv:=2*MEv*Gv/cv^2:

  Surface:= 3189000;  # in meter
```

$$Surface := 3189000 \tag{2}$$

```
> Av:= 1-al/r;
```

$$Av := 1 - \frac{al}{r} \tag{3}$$

```
> evalf(alv);
```

$$0.008870355361295248856668158627529007790505 \tag{4}$$

```
> # The Lagrangian for material points in a Schwarzschild
  gravitational field (in polar coordinates)

  -A*c^2+dr^2/A +r^2*dte^2:
  sqrt(-%):
  L:=%;
```

$$L := \sqrt{A\, c^2 - \frac{dr^2}{A} - r^2\, dte^2} \tag{5}$$

```
> # Conjugate momenta

  diff(L, dr):
  #simplify(%):
  #radsimp(%) assuming r>0, A>0:
  pr:=%;

  diff(L, dte):
  #simplify(%):
  #radsimp(%) assuming r>0, A>0:
  pte:=%;
```

$$pr := -\frac{dr}{\sqrt{A\, c^2 - \frac{dr^2}{A} - r^2\, dte^2}\; A}$$

$$pte := -\frac{r^2\, dte}{\sqrt{A\, c^2 - \frac{dr^2}{A} - r^2\, dte^2}} \tag{6}$$

```
> # Legendre transform: solve momenta for velocities
```

```
[Pr-pr, Pte-pte]:
solve(%, [dr, dte]):
allvalues(%):
[%][1]:
op(%):
simplify(%):
S:=%;
```

$$S := \left[dr = -Pr \sqrt{\frac{A}{A\,Pr^2\,r^2 + Pte^2 + r^2}}\; r\,c\,A,\; dte = -\frac{Pte \sqrt{\frac{A}{A\,Pr^2\,r^2 + Pte^2 + r^2}}\; c}{r} \right] \tag{7}$$

```
> subs(S, dr):
Dr:=%;

subs(S, dte):
Dte:=%;
```

$$Dr := -Pr \sqrt{\frac{A}{A\,Pr^2\,r^2 + Pte^2 + r^2}}\; r\,c\,A$$

$$Dte := -\frac{Pte \sqrt{\frac{A}{A\,Pr^2\,r^2 + Pte^2 + r^2}}\; c}{r} \tag{8}$$

```
> # Hamiltonian

Pr*Dr+ Pte*Dte - L:
subs(S, %):
simplify(%):
radsimp(%) assuming A*Pr^2*r^2 + Pte^2 + r^2>0:
simplify(%):
%;
```

$$-\frac{c\sqrt{\left(\left(A\,Pr^2 + 1\right) r^2 + Pte^2\right) A}}{r} \tag{9}$$

```
> # Total energy first integral

pr*dr+pte*dte-L:
%^2:
simplify(%):
factor(%):
subs([dr^2=dr2, dte^2=dte2], %):
cons1:= %=c^4*ep2;
```

$$cons1 := \frac{A^3\,c^4}{A^2\,c^2 - r^2\,dte2\,A - dr2} = c^4\,ep2 \tag{10}$$

```
> # Angular momentum first integral
```

```
pte^2:
simplify(%):
factor(%):
subs([dr^2=dr2, dte^2=dte2], %):
cons2:=%=k2;
```

$$cons2 := \frac{A\,r^4\,dte2}{A^2\,c^2 - r^2\,dte2\,A - dr2} = k2 \tag{11}$$

```
> # Weierstrass equations

[cons1, cons2]:
solve(%, [dr2, dte2]):
op(%):
factor(%):
%;

subs(%, [dr2, dr2/dte2]):
simplify(%):
%;

subs(A=Av, %):
simplify(%):
%;

wPhi, wPsi := op(%):
wPhi;
wPsi;
```

$$\left[dr2 = -\frac{A^2\left(-c^2\,ep2\,r^2 + A\,r^2 + A\,k2\right)}{r^2\,ep2},\ dte2 = \frac{k2\,A^2}{ep2\,r^4}\right]$$

$$\left[-\frac{\left(\left(r^2 + k2\right)A - c^2\,ep2\,r^2\right)A^2}{r^2\,ep2},\ -\frac{\left(\left(-c^2\,ep2 + A\right)r^2 + A\,k2\right)r^2}{k2}\right]$$

$$\left[\frac{(r - al)^2\left(\left(c^2\,ep2 - 1\right)r^3 + al\,r^2 - k2\,r + k2\,al\right)}{r^5\,ep2},\right.$$
$$\left.\frac{r\left(\left(c^2\,ep2 - 1\right)r^3 + al\,r^2 - k2\,r + k2\,al\right)}{k2}\right]$$

$$\frac{(r - al)^2\left(\left(c^2\,ep2 - 1\right)r^3 + al\,r^2 - k2\,r + k2\,al\right)}{r^5\,ep2}$$

$$\frac{r\left(\left(c^2\,ep2 - 1\right)r^3 + al\,r^2 - k2\,r + k2\,al\right)}{k2} \tag{12}$$

```
> wPhi:
subs(k2=K2*ep2, %):
limit(%, ep2=infinity):
lwPhi:=%;
```

$$lwPhi := \frac{(-r + al)^2\left(c^2\,r^3 + K2\,al - K2\,r\right)}{r^5} \tag{13}$$

```
> wPsi:
  subs(k2=K2*ep2, %):
  limit(%, ep2=infinity):
  lwPsi:=%;
```

$$lwPsi := \frac{r\left(c^2 r^3 + K2\, al - K2\, r\right)}{K2} \tag{14}$$

```
> # now we need to slow down and put Weierstrass in a smart form
  # so that later we can analytically integrate Weisrstrass
  equations.
  # essentially we want to factorize it in first order polynomials
>
  # Remember that:  -c < ep < 0

> numer(wPsi)/r:
  collect(%, r):
  P:=%;
```

$$P := \left(c^2\, ep2 - 1\right) r^3 + al\, r^2 - k2\, r + k2\, al \tag{15}$$

```
> P + (ep2*c^2-1)*(rp-r)*(r-rm)*(r-r0):
  collect(%, r):
  [subs(r=0, %), subs(r=0, diff(%, r)), subs(r=0, diff(%, r, r))]:
  simplify(%):
  solve(%, [ep2, k2, r0]):
  op(%):
  Sol1 := %;
```

$$Sol1 := \left[ep2 = -\frac{al^2\, rm + al^2\, rp - al\, rm^2 - 2\, al\, rm\, rp - al\, rp^2 + rm^2\, rp + rm\, rp^2}{c^2\left(al\, rm^2 + al\, rm\, rp + al\, rp^2 - rm^2\, rp - rm\, rp^2\right)}, k2 = -\frac{al\, rp^2\, rm^2}{al\, rm^2 + al\, rm\, rp + al\, rp^2 - rm^2\, rp - rm\, rp^2}, r0 = -\frac{al\, rm\, rp}{al\, rm + al\, rp - rm\, rp}\right] \tag{16}$$

```
> wPhi:
  subs(Sol1, %):
  simplify(%):
  mPhi := %;
```

$$mPhi := \frac{\left(\left(\left(r + rp\right) rm + r\, rp\right) al - r\, rm\, rp\right)\left(r - al\right)^2 c^2 \left(r - rm\right)\left(r - rp\right) al}{\left(rm + rp\right)\left(al - rp\right)\left(al - rm\right) r^5} \tag{17}$$

```
> wPsi:
  subs(Sol1, %):
  simplify(%):
  mPsi := %;
```

$$mPsi := \frac{\left(r - rm\right) r \left(r - rp\right)\left(\left(\left(al - rp\right) rm + al\, rp\right) r + al\, rm\, rp\right)}{rp^2\, rm^2} \tag{18}$$

```
> A*c^2/Phi - 1/A - r^2/Psi:
  sqrt(%)/c:
  %;
  subs([Phi=mPhi, Psi=mPsi, A=Av], %):
```

```
simplify(%) assuming c=1, al=1, rm=10, rp=30, r=20:
radsimp(%) assuming c=1, al=1, rm=10, rp=30, r=20:
factor(%):
dtau := %;
```

$$\frac{\sqrt{\frac{A\,c^2}{\Phi} - \frac{1}{A} - \frac{r^2}{\Psi}}}{c}$$

$$dtau := \frac{r^{3/2}\sqrt{-al\,rm^2 - al\,rm\,rp - al\,rp^2 + rm^2\,rp + rm\,rp^2}}{\sqrt{al}\,\sqrt{-al\,r\,rm - al\,r\,rp - al\,rm\,rp + r\,rm\,rp}\,\sqrt{r - rm}\,\sqrt{rp - r}\,c} \tag{19}$$

```
> r0:
subs(Sol1, %):
subs([rm=3, rp=5, c=1, al=1], %):
r0v:=%;
```

$$r0v := \frac{15}{7} \tag{20}$$

```
> Sol1:
subs([rm=3, rp=5], %):
subs(%, P):
subs([c=1, al=1], %):
%;


plot(%, r=-5..7,
color=blue,
thickness=2,
view=[-2..7, -1/4..1]):

p1:=%:
```

$$-\frac{7}{71}\,r^3 + r^2 - \frac{225}{71}\,r + \frac{225}{71} \tag{21}$$

```
> r0-rm:
subs(Sol1, %):
subs([c=1, al=1], %):
simplify(%):
solve(%, rm):
rmv:=[%][2];
```

$$rmv := \frac{2\,rp}{rp - 1} \tag{22}$$

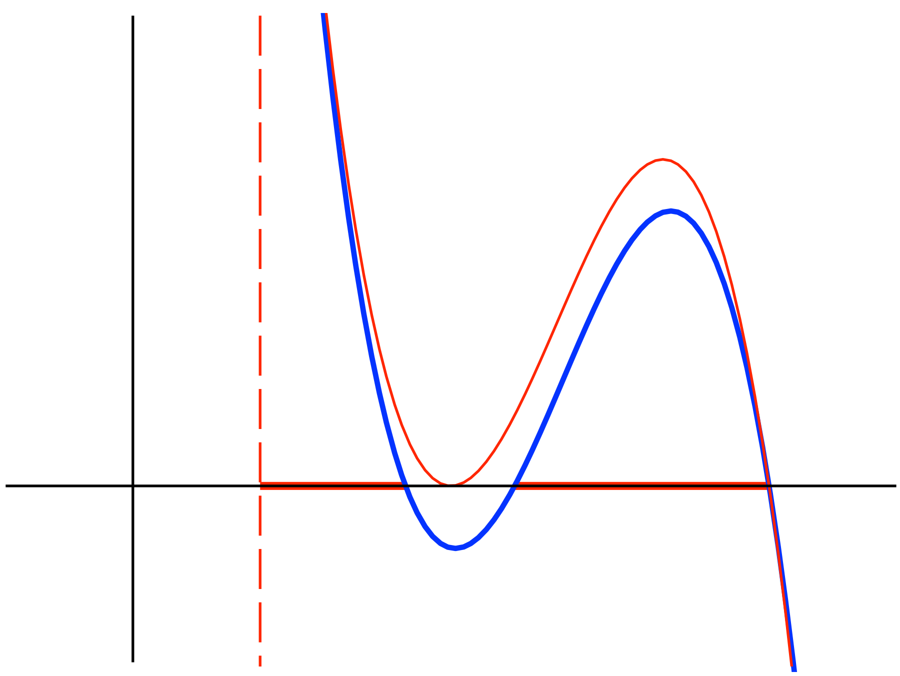
```
> P:
subs(Sol1, %):
subs([rm=rmv], %):
```

```
subs([c=1, al=1], %):
subs([rp=5], %):
%;

plot(%, r=-5..7,
color=red,
thickness=0.5,
view=[-2..7, -1/4..1]):

pt:=%:
```

$$-\frac{1}{10}r^3 + r^2 - \frac{25}{8}r + \frac{25}{8} \quad (23)$$

```
> pal:=plot([1, s, s=-250..250], linestyle=dash, thickness=0.5,
color=red):
A1:=plot([s, 0, s=1..r0v], thickness=3, color=red):
A2:=plot([s, 0, s=3..5], thickness=3, color=red):
```

```
> display(pal, A1, A2,
p1, pt,
view=[-1..6, -1/8..1/3],
tickmarks=[[], []],
labels=["", ""]);
```



```
> Sol1:
subs([rm=3, rp=5, c=1, al=1], %):
```

```
%;

subs(%, [ep2, k2]):
ep2v, k2v:= op(%);
```

$$\left[ep2 = \frac{64}{71}, k2 = \frac{225}{71}, r0 = \frac{15}{7}\right]$$

$$ep2v, k2v := \frac{64}{71}, \frac{225}{71} \qquad (24)$$

```
> P:
subs([ep2=ep2v], %):
subs([c=1, al=1], %):
subs(r=5, %):
solve(%, k2):
k2v:=%;
```

$$k2v := \frac{225}{71} \qquad (25)$$

```
>
P:
subs([ep2=ep2v, k2=k2v-0.05], %):
subs([c=1, al=1], %):
%;

plot(%, r=-5..7,
color=blue,
thickness=2,
view=[-2..7, -1/4..1]):

p3:=%:
```

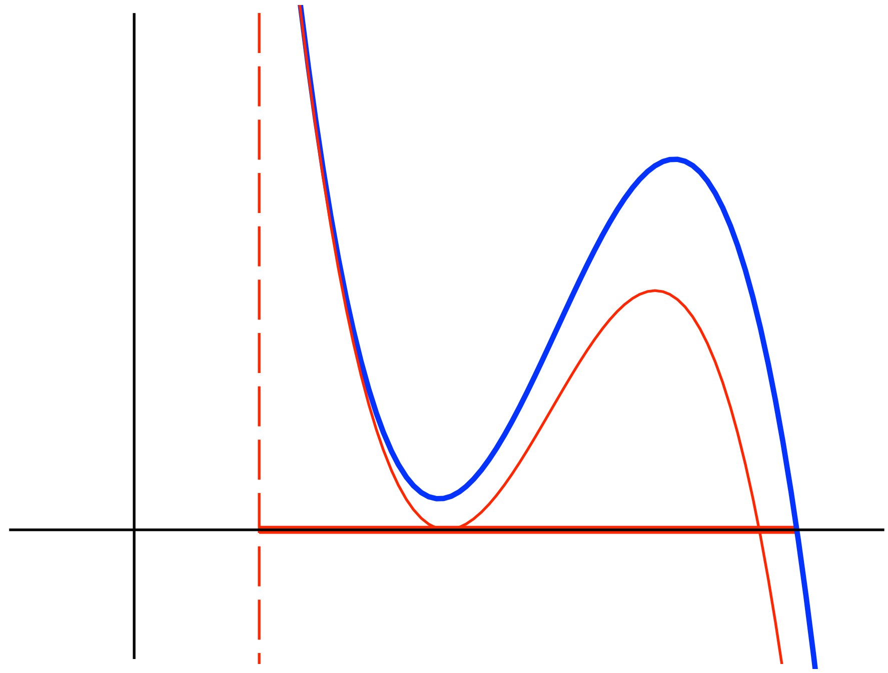
$$-\frac{7}{71} r^3 + r^2 - 3.1190140845070422535211267605633802816 90\, r + 3.1190140845070422535211267605633802816 90 \qquad (26)$$

```
> P:
subs([ep2=ep2v, k2=k2v-0.05], %):
subs([c=1, al=1], %):
solve(%, r):
rrM:=%[1];
```

$$rrM := 5.3002723359028736632670258060131408725 80 \qquad (27)$$

```
> A3:=plot([s, 0, s=1..rrM], thickness=3, color=red):
```

```
> display(pal, A3,
p3, pt,
view=[-1..6, -1/8..1/2],
tickmarks=[[], []],
labels=["", ""]);
```



```
> P:
#subs([ep2=1.2], %):
subs([c=1, al=1], %):
#subs([k2=0.5], %):
pp:=%;
```

$$pp := (ep2 - 1)\, r^3 + r^2 - k2\, r + k2 \tag{28}$$

```
> pp:
diff(%, r):
solve(%, r):
[%][2]:
rrv:=%;
```

$$rrv := -\frac{1 + \sqrt{3\, k2\, ep2 - 3\, k2 + 1}}{3\, (ep2 - 1)} \tag{29}$$

```
> pp:
subs(r=rrv, %):
#subs(ep2=2, %):
simplify(%):
solve(%, k2):
[%]:
simplify(%):
k2v:=%[2];
```

$$k2v := \frac{(9\, ep2 - 8)\sqrt{9\, ep2^2 - 8\, ep2} + 27\, ep2^2 - 36\, ep2 + 8}{8\, ep2 - 8} \tag{30}$$

```
> # ep2 > 8/9:
```

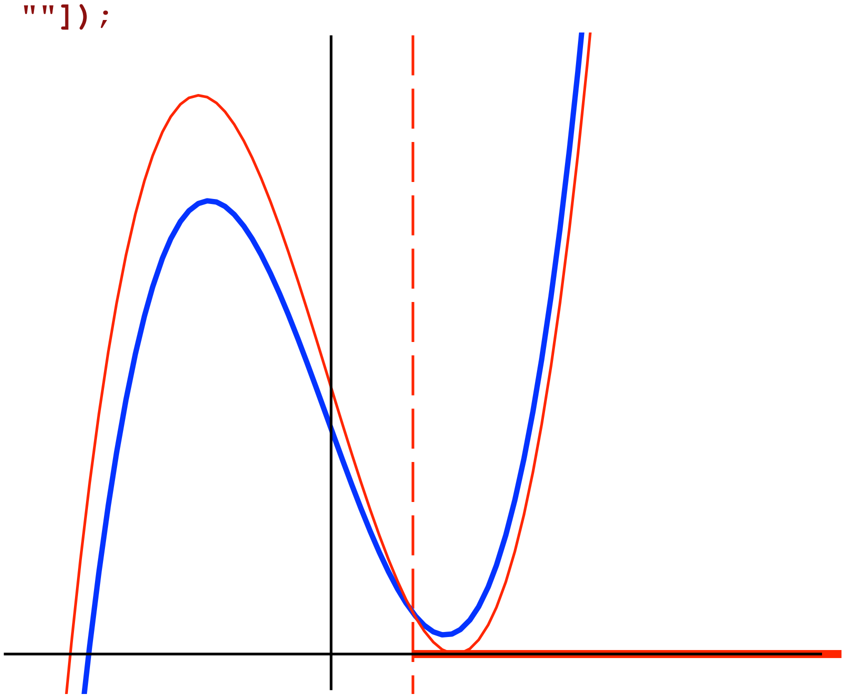
```
> pp:
subs(k2=k2v, %):
subs(ep2=8, %):
simplify(%):
```

```
plot(%, r=-15..7,
color=red,
thickness=0.5,
view=[-7..7, -7..100]):
pt:=%:
```

```
> pp:
subs(k2=k2v-8, %):
subs(ep2=8, %):
simplify(%):


plot(%, r=-15..7,
color=blue,
thickness=2,
view=[-7..7, -7..100]):
p4:=%:
```

```
> A4:=plot([s, 0, s=1..7], thickness=3, color=red):
```

```
> display(pal, A4,
p4,
pt,
view=[-4..6, -7..120],
tickmarks=[[], []],
labels=["", ""]);
```



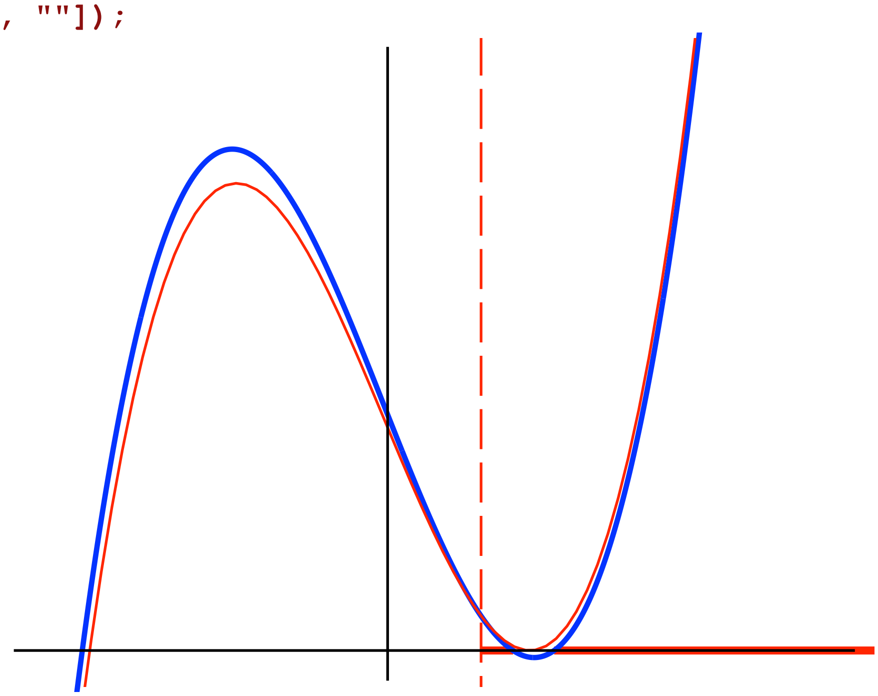
```
> pp:
subs(k2=k2v+3, %):
subs(ep2=8, %):
simplify(%):
solve(%, r):
[%]:
map(Re, %):
evalf(%):
rr2v, rr0v, rr1v:= op(%);
```

$$rr2v, rr0v, rr1v := 1.7821107765117252789573063697165751622 70, -3.2673896934681065278764935148818271772 83, 1.3424217740992383917763300023081091578 71 \tag{31}$$

```
> pp:
  subs(k2=k2v+3, %):
  subs(ep2=8, %):
  simplify(%):


  plot(%, r=-15..7,
  color=blue,
  thickness=2,
  numpoints=2000,
  view=[-7..7, -7..100]):
  p5:=%:
```

```
> A5:=plot([s, 0, s=1..rr1v], thickness=3, color=red):
  A6:=plot([s, 0, s=rr2v..7], thickness=3, color=red):
```

```
> display(pal, A5, A6,
  p5,
  pt,
  view=[-4..5, -7..140],
  tickmarks=[[], []],
  labels=["", ""]);
```



```
> # First satellite
```

```
> rp0v := Surface+20900000;
  rm0v := Surface+20100000;

  t0in := -1500:
  te0in := Pi/6:
  tau0in := -1500:
```

$$rp0v := 24089000$$

$$rm0v := 23289000 \tag{32}$$

```
> Sat0 := [te0in, t0in, tau0in, rm0v, rp0v];
```

$$Sat0 := \left[\frac{\pi}{6}, -1500, -1500, 23289000, 24089000\right] \tag{33}$$

```
>
  # for each satellite we can analytically integrate Weierstrass
  equations
  # This is painfully detailed and sensitive.
  # Small changes end up with Maple find worse representations of the
  integrals.
```

```
> te0in:
  evalf(%);

  t0in:
  evalf(%);

  tau0in:
  evalf(%);


  rp0v:
  evalf(%);

  rm0v:
  evalf(%);
```

$$0.52359877559829887307710723054658381403\,29$$

$$-1500.$$

$$-1500.$$

$$2.4089000 \times 10^{7}$$

$$2.3289000 \times 10^{7} \tag{34}$$

```
> ComputeSat0:= proc()
   global te0in, t0in, tau0in, rm0v, rp0v;
   global t0r, te0r, tau0r, Te0, T0, Tau0;

   mPhi:
   1/%:
   simplify(%)assuming c=cv, al=alv, rm=rm0v, rp=rp0v, r=(rm0v+rp0v)
  /2:
   sqrt(%):
   radsimp(%) assuming c=cv, al=alv, rm=rm0v, rp=rp0v, r=(rm0v+rp0v)
  /2:
   #%;
   subs(r=R, %):
   subs([c=cv, al=alv, rm=rm0v, rp=rp0v], %):
   int(%, R=rm0v..r) assuming  r>rm0v, r<rp0v:
   simplify(%) assuming r>rm0v, r<rp0v:
   radsimp(%) assuming r>rm0v, r<rp0v:
   t0r:= %:

   mPsi:
   1/%:
   simplify(%) assuming c=cv, al=alv, rm=rm0v, rp=rp0v, r=(rm0v+rp0v)
  /2:
   sqrt(%):
   radsimp(%) assuming c=cv, al=alv, rm=rm0v, rp=rp0v, r=(rm0v+rp0v)
  /2:
   #%;
   subs(r=R, %):
   subs([c=cv, al=alv, rm=rm0v, rp=rp0v], %):
   int(%, R=rm0v..r) assuming  r>rm0v, r<rp0v :
   simplify(%) assuming r>rm0v, r<rp0v:
   radsimp(%) assuming r>rm0v, r<rp0v:
   te0r := %:

   dtau :
   subs(r=R, %):
   subs([c=cv, al=alv, rm=rm0v, rp=rp0v], %):
   int(%, R=rm0v..r) assuming  r>rm0v, r<rp0v :
   simplify(%) assuming r>rm0v, r<rp0v:
   radsimp(%) assuming r>rm0v, r<rp0v:
   tau0r := %:

   te0r:
   subs(r=rp0v, %):
   simplify(%):
   Te0:=%:

   t0r:
   subs(r=rp0v, %):
   simplify(%):
   T0:=%:

   tau0r:
   subs(r=rp0v, %):
   simplify(%):
   Tau0:=%:
  end:
```

```
> ComputeSat0():
```

```
>
  # (half-)periods
```

```
>
  evalf(2*T0);

  #  578.22345916140625853458272916314050357 85
  #
  578.223459161406258534582729163140503578589853184028782842424361455
  32288544239669
```

$$36284.717489554504372118273396682444357 32 \quad (35)$$

```
>
  evalf(2*(Te0-Pi)); # Precession (in radiant)

  #   0.85542692730438791614573939149815814 1514
  #
  0.855426927304387916145739391498158141510167447626392396926333 89527
  82391051596014
```

$$3.5301183195832440137091826287 32 \times 10^{-9} \quad (36)$$

```
>
  evalf(2*Tau0);

  #    548.26851078993541052961540893718234388 56
  #
  548.268510789935410529615408937182343885925910474622853044502085646
  92352991626784

  evalf(Tau0/T0); # Slowing time factor

  #    0.948194858065920888428474165398000898 3505
  #
  0.948194858065920888428474165398000898351063614546358766252243 47352
  747293450158832
```

$$36284.717479364384326048917040604600152 66$$
$$0.99999999971916220516246691698241163504 36 \quad (37)$$

```
> cv;
  Surface;
```

$$299792458$$
$$3189000 \quad (38)$$

```
>
  # Second satellite
```

```
> rp1v := Surface+21900000;
```

```
rm1v := Surface+21100000;

t1in := -1600:
te1in := -Pi/6:
tau1in := -1600:
```

$$rp1v := 25089000$$

$$rm1v := 24289000 \tag{39}$$

```
> Sat1 := [te1in, t1in, tau1in, rm1v, rp1v];
```

$$Sat1 := \left[-\frac{\pi}{6}, -1600, -1600, 24289000, 25089000 \right] \tag{40}$$

```
> ComputeSat1:= proc()
  global te1in, t1in, tau1in, rm1v, rp1v;
  global t1r, te1r, tau1r, Te1, T1, Tau1;

  mPhi:
  1/%:
  simplify(%)assuming c=cv, al=alv, rm=rm1v, rp=rp1v, r=(rm1v+rp1v)
 /2:
  sqrt(%):
  radsimp(%) assuming c=cv, al=alv, rm=rm1v, rp=rp1v, r=(rm1v+rp1v)
 /2:
  #%;
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm1v, rp=rp1v], %):
  int(%, R=rm1v..r) assuming  r>rm1v, r<rp1v:
  simplify(%) assuming r>rm1v, r<rp1v:
  radsimp(%) assuming r>rm1v, r<rp1v:
  t1r:= %:

  mPsi:
  1/%:
  simplify(%) assuming c=cv, al=alv, rm=rm1v, rp=rp1v, r=(rm1v+rp1v)
 /2:
  sqrt(%):
  radsimp(%) assuming c=cv, al=alv, rm=rm1v, rp=rp1v, r=(rm1v+rp1v)
 /2:
  #%;
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm1v, rp=rp1v], %):
  int(%, R=rm1v..r) assuming  r>rm1v, r<rp1v :
  simplify(%) assuming r>rm1v, r<rp1v:
  radsimp(%) assuming r>rm1v, r<rp1v:
  te1r := %:

  dtau :
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm1v, rp=rp1v], %):
  int(%, R=rm1v..r) assuming  r>rm1v, r<rp1v :
  simplify(%) assuming r>rm1v, r<rp1v:
  radsimp(%) assuming r>rm1v, r<rp1v:
  tau1r := %:

  te1r:
```

```
  subs(r=rp1v, %):
  simplify(%):
  Te1:=%:

  t1r:
  subs(r=rp1v, %):
  simplify(%):
  T1:=%:

  tau1r:
  subs(r=rp1v, %):
  simplify(%):
  Tau1:=%:
 end:
```

```
> ComputeSat1():
```

```
> evalf(2*T1);

#    1184.261809516394362783565152572288103278
#
1184.26180951639436278356515257228810327784636341212704244297393369
22389943742940
```

$$38606.364226049104666653238610706833527 12 \quad (41)$$

```
> evalf(2*(Te1-Pi)); # Precession (in radiant)

#   0.4339365667617308623247543308139331368 73
#
0.433936566761730862324754330813933136872511908884684488128822156 39
76738953752606
```

$$3.3870582046394034734735459867 12 \times 10^{-9} \quad (42)$$

```
> evalf(2*Tau1);

#   1148.119711746172713886661882040516422037
#
1148.11971174617271388666188204051642203779309198998560051125124650
11723679785136

evalf(Tau1/T1); # Slowing time factor

#   0.969481327963298376788649673225728156380 4
#
0.969481327963298376788649673225728156380773250595058476841176891 44
014037883456925
```

$$38606.364215646126520348513774597079608 00$$

$$0.9999999997305372221690469175902316869746 \quad (43)$$

```
> 2*(T1-Tau1):
```

```
evalf(%*10^5);
```

$$1.040297814630472483610975391912 \quad \textbf{(44)}$$

```
>
# Terzo satellite
```

```
> rp2v := Surface+22900000;
rm2v := Surface+22100000;

t2in := -1400:
te2in := 0:
tau2in := -1400:
```

$$rp2v := 26089000$$
$$rm2v := 25289000 \quad \textbf{(45)}$$

```
> Sat2 := [te2in, t2in, tau2in, rm2v, rp2v];
```

$$Sat2 := [0, -1400, -1400, 25289000, 26089000] \quad \textbf{(46)}$$

```
> OrbitalParameters := [Sat0, Sat1, Sat2];
```

$$OrbitalParameters := \left[\left[\frac{\pi}{6}, -1500, -1500, 23289000, 24089000\right], \left[-\frac{\pi}{6}, -1600, -1600, 24289000, 25089000\right], [0, -1400, -1400, 25289000, 26089000]\right] \quad \textbf{(47)}$$

```
> ComputeSat2:= proc()
  global te2in, t2in, tau2in, rm2v, rp2v;
  global t2r, te2r, tau2r, Te2, T2, Tau2;

  mPhi:
  1/%:
  simplify(%)assuming c=cv, al=alv, rm=rm2v, rp=rp2v, r=(rm2v+rp2v)
 /2:
  sqrt(%):
  radsimp(%) assuming c=cv, al=alv, rm=rm2v, rp=rp2v, r=(rm2v+rp2v)
 /2:
  #%;
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm2v, rp=rp2v], %):
  int(%, R=rm2v..r) assuming  r>rm2v, r<rp2v:
  simplify(%) assuming r>rm2v, r<rp2v:
  radsimp(%) assuming r>rm2v, r<rp2v:
  t2r:= %:

  mPsi:
  1/%:
  simplify(%) assuming c=cv, al=alv, rm=rm2v, rp=rp2v, r=(rm2v+rp2v)
 /2:
  sqrt(%):
  radsimp(%) assuming c=cv, al=alv, rm=rm2v, rp=rp2v, r=(rm2v+rp2v)
 /2:
  #%;
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm2v, rp=rp2v], %):
  int(%, R=rm2v..r) assuming  r>rm2v, r<rp2v :
  simplify(%) assuming r>rm2v, r<rp2v:
```

```
  radsimp(%) assuming r>rm2v, r<rp2v:
  te2r := %:

  dtau :
  subs(r=R, %):
  subs([c=cv, al=alv, rm=rm2v, rp=rp2v], %):
  int(%, R=rm2v..r) assuming  r>rm2v, r<rp2v :
  simplify(%) assuming r>rm2v, r<rp2v:
  radsimp(%) assuming r>rm2v, r<rp2v:
  tau2r := %:

  te2r:
  subs(r=rp2v, %):
  simplify(%):
  Te2:=%:

  t2r:
  subs(r=rp2v, %):
  simplify(%):
  T2:=%:

  tau2r:
  subs(r=rp2v, %):
  simplify(%):
  Tau2:=%:
 end:
```

```
> ComputeSat2():
```

```
> evalf(2*T2);

 #   1924.315710825429852475357649680441260847
 #
 1924.315710825429852475357649680441260848135320413899256564229296504256220925541 3
```

$$40975.51797465026227484852871798838888234 \tag{48}$$

```
> evalf(2*(Te2-Pi)); # Precession (in radiant)

 #    0.294379407208569363342410862227269176774
 #
 0.294379407208569363342410862227269176772596541069278738172817412017122318820428 1
```

$$3.255144378771119121674646905539 \times 10^{-9} \tag{49}$$

```
> evalf(2*Tau2);

 #    1882.602699968465675909535143320371392380
 #
 1882.602699968465675909535143320371392386418967326478455472076312662044750594441 9
```

```
evalf(Tau2/T2); # Slowing time factor

#    0.97832319789195531901590011774974859286 19
#
0.97832319789195531901590011774974859286184191659675804133111562569
483350916747710
```

$$40975.517964038694903734917483370915 11434$$

$$0.99999999974102664479608243471530305 07972 \quad (50)$$

```
> # Again we need to put Weierstrass in a form suitable for later
integration
# even though in this case we have both scattering and infalling
rays.

numer(lwPsi)/r:
lp:=%;

subs(r=al, %):
#subs([al=alv, c=cv], %):
%;
```

$$lp := c^2 r^3 + K2\, al - K2\, r$$

$$al^3 c^2 \quad (51)$$

```
> lp:
diff(%, r):
solve(%, r):
rcrit:=[%][1];
```

$$rcrit := \frac{\sqrt{3}\,\sqrt{K2}}{3\,c} \quad (52)$$

```
> lp:
subs(r=rcrit, %):
subs(K2=K^2, %):
radsimp(%):
%/K^2:
simplify(%):
solve(%, K):
%^2:
K2crit:=%;
```

$$K2crit := \frac{27\, al^2\, c^2}{4} \quad (53)$$

```
> rcrit:
subs(K2=K2crit, %):
simplify(%):
radsimp(%) assuming al>0, c>0:
%;
```

$$\frac{3\, al}{2} \quad (54)$$

```
> lp- c^2*(r-rp)*(r-rh)*(r+rp+rh):
  collect(%, r):
  [subs(r=0, %), subs(r=0, diff(%, r)), subs(r=0, diff(%, r, r))]:
  solve(%, [K2, rh]):
  allvalues(op(%)):
  [%][1]:
  %;

  subs(%, [K2, rh]):
  K2v, rhv := op(%):

  K2v;
  rhv;
```

$$\left[K2 = -\frac{rp^3\,c^2}{al - rp},\, rh = \frac{\left(-al + rp + \sqrt{-3\,al^2 + 2\,al\,rp + rp^2}\right) rp}{2\,(al - rp)}\right]$$

$$-\frac{rp^3\,c^2}{al - rp}$$

$$\frac{\left(-al + rp + \sqrt{-3\,al^2 + 2\,al\,rp + rp^2}\right) rp}{2\,(al - rp)} \quad \textbf{(55)}$$

```
> lp:
  diff(%, r):
  solve(%, r):
  rv:=[%][1];

  lp:
  subs(r=rv, %):
  solve(%, K2):
  K2v:=[%][2];
```

$$rv := \frac{\sqrt{3}\,\sqrt{K2}}{3\,c}$$

$$K2v := \frac{27\,al^2\,c^2}{4} \quad \textbf{(56)}$$

```
> lp:
  subs([K2=K2v], %):
  subs([c=1, al=1], %):
  %;
  plot(%, r=-7..7,
  color=red,
  thickness=0.5,
  view=[-4..7, -10..20]
  ):
  pt:=%:
```

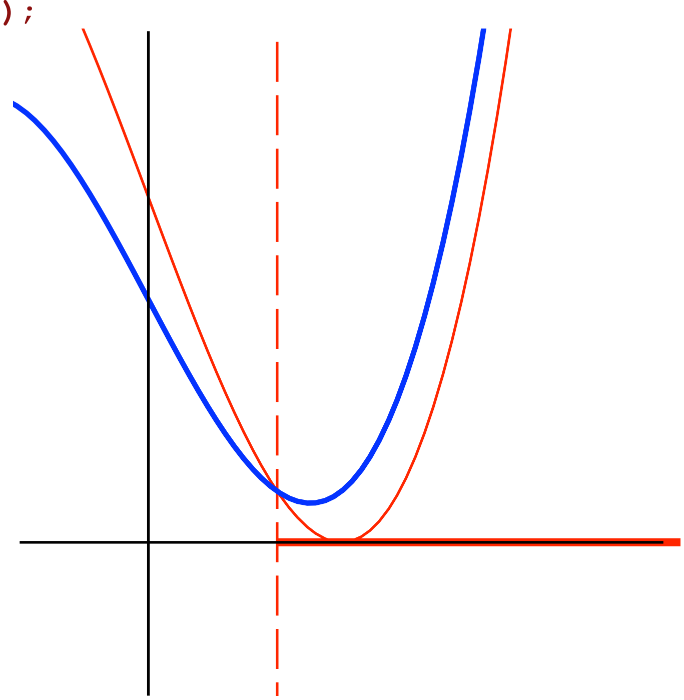
$$\frac{27}{4} - \frac{27}{4}\,r + r^3 \quad \textbf{(57)}$$

```
> A1:=plot([s, 0, s=1..7], thickness=3, color=red):
```

```
> lp:
  subs(K2=K2v-2, %):
  subs([c=1, al=1], %):
  %;
  plot(%, r=-7..7,
  color=blue,
  thickness=2,
  view=[-4..7, -10..20]
  ):
  p1:=%:
```

$$r^3 + \frac{19}{4} - \frac{19}{4} r \quad (58)$$

```
> display(pal, A1,
  pt, p1,
  view=[-1..4, -3..10],
  tickmarks=[[], []],
  labels=["", ""]);
```



```
> lp:
  subs(K2=K2v+1, %):
  subs([c=1, al=1], %):
  solve(%, r):
  [%]:
  evalf(%):
  map(Re, %):
  rr2v, rr0v, rr1v := op(%);
```

$rr2v, rr0v, rr1v :=$ 1.93533708725397031009224104555028320 1789, −3.19047145189944118670562771206716062 7054, 1.25513436464547087661338666651687742 5266 (59)

```
> lp:
```

```
subs(K2=K2v+1, %):
subs([c=1, al=1], %):
plot(%, r=-7..7,
color=blue,
thickness=2,
view=[-4..7, -10..20]
):
p3:=%:
```

```
> A2:=plot([s, 0, s=1..rr1v], thickness=3, color=red):
  A3:=plot([s, 0, s=rr2v..7], thickness=3, color=red):
```

```
> display(pal, A2, A3,
  pt, p3,
  view=[-1..4, -3..10],
  tickmarks=[[], []],
  labels=["", ""]);
```