Relativistic Theories, Gravitational Theories, and General Relativity.

A Relativistic *zibaldone*.

by *L. Fatibene*

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Dedication

This book is dedicated to my students and their unlimited patience and stubbornness.

It is also dedicated to my mentors, all of them.

I hope it will deserve sympathy of the first ones and be a small source of proud for the latter.

Preface

The theory which is commonly known as General Relativity is at least three different things.

First of all, it is a paradigm (in the sense of T.Khun; see[1]) about which is the structure of a physical theory in order to allow a meaningful description of physical reality and its relation with observers. It is a definition of what is *absolute knowledge* in physics together with the claim that physical theories could and should be described in such an absolute fashion.

Second, it is a specific implementation of this paradigm which deals essentially with describing *when* and *where* events happen, especially in presence of a gravitational field. As such, it is essentially a theory of spacetime.

Third, it is a specific dynamical theory of gravitation formulated by Einstein and Hilbert which provides an implementation of these points and which is our current understanding of gravitational physics.

The title of this work refers to these three aspects of General Relativity (GR).

Relativistic theories refer to the first aspect. It deals with observers and their laboratories and how the description of one observer is mapped into the descriptions of other observers so that they altogether can provide an *absolute* description of physical reality. Relativistic theories deal with general covariance principle and how dynamics can be formulated to implement such a principle. It is also based on *background freedom*, in the sense that all structures on spacetime are dynamical, they need to be determined as solutions of field equations.

A theory which implements general covariance principle is called a *natural theory*. We shall also give an extension of such a principle called *gauge-natural covariance principle*. These allow more general symmetries than in natural theories, which are used for gauge theories but also for frame formalism and spinor fields.

Gravitational theories refer to a general dynamical description of gravity which is identified with the geometry of spacetime. These deal with gravitational field, let us say with *possible* models of gravitational field. Dynamics are more general than standard GR. We believe that we currently need at least to discuss which is the most suitable description of the physical gravitational field, despite the great successes produced in the last century by standard GR.

General Relativity refers to a specific gravitational theory, formulated by Einstein and Hilbert which has been used to describe astrophysical and cosmological observations. Usually, hereafter we refer to this specific theory as *standard GR*.

Before starting, a special discussion is also needed for the relation with Special Relativity (SR). Of course, historically speaking, SR came 10 years before GR. Hence Einstein used SR and its world description to elaborate standard GR as well as a motivation. Moreover, SR is definitely better and more generally known, tested and used by physics community than GR. Also for these reasons, often SR is assumed as a foundation for GR; for example, the dynamics of material points in GR is obtained via equivalence principle, which loosely speaking can be formulated by claiming the existence of a class of observers for which SR (approximately and locally) holds true.

However, logically speaking, obviously GR is a more fundamental theory than SR. We believe it would be important to have a framework to introduce GR as much independently of SR as possible. SR should be then obtained as a limit case of GR as a solution of "no gravitational field". Of course, that is not really and exactly SR.

This is not something peculiar of relativity and it happens basically every time a theory is obtained as a limit of a more fundamental one; see[1]. When one obtains Newtonian physics as a low speed limit of SR (or as a formal limit for the speed of light which goes to infinity, whatever that means) one can recover most equations of

Newtonian physics. However, one should be aware that, in this limit theory, absolute time is *approximately* absolute, that most of the forces used in Newtonian fields has no counterpart in SR to be obtained easily, that *space* and *time* have a different meaning with respect to the usual Newtonian framework. Similarly, the classical limit of quantum mechanics is quite well known; however, the meaning of position, velocity and trajectory in such a limit is *approximately* classical.

We shall try to find a logical path to GR which is independent of SR and then obtain SR as a special solution of standard GR with no gravitational field (or weak field approximation).

A second issue is about the role of Mathematics in this work and more generally in Physics. The standard attitude is to regard mathematics as a support for physical intuition. Mathematics is considered a language for expressing and making precise the physical intuition. Accordingly, one develops a (sometimes minimal or simplified) mathematical framework in which one can represent the features of physical systems and predict their behaviour in paradigmatic situations. Paradigmatic predictions are then tested and become the standard for extending the application area of the theory.

We do not share this view for different reasons. First reason is that the mathematics behind GR is relatively old and it is not impossible for students to approach it in its full generality, if not in its full depth and detail. With relatively little effort, one can get a more general and abstract view which sometimes highlights connections and bridges among subjects which seem at first unconnected. Moreover, the things that one needs to know about GR is growing massively and it is difficult to cover it, especially without a basic language which is sufficiently abstract to get the essential issues and delegate the details to computations and details filling which can be long and complex but still standardised.

Second reason is that we believe that it is physical intuition which has to be build on computations, not vice versa. In physics, one studies a physical system. To do that, one sets up a framework, a mathematical framework, which encodes the physical aspects of the system under consideration and defines how the mathematical system behaves in different situations. This is done, e.g., by specifying a dynamics for the system. Then one has a mathematical counterpart of the physical system and can check and test if the two representations behave the same way. On the physical hand side, one has experiments and observations which are something we are not allowed to question. On the mathematical hand side, one needs something similar. Something we cannot bend at will, which is somehow determined when we defined the framework. In other words, we need the mathematical framework to be developed enough so that the behaviour of objects is something out of our control.

As a third reason, the relation between language and intuition is a bootstrap process. Not only new physical insights produce their mathematical representations, but also, the other way around, mathematics defines new physical intuition. There is a big deal of the current mind representation of quantum and gravitational world which is just a translation of the mathematical framework used to describe it. This is not my idea. I believe that that what was meant by Feynman or von Neumann when they said:

You don't understand quantum mechanics, you just get used to it.

We believe they meant that intuition is not there, it must be grown and trained, which is another way to say that it must be built out of the mathematical framework which is the only thing that must be preserved. Now, one should know the mathematical framework in order to be able to compute basically anything, not just what we need to confirm mathematically what we already grasp by our intuition.

All these reasons are motivations to develop somehow standard mathematical tools to a level which is probably annoyingly detailed for standard physics. In order to keep it self contained, we collected all mathematical background material in the fourth Part. Some of the background material

in Physics (standard classical field theories, variational calculus, conservation laws) is collected in Chapter 1. Keeping these parts separate from the relativistic stuff allows the reader to skip them or part of them, or just review notation.

The material is organised as follows.

Part 4 contains a *brief* review (just about 230 pages) of the mathematical *common law* we should share. It goes from basic algebraic constructions (from quotient sets to category theory) to topological spaces (with a focus on topological manifolds), from smooth manifolds (from vector fields to Frobenius theorem) to tensor fields and differential forms, from structures on manifolds (metric, connections, curvature, and symplectic structures—which are essential for Hamiltonian formalism) to Lie groups and Lie algebras (with a brief focus on classification of representations). Finally, we discuss fiber bundles (which are so extensively used in field theories that for a number of years I believed they were invented for that) and a Chapter on natural and gauge-natural bundles which are the mathematical counterpart of generally covariant theories and gauge theories, respectively.

Unfortunately, it contains a good portion of all mathematics we know. This should not be a problem for mathematicians; it just exhibits the connections among different parts of mathematics and mathematical physics. Physicists might be annoyed; they would probably ask whether do we really need all this stuff to deal with GR? The answer is naturally, no.

There are plenty of books about GR which develop a somehow minimal mathematical framework, in which mathematical concepts are introduced under a strong physical motivation, often providing a beautiful intuitive representation of the most important concepts. However, I believe that the concept of connection is there since more than a century ago, that we do not really need to rediscover it from a physical viewpoint, especially if by doing so we obtain a minimal, utilitarian view on it which in principle could even hide relations which may be relevant in the future. I believe it is time to leap ahead, using what we have to set on a firm foundation.

On the other hand, I believe that, having some spare time, a physicist may appreciate the relatively short tour through modern mathematics, considering relativistic theories as a motivation and regarding physics as a tool to show connections between different areas in mathematics.

Chapter 1 also collects background material (for just about 80 pages more). It deals with basic background in classical Lagrangian field theory. Any student should be able to take a Lagrangian, compute field equations, check whether a transformation is a symmetry and compute the Noether current associated to it. By doing this review, we introduce the bundle framework for variational calculus, we discuss globality and transformation rules of mathematical structures and bridge them to their physical meaning. Some of the material of Chapter 1 is then used to discuss in the following Chapters some important issues and features of relativistic theories.

Without a precise definition of Lagrangian symmetry, one could not even define what a relativistic theory is. Without a good grasp on geometry and bundle framework, one could not prove that Noether currents of relativistic theories are necessary exact (i.e. the existence of superpotentials), and discussing under which conditions two field theories are dynamically equivalent (and how one can find out when a theory is just a different theory in disguise) would be impossible without a clear understanding of variational calculus from a geometric perspective.

The rest of Part I, deals with the main features of relativistic theories. I think it is important to realise that *relativistic theories* are somehow *pre-physics*. They are more about how a physical theory should be structured and how to extract information from it, than speaking of a particular theory and its physical meaning.

Chapter 2 is devoted to discuss covariance (general covariance and gauge covariance) and its relation to globality on one hand, and to relativity principles on the other. An important technical issue is determining which dynamics are compatible with covariance, depending on fields allowed. These are the so-called Utiyama-like theorems. Usually, they go as saying that:

> – *if the Lagrangian depends on these fields and their derivatives up to a specific order, then the Lagrangian must be necessarily in this form and obeying these identities.*

Utiyama-like theorems are important also because they are a tool to obtain general properties for covariant theories which then allow to study them in full generality. We shall also set up some basic notion (passive vs. active transformations, initial value problems, hole argument, physical state) which will be useful later for discussions. Finally, we shall consider some examples of field theories in a deeper perspective: Hilbert–Einstein standard GR will be considered and defined as a generally covariant theory and extended to describe interactions with electromagnetism and Klein–Gordon matter fields. We shall also introduce Yang–Mills theories as an extension of Maxwell electromagnetism (which is based on—and entangled with—the interpretation of the electromagnetic potential as a principal connection) as well as *Vielbein* formalism for standard GR and its coupling with spinor fields. We also consider the so-called *conformal gravity*, which is interesting as a relativistic theory because it has extra symmetries (Weyl conformal transformations). Conformal gravity may or may not be a reasonable physical proposal for describing the gravitational field; it is, however, certainly important to understand what is observable in relativistic theories and how that relates to its dynamics.

In Chapter 3 we consider conservation laws in covariant theories. We have to apply the basic tools introduced in Chapter 1 to the specific case of relativistic theories. General and gauge covariance rely on a huge group of symmetries and, because of that, their conservation laws gain more structure. It is important to be technically able to extract such information from a relativistic theories, which is not trivial, and it is the basis for later being able to interpret the physical meaning of these quantities. We shall also consider conservation laws for standard GR, Yang–Mills theories, Chern–Simons and spinor theories, as well as conserved quantities for some solution of Einstein equations.

There is a decades–long argument about whether and how physical quantities such as energy or angular–momentum relate (if at all) to covariant conserved quantities. This is a specific issue in which one really sees how the tradition from SR may turn out to be misleading. In SR one has fixed geometric structures and some terms in conservation laws vanish or not for reasons which do not extend easily to GR. One needs a new perspective, starting from scratch, to deal with GR which then specialises to SR, partially recovering the SR theory, in part giving a new perspective on it.

There are many frameworks, some very popular, for extracting physical quantities from a field theory. Some of them (e.g. *ADM* conserved quantities framework) use extra structures, which are often considered essential for a spacetime to make physical sense. In many ways, these additional structures are equivalent to a Hamiltonian formalism and inspired by a post Newtonian-like attitude. However, these constraints are clearly not dictated by the dynamics, but they are added as *ad hoc* requests. Of course, they may be reasonable and even necessary on the physical stance. However, I think it is interesting to check how far one can go without them, taking general covariance seriously. In a general covariant theory, conservation laws are a tool to extract geometric information which is *absolute*. If one believes that that physics must be absolutely described, then physical quantities must be found, or at least searched, in covariant conservation laws, or doomed to be observer dependent. In any event, one can go pretty far preserving covariance, and recovered extra structures later, on an observer–dependent basis, sometimes in a more general context. I think this shows that extra structures may be just an artefact of our Newtonian background, not really necessary to describe the physical world.

Chapter 5 is about dynamical equivalence between field theories. Two field theories are dynamically equivalent if one has a one-to-one correspondence between solutions. That means that solving either of the theories is equivalent to solve the other. That is similar to what happens with canonical

transformations in mechanics. In view of later applications, one has also to discuss to what extent a dynamical equivalence extends to a complete physical equivalence.

In general, proving dynamical equivalence between two different theories is pretty easy if one knows which transformation to use. If not, it is pretty hard and, sometimes, one still discovers equivalence between very well-known theories which were believed to be inequivalent. We present the case of standard purely metric and metric-affine GR which are usually considered generically inequivalent (though in some cases, depending on the details of matter coupling, they are known to be equivalent). We show that, by designing the transformation properly, one can extend the dynamical equivalence to be generally true. That is a theorem, so no discussion, and it does not come out of the blue. It comes with an extension of Routh transformations from mechanics to field theories. Routh transform may not be the most general mechanism to produce equivalence between theories, though all known cases can be recovered that way. We shall need to discuss dynamical equivalence between different theories in Part II to connect different theories, aiming to reduce the theories to discuss.

In Part II we try to discuss the most general relativistic theory to describe gravitational field. We keep what we learnt from Einstein that the geometry of spacetime is dynamical (it is determined by the matter content as well as it determines the evolution of matter fields) and there is a general identity between geometry of spacetime and the gravitational field. The gravitational field *is* a dynamical theory for the geometry of spacetime and matter. We do not necessarily buy what Einstein was forced to assume in view of the limited knowledge of what a geometry should be. When GR had being developed, *geometry on a manifold* meant essentially a *metric field*. Following Minkowski's work in SR, Lorentzian metrics were considered, but Levi Civita and Ricci were at the same time developing a modern and satisfactory theory for connections. Today we know quite a bit more about what a geometry can be in general, so maybe it is worth starting over from scratch.

In Chapter 6 we review Ehlers–Pirani–Schild (EPS) axiomatic framework for spacetime geometry. It is a wonderful piece of research, purely and beautifully mathematical physics in nature, which has not originally obtained the attention it deserves. They started the project for showing that they can prove that the geometry of spacetime was a Lorentzian geometry, instead of assuming it. They arrived surprisingly closed to their goal, leaving though some extra freedom.

Our review will differ from the original for essentially one issue: we do not base the motion of particles on equivalence principle but we prove it based essentially on covariance. By doing that, we give a new definition of gravitational field (which, to the best of our knowledge, is new as a definition, although it is based on known properties of the gravitational field) as the interaction which is left once every other interaction which can be switched off in an absolute way is gone. This is a good hint about how different gravity is from the other interactions, something of which we have endless evidences here and there (for example, gravitational field has no linear or affine structure as any other known physical fundamental field).

Chapter 7 is devoted to discuss *extended theories of gravitation*, i.e. gravitational theories in which the dynamics is compatible to EPS analysis. In particular, we consider Palatini $f(\mathcal{R})$-theories which are quite generally extended theories. We briefly review some of the typical features of these theories and we shall also discuss a number of dynamically equivalent formalisms for Palatini $f(\mathcal{R})$-theories which are useful to discuss the interpretation of the theory and compare with observations in astrophysics and cosmology.

In Chapter 8, we consider Palatini $f(\mathcal{R})$-cosmologies, i.e. exact solutions which happen to be spatially homogeneous and isotropic. We prefer there to use exact methods over approximations (even when the approximation is well supported and reasonable). Once again, I think one should *define* a good approximation as a situation in which the exact solution and the approximated one are closed together, so it is not particularly wise to base the

general theory on approximations. The motivation for approximations is physical intuition, which is what we would like to support, not something we are born with. Moreover, the exact methods are often not too complicated to be applied.

Also we shall define what *homogeneous and isotropic* means, in an intrinsic way. That is a bit long, though a good example of discussion of global issues and their relation with local physics.

Part III is devoted to generalize (or, better, recover as a special case) the observational protocols of standard GR to the more general geometries described by Weyl frames. This is, on one hand, the basis for being in principle able to observe deviations from standard GR, on the other it provides a better foundation for observations, even if eventually standard GR will go through with no modification as the description of physical gravitational field.

Going through Part I and II, one for sure has the impression of a theory which describes something quite far from our physical experience. Everything is beautifully geometric though our physical world is described as a 4 dimensional spacetime, with no preferred notion of time, no preferred notion of coordinates at all, no affine structure, no spatial objects (force, physical accelerations, physical velocities). This picture is very far from our previous, Newtonian, description of the universe and, honestly, it appears difficult to reconcile with everyday experience as well as with the way we perform scientific observations and experiments. Imagine to ask someone to send a probe to Mars and back without ever referring to spatial velocities, accelerations, or forces (as well as never mentioning rigid bodies and giving a covariant account of chemical forces to describe how fuel burns, or the interaction between the probe and cosmic radiation).

It is common experience to think that although general covariance is beautiful, it needs to be broken eventually to reconcile relativistic theories to physical worldview. Well, in Part III, we shall discuss how this is done, as well as discuss how this procedure is conventional and how it relates back with a covariant world.

The main idea is that the gravitational field, as any other field, is observed through its action on test particles. Hence every observation, should be reduced to observing the worldlines of test particles, i.e. geodesics. In a relativistic theory there are no fundamental preferred time or space, which are not absolute. On the other hand, we have preferred physical devices, namely atomic clocks, which define a time, which may be conventional, still it is what is used in physic practice.

And atomic clocks do not only *conventionally* agree with an arbitrary time we appear to use in our everyday experience. They also have proven to provide a framework — that we call SR — which simply renders and accounts for quantum properties of matter.

In standard GR, one assumes that the time measured by atomic clocks is the same time measured by the metric which describes free fall. EPS analysis shows how this assumption is not forced, even maintaining a pretty conservative attitude. For that reason, we would like to extend observational protocols to (at least integrable) Weyl frames, which capture this freedom and in which atomic time is not assumed to be the gravitational time.

Then one would like to extend all protocols of standard GR, which are given in a Riemannian structure, to a more general Weyl frame, from synchronisation protocols to definition of distances, from cosmological parameters to astronomical distances.

There is also a more subtle reason to *"extend"* observational protocol to Weyl geometries. As far as gravitational theories are proposed as implementations of relativistic theories, i.e. as fundamental theories, they cannot rely on other less fundamental theories to account for observations or whatever. They need to account for how a naive observer can set up its protocols and descrive physical reality around it, before it knows any simpler,

non-relativistic, approximated theory or before assuming some specific dynamics such as standard GR, which eventually force a simplification of geometry from a Weyl frame to a Riemannian structure. Accordingly, the general protocols can (and should) specialise to standard GR, SR, and Newton's physics. In principle though, they should not be defined relying on them.

The observational protocols depend on different conventional structures and hence are easily classified in terms of the additional structures they depend upon.

In Chapter 9 we discuss geodesics, which are covariant and do not depend on additional structures. They are the basis on which the other structures are built. We also discuss special coordinate systems which implement equivalence principle (or we should say *equivalence principles* since we have different formulations).

Chapter 10 we discuss clocks, standard clocks, proper clocks. They implements the notion of *uniform clock* given a Weyl frame. Then we discuss synchronisation of clocks at a distance, as well as the relation between conformal factor and synchronisation. Clock synchronisation in turn defines spatial distances, which come in a number of versions depending on the details (and, pratically, on scales). We also discuss distances in cosmology and astrophysics, which are different from the ones used locally on Earth since one cannot effort to wait a signal to come back from another galaxy. Finally, we shall also discuss relativistic positioning systems which are a standard way to define observable in a relativistic theory and can accordingly be used to compare gravitational theories and test one against the others.

This work is a work in progress. I suppose I'll keep adding to the original core more and more topics.

By now, this material is free to use.

I'd also like to keep track of who is using it and how, so please let me know any comment (to `my.gr.book@gmail.com`).

Let me end this Preface setting some rules about versioning.

This book comes with a version number, its *current version*, denoted in the format: `version:subversion:release`. For example, the original version is `1:0:0`, which is the first edition of the minimal core. Before that version, some of the material appeared as stand alone Chapters, though version `1:0:0` was the first edition where the material was collected together.

Version is increased on every major rearrangement. Usually, to increase the version number we need to add a Chapter. Subversions are issued for adding Sections. Releases are usually issued for correcting typos or small changes.

The book also has a *branch*, which is currently branch 2, of which I am responsible. If you want to start your own branch and reuse some material, write to the manager of the branch you are branching off. I will provide you with your branch number (to track dependencies) and managing instructions for you.

Highlights

Going around, people keep asking me the same question: *what are you working on?*

I never know exactly what to answer, so I decided to write this book to account for what I am working on.

This book is essentially a book on *general relativity* (GR). Since, as such, it is not unique, let us highlight here what is peculiar in it, with respect to the similar literature out there.

Some of the material is pretty standard. We provide detailed introduction for students to get used to techniques in classical field theory. They need to grow confident on how to obtain field equations out of a Lagrangian, how to check if a transformation is a symmetry, to compute its Noether current, to find the most general Lagrangians depending one some fields which has a given group of symmetries, discuss general covariance, compute superpotentials and discuss equivalence among theories. And they need to do it in a general coordinate system.

We append a rich library of mathematical background. Mathematical topics are developed in their generality, usually independently from the physical motivations. We cover manifolds, tensors on manifolds, Lie derivatives, metric structures and connections; we mean natural connection (connections on a manifold, as well as principal bundle connections, a.k.a. *gauge fields*). Then we also introduce fiber bundles which allow us to provide a global view on field theories without forcing intrinsic notation. Bundle theory also allows us to define in full generality physical observers. I do not think one could separate fiber bundles from relativistic theories, which appear really naturally in a bundle framework.

There are a number of detours with respect to what one usually does.

- *Special Relativity and Equivalence Principle*

We often avoid considering GR as an extension of SR to treat gravity. Relativistic theories are more general than SR which is to be considered as a special case. Accordingly, SR is sometimes used as a qualitative motivation (for example not to assume that simultaneity is absolute) or as a simple example to show computations (as in relativistic positioning systems). In the second case, to be honest, that is not even SR! It is rather Minkowski spacetime as a particular solution of GR.

In particular, at some point in Chapter 6, we need to specify how material test particles move. Usually, this is done by using equivalence principle, declaring that in SR they obey the equation $\ddot{x} = 0$ (in a suitable coordinate system), writing it in general coordinates obtaining the geodesic equation (though for Minkowski metric, just in general coordinates) and then declaring that test particles move along the geodesics of an arbitrary Lorentzian metric or connection, allowing curvature.

This is not completely unreasonable, though it is hardly less than assuming a geometry on spacetime by axiom. We prefer to spend few pages discussing in what form the equation for test particles can be assumed in view of covariance principle discussed at that point. We shall prove that geodesic equation of a connection (or better of a projective class of connections) are naturally selected. This discussion better clarifies in my opinion the relation between gravity, covariance, and symmetries.

As far as Equivalence Principle is concerned, it cannot be a *principle*, it is either a theorem or something to be tested in the real world. As a general perspective, we try to avoid foundation on equivalence principle (which, at least in some formulations, calls for SR). After the more than century-long conflict between Equivalence Principle and General Covariance Principle as foundation of GR, I believe one can argue that covariance has been dismissed as an aesthetic and mathematical request with a heuristic role only. It was dismissed too fast, without enough motivation. I think,

among other things, Utiyama-like theorems supports this view. That attitude is not standard, it is also believed to be a wrong perspective. We discuss it in Chapter 2 before applying it.

- *Differential topology vs. differential geometry*

I find particularly confusing many discussion about spacetime and its physical meaning. Sometimes I read people arguing that since coordinates are not covariance (hence non-observable) then points on spacetime are not physically defined. That, at least to the best of my understanding of the argument, is more or less like arguing that since components of a vector depends on the reference frame, then forces are not physically defined. We shall be back on this argument below. As a matter of fact, this is good place to highlight that it depends on which level we are discussing.

At the level of relativistic theories, in view of background freedom, spacetime is a *bare manifold*, not a Riemannian manifold. We start from spacetime with no structure fixed on it, included no geometric structure, and we want do describe evolution of fields. We still have to write down field equations, hence we have no field fixed on spacetime.

That is particularly interesting since, exactly because we have no field fixed on spacetime, it is relatively hard to write PDE for field equations, which is why general covariance does impose strong constraints on allowed dynamics.

A simple example is in order. If we want to describe a wave equation for a scalar field φ, one needs a Lorentzian metric g defined on spacetime to write wave equation $\Box\varphi = 0$. A wave on spacetime alone is not a viable relativistic theory. Of course, it is well defined if we want to describe the interaction between gravitational field and the scalar field.

If spacetime is a bare manifold, there is a mathematical framework already in place for it, it is called *differential topology*. It studies not exactly bare manifolds, but rather classes of manifolds up to diffeomorphisms, which is *exactly* what spacetime will turn out to be in view of the hole argument and the definition of observers' laboratories. At a fundamental level, when we are going to discuss dynamics in relativistic theories, spacetime is exactly such a class of manifolds up to diffeomorphisms and the context in which the theory is developed is differential topology.

It is striking how this simple observation ends up to clarify the origin of many properties (as general covariance itself) and how they relate to physics. For example, one cannot even formulate equivalence principle of a bare manifold. Let us remark that defining bare manifolds before they are endowed with a geometric structure was precisely one of the motivations of Riemann to define manifolds.

Then, we write field equations in a gravitational theory, we discuss solutions, hopefully we found and fix one of them, then we have a manifold with a geometry fixed on it, namely a Riemannian manifold. That is a spacetime with a fixed gravitational field. Here we can precisely formulate the equivalence principle (or derive it) to describe how test particles move, their *free fall*. All that is done in a context of differential geometry.

Accordingly, we need to trace how the theory develop since at different stages we can do different things, even in a purely physical perspective. We need to be aware, when we can use structures and when we need to work on a bare spacetime. That means, more or less, that many physical properties are entangled with what we define to be physics at different levels of the same theory and a lot of relations among different assumption can be made manifest.

That is close enough to *proving* those physical properties, however the proof as usual is mathematical, it is valid in a context. A physical proof, on the other hand, requires experiments. In other words, we are going to highlight relations: if one assumes a context then properties follow, although, of course, there could be a completely different context in which the properties are false, as usual.

- *Killing vectors in Conservation Laws*

In view of what happens in SR, it is often assumed that one needs Killing vector to define physical quantities in GR. Unfortunately, generically spacetimes have no Killing vector at all. On the other hand, in view of general covariance, it is clear that *any* spacetime vector field defines a Lagrangian symmetry for a relativistic theory. In view of Noether theorem, it then defines currents and in view of naturality (or gauge-naturality) it defines superpotentials.

This quantities are geometric and, unlike pseudo-tensors, they define absolute properties of spacetime. Whatever the relation between these quantities and physical conserved quantities such as energy-momentum, angular-momentum and so on, in view of general relativity principle, either physical conservation laws are irrelevant for describing physical laws, or there is a deep relation between them and superpotentials.

Whatever, this relation is, it clearly points to reduce the role of Killing vectors in conservation laws. A detail discussion clearly shows that the fundamental theory of conservation laws is in fact independent of Killing vectors. These only are needed in SR to restore conservation in quantities, which was originally compromised by fixing a metric structure in spacetime instead of treading geometry as a dynamical structure as relativistic theories demand. In a relativistic theory, with no background, there is no need at all of restoring conservation. In SR, one needs to do it *because* general covariance was broken by fixing Minkowski metric as a non-dynamical background.

- *Spatially Homogeneous and Isotropic Spacetimes*

When we turn to cosmology, we need to define spatially homogeneous and isotropic spacetimes. These are usually simply defined as spacetimes for which there exists a coordinate system in which the metric is in FLRW form (or even worse by resorting to non-existing of special points or directions *in space*).

Although it is undeniably longer, we believe it is better to provide an intrinsic characterization of spatially homogeneous and isotropic spacetimes, based on the isometry group instead of coordinates. The real reason for this detour, is that it sheds light on the relation between some structures which appear in cosmology (the space foliation and the comoving motions) which are important later to clarify the measurement protocols used in cosmology.

- *Invariance with respect to re-parameterisations*

Another odd attitude which is maintained along the whole book concerns invariance with respect to changes of parameterisation of worldlines in spacetime. This is argued by trivially noticing that the parameter along worldlines is not a physical time. A trajectory in spacetime already contains all information to obtain the physical motion of a test particle, parameterisations of it are a mathematical add-on, it is something *we add* in order to re-use some of the formalisms and tools introduced for Lagrangian mechanics.

Usually, one fixes proper time along timelike trajectories as a physically preferred time. This is not our way. Leaving parameterisation unfixed in the beginning allows to treat mass test particles and light rays on an equal footing. It also allows to leave the discussion about the role of proper time separate from test particles. Besides, one can always fix proper time later. As a matter of fact, trajectories on spacetimes are more physical in nature than parameterised curved. In a quite precise sense, they are also more geometric, supporting the tight relation gravity has with geometry.

- *Space and time do not exist*

At a fundamental level, in a relativistic theory, space and time are not there. There is no space and no time, only spacetime. Also dynamics does not select space or time.

Physical time emerges when we, conventionally, select a class of clocks (the atomic clocks) as a reference. The choice is arbitrary, though admittedly convenient. To be precise, we are not arguing about the physical choice of atomic clocks, the specific real device, as a reference. What specifically must be chosen is the correspondence of such real devices with some representation in the formal framework, namely their representation as parameterised curves. It is the parameterisation which corresponds to atomic clocks what needs to be selected. While in standard GR the choice is quite (though not completely) canonical, in extended theories it is manifestly a choice, an independent axiom of the model.

Once we have time (along the clock(s)), we can define *space* by synchronisation (as well as, in some other ways, as it happens in cosmology where synchronisation is not a practical choice). Space can be global or local, depending on the situation, though local space-time decomposition can always be defined out of a clock. When one has space and time, we can always define Newtonian-like dynamics, made of forces, spatial accelerations, spatial velocities. However, that framework is still based on the choice of a reference clock.

Do we really need a Newtonian picture? General relativity principle says we do not, still it is particularly satisfying for us. However, we know it is relative and conventional.

Going this way is not particularly efficient. It takes time to build a theory out of thin air, actually bootstrapping it out of very limited and general preconceived ideas of how reality should be described. It is quite a surprise that it is even possible to do. If some readers cannot wait the long road, they are free to leap ahead. However, there is beauty in the road that one misses that way. We try, to the extent we can, to do things as they are supposed to be done. The material is a lot, so sometimes we may fail, though that is what we want to do.

The order of topics is also quite unusual. If we decide to define relativistic theories before having a Newtonian picture of reality, necessarily time and space need to appear much later. We first go for a description of spacetime which is objectively more natural. Accordingly, we are lead to anticipate considerably with respect to what is usually done, a number of issues such as globality of structures and the hole argument. Usually these topics are considered as advanced topics in a primer, here they appear in Chapters 1 and 2. These are also a good motivation for geometerisation of gravity, so we believe it is worth the attempt. The whole dynamics is covariant and it does not need a specific time even when the pointwise causal structure (e.g. the light cones) appears in EPS framework. I think the discussion of covariant issues in a covariant framework is easier and clearer.

We leave the discussion of Newtonian physics to Part III. Relativistic theories are not Newtonian, they do not prefer special coordinates, they do not require one coordinate to be a time and three to be positions in space. Space itself does not exist as an absolute structure. In a relativistic theory, the position of events in spacetime can be given by 4 spatial coordinates or 4 times (as in relativistic positioning systems). Most of Newton fundamental concepts (such as forces, acceleration, and energy) do not even exist in a relativistic theory, in the sense they are not absolute. In view of relativity principle, one can (and should) describe physics in terms of absolute objects only. Accordingly, such concepts emerge eventually in Part III to highlight their conventional nature and to show that they are not really needed in many situations. Proceeding this way at least one sees how one has, e.g., different notions of *time* (the time along a clock, in a neighbourhood of a clock, coordinate, cosmological time) which are conventional in different ways and degree.

Another last issue we have is with approximations. We try to define things and give some examples without approximations. Approximations are often reasonable in view of a physical discussion. However, we have two remarks: first, if we are bootstrapping a description of physical reality, we

cannot rely too much on our physical intuition, which is often based on part of physics (SR or Newton) which are known to be wrong. Once again, these (wrong) frameworks should be recovered later as an approximation of the relativistic model. Our attempt will be to derive physical intuition from mathematics, not vice versa.

 Second, when we approximate, we would like to approximate with respect to some exact framework that should be, at least in principle, known. That means that we should be able to do computations at least in some simple cases before calling for simplifications and approximations. In fact, one thing is solving numerically an integral, which is completely different from neglecting a term in the Lagrangian because it is expected to be small on solutions. The simplest cases at least force one to find the simplest way to solve a problem, e.g. avoiding inverting functions when it is not necessary. As a extra token we get that without approximations, it is easier to consider deviations of one theory from the other and tests are clearer in principle.

 If I had to summarise what we are going to do and why we are doing it this way, I think one could say that we want to introduce extended theories of gravitation (which though are stricter than the usual meaning) as a family of theories of gravitation which encompasses standard GR as a special case and extend the physical interpretation of the theory to this extended framework. Part III appears to be the attempt to build a solid bridge between the mathematical framework and observational protocols, which allows a mathematical, rigorous control on experiments, aiming to set up tests able to in principle discuss differences among different extended theories. One can say we wish to solve, once and for all, at least for extended theories of gravitation, the issue of whether one can physically distinguish between them on an observational stance and see if they can account for current observations in astrophysics and cosmology, besides of course the observations on Earth and in the solar system.

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$\mathrm{Sec}(\pi)$, $\mathrm{Sec}(\mathcal{C})$: sections of a bundle $\mathcal{C} = (C, M, \pi, F)$.

$\mathrm{Diff}(M)$: the group of diffeomorphisms on the manifold M.

$\mathrm{Aut}(\mathcal{B})$: the group of fibered morfisms from a bundle $\mathcal{B}$ onto itself.

$\mathrm{Lor}(M)$: bundle of Lorentzian metrics on the manifold M.

$L(M)$: the bundle of (general) linear frames on the manifold M.

Notation

- Usually manifolds are denoted by capitol letters, M, N, ... and their dimension is usually denoted by the same lower letter, namely $m = \dim(M)$, $n = \dim(N)$.

- We shall use the signature $(3,1)$, i.e. $(-1,1,1,1)$ or, more generally, $(m-1,1)$ for Lorentzian signature on a spacetime M.

- Since we shall use fiber bundles, we shall often use *sections* meaning maps $\sigma : M \to B$. When we refer to Sections of a Chapter, it will be capitalised.

- In equations we shall sometimes color terms. Terms in the same color cancel one with the other. Underlined terms are similar and they sum. Framed terms are zero (usually by symmetry) on their own.

- *Spacetime* will be written with no dash, contrary to English grammar. *Space-time* is reserved to the $3+1$ (or *ADM* splitting) of a spacetime done by one observer (or a class of observers).

- Global forms (of degree at least 1) are denoted by bold letters. For example, the Lagrangian is a horizontal m-form denoted by

$$\boldsymbol{L} = L(q,u)\,dt$$

  where $L(q,u)$ is the scalar density and dt is the basis of horizontal 1-forms induced by coordinates, even when they *are* global.

  The notation is somehow ambiguous since the same object dt can be considered as the differential of a function, thus it could (or should) be denoted by $d(t)$, or, since the coordinate t is global, by $\boldsymbol{dt}$. Accordingly, consider the boldface as a way to emphatise the global character of a form which, usually, we do not want to stress for coordinate induced objects.

- We use grey for universal and coupling constants such as c, G, κ, or g. One can drop them in most theoretical arguments, though they help dimensional analysis and become essential when comparing with observations. Constants are carried over consistently, so, when comparing with other books, one can define them to match a formula and the rest should agree.

  Universal constants, as c, G, are usually equal to 1 in homogeneous unit systems and are denoted in grey boldface.

- For gravitation the coupling constant is denoted by $\kappa := \frac{8\pi G}{c^3}$. Of course, in most cases one could simply understand it to be 1 or 16π depending on the context.

- For electromagnetic and Yang–Mills theories the coupling constant is denoted by $g := \frac{c}{\epsilon_0}$. Of course, in most cases one could simply understand it to be 1 as well as 4π depending on the context.

- Dimensional analysis is noted at a side in grey. Dimensions are standard: M for mass, L for length, T for time, C for electric charges. On spacetime, $[F] = MLT^{-2}$ $x^0 = ct$ is usually a length.

- SR stands for *special relativity*, GR for *general relativity*.

- For definitions, we use $A := B$ which defines A by setting it equal to B. The same thing can also be written as $B =: A$.

That is not to be confused with $A = B$ which claims that two *existing objects* are equal (either as an identity or as an equation if one of the two is parametric).

- $\ldots - [\mu\nu]$ means: $\ldots$ *minus the same terms with the indices* $[\mu\nu]$ *exchanged.* It has higher priority that the sum.

  Thus, for example, $A_{\mu\nu} = B_{\mu\nu} + C_{\mu\nu} - [\mu\nu]$ means:

  $$A_{\mu\nu} = B_{\mu\nu} + C_{\mu\nu} - [\mu\nu] := B_{\mu\nu} + C_{\mu\nu} - \left(B_{\nu\mu} + C_{\nu\mu}\right)$$

  while $A_{\mu\nu} = B_{\mu\nu} + \left(C_{\mu\nu} - [\mu\nu]\right)$ means:

  $$A_{\mu\nu} = B_{\mu\nu} + \left(C_{\mu\nu} - [\mu\nu]\right) := B_{\mu\nu} + C_{\mu\nu} - C_{\nu\mu}$$

- $\ldots + (\mu\nu)$ means: $\ldots$ *plus the same terms with the indices* $(\mu\nu)$ *exchanged.* It has higher priority that the sum.

  Thus, for example, $A_{\mu\nu} = B_{\mu\nu} + C_{\mu\nu} + (\mu\nu)$ means:

  $$A_{\mu\nu} = B_{\mu\nu} + C_{\mu\nu} + (\mu\nu) := B_{\mu\nu} + C_{\mu\nu} + \left(B_{\nu\mu} + C_{\nu\mu}\right)$$

  while $A_{\mu\nu} = B_{\mu\nu} + \left(C_{\mu\nu} + (\mu\nu)\right)$ means:

  $$A_{\mu\nu} = B_{\mu\nu} + \left(C_{\mu\nu} + (\mu\nu)\right) := B_{\mu\nu} + C_{\mu\nu} + C_{\nu\mu}$$

Best observed values

Newton constant	$G = 6.67 \cdot 10^{-11} \, Kg^{-1} \, m^3 \, s^{-2}$
Vacuum speed of light	$c = 2.99 \cdot 10^8 \, m \, s^{-1}$
Planck constant	$\hbar = 6.626 \cdot 10^{-34} \, Kg \, m^2 \, s^{-1}$
Hubble parameter	$H_0 = 2.2 \cdot 10^{-18} \, s^{-1}$
Today abundance of Dark energy	$\Omega_\Lambda^0 = 0.714$
Today abundance of barionic matter	$\Omega_b^0 = 0.046$
Today abundance of cold dark matter	$\Omega_{CDM}^0 = 0.24$
Today abundance of curvature	$\Omega_k^0 = 0$

Best computed values

Newton constant	$\kappa = \frac{8\pi G}{c^3} = 6.27 \cdot 10^{-35} \, Kg^{-1} \, s$
Planck length	$\ell_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35} \, m$
Critical density today	$\rho_0^c = \frac{3}{\kappa c} H_0^2 = 7.74 \cdot 10^{-10} \, Kg \, m^{-1} \, s^{-2}$
Density of visible matter today	$\rho_0 = \Omega_b^0 \cdot \rho_0^c = 3.56 \cdot 10^{-11} \, Kg \, m^{-1} \, s^{-2}$

Part I

Relativistic theories

Uscendo dal tempio dove aveva tanto studiato ed erano passati tanti giovani anni della sua vita, si sentí vecchio.
—Maestro—gridò nel vento—è ora che io conosca il segreto?
—No! Fila a divertirti—rispose una voce lontana.

On his way out of the temple where he had studied so long and where he spent so many years of his young life, he felt old.
— Master—he shouted in the wind—is it time for me to know the secret?
—No! Go and have fun—a far voice answered.

(*Baol*, Stefano Benni)

I. Introduction to Part I

Physics is about inventing mathematical models able to represent physical entities, predicting their evolution and forecasting the results of observations and experiments. Sometimes one wishes to describe the evolution of certain quantities in time, sometime one wishes to describe the values of some quantities somewhere in space and at some time.

In the first case, for example one wants to describe a falling body and forecast its position as time goes by. For the position one fixes some conventions *Mechanical setting* (a reference frame) about which quantities are relevant, where one should start measuring distances, which units are to be used, and so on. One can decide that horizontal position in a given direction and height above the ground are relevant, to measure distances in meters. Let us call x the horizontal distance and y the height above the ground. What one wants to know is how these quantities changes in time, i.e. the functions $x(t)$ and $y(t)$. This kind of setting is called *mechanics*. The first part of fixing conventions is called *kinematics*. The duty of kinematics is to fix enough conventions in order to be able to encode all possible observations (the positions of the falling body) in terms of numbers (the value of x and y at a given t). The collection of all these conventions will be called an *observer*.

In the second case, for example one wants to describe the wind in a room. One can measure a vector (the wind velocity) at any point x in the room *Field theory setting* and at any time t. The vector can be described by its components w^i with respect to a fixed basis (e.g. the one determined by the walls). One would like to predict the functions $w^i(t, x)$. This kind of setting is called a *field theory*. Again what we have done so far is the kinematical specification of the system and the observer.

In both cases, when kinematics is done, one would like to specify some differential equations which, once the initial state of the system is known, *Def: dynamics* are able to (possibly uniquely) determine the evolution of the system. In mechanics, one expects equations to be (a system of) ordinary differential equations (ODE). In field theory, the equations are expected to be (a system of) partial differential equations (PDE). In both cases, the order of the equations (i.e. the highest order of derivatives appearing in the equations) determines how much information one has to specify at $t = 0$ in order to be able to (uniquely) determine the evolution of the system. If equations are first order, in general one expects the evolution to be determined by the initial position only, if the equations are second order initial velocities (first derivatives of positions) will be also needed. In any event, one would like the equations to be sufficiently regular to obey some kind of Cauchy existence and uniqueness theorem. The specification of the equations is called a *dynamics* of the system.

Finally, one would like to check that the evolution of the mathematical model agrees with the evolution of physical quantities they describe. If it does, one can predict the behaviour of the physical world by investigating the behaviour of the mathematical model.

This classical and standard scheme is so simple (and somehow trivial) that things almost never are that clear and simple. We shall see that almost always the equations describing the dynamics are at the same time (which, I know, it is hard to believe) *underdetermined* and *overdetermined*. By *underdetermined* we mean that part of the fields are not determined by dynamics. In some sense, they can be changed at will and in fact they describe the freedom of observers to fix their conventions. At the same time, they are *overdetermined*, meaning that not all initial conditions are allowed. In fact, some of the equations are not evolution equations (they do not contain time derivatives at their higher order) and in fact they become constraints for the initial conditions.

Although field equations could in principle be chosen at will, as a matter of fact they often come from a variational principle. A *variational principle* *Variational setting*
is given by choosing a functional space, usually the space of (global) sections of a bundle called the *configuration bundle*, and fixing a functional over
it, called the *action*. Then field equations are defined and obtained out of the action by requiring that solutions are *critical sections* (usually minima,
but sometimes also maxima or saddle points) for the action. This condition, usually called *principle of stationary action* or *Hamilton principle*, is
equivalent to well-defined equations which describe the dynamics.

Though configurations can in principle be any functional space, often they are section of a fiber bundle, called *configuration bundle*. Though the action *Lagrangian setting*
could in principle be an arbitrary functional of the space of global sections of the configuration bundle, usually it is a so-called *local action*, meaning
that the action is computed by integration of a particular form, called the *Lagrangian*, computed along fields and one assumes that the Lagrangian
depends on the value of fields at a spacetime point, together with their derivatives up to some finite order k, which is also called the *order* of the
Lagrangian. We shall always consider dynamics given in terms of a local action, i.e. a finite order Lagrangian.

Probably, one should remark that why nature decided to describe all its fundamental dynamics (as well as most of its non-fundamental ones) in terms
of variational principles is essentially a mystery. Variational principles were historically discovered in geometry and optics. Since variational principles
are global principles in which the evolution is determined so that an integral property (the action) is minimal that is definitely unclear. For a long
time variational formulations were considered a tool or a curiosity. In the end, variational techniques were recognised to be fundamental in view of
their deep relation with quantum mechanics (and quantum field theory). This relation is very fascinating though beyond the scope of this book.

Since most of the times dynamics is described by a variational principle, then one has as many equations as fields. If fields have components encoding
observer freedom which are not determined by dynamics, usually an equal number of equations become constraints on initial conditions, since the other
field equations are enough to determine the evolutions of the other fields' components.

Another important issue which is simply understood in terms of action principle is that if one chooses the action intrinsically (i.e. the value of the
action depends on the fields, not on the observer conventions to describe the fields) then field equations are guaranteed to be global (i.e. themselves
independent of observers' conventions).

Being independent of observer conventions is utterly important. It is the root of a possible absolute description of physical real world. In fact, any
observer gives a subjective description of the world, where *subjective* means that it is influenced by the observer conventions.

Let us stress that such a description contains particularly few (if any) information. If we receive the news of a supernova in the sky with declination
+9°54′31″, right ascension 13°12′54″ which is reaching its maximal luminosity at 13:58:12, we in fact do not learn anything unless we know where
the observatory is located on the Earth and which clock conventions is using. In physics we use units and systems of units in order to specify classes
of conventions in order to be more informative. Still the report alone is not at all informative and systems of units take the form of maps between
observers. For example, we can force all observatories to report as if they were at the poles.

Let us now remark that our knowledge is almost completely hidden in the conventions, in the sense that without a detailed knowledge of the conventions
we do not learn anything about the physical world. For example, if we received two reports about a supernova in the sky, we are unable even to say if
the two reports are about the same supernova or two different events.

Now, specifying conventions is particularly hard in principle since we should essentially become able to predict the report generated by any event
observed. Of course, there are infinitely many possible events to be reported, but this is not the problem (that is what functions were invented for, for
specifying infinite objects corresponding to infinite objects). The real trouble with it is that, while we control possible reports (specifying a grammar,

reports are essentially mathematical objects), we are not able to control the possible events. Here in fact we are setting up a mathematical way of describing events, while it is precisely such mathematical description which defines what are the possible events to be observed. In other words, one could say that the models are mathematical, i.e. they are under our control in the sense that we *invent* them, while the events themselves are out of our control in the sense that we *discover* them. It is precisely the model which brings the events under control (until something unexpected happens).

We are here discussing how to bootstrap a description of the physical world and probably this is the right place to discuss how to avoid circular reasoning. What saves the bootstrap operation is that we do not need to map reports to events. We already remarked that systems of units take the form of mapping observer conventions into conventions of a canonical observer. The point is that in general what we need is to predict the correlation of reports between pairs of observers. If an observer sees an event and produce a report about it, we have to be able to predict the report of a different observer looking at the same event. In other words, we have to map the reports of an observer into the reports of another.

At this point mathematics kicks in. If the two observers are mapping events on spacetime reporting their coordinates, than the maps between observers reports are *transition functions*. And one can show that, if one knows a family of local descriptions of the spacetime each given by arbitrary (local) observers and their transition functions, then there exists a unique (modulo diffeomorphisms) manifold M which is an absolute model for spacetime. If observers are reporting about values of fields, their transition functions turn out to be changes of fibered coordinates and one can show there is a unique (up to bundle isomorphisms) configuration bundle which provides an absolute description of fields in terms of sections.

It is particularly inspiring that the absolute description of the real world obtained in these cases is encoded essentially in the *relation* between pairs of observers. One should also stress that although we often introduce a spacetime manifold (or a configuration bundle) as an absolute description of the real world, what is to be understood is that spacetime manifold (or configuration bundle) is a convenient representative of the absolute description which is in fact their class of isomorphisms.

> A very similar situation occurred in geometry. Geometry studies properties of surfaces. Though surfaces are defined through parameterisations, different parameterisations can describe the same surface and geometry is interested in the properties of surfaces which do not depend on the parameterisations (exactly in the same sense in which physics is interested in properties which are independent of the observers, though the only description of reality available is the one through the subjective description given by observers). All possible different parameterisations of a surface provide an absolute description of surfaces.

As for parameterisations in geometry, observers are in general *local observers*. Only specific situations allow to be globally described by a single observer. An observer is a set of conventions which allows to map the real world by a set of numbers. Such conventions in general fail to apply to any event *anywhere* and sooner or later fail to do their job.

For example, an observer which rotates (with respect to an inertial one) may set its conventions as it were not rotating. In this way, going away from the rotation axis, sooner or later the velocity of the frame will reach the speed of light and become unphysical. Another example is when curvilinear coordinates are used but at some point the coordinates fail to be a bijection.

These two examples are examples in which the conventions fail to be global. Sometimes it happens that a specific situation simply does not allow *any* global description and local observers are the only ones available. This is what happens on a surface when the surface does not allow global parameterisations and one is forced to use many local parameterisations. This is when conventions fail to be global not because the observer chooses them in a poor way, but because some physical reason dooms any convention to fail somewhere.

Part I will be organised as follows.

In Chapter 1 we shall introduce basic facts about classical field theories by using a *very basic* mathematical framework. We shall show how one can obtain field equations from an action principle and compute conservation laws from symmetries. Since we shall not use fancy geometric framework computation will be rather complicated though somehow elementary. This is meant also to convince the reader that the geometric framework introduced later makes in fact things simpler. We shall start discussing globality of objects which will turn out later to be essential for relativistic theories. This Chapter is for introduction of standard techniques that will be used extensively later. Of course, it should be skipped by readers which are already familiar with geometric framework for field theories, which will be discussed later in Chapter 2. This Chapter 1 will also provide basic examples of field theories which will later used to illustrate properties of relativistic theories.

In Chapter 2 we shall introduce relativistic theories (as natural and gauge-natural field theories) and discuss general covariance. The purely metric formulation of standard GR will be also provided as an example. We shall also show how symmetries constrain compatible dynamics (Utiyama-like arguments) and sketch a framework for later discuss initial value problems in field theories.

Chapter 3 will be devoted to conservation laws, the theory of superpotentials and conserved quantities. The whole Chapter is based on Noether theorem which connects symmetries groups to conserved currents.

In Chapter 4 we shall discuss dynamical equivalence between different theories. We shall also prove that standard GR in purely metric and metric-affine formalism are (always) dynamically equivalent, contrarily to what is usually believed (and provided one suitably defines the correspondence).

In Part II we shall use the material of Part I to study in more details gravitational theories.

In Part III we shall collect a detailed discussion of standard GR and its limit SR.

In Part IV an extended collection of mathematical background is presented.

Chapter 1. Spacetime and field theories

Wax on, right hand. Wax off, left hand. Wax on, wax off.
Breathe in through nose, out the mouth. Wax on, wax off.
Don't forget to breathe, very important.

(Miyagi Sensei)

1. Overture

 In this Chapter, we shall introduce basic facts about classical field theories by using a *very basic* mathematical framework. That means long (though somehow elementary) computations.

 In particular, we shall present a bundle framework for variational calculus, discussing how one can describe a Lagrangian dynamics in field theory and how to extract field equations from it. Even if one needs time to get used to it, we believe this geometric framework leads eventually to a clearer and more general understanding of field theories, especially concerning covariant theories. Although it will be discussed in more details later in Chapter 2, we shall here start discussing globality of field equations. This will turn out to be a key issue in relativistic theories which is strictly entangled with general and gauge covariance. Since some problems are better understood in this way, we start also to introduce Poincaré–Cartan forms in field theories and discuss their globalisation and uniqueness.

 Then we shall discuss symmetries, Noether theorem, and conservation laws in field theories. Again, the whole Chapter 3 will be devoted to cope with this issue. However, here we introduce the basic ideas and techniques which will be there applied to general covariant theories. It is important to understand that, to some extent, there is nothing to be understood in computing field equations for a Lagrangian, in discussing if a transformation is a Lagrangian symmetry, if it is, in computing its Noether current. These are general variational techniques which are used over and over in different contexts. One has to practice them until (s)he is sufficiently confident about them.

 As examples, we shall discuss some basic ugly examples and somehow pathological cases: cases in which its hard to see the geometry behind it, cases in which degeneracy of dynamics produces really awkward field equations, or cases of local Lagrangians still producing global field equations. These examples are important when discussing general theories; they provide counterexamples which will guide us safely.

 We shall also discuss under many viewpoints classical covariant formulation of electromagnetism. This is an example, probably the most basic example, of a covariant theory and we all need to keep it in mind and remark the relation with its original non-covariant formulation. Electromagnetism is very well understood from a physical viewpoint and it is important to keep well in mind how its geometrisation relates to its physical meaning. It is also a good start off in the direction of gauge theories.

We shall also give a first glance to field theories to describe the gravitational field. We shall introduce Hilbert–Einstein standard GR, $f(R)$-theories and Brans–Dicke theories in their purely metric formulations. These will be discussed for the rest of the book in better details. However, I believe it is convenient to start to set the computation aside, even for allowing a later more essential discussion about their physical meaning without getting lost in the analytical computations.

Finally, we review mechanics in its Lagrangian and Hamiltonian formulations. Mechanics is another important corner stone to compare field theories to. Some of its structures survive in the extension to field theories, some do not, some, finally, need to be reformulated. Anyway, they inspire field theories and often thay are a first battleground to experience and test a better formulation.

We shall introduce the reader with no background in classical field theories to standard techniques such as obtaining field equations from an action principle and computing conservation laws from symmetries. By doing it with almost no geometric background should later convince that geometric methods really simplify your life.

Students should go through the examples and check that they can repeat computation. If they cannot, they can fill in the gaps of the mathematics they need from Part IV. It does not really matter at this stage that one grasps the physical meaning in the examples. The physical meaning will come later, now it is a maths exercise so that we shall be able to discuss later the physical content without worrying too much about computational details.

We shall present a number of worked out examples in full detail. The reader who can work them out can skip them entirely, just tuning notation.

Teachers can choose examples to discuss based on their plans. I have to confess I chose the examples I most often use in courses so to have a reference where they are treated in details and in a coherent formalism.

2. Spacetime

A relativistic theory is a field theory on *spacetime* (of course, with some further characteristics). Later on, we shall give a precise definition. However, we need to introduce spacetime and homogeneous formalism.

> The world "spacetime" does not really exist in English, the correct spelling would be "space-time". However, some have used a brand new word "spacetime", which I believe is better than the old "space-time". In fact, each event is a collection of information about *when* and *where* something happens. Relativity taught us that one cannot really split the information about *when* from the information about *where*, not in a way which is the same for all possible observers, anyway. The description of physical reality in terms of events is more fundamental, as we shall see, than the description in terms of space and time. Spacetime provides an entangled description of physical reality which is more fundamental exactly because it entangles space and time together. One should not break space from time too easily and the world "spacetime" reflects this fact.

Essentially, spacetime is the arena where physics happens. Field theories are about some fields which values can be measured at a point in space and at a time. Spacetime takes into account where fields can be measured, i.e. both spatial position and time. A point in spacetime is called an *event*. Unlike in Newtonian physics, we shall not assume that there is a canonical way (i.e. which is the same for any observer) to split spacetime into a space and time. On the contrary, we shall show that different observers give different spatial positions and times to the same event.

Part of this observation is trivial and applies to Newtonian physics as well. Two observers can fix different origins for the spatial reference frame and different clocks and consequently they assign to the same event different numbers for describing its spatial position and time.

Here we are referring to a more fundamental issue. We all know (and eventually we shall show hereafter) that two observers with identical clocks, when they are in motion one with respect to the other, will experience a time dilatation. This, in SR, is a consequence of Lorentz transformations. Even more essentially, different observers have different notion of contemporaneity: two events that are contemporary for an observer are not for another observer.

All these effects are probably known from special relativity (SR) where this happens whenever a relative motion of the observers is taken into account. Any gravitational field has a similar effect, compromising the synchronisation of clocks. We shall eventually describe these effects in detail later on (see here).

It would be nice to discuss (and we shall do) these effects on the physical ground. However, in order to be able to do it we cannot impose unphysical structures on the mathematical stance in the beginning. One has to maintain a cautious attitude and be initially liberal about what it might happen. While it is clear that any observer can report information about where and when an event happened, we have to be more liberal about how the report of an observer relates to the report of another observer.

It is clear that in principle, if one analyses in details all the laboratory conventions, one should be able to determine what would be the report of another laboratory once the report of the first lab is given. In other words, one should be able to map the measurements of one laboratory into the measurements of the other. This mild assumption is in fact enough to assume that reality can be described in an *absolute* (i.e. independent of the laboratory) way on a spacetime manifold M (up to diffeomorphisms) and to identify laboratories with charts on spacetime; see the standard construction here.

Since there is no need for a lab to be a person, for us it will be neutral. A *laboratory* is a collection of conventions which enable it to encode observations in terms of a collection of numbers.

An *observer* will be physical systems, ideally pointlike, able to set up a laboratory, i.e. conventions to describe physics in an openset of spacetime in terms of numbers.

This is called *homogeneous formalism*. In this framework, events are points on the spacetime manifolds and different *(natural) laboratories* (namely, local charts) map it into different coordinates. The description at spacetime level is more fundamental than the description in terms of observers and laboratories. This is both because the spacetime description is global and because the description is absolute. We shall hereafter discuss more in detail the relation between global and absolute; see here.

As far as we know, *a priori* observers need to fix far more conventions than the ones needed to single out events. If one is describing the temperature in a room, a convention about temperature is needed as well. If one is describing the wind in a room, a basis to break a vector into its components is needed. If one is describing the electromagnetic field in terms of the 4-potential, one will need a gauge fixing which singles out a representative potential for each gauge class. In other words, beside the conventions about events in spacetime, one usually needs conventions about fields. This will lead to identify the lab with a fibered chart on the configuration bundle of the system.

Sometimes, there is a natural choice of the field conventions in terms of the chart of spacetime. If the configuration bundle is the tangent to spacetime, then any spacetime vector can be referred to the natural basis induced by the spacetime coordinates. In other words, the spacetime chart does induce a fibered chart on the tangent bundle. Then we define a *natural lab* as a spacetime chart and a chart canonically induces fibered coordinates on the tangent bundle.

Def: Natural observers

In more general cases (for example, in gauge theories), an observer really needs more conventions than spacetime coordinates. We shall consider this case later on. Here we can define a *gauge-natural laboratory* as a system of fiber coordinates on a principal bundle $\mathcal{P}$, which canonically induces a fibered chart in configuration bundle which in these cases is associated to $\mathcal{P}$.

For now and for the sake of simplicity, let us start by considering the cases in which laboratories can be identified with a system of spacetime coordinates, i.e. they are natural labs. This will lead us to the so-called *natural theories*; see here.

Eventually, some field equations will be given to determine the dynamics of the system. These equations in a classical (i.e. not quantum) setting will be (partial) *differential* equations. For derivatives to make sense, one needs spacetime to be a differential manifold. If equations are of order k, one should be able to define k order derivatives. We shall not be strict with regularity conditions; we shall assume spacetime to be a (connected, paracompact) smooth manifold M of dimension $\dim(M) = m$.

We shall also often need to describe the geometry of spacetime. The simplest way to do it is in terms of a Riemannian metric structure on M.

> Here there is possible confusion in notation. Sometimes in the literature *Riemannian* means a definite positive metric structure and *pseudo-Riemannian* refers to definite but not necessarily positive metrics.
> Sometimes, *Riemannian* refers to non-degenerate but not necessarily definite-positive metrics, while a definite positive metric is called a *strictly Riemannian* (or *Euclidean*) metric.
> We shall hereafter follow the second notation.

For many different reasons (to obtain SR and Minkowski geometry as a special case, or for fitting observations, or for *a posteriori* confirmation of the setting by observational evidences, or deriving it from Axioms), the metric structure on spacetime is assumed to be Lorentzian, i.e. of signature $(m - 1, 1)$.

> Some books prefer the signature $(1, m - 1)$ for Lorentzian case. Each choice has pros and cons. However, in the end, it is only a convention and, as for any issue about conventions, one has to do a choice. We shall always use signature $(m - 1, 1)$.

Here is a first beautiful interaction between physics and mathematics. One can show that not all manifolds support a global Lorentzian metric.

> In fact, suppose M allows a global Lorentzian metric g. Then, since the manifold is paracompact, one can always define a Euclidean metric h; see here.
> One can compute the eigenvectors (corresponding to the negative eigenvalue λ) of the quadratic form g with respect to the metric h, namely the (non-zero) vectors $v \in T_x M$ such that
>
> $$g_{\mu\nu} v^\nu = \lambda h_{\mu\nu} v^\nu \tag{1.2.1}$$
>
> This defines a global family of subspaces $V_x \subset T_x M$ on which g induces a definite-negative metric (i.e. for any $v \in V_x$ one has $g(v, v) < 0$). Analogously, the $(m - 1)$ eigenvectors corresponding to positive eigenvalues define a family of g-space-like subspaces $U_x \subset T_x M$ (i.e. $\forall u \in U_x : g(u, u) > 0$). At any point, since the form g is symmetric and thence it can be set to a canonical form, i.e. there exists an h-orthonormal basis of eigenvectors, one has $T_x M = U_x \oplus V_x$.
> In view of the regularity conditions that we are using here, one can define the sub-bundles $U = \cup_{x \in M} U_x$ and $V = \cup_{x \in M} V_x$ in $T(M)$ and one has $T(M) = U \oplus V$.
> Of course, the sub-bundles U and V depend on the auxiliary metric h, but in any event the tangent bundle $T(M)$ splits as the sum of a rank 1 and a rank $(m - 1)$ sub-bundle.
> Now one can show that a sphere S_2 cannot match these requests. In fact, if TS_2 split as $TS_2 = U \oplus V$, then in particular one could choose a h-unit vector at any V_x and define a nowhere vanishing field of directions on the sphere, which is known from topology to be impossible on any even dimensional sphere.
> On a sphere S_2, there is no (continuous) field of directions which is defined everywhere. This is also called the *hairy ball (or no-hairing cat) theorem*.

Hence, as long as we need a global Lorentzian metric on spacetime, we can be mathematically sure that there are manifolds which are allowed as spacetimes and manifolds which are not. If a manifold allows global Lorentzian metrics, it is called a *Lorentz manifold* and spacetimes are necessarily Lorentz manifolds.

The odd thing is that being a Lorentz manifold is something which is decided in differential topology (it is a diffeomorphism invariant property), hence it is a global property. It is odd, though not a unique case in mathematics, in which the existence of differential structures (which are locally expressed

in terms of fields) turns out to impose a global topological constraint (i.e. something which is homeomorphic invariant). For another example, one can define surface integrals only in orientable manifolds. Orientability is equivalent to a nowhere vanishing volume form, which though amounts to a topological condition.

There is also a different attitude about globality in field theory. Let us discuss this point once and for all. Since dynamics produces equations which are local (they involve the values of fields and their derivatives up to some finite order k, all things that are perfectly defined once one knows fields in a neighbourhood of a point x) sometimes *local* fields are allowed. In this context they usually solve the equations locally, determining a local description of the system, and then define the global spacetime by glueing together local solutions which match on the patch overlaps. The glueing procedure depends on the regularity required and on transition maps. If analyticity is used, then the extension is even uniquely determined by the local solution in any open set. If the solution is only required to be smooth (as we usually do) then there is more freedom and the extension is non-unique but still one can define the global spacetime manifold M by glueing together patches.

In the analytical case, the extension is unique. This means that knowing exactly the behaviour of the system around a point will contain enough information to know the system everywhere in the universe. This is not completely unreasonable form a classical viewpoint. It means that anything happening anywhere in the universe produces an effect here.

Not considering philosophical arguments connected to it, this is certainly disturbing, even if we simply allow a certain amount of uncertainty in our knowledge of the system. For this reason, we prefer to assume smooth and not necessarily analytic systems.

In any event, this situation is totally general; *any manifold can be defined by glueing patches*; see here. And in the end, fields will be global fields on that manifold; see here. Hence our approach (starting from a global spacetime M and considering only global fields on it) is exactly equivalent to the local attitude so common in physics.

When we fix a specific manifold M and solve the equations on that specific manifold, we can obtain only the solutions which are global on that manifold. To recover all possible spacetimes that can be considered locally, we only have to repeat the discussion on all possible manifolds. A possible advantage of this splitting of solutions of the local problem into many different global problems comes from the fact that often less solutions can be treated more easily since possibly one can better control their properties. In any event, we shall see that in view of general covariance principle, often one can discuss the situation on a generic spacetime manifold without entering the details until very late when specific solutions are discussed. This makes quite easy to discuss general properties of the theories disentangling almost completely the discussion about the properties of the system from the discussion of specific solutions.

Finally, there is another related variation of attitude about globality which is worth discussing once and for all. Often one reads that bundle theory is used in mathematica theory to discuss globality. Although the motivation is real, the issue is ill posed. It is certainly true that bundles take care of globality of objects as described above, however, this is hardly the reason (not the only reason, anyway) one uses bundles in field theory. There are cases in which bundles happens to be trivial, there is no issue about globality to be enforced, still one uses bundles in field theory.

On a trivial bundle there is no difference between sections defined in a trivialization and global sections. However, one still needs configurations to be independent of the choice of the trivialization which is something chosen by the observer in a laboratory, not a physical property. The bundle setting in field theory, in fact, provides us with transformation laws of configuration of local fields which are then used to glue them together to form a global intrinsic field which in fact is an absolute description of the physical field, exactly because it does not depend on the trivialization.

In a relativistic theory there is no local field of which transformation laws, even in the cases in which there is no issue about globality. To be clear, I am not too fond in the language of bundles in physics. One can discuss the same facts in terms of intrinsic properties or local descriptions together with their transformations laws. The only difference is a difference in languages. The best language depends on the problem and the context. However, there is a undeniable problem about local description of fields: either they need to be described intrinsically or they need to be augmented by transformation rules. Either way, there is no relativistic theories with local fields and without transformation laws.

3. Variational setting

We shall here introduce variational calculus on fiber bundles and their jet prolongations. We aim to give a geometric account of variational principles.

In a field theory, the configurations of the system are global sections of a bundle $\mathcal{C} = (C, M, \pi, F)$ which is called the *configuration bundle*. Let us   *Configuration bundle* consider a system of fibered coordinates (x^μ, y^i); the local expression of a section is a map $\sigma : M \to C : x^\mu \mapsto (x^\mu, y^i(x))$. Hence the local description of a section amounts to provide the fiber coordinates as functions of the coordinates in the basis. For these reasons, the base of the configuration bundle, i.e. the manifold M, is identified with spacetime (which is where the fields can be measured) and the coordinates y^i along the standard fiber are identified with fields (which, in a configuration, are measured at a spacetime event).

> Here the term *configuration* is somehow different with respect to the standard use. In mechanics a configuration is a collection of positions which encodes all possible snapshots of the system. In snapshots there is nothing about velocities.
>
> [Here we are not considering that one can in fact take snapshots of velocities in some circumstances! For example the velocity of an electric charge has an effect through the magnetic field it generates; the magnetic field can affect a compass. In a snapshot, one can infer information about the velocity of the charge from the position of the compass. Let us forget about all this and regard configuration as pure information about positions. All these examples use the strange (well, not so strange) properties of the electromagnetic field which is, in its own nature, a spacetime object as we shall argue below.]
>
> Then one has the evolution of the system which assigns a configuration as a function of time. This is called a *history* of the system. If the space of all possible configurations is called the *configuration space*, then a history is a curve in the configuration space.
> In homogeneous formalism, the *configuration* of a moving particle is a curve (well, actually, a trajectory) in spacetime and, as such, it represents the history of the system. A field over spacetime collects all possible measures of fields in space *and* time; accordingly, it encodes the evolution of the system as well, i.e. its history.

The configuration bundle is a more fundamental description than the local fields (exactly in the same sense in which the description in spacetime is more fundamental than the coordinate description). A section $X : M \to TM$ in the tangent bundle $T(M)$ associates a vector $X(x) \in T_x M$ to any point $x \in M$. This is an *absolute* fact; any observer will agree on that. Then any observer chooses different coordinates on M, they induce a different natural basis and define different components $X^\mu(x)$ for the vector field. Then any lab describes the configuration using different component functions $X^\mu(x)$, though they all agree on which vector they define.

> As discussed for the manifold structure on spacetime, if one just assumes to be able to map the local description of fields of one observer into the local description of fields of another observer, that eventually allows to define the configuration bundle and gives an absolute description of the physical situation. The map above do in fact define the transition maps of the configuration bundle and any bundle can be defined by glueing tubes together. We are not adding global information which may be unphysical. The global information is already encoded by the relation among local labs.

The configuration bundle encodes all kinematic information about the system. Once the configuration bundle is given, its global sections are in one-to-one correspondence with system configurations (on that spacetime M).

We shall always assume that our systems are variational. Hence dynamics is specified by choosing a Lagrangian and assuming Hamilton principle of stationary action.

Let us start defining the Lagrangian of a field theory. There are essentially two ways: one is defining the Lagrangian functional locally, the other is using jet bundles.

Of course, the functional approach seems easier at first sight (see[2]). A Lagrangian is defined as a map from the functional space of configurations into local m-forms on M which can be integrated on a spacetime volume D to define the action. One has

$$A_D[\sigma] = \int_D \boldsymbol{L}[\sigma] \tag{1.3.1}$$

Notice that $\boldsymbol{L}[\sigma]$ is a m-form over M, i.e. a suitable object to be integrated over m-dimensional $D \subset M$.

> Let us define a *k-region* of spacetime to be a compact sub-manifold $D \subset M$ of dimension k with boundary $\partial D \subset M$ which is a compact sub-manifold of dimension $(k-1)$. Of course k is an integer $k \geq 1$.
>
> The notion of k-region D will be useful hereafter since it is a good integration region for k-forms and in view of Stokes theorem it prevents problems when one wants to integrate over the boundary ∂D. In particular, we shall use m-regions to integrate the Lagrangian to define the action.

Although the Lagrangian functional seems easier to be defined, one soon finds out a lot of problems. We shall need to specify some regularity conditions on the functional. For it, one would need a detailed knowledge of its domain, i.e. the functional space of configurations. It is well known that, depending on the regularity conditions required, functional spaces have different structures. It is not always easy to guarantee that the action integral is well defined, especially considering that we are not working with the relatively easy case of functions in $\mathbb{R}^m$ but often we shall need to consider tensor fields over a manifold (which, even if one decides to work locally, are equivalence classes of functions). Moreover, often one needs the fields to be a vector space in order to discuss a topological and differential structure on them. This is not always the case, e.g. metrics of a given manifold are not a vector space or an affine space.

All this preliminary work is necessary when one is interested in applying the analytical techniques which are necessary to obtain existence results. Although this is an important mathematical problem, in the cases we shall consider hereafter, it is hardly ever the case.

Moreover, in all physical cases we shall consider, the actions are definitely not generic functionals. For example, the action integrand (i.e. the Lagrangian functional) does depend on fields only locally. For example, there are no Lagrangians which depend on integrals of configurations or feel the values of fields at far away points. Even local quantities enter in a finite number: Lagrangians depending on all infinite derivatives of fields (which would be somehow *local*) will never be considered. In other words, the Lagrangian functionals of physical interest in fact depend only on a finite number of parameters (at any spacetime point), not on the infinitely many degrees of freedom encoded into the functional space of configurations!

The second strategy is using jet bundles. The advantage is that most of the objects to be defined are maps between smooth finite dimensional ▷*Jet Bundles* manifolds and, in any event, one finally obtains the existence of the action integral without resorting to "exotic" functional spaces. This produces the correct field equations and allow a detailed discussion of conservation laws. Locally, it reduces to the *naive* discussion (which ignores at first all problems from regularity conditions). Of course, jet bundles need some preliminary work to be defined as well.

We shall follow this second strategy. We shall also stress when most of the following results in this Chapter can be obtained using local expressions forgetting about the global geometry of jet bundles.

A *k-order Lagrangian* is a horizontal *m*-form $\boldsymbol{L}$ on $J^k\mathcal{C}$. Here the bundle $J^k\mathcal{C}$ is the *k*-jet prolongation of the configuration bundle $\mathcal{C}$. If (x^μ, y^i) are fibered coordinates on the configuration bundle $\mathcal{C}$, they induce fibered *coordinates* $(x^\mu, y^i, y^i_\mu, \ldots, y^i_{\mu_1\ldots\mu_k})$ on $J^k\mathcal{C}$. The coordinates $y^i_{\mu_1\ldots\mu_p}$ represent the *p*-order partial detivatives of fields and are hence assumed to be symmetric with respect to lower indices.

Locally, a *k*-order Lagrangian is expressed by

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu, \ldots, y^i_{\mu_1\ldots\mu_k})\, d\sigma \tag{1.3.2}$$

We shall write shortly $\boldsymbol{L} = L(j^k\sigma)\, d\sigma$ where $j^k\sigma : M \to J^k\mathcal{C}$ denotes the *k*-order prolongation of a configuration σ to $J^k\mathcal{C}$.

> There are other frameworks to describe the same thing. The Lagrangian can be defined as a bundle map $L : J^kC \to A_m(M)$ that associates an *m*-form to a point in the jet bundle[3]. Or, in other settings, a Lagrangian is a cohomology class of *m*-forms[4]. Or, finally, one can choose different forms as representatives of the cohomology class, e.g. a representative which is not horizontal and still defines the same action. An example of this is the Poincaré-Cartan form setting we also discuss below since it highlights some properties which are useful to discuss Hamiltonian structure of field theory.
> By now we discuss simple issues for which different approaches are equivalent. Hence we get stuck to one definition of Lagrangian, only.

In what will follow, we shall always deal with first and second order Lagrangians. A first order Lagrangian is locally simply

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu)\, d\sigma \tag{1.3.3}$$

while a second order Lagrangian is locally

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu})\, d\sigma \tag{1.3.4}$$

Let us stress that, unless we explicitly say differently, we shall always consider *global* Lagrangians. A global Lagrangian produces global field equations. There are examples of local Lagrangians which produce global equations, though these cases need extra care to be dealt. When the Lagrangian is global, there is nothing to be checked and the equations will be global. We shall soon see some explicit examples.

Before discussing field equations, let us discuss Hamilton stationary action principle from a global viewpoint. Let us start by defining the action on a *Hamilton principle* *m*-region $D \subset M$ along a configuration σ

$$A_D[\sigma] := \int_D (j^k\sigma)^* \boldsymbol{L} \tag{1.3.5}$$

Notice that $(j^k\sigma)^*\boldsymbol{L}$ is in fact an *m*-form on M that can be integrated on D.

> We shall not even really need the action integral to exist! As far as we are concerned, one just needs to specify the dependence of the action from fields and their derivatives. The quantity $A_D[\sigma]$ does not need (by now) to be a number. If one defines the Lagrangian as a formal series, then compositions and integrals are well defined and we can go through all the arguments hereafter even if $A_D[\sigma]$ is a formal series and even if one could not control its convergence. At a classical level, one has to stress that the *value* of the action does not play any relevant role, not if one is interested in the Lagrangian to find field equations.

In any event, the m-region D is compact so that the action is well defined as a number.

Let us now define a *deformation* (or a *variation*) to be a vertical vector field X on the configuration bundle $\mathcal{C}$. Its prolongation to the k-order jet bundle $J^k\mathcal{C}$ will be denoted by $j^k X$. The support of the deformation X will be defined as the closure of the projection in M of set where X is non-zero. It will be denoted by $\mathrm{supp}(X)$. Let us stress that $\mathrm{supp}(X) \subset M$ is in the spacetime.

A deformation is locally given by

$$X = X^i(x, y)\partial_i \tag{1.3.6}$$

which is prolonged to

$$j^k X = X^i(x, y)\partial_i + d_\mu X^i(x, y)\partial_i^\mu + d_{\mu\nu} X^i(x, y)\partial_i^{\mu\nu} + \ldots \tag{1.3.7}$$

For notational convenience (and to adapt to standard notation), the components of a deformation along a configuration σ (locally given by $y^i(x)$) are also denoted by $\delta y^i(x) := X^i(x, y(x))$. For the prolongation, we have

$$\delta y^i := X^i \qquad \delta y^i_\mu := d_\mu \delta y^i = d_\mu X^i =: X^i_\mu \qquad \delta y^i_{\mu\nu} := d_{\mu\nu}\delta y^i = d_{\mu\nu} X^i =: X^i_{\mu\nu} \tag{1.3.8}$$

We have to stress that for us δy^i is here one object (namely, a function of x), not an operator δ acting on the fields y^i, as it is usual in literature.

Once a deformation X is given, one has its flow Φ_s; the prolongation $j^k\Phi_s$ is also the flow of the prolonged deformation $j^k X$. As usual, the vector field X contains the same information as its flow Φ_s. With it we can define a lot of things.

First of all, we can drag any configuration σ to define a 1-parameter family of configurations

$$\sigma_s = \Phi_s \circ \sigma \tag{1.3.9}$$

Since everything is global, then the sections σ_s are global at least for s in some neighbourhood of $s = 0 \in \mathbb{R}$.

Even if the vector field X is global, there is nothing guaranteeing that its flow extends to all $s \in \mathbb{R}$. For example, on $M = \mathbb{R}$ with coordinate x one can consider a global vector field $X = f(x)\partial$. A curve $\gamma : \mathbb{R} \to M : s \mapsto x(s)$ is an integral curve of X iff

$$\frac{dx}{ds} = f(x) \qquad \Rightarrow s - s_0 = \int_{x_0}^x \frac{dx}{f(x)} \tag{1.3.10}$$

In the special case $f(x) = x^2$ then $s - s_0 = -\frac{1}{x}$ from which one obtains the integral curve

$$x(s) = -\frac{1}{s - \frac{1}{x_0}} = \frac{x_0}{1 - s x_0} \tag{1.3.11}$$

where the integration constant s_0 has been fixed so that one has $x(0) = x_0$. Of course, the integral curve is defined in the range $s \in (-\infty, \frac{1}{x_0})$ and when $s \to \frac{1}{x_0}$ then $x(s) \to +\infty$. The integral curve goes from $x = x_0$ to $x = +\infty$ in a finite time s.

Then we can define

$$\delta_X A_D[\sigma] := \frac{d}{ds}\left(A_D[\sigma_s]\right)\Big|_{s=0} = \int_D \frac{d}{ds}\left((j^k\sigma_s)^*\boldsymbol{L}\right)\Big|_{s=0} \tag{1.3.12}$$

We are finally able to state *Hamilton principle*. Mathematically speaking, the principle has the form of a definition for *critical configurations*.

A configuration σ is *critical* iff for any m-region D and any deformation X compactly supported in D, one has

$$\delta_X A_D[\sigma] = 0 \tag{1.3.13}$$

We have to stress that Hamilton principle is completely expressed intrinsically, i.e. without any reference to coordinates or local objects.

Local Euler–Lagrange equations

The condition for a configuration to be critical can now be expanded in coordinates. Let us first do it locally, expanding everything in a coordinate system. Let us consider a second order Lagrangian.

The condition of Hamilton principle expands as

$$\delta_X A_D[\sigma] = \int_D \left(\frac{\partial L}{\partial y^i} \delta y^i + \frac{\partial L}{\partial y^i_\mu} d_\mu \delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_{\mu\nu} \delta y^i \right) \, d\sigma = 0 \tag{1.3.14}$$

> Notice as *formally* δ_X acts as a derivative and, by an abuse of notation, it is often denoted by δ for short. Equation (1.3.14) can be regarded as the chain rule. All in all, we shall see that the integrand will be recast as $j^k X \lrcorner d\boldsymbol{L}$ and the contraction $j^k X \lrcorner (\cdot)$ is well known to be a derivative of the exterior algebra.

One can do some integrations by parts

$$
\begin{aligned}
\delta_X A_D(\sigma) &= \int_D \left(\frac{\partial L}{\partial y^i} \delta y^i - d_\mu \frac{\partial L}{\partial y^i_\mu} \delta y^i - d_\mu \frac{\partial L}{\partial y^i_{\mu\nu}} d_\nu \delta y^i \right) \, d\sigma + \int_D d_\mu \left(\frac{\partial L}{\partial y^i_\mu} \delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_\nu \delta y^i \right) \, d\sigma = \\
&= \int_D \left(\frac{\partial L}{\partial y^i} - d_\mu \frac{\partial L}{\partial y^i_\mu} + d_{\mu\nu} \frac{\partial L}{\partial y^i_{\mu\nu}} \right) \delta y^i \, d\sigma + \int_{\partial D} \left(\left(\frac{\partial L}{\partial y^i_\mu} - d_\nu \frac{\partial L}{\partial y^i_{\mu\nu}} \right) \delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_\nu \delta y^i \right) \, d\sigma_\mu = 0
\end{aligned}
\tag{1.3.15}
$$

Let us define the *naive momenta*

$$p_i := \frac{\partial L}{\partial y^i} \qquad p_i^\mu := \frac{\partial L}{\partial y^i_\mu} \qquad p_i^{\mu\nu} := \frac{\partial L}{\partial y^i_{\mu\nu}} \tag{1.3.16}$$

and the *formal momenta*

$$f_i^\mu := p_i^\mu - d_\nu p_i^{\mu\nu} \qquad f_i^{\mu\nu} := p_i^{\mu\nu} \tag{1.3.17}$$

With this notation the above formula reads as

$$\delta_X A_D[\sigma] = \int_D \left(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} \right) \delta y^i \, d\sigma + \int_{\partial D} \left(f_i^\mu \delta y^i + f_i^{\mu\nu} d_\nu \delta y^i \right) \, d\sigma_\mu = 0 \tag{1.3.18}$$

Since the deformation X is compactly supported in D (i.e. it vanishes on ∂D together with all its derivatives) one has

$$\delta_X A_D(\sigma) = \int_D \left(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} \right) \delta y^i \, d\sigma = 0 \tag{1.3.19}$$

Since the m-region is arbitrary the integrand must be zero

$$\left(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu}\right) \delta y^i \, d\sigma = 0 \tag{1.3.20}$$

Since $d\sigma$ is a basis of the m-forms then the coefficient must vanish. Since the deformation is arbitrary one has

$$\mathbb{E}_i(L) := p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} = 0 \tag{1.3.21}$$

These are PDE generically of order 4. They are called *Euler–Lagrange equations* or *field equations*.

 For a first order Lagrangian, Euler–Lagrange equations are obtained setting $p_i^{\mu\nu} = 0$

$$\mathbb{E}_i(L) := p_i - d_\mu p_i^\mu = 0 \tag{1.3.22}$$

 which are PDE generically of order 2.

 Similar formula can be obtained at any order.

 These are quite useless formulae. They are interesting because they are general: for example, one can easily see that a k-order Lagrangian necessarily induces equations of order at most $2k$. However, often the Lagrangian depends on relatively complicated functions of the derivatives of fields and it is easier to follow this procedure from scratch rather than computing the naive momenta.

 Moreover, often one wishes to write equations in terms of covariant combinations to maintain manifest general covariance and such combinations may be difficult to be collected in equations written in terms of naive momenta.

Global Euler–Lagrange equations

 As we noticed, the variation of the action can be written intrinsically as

$$\delta_X A_D[\sigma] = \int_D (j^k\sigma)^* \left((j^k X)\lrcorner d\boldsymbol{L}\right) \tag{1.3.23}$$

This formula can be given a beautiful geometric meaning. If one notices that, since $j^k X$ is vertical, one has $j^k X \lrcorner \boldsymbol{L} = 0$. Then the action variation (1.3.23) can be written as

$$\delta_X A_D[\sigma] = \int_D (j^k\sigma)^* \left((j^k X)\lrcorner d\boldsymbol{L} + d\left((j^k X)\lrcorner \boldsymbol{L}\right)\right) = \int_D (j^k\sigma)^* \left(\pounds_{j^k X} \boldsymbol{L}\right) \tag{1.3.24}$$

and Hamilton principle can be stated by saying that, at critical sections, the Lagrangian is invariant under deformations.

 If the Lagrangian is global then field equations are global.

 We could prove it at any order. For the sake of simplicity, let us prove it for second order Lagrangians (which include also first order ones just by setting the second order naive momenta $p_i^{\alpha\beta}$ everywhere to zero). The configuration bundle is covered by local laboratories, each identified with a fibered chart. Two nearby charts are related by transition maps in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = y'^i(x,y) \end{cases} \Rightarrow \begin{cases} y'^i_\mu = \bar{J}^\alpha_\mu \left(J^i_\alpha + J^i_k y^k_\alpha\right) = \bar{J}^\alpha_\mu d_\alpha y'^i \\ y'^i_{\mu\nu} = \bar{J}^\alpha_\mu d_\alpha \left(\bar{J}^\beta_\nu d_\beta y'^i\right) \end{cases} \tag{1.3.25}$$

Let us assume standard notation for Jacobians. The Lagrangian is global iff we have

$$L'(x'^{\mu}, y'^i, y'^i_\mu, y'^i_{\mu\nu})\, d\sigma' = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu})\, d\sigma \tag{1.3.26}$$

This can be recast in terms of the Lagrangian density

$$J L'(x'^\mu, y'^i, y'^i_\mu, y'^i_{\mu\nu}) = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu}) \tag{1.3.27}$$

from which it follows

$$\begin{cases} J\left(p'_k J^k_i + p'^\alpha_k d'_\alpha J^k_i + p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i\right) = p_i \\ J\left(\bar J^\mu_\alpha p'^\alpha_k J^k_i + p'^{\alpha\beta}_k \left(d'_\alpha\left(\bar J^\mu_\beta J^k_i\right) + \bar J^\mu_\alpha d'_\beta J^k_i\right)\right) = p^\mu_i \\ J \bar J^\mu_\alpha \bar J^\nu_\beta p'^{\alpha\beta}_k J^k_i = p^{\mu\nu}_i \end{cases} \tag{1.3.28}$$

The formal momenta transform as

$$\begin{cases} f^\mu_i = p^\mu_i - d_\nu p^{\mu\nu}_i = J \bar J^\mu_\alpha p'^\alpha_k J^k_i + J p'^{\alpha\beta}_k \left(d'_\alpha\left(\bar J^\mu_\beta J^k_i\right) + \bar J^\mu_\alpha d'_\beta J^k_i\right) - d_\nu\left(J \bar J^\mu_\alpha \bar J^\nu_\beta p'^{\alpha\beta}_k J^k_i\right) = \\ \quad = \underline{J \bar J^\mu_\alpha p'^\alpha_k J^k_i} + J p'^{\alpha\beta}_k \left(d'_\alpha\left(\bar J^\mu_\beta J^k_i\right) + \bar J^\mu_\alpha d'_\beta J^k_i\right) - d_\nu\left(J \bar J^\nu_\beta\right) \bar J^\mu_\alpha J^k_i p'^{\alpha\beta}_k - J d'_\alpha\left(\bar J^\mu_\beta J^k_i\right)p'^{\alpha\beta}_k - \underline{J \bar J^\mu_\alpha d_\beta p'^{\alpha\beta}_k J^k_i} = \\ \quad = J \bar J^\mu_\alpha f'^\alpha_k J^k_i + J p'^{\alpha\beta}_k \bar J^\mu_\alpha d'_\beta J^k_i + {\color{blue}J J^\rho_{\sigma\beta} J^\sigma_\rho \bar J^\mu_\alpha J^k_i p'^{\alpha\beta}_k} - {\color{blue}J \bar J^\nu_{\beta\lambda} J^\lambda_\nu \bar J^\mu_\alpha J^k_i p'^{\alpha\beta}_k} = J \bar J^\mu_\alpha\left(f'^\alpha_k J^k_i + f'^{\alpha\beta}_k d'_\beta J^k_i\right) \\ f^{\mu\nu}_i = J \bar J^\mu_\alpha \bar J^\nu_\beta f'^{\alpha\beta}_k J^k_i \end{cases} \tag{1.3.29}$$

Let us also compute transformation rules for the contact forms

$$\begin{cases} \omega'^i = dy'^i - y'^i_\mu dx'^\mu = J^i_k\, dy^k + J^i_\nu\, dx^\nu - \bar J^\lambda_\mu\left(J^i_\lambda + J^i_k y^k_\lambda\right) dx'^\mu = J^i_k\, dy^k + {\color{red}J^i_\nu\, dx^\nu - J^i_\lambda\, dx^\lambda} - J^i_k y^k_\lambda\, dx^\lambda = J^i_k \omega^k \\ \omega'^i_\alpha = dy'^i_\alpha - y'^i_{\alpha\mu} dx'^\mu = \bar J^\lambda_\alpha d\left(J^i_\lambda + J^i_k y^k_\lambda\right) - \bar J^\lambda_\alpha d_\beta\left(J^i_\lambda + J^i_k y^k_\lambda\right) dx^\beta = \\ \quad = \bar J^\lambda_\alpha\left(J^i_{k\lambda} dy^k + J^i_{lk} y^k_\lambda dy^l + J^i_k\, dy^k_\lambda\right) - \bar J^\lambda_\alpha\left(J^i_{k\lambda} y^k_\beta + J^i_{lk} y^l_\beta y^k_\lambda + J^i_k y^k_{\beta\lambda}\right) dx^\beta = \\ \quad = d'_\alpha J^i_l\, \omega^l + J^i_k\, \omega^k_\lambda\, \bar J^\lambda_\alpha \end{cases} \tag{1.3.30}$$

Finally, let us prove the following formula

$$d_\mu\left(J \bar J^\mu_\alpha\right) = J\left(\bar J^\rho_\beta d_\mu J^\beta_\rho \bar J^\mu_\alpha + \bar J^\mu_{\alpha\beta} J^\beta_\mu\right) = J\left({\color{red}-\bar J^\rho_{\alpha\beta} J^\beta_\rho} + {\color{red}\bar J^\mu_{\alpha\beta} J^\beta_\mu}\right) = 0 \tag{1.3.31}$$

Then one has

$$\mathbb{E}_i(L) = p_i - d_\mu p^\mu_i + d_{\mu\nu} p^{\mu\nu}_i = J p'_k J^k_i + J p'^\alpha_k d'_\alpha J^k_i + J p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i - d_\mu\left(J\left(\bar J^\mu_\alpha p'^\alpha_k J^k_i + p'^{\alpha\beta}_k\left(d'_\alpha\left(\bar J^\mu_\beta J^k_i\right) + \bar J^\mu_\alpha d'_\beta J^k_i\right)\right)\right) +$$
$$+ d_{\mu\nu}\left(J \bar J^\mu_\alpha \bar J^\nu_\beta p'^{\alpha\beta}_k J^k_i\right) = \underline{J p'_k J^k_i} + {\color{red}J p'^\alpha_k d'_\alpha J^k_i} - d_\mu J \bar J^\mu_\alpha p'^\alpha_k J^k_i - J J^\beta_\mu \bar J^\mu_{\alpha\beta} p'^\alpha_k J^k_i - \underline{J d'_\alpha p'^\alpha_k J^k_i} - {\color{red}J p'^\alpha_k d'_\alpha J^k_i} +$$
$$+ J p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i - d_\mu\left({\color{blue}J p'^{\alpha\beta}_k \bar J^\nu_\beta d_\nu\left(\bar J^\mu_\alpha J^k_i\right)} + J p'^{\alpha\beta}_k \bar J^\mu_\alpha d'_\beta J^k_i\right) + d_\mu\left(d_\nu\left(J \bar J^\nu_\beta p'^{\alpha\beta}_k\right) \bar J^\mu_\alpha J^k_i + {\color{blue}J \bar J^\nu_\beta p'^{\alpha\beta}_k d_\nu\left(\bar J^\mu_\alpha J^k_i\right)}\right) =$$
$$= J\left(p'_k - d'_\alpha p'^\alpha_k\right) J^k_i - J J^\beta_\lambda J^\lambda_{\beta\mu} \bar J^\mu_\alpha p'^\alpha_k J^k_i - J J^\beta_\mu \bar J^\mu_{\alpha\beta} p'^\alpha_k J^k_i + J p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i - d_\mu\left(J p'^{\alpha\beta}_k \bar J^\mu_\alpha d'_\beta J^k_i\right) +$$
$$+ d_\mu\left({\color{blue}-J J^\rho_{\sigma\beta} J^\sigma_\rho p'^{\alpha\beta}_k \bar J^\mu_\alpha J^k_i} + {\color{blue}J J^\sigma_\rho \bar J^\rho_{\sigma\beta} p'^{\alpha\beta}_k \bar J^\mu_\alpha J^k_i}\right) + d_\mu\left(J \bar J^\mu_\alpha J^k_i\right) d'_\beta p'^{\alpha\beta}_k + J d'_\alpha p'^{\alpha\beta}_k J^k_i =$$
$$= J\left(p'_k - d'_\alpha p'^\alpha_k + d'_{\alpha\beta} p'^{\alpha\beta}_k\right) J^k_i + {\color{red}J \bar J^\mu_{\beta\alpha} J^\beta_\mu p'^\alpha_k J^k_i} - {\color{red}J J^\beta_\mu \bar J^\mu_{\alpha\beta} p'^\alpha_k J^k_i} + {\color{green}J p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i} - {\color{green}J p'^{\alpha\beta}_k d'_{\alpha\beta} J^k_i} +$$
$$- {\color{green}J d'_\alpha p'^{\alpha\beta}_k d'_\beta J^k_i} - \boxed{d_\mu\left(J \bar J^\mu_\alpha\right) p'^{\alpha\beta}_k d'_\beta J^k_i} + \boxed{\color{red}d_\mu\left(J \bar J^\mu_\alpha\right) J^k_i d'_\beta p'^{\alpha\beta}_k} + {\color{green}J d'_\beta J^k_i d'_\alpha p'^{\alpha\beta}_k} =$$
$$= J\left(p'_k - d'_\alpha p'^\alpha_k + d'_{\alpha\beta} p'^{\alpha\beta}_k\right) J^k_i = J \mathbb{E}'_k(L') J^k_i \tag{1.3.32}$$

Then we can define the form

$$\mathbb{E}(L) := \mathbb{E}_i(L)\, \omega^i \wedge\, d\sigma \tag{1.3.33}$$

which is, in fact, global since

$$\mathbb{E}(L) = \mathbb{E}_i(L)\omega^i \wedge\, d\sigma = \mathbb{E}'_k(L')J^k_i\omega^i \wedge\, d\sigma' = \mathbb{E}'_i(L')\omega'^k \wedge\, d\sigma' = \mathbb{E}'(L') \tag{1.3.34}$$

One can also define the global form

$$\mathbb{F}(L) := f^\mu_i \omega^i \wedge\, d\sigma_\mu + f^{\mu\nu}_i \omega^i_\nu \wedge\, d\sigma_\mu \tag{1.3.35}$$

which is global since

$$\begin{aligned}
\mathbb{F}(L) =& f^\mu_i \omega^i \wedge\, d\sigma_\mu + f^{\mu\nu}_i \omega^i_\nu \wedge\, d\sigma_\mu = \\
=& J\bar{J}^\mu_\alpha f'^\alpha_k J^k_i\omega^i \wedge\, d\sigma_\mu + J\bar{J}^\mu_\alpha f'^{\alpha\beta}_k d'_\beta J^k_i\omega^i \wedge\, d\sigma_\mu + J\bar{J}^\mu_\alpha \bar{J}^\nu_\beta f'^{\alpha\beta}_k J^k_i\omega^i_\nu \wedge\, d\sigma_\mu = \\
=& f'^\alpha_k \omega'^i \wedge\, d\sigma'_\alpha + f'^{\alpha\beta}_k \left(d'_\beta J^k_i\omega^i + \bar{J}^\nu_\beta J^k_i\omega^i_\nu\right) \wedge\, d\sigma'_\alpha = f'^\alpha_k \omega'^i \wedge\, d\sigma'_\alpha + f'^{\alpha\beta}_k \omega'^k_\beta \wedge\, d\sigma'_\alpha = \mathbb{F}'(L')
\end{aligned} \tag{1.3.36}$$

Notice that we have the following equation

$$(j^k X)\lrcorner d\boldsymbol{L} = X\lrcorner\mathbb{E}(L) + d\left((j^{k-1}X)\lrcorner\mathbb{F}(L)\right) \tag{1.3.37}$$

This equation holds true *before* being computed along a configurarion, it is a property of objects on jet prolongations of the configuration bundle.

This is called the *first variation formula* and it encodes (globally) many properties of variational principles. The form $\mathbb{E}(L)$ is called the *Euler–Lagrange part* of $\boldsymbol{L}$. A form $\mathbb{F}(L)$ for which the first variation formula holds true is called the *Liouville* part of $\boldsymbol{L}$. The first variation formula encodes global field equations which are intrinsically written in terms of the Euler–Lagrange part of $\boldsymbol{L}$. A configuration σ is critical iff one has

$$\mathbb{E}_i(L) = 0 \tag{1.3.38}$$

Uniqueness results

Given a Lagrangian, we look for forms which satisfy the first variation formula. If we consider a form which has contact degree 2 (or higher), e.g. $\boldsymbol{e} = e^{\mu\nu}_{ij}\omega^i \wedge \omega^j \wedge\, d\sigma_{\mu\nu}$, and we modify by it the Euler–Lagrange part $\mathbb{E}'(L) = \mathbb{E}(L) + \boldsymbol{e}$ then the form $\mathbb{E}'(L)$ satisfies the first variation formula as well. In fact, the contraction along the vertical vector field X leaves a contact form, which will be eventually pulled back along a holonomic section to give zero. The same argument for adding $\boldsymbol{f} = f^{\lambda\mu\nu}_{ij}\omega^i \wedge \omega^j \wedge\, d\sigma_{\mu\nu\lambda}$ to $\mathbb{F}(L)$.

A similar argument applies to horizontal forms (i.e. forms of contact degrees 0). Both $\mathbb{E}(L)$ and $\mathbb{F}(L)$ appear everywhere above and below contracted with a vertical vector field. Thus the addition of any horizontal term is irrelevant. As a consequence both $\mathbb{E}(L)$ and $\mathbb{F}(L)$ are to be considered as forms of contact degree 1 and we shall denote by $\approx$ the equality up to horizontal and contact forms of degree 2 or higher.

As a matter of fact, the first variation formula has to be understood as an equality up to forms which have contact degree other than 1. Accordingly, the Euler–Lagrange and Liouville parts are defined up to forms which have contact degree other than 1. This is true because of the usage done of such forms. Neither field equations or conservation laws we shall discuss hereafter will be affected by forms which have contact degree other than 1.

If one adds a term with contact degree 1 to $\mathbb{E}(L)$ then the first variation formula does not hold true anymore. Thus the Euler–Lagrange part of the Lagrangian is uniquely determined by the first variation formula up to forms of contact degree other than 1.

If one considers instead $\mathbb{F}'(L) = \mathbb{F}(L) - d\boldsymbol{\alpha}$ with, e.g., $\boldsymbol{\alpha} = \frac{1}{2}\alpha_i^{\mu\nu}\omega^i \wedge d\sigma_{\mu\nu}$ then we have

$$
\begin{aligned}
d\boldsymbol{\alpha} &\approx -d_\nu\alpha_i^{\mu\nu}\omega^i \wedge d\sigma_\mu - \alpha_i^{\mu\nu}\omega_\nu^i \wedge d\sigma_\mu \\
(j^1X)\lrcorner d\boldsymbol{\alpha} &= -d_\nu\alpha_i^{\mu\nu}\delta y^i \, d\sigma_\mu - \alpha_i^{\mu\nu}d_\nu\delta y^i \, d\sigma_\mu = -d_\nu\left(\alpha_i^{\mu\nu}\delta y^i\right)d\sigma_\mu \approx -d\left(\frac{1}{2}\alpha_i^{\mu\nu}\delta y^i \, d\sigma_{\mu\nu}\right) \qquad \Rightarrow d\left((j^1X)\lrcorner d\boldsymbol{\alpha}\right) \approx 0
\end{aligned}
\tag{1.3.39}
$$

Thus $\mathbb{F}'(L)$ and $\mathbb{F}(L)$ both define the same first variation formula and field equations, while, as we shall see below, they define two different Noether currents. For this reason, one has to consider $(\mathbb{E}(L), \mathbb{F}(L))$ and $(\mathbb{E}(L), \mathbb{F}'(L))$ as two different pairs for which the first variation formula holds true.

While the Euler–Lagrange part appearing in the first variation formula is essentially unique (i.e. unique up to forms of contact degree other than 1), one can add to the Liouville part any exact form and the first variation formula still holds true. However, one can consider Liouville parts with the additional properties of being symmetric with respect to upper indices, namely to be in the form

$$
\tilde{\mathbb{F}}(L) = \tilde{f}_i^\mu\omega^i \wedge d\sigma_\mu + \tilde{f}_i^{\mu\nu}\omega_\nu^i \wedge d\sigma_\mu
\tag{1.3.40}
$$

with $\tilde{f}_i^{\mu\nu} = \tilde{f}_i^{(\mu\nu)}$. Such Liouville parts are called *reduced*. While Liouville parts are essentially non-unique, one can show that reduced Liouville parts are unique.

Let us suppose that $\mathbb{F}(L) = f_i^\mu\omega^i \wedge d\sigma_\mu + f_i^{\mu\nu}\omega_\nu^i \wedge d\sigma_\mu$ is a Liouville part for the Lagrangian $\boldsymbol{L}$. Then one can split $\mathbb{F}(L)$ as $\mathbb{F}(L) = \tilde{\mathbb{F}}(L) + \Delta$ by setting

$$
\tilde{\mathbb{F}}(L) = \left(f_i^\mu - d_\nu f_i^{[\mu\nu]}\right)\omega^i \wedge d\sigma_\mu + f_i^{(\mu\nu)}\omega_\nu^i \wedge d\sigma_\mu \qquad \Delta = f_i^{[\mu\nu]}\omega_\nu^i \wedge d\sigma_\mu + d_\nu f_i^{[\mu\nu]}\omega^i \wedge d\sigma_\mu
\tag{1.3.41}
$$

where $\tilde{\mathbb{F}}(L)$ is now reduced. Hence let us set $\tilde{f}_i^\mu := f_i^\mu - d_\nu f_i^{[\mu\nu]}$ and $\tilde{f}_i^{\mu\nu} := f_i^{(\mu\nu)}$ so that one has

$$
\tilde{\mathbb{F}}(L) = \tilde{f}_i^\mu\omega^i \wedge d\sigma_\mu + \tilde{f}_i^{\mu\nu}\omega_\nu^i \wedge d\sigma_\mu
\tag{1.3.42}
$$

Since we can always write Δ as

$$
d\left(\frac{1}{2}f_i^{[\mu\nu]}\omega^i \wedge d\sigma_{\mu\nu}\right) \approx \frac{1}{2}d_\lambda f_i^{[\mu\nu]}dx^\lambda \wedge \omega^i \wedge d\sigma_{\mu\nu} - \frac{1}{2}f_i^{[\mu\nu]}\omega_\alpha^i \wedge dx^\alpha \wedge d\sigma_{\mu\nu} = -d_\nu f_i^{[\mu\nu]}\omega^i \wedge d\sigma_\mu - f_i^{[\mu\nu]}\omega_\nu^i \wedge d\sigma_\mu = -\Delta
\tag{1.3.43}
$$

i.e. Δ is a pure divergence and both $\mathbb{F}(L)$ and $\tilde{\mathbb{F}}(L)$ define the same first variation formula.

Thus any Liouville part $\mathbb{F}(L)$ can be replaced by a reduced one $\tilde{\mathbb{F}}(L)$.

Moreover, let us suppose that $\tilde{\mathbb{F}}'(L)$ is another reduced Liouville part equivalent to $\mathbb{F}(L)$. That means that $\tilde{\mathbb{F}}'(L) = \tilde{\mathbb{F}}(L) - d\alpha$ for some $(m-1)$-form $\boldsymbol{\alpha} = \frac{1}{2}\alpha_i^{\mu\nu}\omega^i \wedge d\sigma_{\mu\nu}$.

Thus one has:

$$
\tilde{f}_i'^\mu\omega^i \wedge d\sigma_\mu + \tilde{f}_i'^{\mu\nu}\omega_\nu^i \wedge d\sigma_\mu = \left(\tilde{f}_i^\mu + d_\nu\alpha_i^{\mu\nu}\right)\omega^i \wedge d\sigma_\mu + \left(\tilde{f}_i^{\mu\nu} + \alpha_i^{\mu\nu}\right)\omega_\nu^i \wedge d\sigma_\mu
\tag{1.3.44}
$$

However, since both $\tilde{\mathbb{F}}'(L)$ and $\tilde{\mathbb{F}}(L)$ have to be reduced one has

$$
0 = \tilde{f}'^{[\mu\nu]} = \tilde{f}_i^{[\mu\nu]} + \alpha_i^{[\mu\nu]} = \alpha_i^{[\mu\nu]}
\tag{1.3.45}
$$

Hence $\boldsymbol{\alpha} = 0$ and $\tilde{\mathbb{F}}'(L) = \tilde{\mathbb{F}}(L)$.

Thus any Lagrangian $\boldsymbol{L}$ defines two forms $\mathbb{E}(L)$ and $\mathbb{F}(L)$ with $\mathbb{F}(L)$ reduced, which are both uniquely determined and such that the first variation formula holds true.

Non-globality of Liouville part at order 3

Although we shall always consider field theories of order at most 2, let us remark that the situation with the Liouville part becomes involved at higher orders. Let us consider a third order Lagrangian

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu}, y^i_{\mu\nu\rho})\, d\sigma \tag{1.3.46}$$

Then by standard variation we have

$$
\begin{aligned}
\delta \boldsymbol{L} :=& \delta_X \boldsymbol{L} = \left(p_i \delta y^i + p_i^\mu d_\mu \delta y^i + p_i^{\mu\nu} d_{\mu\nu}\delta y^i + p_i^{\mu\nu\rho} d_{\mu\nu\rho}\delta y^i \right)\, d\sigma = \\
=& \left(p_i - d_\mu p_i^\mu + d_{\mu\nu}p_i^{\mu\nu} - d_{\mu\nu\rho}p_i^{\mu\nu\rho} \right) \delta y^i\, d\sigma + d\left[\left(\left(p_i^\mu - d_\nu p_i^{\mu\nu} + d_{\nu\rho}p_i^{\mu\nu\rho} \right) \delta y^i + \left(p_i^{\mu\nu} - d_\rho p_i^{\mu\nu\rho} \right) d_\nu \delta y^i + p_i^{\mu\nu\rho} d_{\nu\rho}\delta y^i \right)\, d\sigma_\mu \right]
\end{aligned}
\tag{1.3.47}
$$

and we should define

$$
\begin{aligned}
\mathbb{E}(L) &= \left(p_i - d_\mu p_i^\mu + d_{\mu\nu}p_i^{\mu\nu} - d_{\mu\nu\rho}p_i^{\mu\nu\rho} \right) \omega^i \wedge\, d\sigma \\
\mathbb{F}(L) &= \left(p_i^\mu - d_\nu p_i^{\mu\nu} + d_{\nu\rho}p_i^{\mu\nu\rho} \right) \omega^i \wedge\, d\sigma_\mu + \left(p_i^{\mu\nu} - d_\rho p_i^{\mu\nu\rho} \right) \omega^i_\nu \wedge\, d\sigma_\mu + p_i^{\mu\nu\rho}\omega^i_{\nu\rho} \wedge\, d\sigma_\mu
\end{aligned}
\tag{1.3.48}
$$

Let us define $\mathbb{E}_i(L) = p_i - d_\mu p_i^\mu + d_{\mu\nu}p_i^{\mu\nu} - d_{\mu\nu\rho}p_i^{\mu\nu\rho}$ and one can show that $\mathbb{E}_i(L) = J\mathbb{E}'_k(L')J_i^k$. Accordingly, the Euler–Lagrange part $\mathbb{E}(L)$ is a global form. On the other hand, let us define the formal momenta as

$$f_i^\mu := p_i^\mu - d_\nu p_i^{\mu\nu} + d_{\nu\rho}p_i^{\mu\nu\rho} \qquad f_i^{\mu\nu} := p_i^{\mu\nu} - d_\rho p_i^{\mu\nu\rho} \qquad f_i^{\mu\nu\rho} := p_i^{\mu\nu\rho} \tag{1.3.49}$$

One can show that these objects transform differently from the coefficients of a form as $\mathbb{F} = f_i^\mu \omega^i \wedge\, d\sigma_\mu + f_i^{\mu\nu}\omega^i_\nu \wedge\, d\sigma_\mu + f_i^{\mu\nu\rho}\omega^i_{\nu\rho} \wedge\, d\sigma_\mu$. Thus the Liouville part as defined above in (1.3.48) is not a global form.

Two nearby charts are related by transition maps in the form

$$
\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = y'^i(x,y) \end{cases}
\Rightarrow
\begin{cases}
y'^i_\mu = \bar{J}^\alpha_\mu \left(J^i_\alpha + J^i_k y^k_\alpha \right) = \bar{J}^\alpha_\mu d_\alpha y'^i \\
y'^i_{\mu\nu} = \bar{J}^\alpha_\mu d_\alpha \left(\bar{J}^\beta_\nu d_\beta y'^i \right) = \bar{J}^\beta_{\mu\nu} \left(J^i_\beta + J^i_k y^k_\beta \right) + \bar{J}^\alpha_\mu \bar{J}^\beta_\nu \left(J^i_{\alpha\beta} + J^i_{\alpha k}y^k_\beta + J^i_{kj}y^j_\alpha y^k_\beta + J^i_k y^k_{\alpha\beta} \right) \\
y'^i_{\mu\nu\rho} = \bar{J}^\sigma_\rho d_\sigma \left(\bar{J}^\alpha_\mu d_\alpha \left(\bar{J}^\beta_\nu d_\beta y'^i \right) \right)
\end{cases}
\tag{1.3.50}
$$

Let us assume standard notation for Jacobians and compute transformation rules for the contact forms ▷*Standard Jacobians*

$$
\begin{cases}
\omega'^i \wedge\, d\sigma'_\mu = J J^i_k \bar{J}^\rho_\mu \omega^k \wedge\, d\sigma_\rho \\
\omega'^i_\alpha \wedge\, d\sigma'_\mu = J d'_\alpha J^i_k \bar{J}^\rho_\mu \omega^k \wedge\, d\sigma_\rho + J J^i_k \bar{J}^\lambda_\alpha \bar{J}^\rho_\mu \omega^k_\lambda \wedge\, d\sigma_\rho \\
\omega'^i_{\alpha\beta} \wedge\, d\sigma'_\mu = J \bar{J}^\rho_\mu dy'^i_{\alpha\beta} \wedge\, d\sigma_\rho - J y'^i_{\alpha\beta\lambda} \bar{J}^\rho_\mu dx'^\lambda \wedge\, d\sigma_\rho = J \bar{J}^\rho_\mu dy'^i_{\alpha\beta} \wedge\, d\sigma_\rho - J d_\lambda y'^i_{\alpha\beta} \bar{J}^\rho_\mu dx^\lambda \wedge\, d\sigma_\rho = \\
= J \bar{J}^\rho_\mu \left[\bar{J}^\epsilon_{\alpha\beta} \left(J^i_{k\epsilon} + J^i_{kl}y^l_\epsilon \right) + \bar{J}^\epsilon_\alpha \bar{J}^\sigma_\beta \left(J^i_{k\epsilon\sigma} + J^i_{\epsilon kl}y^l_\sigma + J^i_{klj}y^j_\epsilon y^l_\sigma + J^i_{kl}y^l_{\epsilon\sigma} \right) \right] \omega^k \wedge\, d\sigma_\rho + J \bar{J}^\rho_\mu \left(\bar{J}^\lambda_{\alpha\beta} J^i_k + \bar{J}^\epsilon_\alpha \bar{J}^\lambda_\beta J^i_{\epsilon k} + 2\bar{J}^\epsilon_\alpha \bar{J}^\lambda_\beta J^i_{kj}y^j_\epsilon \right) \omega^k_\lambda \wedge\, d\sigma_\rho + J \bar{J}^\rho_\mu \bar{J}^\sigma_\beta \bar{J}^\lambda_\alpha J^i_k \omega^k_{\lambda\sigma} \wedge\, d\sigma_\rho
\end{cases}
\tag{1.3.51}
$$

Thus the coefficients transform as

$$
\begin{cases}
f^\rho_k = J \left[f'^\mu_i J^i_k \bar{J}^\rho_\mu + f'^{\mu\alpha}_i d'_\alpha J^i_k \bar{J}^\rho_\mu + f'^{\mu\alpha\beta}_i \bar{J}^\rho_\mu \left(\bar{J}^\epsilon_{\alpha\beta} \left(J^i_{k\epsilon} + J^i_{kl}y^l_\epsilon \right) + \bar{J}^\epsilon_\alpha \bar{J}^\sigma_\beta \left(J^i_{k\epsilon\sigma} + J^i_{\epsilon kl}y^l_\sigma + J^i_{klj}y^j_\epsilon y^l_\sigma + J^i_{kl}y^l_{\epsilon\sigma} \right) \right) \right] \\
f^{\rho\lambda}_k = J \left[f'^{\mu\alpha}_i J^i_k \bar{J}^\lambda_\alpha \bar{J}^\rho_\mu + f'^{\mu\alpha\beta}_i \bar{J}^\rho_\mu \left(\bar{J}^\lambda_{\alpha\beta} J^i_k + \bar{J}^\epsilon_\alpha \bar{J}^\lambda_\beta J^i_{\epsilon k} + 2\bar{J}^\epsilon_\alpha \bar{J}^\lambda_\beta J^i_{kj}y^j_\epsilon \right) \right] \\
f^{\rho\lambda\sigma}_k = J J^i_k f'^{\mu\alpha\beta}_i \bar{J}^\rho_\mu \bar{J}^\sigma_\beta \bar{J}^\lambda_\alpha
\end{cases}
\tag{1.3.52}
$$

:Index: :AIndex: :Symbols: :Notation:

The transformation laws for $f_k^{\rho\lambda\sigma}$ are compatible with the fact that this coefficient turns out to be totally symmetric. On the contrary, the transformation laws for $f_k^{\rho\lambda}$ are not compatible with the symmetry of coefficients; if we assume that $f_i'^{\mu\alpha}$ is symmetric, this does not guarantee that $f_k^{\rho\lambda}$ is. In fact, we have

$$f_k^{[\rho\lambda]} = J f_i'^{\mu\alpha\beta} \bar{J}_\mu^{[\rho} \bar{J}_{\alpha\beta}^{\lambda]} J_k^i \tag{1.3.53}$$

which has no reason to vanish even in view of the total symmetry of the coefficient $f_i'^{\mu\alpha\beta}$.

One can show that, in order to define a global Liouville part for a third order Lagrangian, a connection on the base manifold is needed and in fact there are infinitely many Liouville parts parameterised exactly by connections on the manifold M, in the sense that, chosen a connection, one can single out a *reduced* Liouville part (which is reduced with respect to that connection but not with respect to other connections). Changing the connection amounts to change the reduced representative. Since we shall not consider Lagrangians of order 3, we refer to[3].

4. Symmetries and Noether theorem

Another variational technique is about symmetries and conservation laws. This will be extremely important for relativistic theories. Let us hereafter review the basics in a generic field theory first. There are different notions of symmetries, from the sharpest to the most general. Let us start from the strictest ones and then consider some generalisations.

Lagrangian symmetries

Let us consider a field theory defined on a configuration bundle $\mathcal{C} = (C, M, \pi, F)$. Dynamics is described by a k-order Lagrangian $\boldsymbol{L} = L(j^k\sigma)\,d\sigma$.

An isomorphism $\Phi : C \to C$ of the configuration bundle is called a *Lagrangian symmetry* if its prolongation $j^k\Phi : J^kC \to J^kC$ leaves the Lagrangian invariant, i.e.

$$(j^k\Phi)^* \boldsymbol{L} = \boldsymbol{L} \tag{1.4.1}$$

One can also say that the Lagrangian $\boldsymbol{L}$ is *covariant* with respect to the transformation Φ.

A 1-parameter flow of Lagrangian symmetries is called an *infinitesimal Lagrangian symmetry*. An infinitesimal Lagrangian symmetry is also represented by the vector field Ξ which generates the flow Φ_s on the configuration bundle.

If Φ_s is an infinitesimal Lagrangian symmetry generated by the vector field Ξ, then

$$(j^k \Phi_s)^* \boldsymbol{L} = \boldsymbol{L} \qquad \Longleftrightarrow \qquad \pounds_{j^k\Xi} \boldsymbol{L} = 0 \qquad\qquad (1.4.2)$$

Lagrangian symmetries Φ_s are fibered transformations, and consequently the generator Ξ is projectable. Let us denote by $\xi := \pi_* \Xi$ the induced spacetime vector field. Then we have the so-called *covariance identity*

$$\left(j^k \pounds_\Xi\right) \lrcorner d\boldsymbol{L} = d\left(\xi \lrcorner \boldsymbol{L}\right) \qquad\qquad (1.4.3)$$

Let us consider a system of fibered coordinates (x^μ, y^i) on configuration bundle. The Lie derivative along $\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i$ induces a generalised vector field $\pounds_\Xi = \left(\xi^\mu y^i_\mu - \xi^i\right)\partial_i$. Let us denote by Φ_s the flow of Ξ.

The condition (1.4.1) for $\Phi_s : C \to C : (x^\mu, y^i) \mapsto (x'^\mu, y'^i)$ to be a Lagrangian symmetry, with the usual conventions about Jacobians, reads locally as

$$J L(x'^\mu, y'^i, y'^i_\mu, \ldots) = L(x^\mu, y^i, y^i_\mu, \ldots) \qquad\qquad (1.4.4)$$

which is equivalent to its infinitesimal version obtained by taking the derivative with respect to the family parameter s, i.e.

$$\partial_\mu \xi^\mu L + \partial_\mu L \xi^\mu + p_i \xi^i + p^\mu_i \xi^i_\mu + \ldots = 0 \quad \Rightarrow d_\mu \xi^\mu L + d_\mu L \xi^\mu + p_i(\xi^i - y^i_\lambda \xi^\lambda) + p^\mu_i(\xi^i_\mu - y^i_{\mu\lambda}\xi^\lambda) + \ldots = 0 \qquad (1.4.5)$$

Taking into account that for the prolongation of a vector field one has $\xi^i_\mu = d_\mu \xi^i - d_\mu \xi^\lambda y^i_\lambda$ and

$$\xi^i_\mu - y^i_{\mu\lambda}\xi^\lambda = d_\mu \xi^i - d_\mu \xi^\lambda y^i_\lambda - y^i_{\mu\lambda}\xi^\lambda = d_\mu\left(\xi^i - \xi^\lambda y^i_\lambda\right) = -d_\mu\left(\pounds_\Xi\right)^i \qquad\qquad (1.4.6)$$

then one has

$$p_i(\pounds_\Xi)^i + p^\mu_i d_\mu(\pounds_\Xi)^i + \ldots = d_\mu\left(\xi^\mu L\right) \qquad\qquad (1.4.7)$$

which is exactly the local expression for the intrinsic expression

$$\left(j^k \pounds_\Xi\right) \lrcorner d\boldsymbol{L} = d\left(\xi \lrcorner \boldsymbol{L}\right) \qquad\qquad (1.4.8)$$

A (projectable) vector field Ξ on the configuration bundle is an infinitesimal symmetry iff it satisfies the identity (1.4.3).

The reader is urged to compare the condition for a Lagrangian to be global, namely (1.3.27), with the condition to be covariant, namely (1.4.4). Be careful.

Noether theorem

Let us consider a Lagrangian field theory with a 1-parameter flow of symmetries represented by the generator Ξ. The covariance identity (1.4.3) holds true. Since the Lie derivative $\pounds_\Xi = \pounds_\Xi y^i \partial_i$ is a vertical vector, for it the first variation formula holds true as well. Hence it is easy to recast the covariance identity into the following form

$$\left(j^k \pounds_\Xi\right) \lrcorner d\boldsymbol{L} = (\pounds_\Xi)\lrcorner \mathbb{E}(L) + d\left((j^{k-1}\pounds_\Xi)\lrcorner \mathbb{F}(L)\right) = d\left(\xi \lrcorner \boldsymbol{L}\right) \qquad \Rightarrow d\left((j^{k-1}\pounds_\Xi)\lrcorner \mathbb{F}(L) - \xi \lrcorner \boldsymbol{L}\right) = -(\pounds_\Xi)\lrcorner \mathbb{E}(L)$$

$$(1.4.9)$$

This has to be considered as an identity between horizontal forms. Let us set $\mathcal{E}(L, \Xi) := (j^{k-1}\pounds_\Xi)\lrcorner\mathbb{F}(L) - \xi\lrcorner\boldsymbol{L}$, which is called the *Noether current*, and $\mathcal{W}(L, \Xi) := -(\pounds_\Xi)\lrcorner\mathbb{E}(L)$, which is called the *work current*. Let us stress that the Noether current is a global horizontal $(m-1)$-form on $J^{2k-1}\mathcal{C}$, while the work current is a global horizontal m-form on $J^{2k}\mathcal{C}$.

For any configuration σ, one can pull back both these quantities on spacetime to obtain

$$d\left((j^{2k-1}\sigma)^*\mathcal{E}(L, \Xi)\right) = (j^{2k}\sigma)^*\mathcal{W}(L, \Xi) \tag{1.4.10}$$

If σ is a solution of field equations, then $(j^{2k}\sigma)^*\mathcal{W}(L, \Xi) \equiv 0$ and the Noether current is closed. If the Noether current can be split as

$$\mathcal{E}(L, \Xi) = \tilde{\mathcal{E}}(L, \Xi) + d\mathcal{U}(L, \Xi) \tag{1.4.11}$$

and the form $\tilde{\mathcal{E}}(L, \Xi)$ vanishes along solutions of field equations then the form $\tilde{\mathcal{E}}(L, \Xi)$ is called the *reduced Noether current* and the form $\mathcal{U}(L, \Xi)$ is called a *superpotential*.

In a second order field theory, the local expressions for Noether current and work current are

$$\begin{cases} \mathcal{E}(L, \Xi) = \left(f_i^\mu \pounds_\Xi y^i + f_i^{\mu\nu} d_\nu \pounds_\Xi y^i - \xi^\mu L\right)\; d\sigma_\mu \\ \mathcal{W}(L, \Xi) = -\left(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu}\right) \pounds_\Xi y^i\; d\sigma \end{cases} \tag{1.4.12}$$

which are of course global since they have been first obtained intrinsically.

Notice that Noether currents in principle carry information about field equations. If one knows a Noether current $\mathcal{E}(L, \Xi)$, its differential $d\mathcal{E}(L, \Xi) = \mathcal{W}(L, \Xi)$ gives the work current which contains information about field equations.

In fact, one obtains in this way $\mathcal{W}(L, \Xi) = -(\pounds_\Xi)\lrcorner\mathbb{E}(L)$ and to obtain field equations one would need either extra information about the symmetry (basically should know how to isolate the factor $\pounds_\Xi$) or to have enough symmetries so that the Lie derivatives $\pounds_\Xi$ span a basis of vertical vectors and the system $\mathcal{W}(L, \Xi) = 0$ for all symmetries Ξ implies the vanishing of the factor $\mathbb{E}_i(L) = 0$. We shall see examples of this situation.

Of course, the Noether current is an $(m-1)$-form. In mechanics $m = 1$, hence the Noether current is a 0-form, i.e. a function. In that case it is called a *first integral*. A 0-form is closed iff it is constant, so a first integral is constant along solutions of Euler-Lagrange equations of a holonomic mechanical system.

Continuity equations

Let us set $(j^{2k-1}\sigma)^*\mathcal{E}(L, \Xi) = \mathcal{E}^\mu(x)\; d\sigma_\mu$ for the local expression of Noether current evaluated along a solution σ of field equations. The global conservation law (1.4.10) is locally expressed as

$$d\left(\mathcal{E}^\mu\; d\sigma_\mu\right) = d_\nu \mathcal{E}^\mu\, dx^\nu \wedge\; d\sigma_\mu = d_\mu \mathcal{E}^\mu\; d\sigma = 0 \qquad \Rightarrow \quad d_\mu \mathcal{E}^\mu = 0 \tag{1.4.13}$$

The quantities $\mathcal{E}^\mu$ form a vector density of weight 1 (since $(j^{2k-1}\sigma)^*\mathcal{E}(L, \Xi)$ is a global $(m-1)$-form).

The equation $d_\mu \mathcal{E}^\mu = 0$ is called a *continuity equation* and it is related to conservation laws.

Suppose M is Minkowski spacetime with Cartesian coordinates (x^0, x^i). The continuity equation is in fact

$$d_0 \mathcal{E}^0 + \text{div} \vec{\mathcal{E}} = 0 \tag{1.4.14}$$

The component $\mathcal{E}^0$ is called the *density* while the "spatial vector" $\vec{\mathcal{E}} = \mathcal{E}^i \partial_i$ is called the *flow*. Let us define

$$Q = \int_D \mathcal{E}^0 \, d\sigma_0 \tag{1.4.15}$$

on a $(m-1)$-spacetime region D lying on a $x^0 = const$ surface. Then one has

$$d_0 Q = \int_D d_0 \mathcal{E}^0 \, d\sigma_0 = -\int_D d_i \mathcal{E}^i \, d\sigma_0 = -\int_D d_j \mathcal{E}^i \, dx^j \wedge d\sigma_{0i} = -\int_D d(\mathcal{E}^i \, d\sigma_{0i}) = -\int_{\partial D} \mathcal{E}^i \, d\sigma_{0i} \tag{1.4.16}$$

In other words, the change in time of the quantity Q is controlled *only* by the flow of $\vec{\mathcal{E}}$ through ∂D. This means that within D there is no source of the quantity Q which can change only by entering or leaving the region D by crossing the boundary ∂D.

More generally, in a spacetime, the integral

$$\int_{\partial \Omega} \mathcal{E}^\mu \, d\sigma_\mu = \int_\Omega d\left(\mathcal{E}^\mu \, d\sigma_\mu\right) = 0 \tag{1.4.17}$$

along any m-region $\Omega \subset M$.

Consider now the case in which one has two $(m-1)$-regions D_1 and D_2 which together form the boundary of an m-region Ω, i.e. $\partial \Omega = D_2 - D_1$. Then one has

$$\int_{\partial \Omega} \mathcal{E}^\mu \, d\sigma_\mu = \int_{D_2 - D_1} \mathcal{E}^\mu \, d\sigma_\mu = \int_{D_2} \mathcal{E}^\mu \, d\sigma_\mu - \int_{D_1} \mathcal{E}^\mu \, d\sigma_\mu = 0 \qquad \Rightarrow \qquad \int_{D_2} \mathcal{E}^\mu \, d\sigma_\mu = \int_{D_1} \mathcal{E}^\mu \, d\sigma_\mu \tag{1.4.18}$$

Hence the integral of a Noether current is a homological invariant. It is an invariant characteristic of the solution which will be called a *covariantly conserved quantity*. The equation $d\mathcal{E} = 0$ is called a *covariant conservation law*.

When in a spacetime $M = \Omega - \{p\}$ fields are defined on an m-region but with a singularity at a point p (which of course is removed from spacetime since fields on spacetime have to be regular), one can consider an $(m-1)$-region D wrapping around the point p and define

$$Q = \int_D \mathcal{E}^\mu \, d\sigma_\mu \tag{1.4.19}$$

This is a number, covariantly determined, which is independent of the particular choice of D. It is characteristic of field singularity at p, as a kind of residual at p.

Covariantly conserved quantities are the analogous of conservation laws in a relativistic theory in which there is no preferred notion of time. We shall later on investigate the relation between covariantly conserved quantities and quantities which are conserved in time.   ▷ *Conserved quantities*

5. Pure divergences

We can discuss what happens when a Lagrangian is the differential of a horizontal $(m-1)$-form $\boldsymbol{\alpha}$ on some jet prolongation of the configuration $\triangleright$*Next Section* bundle. In that case, we call it a *pure divergence* since, e.g. on the first jet bundle, we have

$$\boldsymbol{\alpha} = \alpha^\mu(j^1 y)\, d\sigma_\mu \qquad \Rightarrow \boldsymbol{L} = d\boldsymbol{\alpha} = d_\lambda \alpha^\mu(j^1 y)\, dx^\lambda \wedge\, d\sigma_\mu = d_\mu \alpha^\mu(j^1 y)\, d\sigma \tag{1.5.1}$$

We have to show that these Lagrangians are trivial, in the sense that they produce identically satisfied field equations, as they have trivial (though non-zero) conservation laws. Hence, if we have two Lagrangians which differ by a pure divergence, they have the same field equations and somehow equivalent conservation laws. More on conservation laws in this case will follow in Section 3.5.

A first order pure-divergence Lagrangian is

$$L = d_\mu \alpha^\mu = \partial_\mu \alpha^\mu + \partial_i \alpha^\mu y^i_\mu + \partial^\nu_i \alpha^\mu y^i_{\mu\nu} \tag{1.5.2}$$

Its variation is

$$\delta L = d_\mu \partial_i \alpha^\mu \delta y^i + d_\mu \partial^\nu_i \alpha^\mu d_\nu \delta y^i + \partial_i \alpha^\mu d_\mu \delta y^i + \partial^\nu_i \alpha^\mu d_{\mu\nu} \delta y^i = \color{red}{d_\mu \partial_i \alpha^\mu \delta y^i + d_\mu \partial^\nu_i \alpha^\mu d_\nu \delta y^i - d_\mu \partial_i \alpha^\mu \delta y^i - d_\mu \partial^\nu_i \alpha^\mu d_\nu \delta y^i} + d_\mu \left(\partial_i \alpha^\mu \delta y^i + \partial^\nu_i \alpha^\mu d_\nu \delta y^i \right) \tag{1.5.3}$$

Hence the Euler-Lagrange part $\mathbb{E}(L) = 0$ is identically vanishing, i.e. any configuration is a solution. The Liouville part is

$$\mathbb{F}(L) = \partial_i \alpha^\mu \omega^i \wedge\, d\sigma_\mu + \partial^\nu_i \alpha^\mu \omega^i_\nu \wedge\, d\sigma_\mu \tag{1.5.4}$$

where we set $\omega^i = dy^i - y^i_\alpha\, dx^\alpha$ and $\omega^i_\nu = dy^i_\nu - y^i_{\nu\alpha}\, dx^\alpha$ for the relevant contact forms.

For conservation laws, we can show that any transformation Ξ is an (infinitesimal) symmetry for a pure divergence Lagrangian.

Let us consider a generic projectable vector field $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$ on the configuration bundle. The covariance identity reads as

$$p_i \pounds_\Xi y^i + p^\mu_i \pounds_\Xi y^i_\mu + p^{\alpha\mu}_i \pounds_\Xi y^i_{\alpha\mu} = d_\alpha \left(d_\mu \alpha^\mu \xi^\alpha \right) \tag{1.5.5}$$

The relevant Lie derivatives are $\pounds_\Xi y^i = y^i_\alpha \xi^\alpha - \xi^i$ as well as

$$\pounds_\Xi y^i_\mu = d_\mu \pounds_\Xi y^i \qquad \pounds_\Xi y^i_{\alpha\mu} = d_{\alpha\mu} \pounds_\Xi y^i \tag{1.5.6}$$

The covariance identity is identically satisfied since we have for the left hand side

$$p_i \pounds_\Xi y^i + p^\mu_i \pounds_\Xi y^i_\mu + p^{\alpha\mu}_i \pounds_\Xi y^i_{\alpha\mu} = \underline{d_\mu \partial_k \alpha^\mu \pounds_\Xi y^k} + \underline{d_\mu \partial^\alpha_k \alpha^\mu d_\alpha \pounds_\Xi y^k} + \underline{\partial_i \alpha^\mu d_\mu \pounds_\Xi y^i} + \underline{\partial^\alpha_i \alpha^\mu d_\mu \pounds_\Xi y^i_\alpha} = d_\mu \left(\partial_k \alpha^\mu \pounds_\Xi y^k + \partial^\alpha_k \alpha^\mu \pounds_\Xi y^k_\alpha \right) = \pounds_\Xi d_\mu \alpha^\mu = d_\epsilon \left(\xi^\epsilon d_\mu \alpha^\mu \right) \tag{1.5.7}$$

The Noether current is

$$\begin{aligned}
\mathcal{E} &= \left(\partial_k \alpha^\mu \pounds_\Xi y^k + \partial^\alpha_k \alpha^\mu d_\alpha \pounds_\Xi y^k \right) d\sigma_\mu - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \pounds_\Xi \boldsymbol{\alpha} - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu + d \left(\xi^\nu \alpha^\mu\, d\sigma_{\mu\nu} \right) - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \\
&= \left(\color{red}{\xi^\mu d_\nu \alpha^\nu} + d_\nu \left(\xi^\nu \alpha^\mu \right) - d_\nu \left(\xi^\mu \alpha^\nu \right) - \color{red}{\xi^\mu d_\nu \alpha^\nu} \right) d\sigma_\mu = 2 d_\nu \left(\xi^{[\nu} \alpha^{\mu]} \right) d\sigma_\mu
\end{aligned} \tag{1.5.8}$$

which can be recast as

$$\mathcal{E} = d\left(\xi^{[\nu}\alpha^{\mu]}\,d\sigma_{\mu\nu}\right) = d_\lambda\left(\xi^{[\nu}\alpha^{\mu]}\right)\,dx^\lambda \wedge\,d\sigma_{\mu\nu} = 2d_\nu\left(\xi^{[\nu}\alpha^{\mu]}\right)\,d\sigma_\mu \tag{1.5.9}$$

where the reduced current $\tilde{\mathcal{E}}$ is identically zero (as it should be since it should be zero on shell, which in this case means along any configuration, i.e. it coincides with the off-shell vanishing) and the superpotential is

$$\mathcal{U} = \xi^{[\nu}\alpha^{\mu]}\,d\sigma_{\mu\nu} \tag{1.5.10}$$

Symmetries of the action

A Lagrangian symmetry leaves the Lagrangian invariant. Hence it leaves the action functional invariant. The same action implies the same field equations and the same solutions. However, one can preserve the action without necessarily preserving the Lagrangian. Let us consider a global horizontal $(m-1)$-form $\boldsymbol{\alpha}$. Consider a transformation $\Phi : C \to C$ such that

$$(j^k\Phi)^*\boldsymbol{L} = \boldsymbol{L} + d\boldsymbol{\alpha} \tag{1.5.11}$$

This will be called an *action symmetry*. The transformation Φ projects over a spacetime diffeomorphism $\phi : M \to M$ and it maps D into $D' = \phi(D)$. Then one has

$$A'_{D'}[\sigma] = \int_{D'}(j^k\sigma)^*(j^k\Phi)^*\boldsymbol{L} = \int_D(j^k\sigma)^*\boldsymbol{L} + \int_D(j^k\sigma)^*d\boldsymbol{\alpha} = A_D[\sigma] + \int_{\partial D}(j^k\sigma)^*\boldsymbol{\alpha} \tag{1.5.12}$$

However, the boundary contribution $\int_{\partial D}\boldsymbol{\alpha}$ is constant with respect to field deformations. Hence it does not contribute to the action variation and to field equations. The covariance identity for action symmetries is

$$\left(j^k\pounds_\Xi\right)\lrcorner d\boldsymbol{L} = d\left(\xi\lrcorner\boldsymbol{L}\right) + d\boldsymbol{\alpha} \tag{1.5.13}$$

where Ξ is the infinitesimal generator of a 1-parameter flow of action symmetries Φ_s, and $\boldsymbol{\alpha}$ has to be linear with respect to the symmetry generator, namely $\boldsymbol{\alpha} = a(\Xi) = \xi^\mu a_\mu + \xi^i a_i$ with the coefficients (a_μ, a_i) functions of the points in J^kC. Also in this case one has a kind Noether theorem. In fact, by the usual integration by parts, one has

$$\mathcal{E}(L, \Xi) = (j^{k-1}\pounds_\Xi)\lrcorner\mathbb{F}(L) - \xi\lrcorner\boldsymbol{L} + \boldsymbol{\alpha} \tag{1.5.14}$$

which is the corresponding *generalised Noether current*.

One has examples of generalized Noether currents in mechanics already. Consider the Lagrangian describing the motion of a charge in an electromagnetic field. A gauge transformations changes the potential $V = eA_\mu u^\mu$ into

$$V' = eA'_\mu u^\mu = e\left(A_\mu + \partial_\mu f(x)\right)u^\mu = V + e\partial_\mu f u^\mu = V + \tfrac{d}{dt}(ef) \tag{1.5.15}$$

Accordingly, a gauge transformation affects the Lagrangian changing it into itself plus a total derivative, which in mechanics is precisely a total divergence. In other words, gauge symmetries acts a symmetry of the action.

Generalised symmetries

Noether currents are linear in momenta. Of course, there are conserved quantities which are not linear in the momenta. Noether theorem can be generalised to transformations which depend on field derivatives, e.g. Φ : $J^1C \to C$. Forms on jet bundles can be lift to any order of prolongation along the jet bundles projections. Then one can define *generalised symmetries* when they preserve the Lagrangian (or the action), i.e.

$$(j^k\Phi)^*\boldsymbol{L} = (\pi_k^{k+1})^*\boldsymbol{L} \tag{1.5.16}$$

An *infinitesimal generalised symmetry* is a generalised field, e.g.

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x^\lambda, y^i, y^i_\lambda)\partial_i \tag{1.5.17}$$
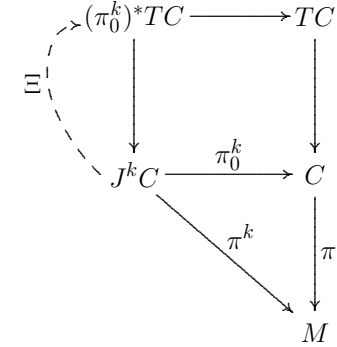
These objects are global sections in a suitably defined bundle, namely the one shown in the commutative diagram here beside.

Let us stress that original papers by Noether do considered these cases (in mechanics).

Generalised symmetries produces conservation laws in mechanics which can be more than linear in momenta, as for example the Runge–Lenz vector.

Symmetries of a differential equation

A symmetry of a differential equation is a transformation mapping solutions into solutions.

An action symmetry leaves the equations invariant and hence are symmetries of the field equations.

$$A_{D'}[\sigma'] = \int_{D'}(j^k\sigma')^*\boldsymbol{L} = \int_D (j^k\sigma)^*(j^k\Phi)^*\boldsymbol{L} = \int_D (j^k\sigma)^*\boldsymbol{L} + \int_{\partial D}(j^k\sigma)^*\boldsymbol{\alpha} = A_D[\sigma] + \int_{\partial D}(j^k\sigma)^*\boldsymbol{\alpha} \tag{1.5.18}$$

If σ is a critical point for the action functional, then σ' is a critical point as well.

In a field theory with linear field equations, linear symmetries of equations are represented by operators which commute with the operator $\mathbb{E}$ that represents field equations.

6. Inverse problem (TBA)

The inverse problem in variational calculus is to find out is an equation $\mathbb{E} \subset J^kC$ can be generated by a variational principle. Traditionally, that is solved by *Helmholtz conditions*.

TBD Facciamo almeno first order Lagrangians

Inverse problem is one of those issues which are easily seen and solved in the framework if variational complexes in which variational calculus is restated in cohomological terms in a suitable sequence of forms. In such a framework, Helmholtz conditions turn out to be sufficient conditions as the ones for existence of a potential given the force.

7. Variational principles on composite bundles

In this Section we shall extend variational calculus to composite bundles. The process is quite easy since, as we noticed, a composite bundle $[C \to B \to M]$ is just a bundle $[C \to M]$ over M with stratified transition maps. We can use fibered coordinates $(x^\mu; w^A) = (x^\mu; y^i, z^a)$. A composite section (ρ, σ) defines a section $\rho \circ \sigma : x \mapsto (x; w(x)) = (x, y(x), z(x, y(x)))$.

The jet prolongation $[\mathbb{P}_2 \to \mathbb{P}_1 \to C \to B \to M]$ has coordinates $(x^\mu, y^i, z^a, y^i_\mu, z^a_\mu, y^i_{\mu\nu}, z^a_{\mu\nu}, z^i_{i\mu}, , z^a_{ij})$ is a composite bundle and the composite section (ρ, σ) prolongs to $j^2(\rho, \sigma)$. Let us consider a Lagrangian

$$\boldsymbol{L} = L(x^\mu, y^i, z^a, y^i_\mu, z^a_\mu, y^i_{\mu\nu}, z^a_{\mu\nu}) \, d\sigma \tag{1.7.1}$$

where, of course, we set $d_\mu z^a := z^a_\mu + y^k_\mu z^a_k$ and $d_{\mu\nu} z^a = z^a_{\mu\nu} + 2z^a_{i(\nu} y^i_{\nu)} + y^k_\mu y^i_\nu z^a_{ik} + y^k_{\mu\nu} z^a_k$ for the (first and second order) total derivatives, and a deformation is a vertical vector field on C, i.e. a section of $[V(C) \to C]$, namely

$$X \circ \rho \circ \sigma = \delta y^i \frac{\partial}{\partial y^i} + \delta z^a \frac{\partial}{\partial z^a} \tag{1.7.2}$$

As for the ordinary case, we take the variation of the Lagrangian

$$\begin{aligned}
\delta \boldsymbol{L} =& p_i \delta y^i + p_i^\mu d_\mu \delta y^i + p_i^{\mu\nu} d_{\mu\nu} \delta y^i + p_a \delta z^a + p_a^\mu d_\mu \delta z^a + p_a^{\mu\nu} d_{\mu\nu} \delta z^a = \\
=& \left(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} \right) \delta y^i + \left(p_a - d_\mu p_a^\mu + d_{\mu\nu} p_a^{\mu\nu} \right) \delta z^a + d_\mu \left(p_i^\mu \delta y^i + p_i^{\mu\nu} d_\nu \delta y^i - d_\mu p_i^{\mu\nu} \delta y^i + p_a^\mu \delta z^a + p_a^{\mu\nu} d_\nu \delta z^a - d_\mu p_a^{\mu\nu} \delta z^a \right)
\end{aligned} \tag{1.7.3}$$

The proof of globality given in the ordinary case goes through also in the case of composite bundles. Accordingly, we set

$$\mathbb{E}_i := p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} \qquad \mathbb{E}_a := p_a - d_\mu p_a^\mu + d_{\mu\nu} p_a^{\mu\nu} \qquad f_i^\mu := p_i^\mu - d_\mu p_i^{\mu\nu} \qquad f_i^{\mu\nu} := p_i^{\mu\nu} \qquad f_a^\mu := p_a^\mu - d_\mu p_a^{\mu\nu} \qquad f_a^{\mu\nu} := p_a^{\mu\nu} \tag{1.7.4}$$

and we define the global forms

$$\mathbb{E}(L) = \left(\mathbb{E}_i \, \omega^i + \mathbb{E}_a \, \omega^a \right) \wedge d\sigma \qquad \mathbb{F}(L) = \left(f_i^\mu \, \omega^i + f_i^{\mu\nu} \, \omega^i_\nu + f_a^\mu \, \omega^a + f_a^{\mu\nu} \, \omega^a_\nu \right) \wedge d\sigma_\mu \tag{1.7.5}$$

The first variation formula is

$$j^2 X \lrcorner d\boldsymbol{L} = X \lrcorner \mathbb{E}(L) + d(j^1 X \lrcorner \mathbb{F}(L)) \tag{1.7.6}$$

8. Linearized equations

Let us now consider an ordinary variational principle, defined on a configuration bundle $[B \to M]$ and described by a second order Lagrangian $\boldsymbol{L}$. As a matter of fact, the variation of the Lagrangian $\delta\boldsymbol{L} : J^2B \to V^*(J^2B) \otimes A_m(M)$ induces an auxiliary second order Lagrangian

$$J(\boldsymbol{L}) = j^2X \lrcorner d\boldsymbol{L} = (p_iX^i + p_i^\mu d_\mu X^i + p_i^{\mu\nu}d_{\mu\nu}X^i)d\sigma \tag{1.8.1}$$

which is called the *Jacobi Lagrangian*, on the composite bundle $[V(B) \to B \to M]$. Let us stress that $J(\boldsymbol{L})$ is an m-form in $J^2V(B)$, i.e. in the form

$$J(\boldsymbol{L}) = L_J(x, y, X, dy, dX, d^2y, d^2X)\, d\sigma \tag{1.8.2}$$

Notice that the Jacobi Lagrangian $J(\boldsymbol{L})$ happens to be linear in the vertical vector fields X.

Then we can consider a deformation $\Xi = \delta y^i \frac{\partial}{\partial y^i} + \delta X^i \frac{\partial}{\partial X^i}$ on $V(B)$ and the variation of the Lagrangian $J(\boldsymbol{L})$.

We have

$$
\begin{aligned}
j^2\Xi\lrcorner d(J(\boldsymbol{L})) =& (p_i\delta X^i + p_i^\mu d_\mu\delta X^i + p_i^{\mu\nu}d_{\mu\nu}\delta X^i) + (\partial_k p_i\delta y^k + \partial_k^\alpha p_i d_\alpha\delta y^k + \partial_k^{\alpha\beta}p_i d_{\alpha\beta}\delta y^k)X^i + (\partial_k p_i^\mu\delta y^k + \partial_k^\alpha p_i^\mu d_\alpha\delta y^k + \partial_k^{\alpha\beta}p_i^\mu d_{\alpha\beta}\delta y^k)d_\mu X^i + \\
&+ (\partial_k p_i^{\mu\nu}\delta y^k + \partial_k^\alpha p_i^{\mu\nu}d_\alpha\delta y^k + \partial_k^{\alpha\beta}p_i^{\mu\nu}d_{\alpha\beta}\delta y^k)d_{\mu\nu}X^i = \\
=& (p_i - d_\mu p_i^\mu + d_{\mu\nu}p_i^{\mu\nu})\,\delta X^i + \\
&+ \left(\partial_k p_i X^i + \partial_k p_i^\mu d_\mu X^i + \partial_k p_i^{\mu\nu}d_{\mu\nu}X^i - d_\alpha\left(\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i + \partial_k^\alpha p_i^{\mu\nu}d_{\mu\nu}X^i\right) + d_{\alpha\beta}\left(\partial_k^{\alpha\beta}p_i X^i + \partial_k^{\alpha\beta}p_i^\mu d_\mu X^i + \partial_k^{\alpha\beta}p_i^{\mu\nu}d_{\mu\nu}X^i\right)\right)\delta y^k + \\
&+ d_\mu((p_i^\mu - d_\nu p_i^{\mu\nu})\,\delta X^i + p_i^{\mu\nu}d_\nu\delta X^i + \left(\partial_k^\mu p_i X^i + \partial_k^\mu p_i^\lambda d_\lambda X^i + \partial_k^\mu p_i^{\lambda\nu}d_{\lambda\nu}X^i\right)\delta y^k - d_\alpha\left(\partial_k^{\alpha\mu}p_i X^i + \partial_k^{\alpha\mu}p_i^\lambda d_\lambda X^i + \partial_k^{\alpha\mu}p_i^{\lambda\nu}d_{\lambda\nu}X^i\right)\delta y^k + \\
&+ \left(\partial_k^{\mu\beta}p_i X^i + \partial_k^{\mu\beta}p_i^\lambda d_\lambda X^i + \partial_k^{\mu\beta}p_i^{\lambda\nu}d_{\lambda\nu}X^i\right)d_\beta\delta y^k)
\end{aligned}
\tag{1.8.3}
$$

We obtain as a first field equation $\mathbb{E}_i = p_i - d_\mu p_i^\mu + d_{\mu\nu}p_i^{\mu\nu} = 0$, i.e. the same field equations as for the original Lagrangian $\boldsymbol{L}$. For the second field equation we can write it as

$$
\begin{aligned}
0 =& \underline{X^i\partial_i p_k} + d_\mu X^i\partial_i^\mu p_k + d_{\mu\nu}X^i\partial_i^{\mu\nu}p_k - \left(d_\alpha X^i\partial_i p_k^\alpha + d_{\mu\alpha}X^i\partial_i^\mu p_k^\alpha + d_{\mu\nu\alpha}X^i\partial_i^{\mu\nu}p_k^\alpha + \underline{X^i\partial_i d_\alpha p_k^\alpha} + d_\mu X^i d_\alpha\partial_i^\mu p_k^\alpha + d_{\mu\nu}X^i d_\alpha\partial_i^{\mu\nu}p_k^\alpha\right) + \\
&+ d_{\alpha\beta}X^i\partial_i p_k^{\alpha\beta} + d_{\mu\alpha\beta}X^i\partial_i^\mu p_k^{\alpha\beta} + d_{\mu\nu\alpha\beta}X^i\partial_i^{\mu\nu}p_k^{\alpha\beta} + 2d_\alpha X^i\partial_i d_\beta p_k^{\alpha\beta} + 2d_{\mu\alpha}X^i d_\beta\partial_i^\mu p_k^{\alpha\beta} + 2d_{\mu\nu\alpha}X^i d_\beta\partial_i^{\mu\nu}p_k^{\alpha\beta} + \\
&+ \underline{X^i\partial_i d_{\alpha\beta}p_k^{\alpha\beta}} + d_\mu X^i d_{\alpha\beta}\partial_i^\mu p_k^{\alpha\beta} + d_{\mu\nu}X^i d_{\alpha\beta}\partial_i^{\mu\nu}p_k^{\alpha\beta} = \\
=& X^i\partial_i\left(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}\right) + \underline{d_\mu X^i\partial_i^\mu p_k} + \underline{\underline{d_{\mu\nu}X^i\partial_i^{\mu\nu}p_k}} + \\
&- \left(\textcolor{blue}{d_\alpha X^i\partial_i p_k^\alpha} + \textcolor{blue}{d_{\mu\alpha}X^i\partial_i^\mu p_k^\alpha} + d_{\mu\nu\alpha}X^i\partial_i^{\mu\nu}p_k^\alpha + \underline{d_\mu X^i\partial_i^\mu d_\alpha p_k^\alpha} - \textcolor{red}{d_\mu X^i\partial_i p_k^\mu} + \underline{\underline{d_{\mu\nu}X^i\partial_i^{\mu\nu}d_\alpha p_k^\alpha}} - \textcolor{blue}{d_{\mu\alpha}X^i\partial_i^\mu p_k^\alpha}\right) + \\
&+ d_{\alpha\beta}X^i\partial_i p_k^{\alpha\beta} + d_{\mu\alpha\beta}X^i\partial_i^\mu p_k^{\alpha\beta} + d_{\mu\nu\alpha\beta}X^i\partial_i^{\mu\nu}p_k^{\alpha\beta} + 2d_\alpha X^i\partial_i d_\beta p_k^{\alpha\beta} + 2d_{\mu\alpha}\underline{X^i d_\beta\partial_i^\mu p_k^{\alpha\beta}} + 2d_{\mu\nu\alpha}X^i d_\beta\partial_i^\mu p_k^{\alpha\beta} + \\
&+ d_\mu X^i d_\alpha(\partial_i^\mu d_\beta p_k^{\alpha\beta}) - d_\mu X^i d_\alpha\partial_i p_k^{\alpha\mu} + d_{\mu\nu}X^i d_\alpha(\partial_i^{\mu\nu}d_\beta p_k^{\alpha\beta}) - d_{\mu\nu}X^i d_\alpha\partial_i^\mu p_k^{\alpha\nu} = \\
=& X^i\partial_i\mathbb{E}_k + d_\mu X^i\partial_i^\mu\left(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}\right) + d_{\mu\nu}X^i\partial_i^{\mu\nu}\left(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}\right) - d_{\mu\nu\alpha}X^i\partial_i^{\mu\nu}p_k^\alpha + \\
&+ \textcolor{red}{\underline{d_{\alpha\beta}X^i\partial_i p_k^{\alpha\beta}}} + d_{\mu\alpha\beta}X^i\partial_i^\mu p_k^{\alpha\beta} + d_{\mu\nu\alpha\beta}X^i\partial_i^{\mu\nu}p_k^{\alpha\beta} + \textcolor{blue}{d_\alpha X^i\partial_i d_\beta p_k^{\alpha\beta}} + \textcolor{blue}{d_{\mu\alpha}X^i\partial_i^\mu d_\beta p_k^{\alpha\beta}} - \textcolor{red}{\underline{d_{\mu\alpha}X^i\partial_i p_k^{\alpha\mu}}} + 2d_{\mu\nu\alpha}X^i d_\beta\partial_i^{\mu\nu}p_k^{\alpha\beta} - \textcolor{red}{d_\mu X^i\partial_i d_\beta p_k^{\mu\beta}} - d_{\mu\nu}X^i\partial_i^\mu d_\beta p_k^{\nu\beta}
\end{aligned}
\tag{1.8.4}
$$

where we used that $[d_\alpha, \partial_k] = 0$, $[d_\alpha, \partial_k^\mu] = -\delta_\alpha^\mu\partial_k$, $[d_\alpha, \partial_k^{\mu\nu}] = -\delta_\alpha^{(\nu}\partial_k^{\mu)}$, and $-[d_\alpha, \partial_k^{\mu\nu\epsilon}] = \partial_k^{\mu\nu\epsilon}d_\alpha = \delta_\alpha^{(\epsilon}\partial_k^{\mu\nu)}$. We obtain:

$$X^i\partial_i\mathbb{E}_k + d_\mu X^i\partial_i^\mu\mathbb{E}_k + d_{\mu\nu}X^i\partial_i^{\mu\nu}\mathbb{E}_k + d_{\mu\nu\alpha}X^i\left(-\partial_i^{\mu\nu}p_k^\alpha + 2\partial_i^{\mu\nu}d_\beta p_k^{\alpha\beta} - \partial_i^\mu p_k^{\alpha\nu}\right) + d_{\mu\nu\alpha\beta}X^i\partial_i^{\mu\nu}p_k^{\alpha\beta} = 0 \tag{1.8.5}$$

In order to further simplify the second field equation, let is compute on a side

$$
\begin{aligned}
d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta\epsilon\lambda}\mathbb{E}_k &= d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta\epsilon\lambda}(p_k - d_\mu p_k^\mu + d_{\mu\nu}p_k^{\mu\nu}) = d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta\epsilon\lambda}d_{\mu\nu}p_k^{\mu\nu} = d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta}p_k^{\epsilon\lambda} \\
d_{\alpha\beta\epsilon}X^i\partial_i^{\alpha\beta\epsilon}\mathbb{E}_k &= d_{\alpha\beta\epsilon}X^i\partial_i^{\alpha\beta\epsilon}(p_k - d_\mu p_k^\mu + d_{\mu\nu}p_k^{\mu\nu}) = d_{\alpha\beta\epsilon}X^i\left(-\partial_i^{\beta\epsilon}p_k^\alpha - \partial_i^\alpha p_k^{\beta\epsilon} + 2\partial_i^{\alpha\beta}d_\mu p_k^{\mu\epsilon}\right)
\end{aligned}
\tag{1.8.6}
$$

where we used the identities

$$
d_{\alpha\beta\epsilon}X^i\partial_i^{\alpha\beta\epsilon}d_{\mu\nu}p_k^{\mu\nu} = d_{\alpha\beta\epsilon}X^i(d_\mu\partial_i^{\alpha\beta\epsilon}d_\nu p_k^{\mu\nu} + \partial_i^{\alpha\beta}d_\nu p_k^{\epsilon\nu}) = d_{\alpha\beta\epsilon}X^i(d_\mu\partial_i^{\alpha\beta}p_k^{\mu\epsilon} + \partial_i^{\alpha\beta}d_\mu p_k^{\mu\epsilon}) = d_{\alpha\beta\epsilon}X^i(-\partial_i^\alpha p_k^{\beta\epsilon} + 2\partial_i^{\alpha\beta}d_\mu p_k^{\mu\epsilon})
\tag{1.8.7}
$$

$$
d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta\epsilon\lambda}d_{\mu\nu}p_k^{\mu\nu} = d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta}p_k^{\epsilon\lambda}
\tag{1.8.8}
$$

In view of these results, the second field equation is recast as

$$
X^i\partial_i\mathbb{E}_k + d_\mu X^i\partial_i^\mu\mathbb{E}_k + d_{\mu\nu}X^i\partial_i^{\mu\nu}\mathbb{E}_k + d_{\alpha\beta\epsilon}X^i\partial_i^{\alpha\beta\epsilon}\mathbb{E}_k + d_{\alpha\beta\epsilon\lambda}X^i\partial_i^{\alpha\beta\epsilon\lambda}\mathbb{E}_k = 0 \qquad \Rightarrow j^4X(\mathbb{E}_i) = 0
\tag{1.8.9}
$$

which means the jet prolongation of the vector field X (vertical on the original configuration bundle $[B \to M]$) is tangent to the original field equations thought as a sub-manifold into J^4B.

Theorem: *as a consequence, the flow of X drags solution of $\boldsymbol{L}$ into solutions.*

Because of that, the second field equation is called the *linearized equations* (or the *Jacobi equation*) of equations $\mathbb{E}_k = p_k - d_\mu p_k^\mu + d_{\mu\nu}p_k^{\mu\nu} = 0$. A solution X of Jacobi equation drags solutions of $\mathbb{E}_k = 0$ into solutions, meaning *at first order* but *also at any higher order*. Accordingly, a solution X of linearized equations is called a *perturbation* (or a *Jacobi field*) of the solutions of the original theory.

Let us stress that the linearized equations come with a variational origin, they are generated by the Jacobi Lagrangian $J(\boldsymbol{L})$.

Usually, in perturbation theory, one expands the original field equations at first order about a solution $\bar{y}^i(x)$. That means that one considers $y^i(x) \simeq \bar{y}^i(x) + \epsilon X^i(x)$, replaces it into the field equations $\mathbb{E}_k = 0$, keeps first order terms in ϵ, and interprets it as equations of X^i. When these equations are satisfied $y^i(x)$ is still a solution *at first order*.

Jacoby equations are more complicated, however, the configurations generated by a flow along a Jacobi field of a solution are *exact* solutions.

To check that the two prescriptions produce the same linearized equations, we restrict, for the sake of simplicity, to a first order Lagrangian $\boldsymbol{L} = L(x, y, dy)d\sigma$ which produces equations $E_k = p_k - d_\mu p_k^\mu = 0$. At first order in the expansion in ϵ, we have

$$
p_k(x, \bar{y}^i + \epsilon X^i, \bar{y}_\mu^i + \epsilon X_\mu^i) - d_\mu p_k^\mu(x, \bar{y}^i + \epsilon X^i, \bar{y}_\mu^i + \epsilon X_\mu^i) \simeq p_k(x, \bar{y}^i, \bar{y}_\mu^i) - d_\mu p_k^\mu(x, \bar{y}^i, \bar{y}_\mu^i) + \epsilon\left(\partial_i p_k X^i + \partial_i^\mu p_k X_\mu^i - d_\mu(X^i\partial_i p_k^\mu + X_\epsilon^i\partial_i^\epsilon p_k^\mu)\right)
\tag{1.8.10}
$$

The zero order term is identically satisfied since we assumed $\bar{y}^i$ to be a solution of the original equations. The first order term is

$$
X^i\partial_i p_k + X_\mu^i\partial_i^\mu p_k - d_\mu(X^i\partial_i p_k^\mu + X_\epsilon^i\partial_i^\epsilon p_k^\mu) = 0
\tag{1.8.11}
$$

On the other hand, when we specialize the Jacobi equations to a first order Lagrangian, and we expand them at first order, we obtain

$$
\begin{aligned}
X^i\partial_i p_k + X_\mu^i\partial_i^\mu p_k - X^i d_\mu\partial_i p_k^\mu - X_\mu^i\partial_i^\mu d_\alpha p_k^\alpha - X_{\mu\nu}^i\partial_i^\nu p_k^\mu &= X^i\partial_i p_k + X_\mu^i\partial_i^\mu p_k - d_\mu(X^i\partial_i p_k^\mu) + \textcolor{red}{X_\mu^i\partial_i p_k^\mu} - X_\mu^i d_\alpha\partial_i^\mu p_k^\alpha - \textcolor{red}{X_\mu^i\partial_i p_k^\mu} - X_{\mu\nu}^i\partial_i^\nu p_k^\mu = \\
= X^i\partial_i p_k + X_\mu^i\partial_i^\mu p_k - d_\mu(X^i\partial_i p_k^\mu) - d_\alpha\left(X_\mu^i\partial_i^\mu p_k^\alpha\right) + \textcolor{red}{X_{\alpha\mu}^i\partial_i^\mu p_k^\alpha} - \textcolor{red}{X_{\mu\nu}^i\partial_i^\mu p_k^\mu} &= X^i\partial_i p_k + X_\mu^i\partial_i^\mu p_k - d_\mu(X^i\partial_i p_k^\mu + X_\nu^i\partial_i^\nu p_k^\mu) = 0
\end{aligned}
\tag{1.8.12}
$$

Thus we see we obtain the same equations.

Finally, we have to point out that Jacobi fields carry more information than linear perturbations. That is the same difference on a manifold between a tangent vector v and a vector field X (just not on a manifold but on the space of solutions). Linear perturbations are first order curve $v \simeq \bar{y} + tv$ at a point $\bar{y}$. A vector field X instead is a vector at *any* point. The vector field has a flow Φ_s, the flow drags a point $\bar{y}$ into an integral curve $\gamma_{\bar{y}}$ which is on the manifold, not at first order, it is an exact integral curve, one takes the integral curve and expands it at any order.

In field theories, first order perturbations are exact solutions of the linearized equations at the unperturbed reference solution, although they are an approximate solutions of the original equations, an approximation at first order. A Jacobi field, instead, drags a solution into exact solutions.

Jacobi field equations (1.8.9) are linear in X, therefore Jacobi fields form a real vector space $\mathcal{J}$. Of course, that does not mean it is easy to find Jacobi fields.

9. Second variation and stability

For linearized equations, we started on B, we considered a deformation X in $V(B)$, then we considered a new Lagrangian $J(\boldsymbol{L})$ on $V(B)$, a new deformation Ξ on $V(B)$ and considered the variation of $J(\boldsymbol{L})$ with respect to the deformation Ξ. Let us remark that there is no direct relation between X and Ξ.

Now, for second order variation, we act differently. We consider a theory on B, consider a variation $X = X^i \partial_i$ (i.e. a vertical vector field on B). By the flow Φ_s of X, we can drag a section σ as $\sigma_s = \Phi_s \circ \sigma$, which is locally represented by $y_s^i(x)$ and expand to second order

$$y_s^i(x) = y^i(x) + s\delta y^i(x) + \tfrac{1}{2}s^2\delta^2 y^i(x) + O(s^3) \tag{1.9.1}$$

where we set $\delta y^i(x) := X^i(x, y(x))$ and $\delta^2 y^i(x) := \partial_k X^i(x, y(x))\delta y^k(x)$.

We start from a first order Lagrangian $L(x^\mu, y_s^i(x), \partial_\mu y_s^i(x))$. For $s = 0$, we get L. The first derivative is

$$\tfrac{dL}{ds}(x^\lambda, y_s^k(x), \partial_\lambda y_s^k(x)) = p_i(x^\lambda, y_s^k(x), \partial_\lambda y_s^k(x))X^i(x, y_s(x)) + p_i^\mu(x^\lambda, y_s^k(x), \partial_\lambda y_s^k(x))d_\mu X^i(x, y_s(x)) \tag{1.9.2}$$

which then specializes, for $s = 0$, to $\delta L = J(L)$. The second derivative is then

$$\tfrac{d^2 L}{ds^2}(x^\lambda, y_s^k(x), \partial_\lambda y_s^k(x))\Big|_{s=0} = \left(p_i\,\delta^2 y^i + p_i^\mu\,d_\mu\delta^2 y^i\right) + \left(\partial_k p_i\delta y^k + \partial_k^\alpha p_i d_\alpha\delta y^k\right)\delta y^i + \left(\partial_k p_i^\mu\delta y^k + \partial_k^\alpha p_i^\mu d_\alpha\delta y^k\right)d_\mu\delta y^i \tag{1.9.3}$$

Notice how that corresponds to set $X^i = \delta y^i$ (and consequently $\delta X^i = \delta^2 y^i$) in the variation $\delta\left(J(L)\right)$ of the Jacobi Lagrangian.

For the variation of the Jacobi Lagrangian (of a first order Lagrangian) we have

$$\delta\left(J(L)\right) = (p_i\delta^2 y^i + p_i^\mu d_\mu\delta^2 y^i) + (\partial_k p_i\delta y^k + \partial_k^\alpha p_i d_\alpha\delta y^k)\delta y^i + (\partial_k p_i^\mu\delta y^k + \partial_k^\alpha p_i^\mu d_\alpha\delta y^k)d_\mu\delta y^i = \tfrac{d^2 L}{ds^2} \tag{1.9.4}$$
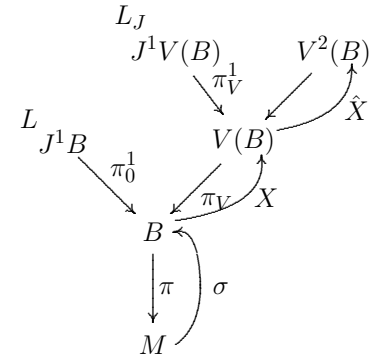
The Lagrangian can be expanded at second order.

$$L(x^\mu, y^i_s, y^i_{s\mu}) = L(x^\mu, y^i, y^i_\mu) + s\left(p_i \delta y^i + p^\mu_i d_\mu \delta y^i\right) +$$
$$+ \frac{s^2}{2}\left(\left(p_i\, \delta^2 y^i + p^\mu_i\, d_\mu \delta^2 y^i\right) + \left(\partial_k p_i\, \delta y^i \delta y^k + 2\partial_i p^\alpha_k\, \delta y^i d_\alpha \delta y^k + \partial^\alpha_k p^\mu_i\, d_\mu \delta y^i d_\alpha \delta y^k\right)\right) + O(s^3) \tag{1.9.5}$$

To summarize, we have

$$L(j^1 y_s) = L + s\delta L + \tfrac{1}{2}s^2\delta^2 L + O(s^3) \tag{1.9.6}$$

where we set

$$\delta^2 \boldsymbol{L} := \left(\left(p_i\, \delta^2 y^i + p^\mu_i\, d_\mu \delta^2 y^i\right) + \left(\partial_k p_i\, \delta y^i \delta y^k + 2\partial_i p^\alpha_k\, \delta y^i d_\alpha \delta y^k + \partial^\alpha_k p^\mu_i\, d_\mu \delta y^i d_\alpha \delta y^k\right)\right) d\sigma \tag{1.9.7}$$

Let us check that the second variation $\delta^2 \boldsymbol{L}$ of a global first order Lagrangian $\boldsymbol{L}$ is a geometric object.

We consider a global (first order) Lagrangian on B, i.e.

$$\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu)d\sigma \tag{1.9.8}$$

For globality, we have $JL'(x', y'^i, y'^i_\mu) = L(x^\mu, y^i, y^i_\mu)$ and consequently

$$p^\alpha_k = J\bar{J}^\alpha_\mu p'^\mu_i J^i_k \qquad p_k = Jp'_i J^i_k + J\bar{J}^\mu_\alpha p'^\alpha_i \left(J^i_{\mu k} + J^i_{jk}y^j_\mu\right) = J\left(p'_i J^i_k + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_k)\right) \tag{1.9.9}$$

For second partial derivatives of the Lagrangian density, we have

$$\partial^\mu_i p^\alpha_k = J\bar{J}^\alpha_\beta \bar{J}^\mu_\nu \partial^{\prime\nu}_j p'^\beta_h J^h_k J^j_i \tag{1.9.10}$$

$$\partial_i p^\alpha_k = J\bar{J}^\alpha_\mu(\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik}) = \partial^\alpha_k p_i \tag{1.9.11}$$

Finally, we have

$$\partial_j p_k = J\left(\partial'_l p'_i J^l_j J^i_k + 2\bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk}\right) \tag{1.9.12}$$

We see that, although p^α_k and $\partial^\mu_i p^\alpha_k$ are well-defined geometric objects, the others are not. However, there are a number of geometric objects defined out of them which should be pointed out, since we expect to provide a good intrinsic formalism for variational calculus.

Let us check transformation rules for second variation.

$$\delta^2 \boldsymbol{L} := \left(p_i\, \delta^2 y^i + p^\mu_i\, d_\mu \delta^2 y^i\right) + \left(\partial_k p_i\, \delta y^i \delta y^k + 2\partial_i p^\alpha_k\, \delta y^i d_\alpha \delta y^k + \partial^\alpha_k p^\mu_i\, d_\mu \delta y^i d_\alpha \delta y^k\right) d\sigma =$$

$$= J\Big[\left(\left(p'_i J^i_k + \bar{J}^\lambda_\mu p'^\mu_i d_\lambda J^i_k\right)\,\delta^2 y^k + \bar{J}^\alpha_\mu p'^\mu_i J^i_k\, d_\alpha \delta^2 y^k\right) + \left(\partial'_l p'_i J^l_j J^i_k + 2\bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk}\right)\,\delta y^k \delta y^j +$$

$$+ 2\bar{J}^\alpha_\mu \left(\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik}\right)\,\delta y^i d_\alpha \delta y^k + \bar{J}^\alpha_\beta \bar{J}^\mu_\nu \partial^{\prime\nu}_j p'^\beta_h J^h_k J^j_i\, d_\alpha \delta y^k d_\mu \delta y^i\Big]d\sigma =$$

$$= \Big[\left(p'_i\,\left(\delta^2 y'^i - J^i_{kj}\delta y^j \delta y^k\right) + p'^\mu_i\,\left(\bar{d}'_\mu \delta^2 y'^i - \bar{J}^\sigma_\mu d_\alpha\left(J^i_{kj}\delta y^j \delta y^k\right)\right)\right) + \left(\partial'_j p'_i \delta y'^i \delta y'^j\right) + \left(2\bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk}\right)\,\delta y^k \delta y^j +$$

$$+ 2\bar{J}^\alpha_\mu \left(\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik}\right)\,\delta y^i \bar{J}^k_i\left(J^\mu_\alpha \delta y'^i_\mu - d_\alpha J^i_j\,\delta y^j\right) + \partial^{\prime\nu}_j p'^\beta_l\,\left(\delta y'^l_\beta \delta y'^i_\nu - \bar{J}^\alpha_\beta d_\alpha J^l_k\,\delta y^k \delta y'^i_\nu - \bar{J}^\mu_\nu d_\mu J^j_n\,\delta y^n \delta y'^l_\beta + \bar{J}^\alpha_\beta d_\alpha J^l_k \bar{J}^\mu_\nu d_\mu J^j_n\,\delta y^k \delta y^n\right)\Big]d\sigma' =$$

$$= \Big[\left(p'_i\,\delta^2 y'^i + p'^\mu_i \bar{d}'_\mu \delta^2 y'^i\right) + \left(\partial'_j p'_i \delta y'^i \delta y'^j + \partial^{\prime\nu}_j p'^\beta_l\,\delta y'^l_\beta \delta y'^j_\nu\right) - \underline{\bar{J}^\alpha_\mu p'^\mu_i (d_\alpha J^i_{kj})\delta y^j \delta y^k} - 2p'^\mu_j J^j_{ki}\bar{J}^k_n J^n_i \delta y^j \delta y'^m_\mu + 2\bar{J}^\alpha_\mu p'^\mu_j J^j_{ki}\bar{J}^k_l d_\alpha J^l_h\,\delta y^i \delta y^h +$$

$$+ \left(\underline{2\bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'_i J^i_k (d_\sigma J^l_j)} + \underline{\bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k)} + \underline{\bar{J}^\mu_\mu p'^\mu_i (d_\lambda J^i_{jk})}\right)\,\delta y^k \delta y^j + 2\left(\partial'_l p'^\mu_n J^l_i + \underline{\bar{J}^\sigma_\rho \partial'_l p'^\mu_n (d_\sigma J^l_i)} + p'^\mu_j J^j_{ik}\bar{J}^k_n\right)\,\delta y^i \delta y'^m_\mu +$$

$$- 2\left(\underline{\bar{J}^\sigma_\rho \partial'_l p'^\rho_n J^l_i (d_\sigma J^n_h)} + \underline{\bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial^{\prime\rho}_l p'^\mu_n (d_\sigma J^l_i)(d_\lambda J^n_h)} + \bar{J}^\alpha_\mu p'^\mu_j J^j_{ik}\bar{J}^k_l d_\alpha J^l_h\right)\,\delta y^i \delta y^h - \underline{\bar{J}^\sigma_\rho \partial^{\prime\mu}_n p'^\rho_l (d_\sigma J^l_i)\,\delta y^i \delta y'^m_\mu} - \underline{\bar{J}^\sigma_\nu \partial^{\prime\nu}_j p'^\mu_n (d_\sigma J^j_i)\,\delta y^i \delta y'^m_\mu} +$$

$$+ \underline{\bar{J}^\sigma_\rho \bar{J}^\lambda_\mu \partial^{\prime\mu}_i p'^\rho_l (d_\sigma J^l_j)(d_\lambda J^i_k)\,\delta y^k \delta y^j}\Big]d\sigma' = \Big[\left(p'_i\,\delta^2 y'^i + p'^\mu_i \bar{d}'_\mu \delta^2 y'^i\right) + \left(\partial'_j p'_i \delta y'^i \delta y'^j + 2\partial_i p'^\alpha_k \delta y'^i \delta y'^k_\alpha + \partial^{\prime\nu}_j p'^\beta_l\,\delta y'^l_\beta \delta y'^j_\nu\right)\Big]d\sigma' = \delta^2 \boldsymbol{L}' \tag{1.9.13}$$

We can also expand the deformed action at second order, i.e. as

$$A_D[\sigma_s] = \int_D (j^1\sigma)^* \left(L + s\,\delta L + \tfrac{1}{2}s^2\,\delta^2 L + O(s^3) \right) \tag{1.9.14}$$

Suppose σ is a critical section for L. We say that it is *stable* (with respect to the action functional) if it is a minimum of the action, *unstable* otherwise. To decide that, we have to check how the action changes when we deform σ. For that we need to consider the second variation.

$$A_D[\sigma_s] - A_D[\sigma] = \frac{s^2}{2}\int_D \delta^2 L + O(s^3) =$$
$$= \frac{s^2}{2}\left(\int_D (p_i - d_\mu p_i^\mu)\delta^2 y^i d\sigma + \int_{\partial D} p_i^\mu \delta^2 y^i d\sigma_\mu \right) + \frac{s^2}{2}\int_D \left(\partial_k p_i \delta y^i \delta y^k + 2\partial_i p_k^\alpha \delta y^i d_\alpha \delta y^k + \partial_k^\alpha p_i^\mu d_\mu \delta y^i d_\alpha \delta y^k \right) d\sigma + O(s^3) \tag{1.9.15}$$

Practically speaking, if the bilinear form

$$\mathbb{H} = \begin{pmatrix} \delta y^i & \delta y^i_\mu \end{pmatrix} \begin{pmatrix} \partial_k p_i & \partial_i p_k^\alpha \\ \partial_k p_i^\mu & \partial_k^\alpha p_i^\mu \end{pmatrix} \begin{pmatrix} \delta y^k \\ \delta y^k_\alpha \end{pmatrix} \tag{1.9.16}$$

is definite-positive, then the critical section is in a *minimum* of the action functional and it is called a *stable* solution. Let us remark, however, that the bilinear form $\mathbb{H}$ is applied to a vector $(\delta y^k, \delta y^k_\alpha)$ which is holonomic, not generic. Thus, when the form is not definite positive, we need to discuss the sign of the integral. We shall show that in an example, in the next SubSection.

Stable geodesics on spheres

Let us consider, as a simple example, the Lagrangian for geodesics on $[B := \mathbb{R} \times Q \to \mathbb{R}]$ given by

$$\boldsymbol{L} = \tfrac{1}{2}g_{\mu\nu}u^\mu u^\nu ds \tag{1.9.17}$$

By variation, we obtain

$$\delta L = g_{\mu\nu}u^\mu \tfrac{d}{dt}\delta x^\nu + \tfrac{1}{2}\partial_\lambda g_{\mu\nu}u^\mu u^\nu \delta x^\lambda = u^\mu g_{\mu\epsilon}\tfrac{d}{dt}\delta x^\epsilon + \tfrac{1}{2}\left(-\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} + \partial_\lambda g_{\mu\nu} \right) u^\mu u^\nu \delta x^\lambda = u^\mu g_{\mu\epsilon}\left(\tfrac{d}{dt}\delta x^\epsilon + \{g\}^\epsilon_{\nu\lambda}u^\nu \delta x^\lambda \right) \tag{1.9.18}$$

Hence, the auxiliary Lagrangian on $[\mathbb{R} \times TQ \to \mathbb{R} \times Q \to \mathbb{R}]$ is

$$J(\boldsymbol{L}) := \delta L ds = u^\mu g_{\mu\alpha}\left(\tfrac{dX^\alpha}{ds} + \{g\}^\alpha_{\lambda\nu}X^\lambda u^\nu \right) ds \tag{1.9.19}$$

which is regarded as a Lagrangian on $V(B) = \mathbb{R} \times TQ$, which happens to be linear in $X = X^\mu(t,x)\partial_\mu$ which is assumed to be a vertical vector on $\mathbb{R} \times Q$, or simply a vector field $X = X^\mu(x)\partial_\mu$ on Q. The variation of the auxiliary Lagrangian $J(\boldsymbol{L})$ defines Jacobi equation for X.

By variation with respect to δx^μ and δX^μ, we obtain

$$
\begin{aligned}
\delta(J(L)) =& \delta u^\mu g_{\mu\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right) + u^\mu \partial_\epsilon g_{\mu\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\delta x^\epsilon + u^\mu g_{\mu\alpha}\left(\tfrac{d}{ds}\delta X^\alpha + \{g\}^\alpha_{\sigma\nu}\delta X^\sigma u^\nu + \partial_\epsilon\{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\delta x^\epsilon + \{g\}^\alpha_{\sigma\nu}X^\sigma \delta u^\nu\right) = \\
=& \tfrac{d}{ds}\delta x^\epsilon g_{\epsilon\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right) + u^\mu g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}X^\sigma\tfrac{d}{ds}\delta x^\epsilon + u^\mu\partial_\epsilon g_{\mu\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\delta x^\epsilon + u^\mu g_{\mu\epsilon}\tfrac{d}{ds}\delta X^\epsilon + u^\mu g_{\mu\alpha}\{g\}^\alpha_{\epsilon\nu}\delta X^\epsilon u^\nu + u^\mu g_{\mu\alpha}\partial_\epsilon\{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\delta x^\epsilon = \\
=& -\tfrac{d}{ds}\left(g_{\epsilon\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\right)\delta x^\epsilon - \tfrac{d}{ds}\left(u^\mu g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}X^\sigma\right)\delta x^\epsilon + u^\mu(g_{\mu\beta}\{g\}^\beta_{\epsilon\alpha} + g_{\alpha\beta}\{g\}^\beta_{\mu\epsilon})\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\delta x^\epsilon + u^\mu g_{\mu\alpha}\partial_\epsilon\{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\delta x^\epsilon + \\
&+ \left(\textcolor{red}{u^\mu g_{\mu\alpha}\{g\}^\alpha_{\epsilon\nu}u^\nu} - g_{\beta\epsilon}\dot{u}^\beta - \textcolor{red}{u^\mu g_{\mu\beta}\{g\}^\beta_{\alpha\epsilon}u^\alpha} - g_{\epsilon\beta}\{g\}^\beta_{\mu\alpha}u^\mu u^\alpha)\right)\delta X^\epsilon + \tfrac{d}{ds}\left(\left(g_{\epsilon\alpha}\dot{X}^\alpha + g_{\epsilon\alpha}\{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu + u^\mu g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}X^\sigma\right)\delta x^\epsilon + u^\mu g_{\mu\epsilon}\delta X^\epsilon\right) = \\
=& \left(\left(-u^\rho g_{\epsilon\beta}\{g\}^\beta_{\rho\alpha} - \textcolor{red}{u^\rho g_{\alpha\beta}\{g\}^\beta_{\epsilon\rho}} + u^\rho g_{\rho\beta}\{g\}^\beta_{\epsilon\alpha} + \textcolor{red}{u^\rho g_{\alpha\beta}\{g\}^\beta_{\rho\epsilon}}\right)\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right) - g_{\epsilon\alpha}\tfrac{d}{ds}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\right)\delta x^\epsilon + u^\mu g_{\mu\alpha}d_\epsilon\{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\delta x^\epsilon + \\
&+ \left(-\dot{u}^\mu g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}X^\sigma - u^\mu u^\beta d_\beta g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}X^\sigma - u^\mu g_{\mu\alpha}u^\beta d_\beta\{g\}^\alpha_{\sigma\epsilon}X^\sigma - u^\mu g_{\mu\alpha}\{g\}^\alpha_{\sigma\epsilon}\dot{X}^\sigma\right)\delta x^\epsilon + \\
&- g_{\beta\epsilon}\left(\dot{u}^\beta + \{g\}^\beta_{\mu\alpha}u^\mu u^\alpha\right)\delta X^\epsilon + \tfrac{d}{ds}\left(\left(g_{\epsilon\alpha}\dot{X}^\alpha + (g_{\epsilon\alpha}\{g\}^\alpha_{\sigma\nu} + g_{\nu\alpha}\{g\}^\alpha_{\sigma\epsilon})u^\nu X^\sigma\right)\delta x^\epsilon + u^\mu g_{\mu\epsilon}\delta X^\epsilon\right) = \\
=& \left(u^\rho\left(\textcolor{red}{g_{\rho\beta}\{g\}^\beta_{\epsilon\alpha}} - g_{\epsilon\beta}\{g\}^\beta_{\rho\alpha}\right)\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right) - g_{\epsilon\alpha}\tfrac{d}{ds}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\right)\delta x^\epsilon + u^\mu u^\nu R_{\mu\sigma\epsilon\nu}X^\sigma\delta x^\epsilon + \\
&+ \left(-g_{\alpha\beta}\left(\dot{u}^\beta + \{g\}^\beta_{\mu\nu}u^\mu u^\nu\right)\{g\}^\alpha_{\sigma\epsilon}X^\sigma - \textcolor{red}{u^\rho g_{\rho\beta}\{g\}^\beta_{\alpha\epsilon}}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\right)\delta x^\epsilon + \\
&- g_{\beta\epsilon}\left(\dot{u}^\beta + \{g\}^\beta_{\mu\alpha}u^\mu u^\alpha\right)\delta X^\epsilon + \tfrac{d}{ds}\left(\left(g_{\epsilon\alpha}\dot{X}^\alpha + (g_{\epsilon\alpha}\{g\}^\alpha_{\sigma\nu} + g_{\nu\alpha}\{g\}^\alpha_{\sigma\epsilon})u^\nu X^\sigma\right)\delta x^\epsilon + u^\mu g_{\mu\epsilon}\delta X^\epsilon\right) = \\
=& \left(R_{\epsilon\mu\nu\sigma}X^\sigma u^\mu u^\nu - \left(\dot{u}^\beta + \{g\}^\beta_{\mu\nu}u^\mu u^\nu\right)g_{\alpha\beta}\{g\}^\alpha_{\sigma\epsilon}X^\sigma - g_{\epsilon\beta}\left(\tfrac{d}{ds}\left(\dot{X}^\beta + \{g\}^\beta_{\sigma\nu}X^\sigma u^\nu\right) + \{g\}^\beta_{\alpha\rho}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)u^\rho\right)\right)\delta x^\epsilon - g_{\epsilon\alpha}\left(\dot{u}^\alpha + \{g\}^\alpha_{\mu\nu}u^\mu u^\nu\right)\delta X^\epsilon + \\
&+ \tfrac{d}{ds}\left(g_{\epsilon\alpha}\left(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu\right)\delta x^\epsilon + u^\mu g_{\mu\alpha}\left(\delta X^\epsilon + \{g\}^\alpha_{\sigma\epsilon}X^\sigma\delta x^\epsilon\right)\right)
\end{aligned}
\tag{1.9.20}
$$

Since a section is given by $(x^\mu(t), X^\mu(t,x))$ we have

$$
\begin{aligned}
\dot{X}^\alpha =& \partial X^\alpha + u^\beta d_\beta X^\alpha \qquad\qquad \Rightarrow \quad \dot{X}^\alpha + \{g\}^\alpha_{\sigma\nu}X^\sigma u^\nu = \partial X^\alpha + u^\beta\nabla_\beta X^\alpha \\
\ddot{X}^\alpha =& \partial^2 X^\alpha + \dot{u}^\beta d_\beta X^\alpha + 2u^\beta d_\beta\partial X^\alpha + u^\mu u^\nu(\nabla_{\mu\nu}X^\alpha - \{g\}^\alpha_{\rho\mu}\nabla_\nu X^\rho + \{g\}^\rho_{\mu\nu}\nabla_\rho X^\alpha) - u^\mu u^\nu d_\mu\{g\}^\alpha_{\sigma\nu}X^\sigma - u^\mu u^\nu\{g\}^\alpha_{\sigma\nu}(\nabla_\mu X^\sigma - \{g\}^\sigma_{\rho\mu}X^\rho) = \\
=& \partial^2 X^\alpha + 2u^\beta\nabla_\beta\partial X^\alpha + (\dot{u}^\beta + \{g\}^\beta_{\mu\nu}u^\mu u^\nu)(\nabla_\beta X^\alpha - \{g\}^\alpha_{\sigma\beta}X^\sigma) - 2u^\beta\{g\}^\alpha_{\sigma\beta}\partial X^\sigma + u^\mu u^\nu(\nabla_{\mu\nu}X^\alpha - 2\{g\}^\alpha_{\sigma\mu}\nabla_\nu X^\sigma - d_\mu\{g\}^\alpha_{\sigma\nu}X^\sigma + \{g\}^\alpha_{\beta\nu}\{g\}^\beta_{\sigma\mu}X^\sigma + \{g\}^\beta_{\mu\nu}\{g\}^\alpha_{\sigma\beta}X^\sigma) \\
\Rightarrow \quad & \tfrac{d}{ds}\left(\dot{X}^\beta + \{g\}^\beta_{\sigma\nu}X^\sigma u^\nu\right) = \ddot{X}^\beta + (\dot{u}^\nu + \{g\}^\nu_{\alpha\rho}u^\alpha u^\rho)\{g\}^\beta_{\sigma\nu}X^\sigma + \{g\}^\beta_{\alpha\nu}(\dot{X}^\alpha + \{g\}^\alpha_{\sigma\mu}X^\sigma u^\mu)u^\nu + u^\mu u^\nu\left(d_\mu\{g\}^\beta_{\sigma\nu}X^\sigma - \{g\}^\rho_{\mu\nu}\{g\}^\beta_{\sigma\rho}X^\sigma - \{g\}^\beta_{\alpha\nu}\{g\}^\alpha_{\sigma\mu}X^\sigma\right) = \\
& = \partial^2 X^\alpha + 2u^\beta\nabla_\beta\partial X^\alpha - u^\beta\{g\}^\alpha_{\sigma\beta}\partial X^\sigma + (\dot{u}^\beta + \{g\}^\beta_{\mu\nu}u^\mu u^\nu)\nabla_\beta X^\alpha + u^\mu u^\nu\left(\nabla_{\mu\nu}X^\alpha - \{g\}^\alpha_{\sigma\mu}\nabla_\nu X^\sigma\right)
\end{aligned}
\tag{1.9.21}
$$

Then, in view of the original equation, the Jacobi equation becomes

$$
\partial^2 X^\epsilon + 2u^\beta\nabla_\beta\partial X^\epsilon + u^\mu u^\nu\nabla_{\mu\nu}X^\epsilon = u^\mu u^\nu R^\epsilon_{\ (\mu\nu)\sigma}X^\sigma
\tag{1.9.22}
$$

This is the Jacobi equation for a time-dependent Jacobi field X. In the literature, often one finds the Jacobi equations for vector fields on the manifold M, i.e. with $\partial X^\alpha = 0$, which in fact is

$$
\nabla_{\mu\nu}X^\epsilon = R^\epsilon_{\ \mu\nu\sigma}X^\sigma
\tag{1.9.23}
$$

Therefore, Euler-Lagrange equations for $J(\boldsymbol{L})$, i.e. linearized equation for L, are

$$
\dot{u}^\lambda + \{g\}^\lambda_{\alpha\beta}u^\alpha u^\beta = 0 \qquad\qquad \nabla_{\mu\nu}X^\alpha = R^\alpha_{\ (\nu\mu)\lambda}X^\lambda
\tag{1.9.24}
$$

and the solution vector fields X are called *Jacobi fieds* on (M, g) and their flow sends (affinely parameterised) geodesics into (affinely parameterised) geodesics.

The variation $\delta J(\boldsymbol{L})$ is something on $V(V(B))$ which can be restricted on $V^2(B)$ to get the second variation of the original Lagrangian L, evaluated along a critical configuration, and neglecting boundary term that will vanish for boundary conditions, that is

$$\delta^2 L = - \left(u^\mu u^\nu g_{\epsilon\alpha} \nabla_\mu \nabla_\nu X^\alpha - u^\mu u^\nu R_{\epsilon\nu\mu\lambda} X^\lambda \right) X^\epsilon = \left(g_{\epsilon\alpha} \hat{X}^\alpha \hat{X}^\epsilon + u^\mu u^\nu R_{\epsilon\nu\mu\lambda} X^\lambda X^\epsilon \right) - \tfrac{d}{ds} \left(u^\nu g_{\epsilon\alpha} \nabla_\nu X^\alpha X^\epsilon \right) \tag{1.9.25}$$

where we set $\hat{X}^\alpha := \dot{X}^\alpha + u^\mu \{g\}^\alpha_{\mu\nu} X^\nu = u^\mu \nabla_\mu X^\alpha$.

The bilinear form $\mathbb{H}$ for the geodesic Lagrangian specializes to

$$\mathbb{H} = \begin{pmatrix} X^\alpha & \hat{X}^\alpha \end{pmatrix} \begin{pmatrix} R_{\alpha\mu\nu\beta} u^\mu u^\nu & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix} \begin{pmatrix} X^\beta \\ \hat{X}^\beta \end{pmatrix} \tag{1.9.26}$$

The curvature, top-left block, on the equator of the sphere in stereographic coordinates, reads as

$$\begin{pmatrix} X^1 & X^2 \end{pmatrix} \begin{pmatrix} -\cos^2(t) & -\cos(t)\sin(t) \\ -\cos(t)\sin(t) & -\sin^2(t) \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \tag{1.9.27}$$

which has eigenvalues $\lambda = 0, -1$. Since the metric, bottom-right block $(g_{\alpha\beta})$ is definite positive if we hope to get a negative value to prove instability we need to focus on the negative eigenvector, which in this case is $X = \cos(t)\partial_x + \sin(t)\partial_y$, i.e. it is orthogonal to the equator as one could geometrically guess.

Of course, the components $\hat{X}^\alpha$ are not independent of the component X^α (otherwise we could simply look for negative eigenvectors). Hence we select a vector $\xi = f(t) \left(\cos(t)\partial_x + \sin(t)\partial_y \right)$ along the equator in the direction of a negative eigenvector and $\hat{\xi} = \dot{f}(t) \left(\cos(t)\partial_x + \sin(t)\partial_y \right)$. Then we can compute the bilinear form along ξ to be

$$\mathbb{H} = \dot{f}^2 - f^2 \tag{1.9.28}$$

We have to determine f so that $\int \mathbb{H} \, dt < 0$, which proves instability of the equator. That depends on the boundary points; suppose we integrate between $t = 0$ and $t = t_0$. We do not need ξ to be a Jacobi field, we still need it to be fixed at the boundary, $f(0) = f(t_0) = 0$.

If $t_0 = a\pi$, we can set

$$f(t) = \sin\left(\tfrac{t}{a}\right) \tag{1.9.29}$$

In fact, we have $f(0) = f(a\pi) = 0$, and

$$\int_0^{t_0} \mathbb{H} dt = \tfrac{1}{2a}(1 - a^2) \tag{1.9.30}$$

If $a < 1$ (i.e. $t_0 < \pi$) that is positive and action functional in that direction is growing, which is compatible with the configuration being stable. When $a > 1$ (i.e. $t_0 > \pi$) that is negative and we found one deformation along which the action functional is not increasing which is enough to prove that the solution is unstable.

10. Examples of field theories

We shall hereafter provide some examples of field theories in order to get familiar with variational principles before introducing relativistic theories
and gravitation. We are also going to fix some notation that will be useful later on.

Vibrating flat plate

Let us consider a vibrating 2-dimensional homogeneous planar continuum. This can be seen to be described by a function $\varphi(t, x)$ which measures the vertical displacement at the time t of the point x from the equilibrium. The dynamics is given by the Lagrangian

$$\boldsymbol{L} = \tfrac{\sqrt{g}}{2} \left(\rho \varphi_0^2 - (\Delta\varphi)^2 + 2(1 - \mu)\det(\nabla_{ij}\varphi) - 2f(t, x)\varphi \right) \, d\sigma \tag{1.10.1}$$

where we set $\Delta\varphi = g^{ij}\nabla_{ij}\varphi$ for the Laplacian and $f(t, x)$ is the density of external forces acting on the plate, ρ and μ are constants representing the mass density of the plate and the Poisson's ratio of the material, respectively; see[2].

> In this example, we are understanding the dimensions of some of the coefficients so we cannot use dimensional analysis. In fact, one can see that the terms in the Lagrangian have different dimensions, as they are written.

This is a second order Lagrangian for the field φ, which is a section of the bundle $M \times \mathbb{R}$ with $M = \Sigma \times \mathbb{R}$ and Σ the two dimensional manifold modeling the plate. The Lagrangian also depends on some functions g_{ij} which represent the coefficients of the metric induced on the plate which depend on coordinates x^i used on the plate Σ. However, whatever the coordinates x^i on the plate are, the curvature R of the metric g is always vanishing as long as the plate is planar (which is of course an approximation during *small* vibrations). Here the point is that there is no reason to change coordinates x^i on time (thus g_{ij} just depend on x^i but not on time t) and the metric g_{ij} (as well as its Levi Civita connection) is assigned once and for all. It is not a field to be varied (as φ is) since one is not required to *determine* it from some field equations.

The variation of the Lagrangian is

$$\begin{aligned}
\delta\boldsymbol{L} = & \sqrt{g}\left(\rho\varphi_0\partial_0\delta\varphi - (\Delta\varphi)g^{ij}\nabla_{ij}\delta\varphi + (1-\mu)(\nabla_{22}\varphi\nabla_{11}\delta\varphi + \nabla_{11}\varphi\nabla_{22}\delta\varphi - 2\nabla_{12}\varphi\nabla_{12}\delta\varphi) - f(t, x)\delta\varphi\right)\,d\sigma = \\
= & -\sqrt{g}\left(\rho\varphi_{00} + \Delta^2\varphi + f(t, x) - (1-\mu)(\nabla_{11}\nabla_{22}\varphi + \nabla_{22}\nabla_{11}\varphi - 2\nabla_2\nabla_1\nabla_{12}\varphi)\right)\delta\varphi\,d\sigma + \\
& + d\Big[\sqrt{g}\Big(\rho\varphi_0\delta\varphi\,d\sigma_0 + \left(g^{ij}\nabla_j\left(\Delta\varphi\right)\delta\varphi - g^{ij}\Delta\varphi\nabla_j\delta\varphi\right)\,d\sigma_i + \\
& + (1-\mu)\left[(\nabla_{22}\varphi\nabla_1\delta\varphi - \nabla_1(\nabla_{22}\varphi)\delta\varphi - 2\nabla_{12}\varphi\nabla_2\delta\varphi)\,d\sigma_1 + (\nabla_{11}\varphi\nabla_2\delta\varphi + (2\nabla_1(\nabla_{12}\varphi) - \nabla_2(\nabla_{11}\varphi))\delta\varphi)\right]\Big)\,d\sigma_2\Big]
\end{aligned} \tag{1.10.2}$$

where we set $\Delta^2\varphi = g^{kl}\nabla_{kl}(\Delta\varphi)$ for the double Laplacian with respect to the metric g.

> The tensor
>
> $$\nabla_{11}\nabla_{22}\varphi + \nabla_{22}\nabla_{11}\varphi - 2\nabla_2\nabla_1\nabla_{12}\varphi \tag{1.10.3}$$
>
> is vanishing in Cartesian coordinates, thus it is vanishing in any coordinate system.

One can also compute it directly

$$
\begin{aligned}
\nabla_{11}\nabla_{22}\varphi + \nabla_{22}\nabla_{11}\varphi - 2\nabla_2\nabla_1\nabla_{12}\varphi &= \\
&= \nabla_{11}\nabla_{22}\varphi + \nabla_{22}\nabla_{11}\varphi - \nabla_2\nabla_1\nabla_1\nabla_2\varphi - \nabla_2\nabla_1\nabla_2\nabla_1\varphi = \\
&= \nabla_{11}\nabla_{22}\varphi + \nabla_{22}\nabla_{11}\varphi - (\nabla_1\nabla_2\nabla_1\nabla_2\varphi - R^k{}_{121}\nabla_k\nabla_2\varphi - R^k{}_{221}\nabla_1\nabla_k\varphi) - \nabla_2(\nabla_2\nabla_1\nabla_1\varphi - R^k{}_{112}\nabla_k\varphi) = \\
&= \nabla_{11}\nabla_{22}\varphi - \nabla_1\nabla_2\nabla_1\nabla_2\varphi + R^2{}_{121}\nabla_2\nabla_2\varphi + R^1{}_{221}\nabla_1\nabla_1\varphi + \nabla_2(R^2{}_{112}\nabla_2\varphi) = \\
&= \nabla_{11}\nabla_{22}\varphi - \nabla_1(\nabla_1\nabla_2\nabla_2\varphi - R^k{}_{221}\nabla_k\varphi) - R^2{}_{112}\nabla_2\nabla_2\varphi + R^1{}_{221}\nabla_1\nabla_1\varphi + \nabla_2 R^2{}_{112}\nabla_2\varphi + R^2{}_{112}\nabla_2\nabla_2\varphi = \\
&= \nabla_1 R^1{}_{221}\nabla_1\varphi + R^1{}_{221}\nabla_1\nabla_1\varphi + R^1{}_{221}\nabla_1\nabla_1\varphi + \nabla_2 R^2{}_{112}\nabla_2\varphi
\end{aligned}
\tag{1.10.4}
$$

which vanishes on flat surfaces.

The field equations are thence

$$
\rho\,\partial_t^2\varphi + \Delta^2\varphi + f(t,x) = 0 \tag{1.10.5}
$$

This example is ugly. We chose it just because it is so different from a relativistic theory. Here time and space variables are not treated on equal footing at all. This is also an example of a global Lagrangian with global field equations which is not covariant.

Degenerate Lagrangians

In the previous example, one starts from a second order Lagrangian and obtains, as expected, fourth order field equations. However, there are Lagrangians which are *degenerate*, in the sense that their field equations happen to be of order less that the expected one or, more generally, one cannot solve the momenta for the derivatives of fields. Unfortunately, this is quite a common feature in fundamental physics since all Yang–Mills Lagrangians as well as the Hilbert Lagrangian happen to be degenerate. Unlike what usually happens, in this example one has field equations which are not even quasi-linear (i.e. they are not affine in the higher order derivatives appearing in field equations).

Let us consider the configuration bundle $\mathcal{C} = (\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2, \pi, \mathbb{R})$ with coordinates (x^1, x^2, y).

Let us set $l^2 := (y_1)^2 + (y_2)^2$ and $\Delta := (y_{11}y_{22} - (y_{12})^2)$ and consider the Lagrangian

$$
L = \frac{y\Delta}{l^2} \tag{1.10.6}
$$

> By the way, computing field equation for this Lagrangian is a perfect motivation for introducing advanced notation. The reader is urged to do the computation and compare with the way we shall do it below, which turns out to be much easier (though still not trivial).

Let us assume one has a metric on $\mathbb{R}^2$ in which the coordinates x^μ are orthonormal. The metric $g_{\mu\nu}$ is fixed and we shall not deform it in variational calculus. Then one has

$$
\Delta = \tfrac{1}{2\sqrt{g}}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}y_{\mu\alpha}y_{\nu\beta} \qquad l^2 = g^{\mu\nu}y_\mu y_\nu \tag{1.10.7}
$$

where $\epsilon^{\mu\nu}$ denotes the canonical Levi Civita antisymmetric tensor density in dimension 2. Let us stress that l^2 is a scalar, while Δ is a scalar density ▷*Levi Civita tensor densities* of weight 1 (since $\epsilon^{\alpha\beta}$ are tensor densities of weight 1) so that the Lagrangian density L is a scalar density of the correct weight as well.

One can also define

$$y^{\mu\nu} := \tfrac{1}{\sqrt{g}}\epsilon^{\mu\alpha}\epsilon^{\nu\beta}y_{\alpha\beta} \qquad y^\mu := g^{\mu\alpha}y_\alpha \tag{1.10.8}$$

so that one has

$$\Delta = \tfrac{1}{2}y^{\mu\nu}y_{\mu\nu} \qquad l^2 = y^\mu y_\mu \tag{1.10.9}$$

Notice that, for symmetry reasons, one readily has

$$d_\mu y^{\mu\nu} = \tfrac{1}{\sqrt{g}}\epsilon^{\mu\alpha}\epsilon^{\nu\beta}y_{\alpha\beta\mu} - \tfrac{1}{2\sqrt{g}}g^{\rho\sigma}d_\mu g_{\rho\sigma}\epsilon^{\mu\alpha}\epsilon^{\nu\beta}y_{\alpha\beta} = -\{g\}^\rho_{\mu\rho}y^{\mu\nu} \tag{1.10.10}$$

which vanishes in Cartesian coordinates.

The variation of such a Lagrangian, neglecting pure divergence terms (as well as terms which vanish in Cartesian coordinates by equation (1.10.10)), is then

$$\begin{aligned}
\delta L =& \frac{\Delta}{l^2}\delta y + \frac{y}{l^2}y^{\mu\nu}\delta y_{\mu\nu} - 2\frac{y\Delta}{l^4}y^\mu\delta y_\mu = \frac{\Delta}{l^2}\delta y - \frac{y_\nu}{l^2}y^{\mu\nu}\delta y_\mu + 2\frac{y}{l^4}y^\alpha y_{\alpha\nu}y^{\mu\nu}\delta y_\mu - \frac{y}{l^4}y^{\alpha\beta}y_{\alpha\beta}y^\mu\delta y_\mu + \mathrm{div} = \\
=& \left[\frac{\Delta}{l^2} + d_\mu\left(\frac{y_\nu}{l^2}\right)y^{\mu\nu} - 2d_\mu\left(\frac{y}{l^4}y^\alpha y_{\alpha\nu}\right)y^{\mu\nu} + d_\mu\left(\frac{y}{l^4}y^{\alpha\beta}y_{\alpha\beta}y^\mu\right)\right]\delta y + \mathrm{div}
\end{aligned} \tag{1.10.11}$$

where div stands for divengence terms that we should keep track of to compute $\mathbb{F}$ (while here we do not since we are focusing on $\mathbb{E}$ only).

Then field equations are then obtained as

$$\begin{aligned}
\mathbb{E} :=& \frac{\Delta}{l^2} + d_\mu\left(\frac{y_\nu}{l^2}\right)y^{\mu\nu} - 2d_\mu\left(\frac{y}{l^4}y^\alpha y_{\alpha\nu}\right)y^{\mu\nu} + d_\mu\left(\frac{y}{l^4}y^{\alpha\beta}y_{\alpha\beta}y^\mu\right) = \\
=& \underline{\frac{\Delta}{l^2}} + 2\underline{\frac{\Delta}{l^2}} - \underline{\underline{\frac{2}{l^4}y^\alpha y_{\alpha\mu}y_\nu y^{\mu\nu}}} - \underline{\underline{\frac{2}{l^4}y_\mu y^\alpha y_{\alpha\nu}y^{\mu\nu}}} - 2yd_\mu\left(\frac{y^\alpha}{l^4}y_{\alpha\nu}\right)y^{\mu\nu} + \underline{\frac{2}{l^4}l^2\Delta} + yd_\mu\left(\frac{1}{l^4}y^{\alpha\beta}y_{\alpha\beta}y^\mu\right) = \\
=& 5\frac{\Delta}{l^2} - \frac{4}{l^4}y^\alpha y_{\alpha\mu}y_\nu y^{\mu\nu} - 2yd_\mu\left(\frac{y^\alpha}{l^4}\right)y_{\alpha\nu}y^{\mu\nu} - {\color{red}2y\frac{y^\alpha}{l^4}y_{\alpha\mu\nu}y^{\mu\nu}} + yd_\mu\left(\frac{y^\mu}{l^4}\right)y^{\alpha\beta}y_{\alpha\beta} + {\color{red}2y\frac{y^\mu}{l^4}y^{\alpha\beta}y_{\mu\alpha\beta}} = \\
=& 5\frac{\Delta}{l^2} - \frac{4}{l^4}y^\alpha y_{\alpha\mu}y_\nu y^{\mu\nu} - 2yd_\mu\left(\frac{y^\alpha}{l^4}\right)y_{\alpha\nu}y^{\mu\nu} + yd_\mu\left(\frac{y^\mu}{l^4}\right)y^{\alpha\beta}y_{\alpha\beta}
\end{aligned} \tag{1.10.12}$$

Already at this stage, we can see that field equation will be second order instead of fourth. However, it is worth going on to simplify their expression to obtain

$$\begin{aligned}
\mathbb{E} =& 5\frac{\Delta}{l^2} - \frac{4}{l^4}y^\alpha y_{\alpha\mu}y_\nu y^{\mu\nu} - 2yd_\mu\left(\frac{y^\alpha}{l^4}\right)y_{\alpha\nu}y^{\mu\nu} + yd_\mu\left(\frac{y^\mu}{l^4}\right)y^{\alpha\beta}y_{\alpha\beta} = \\
=& 5\frac{\Delta}{l^2} - \frac{4}{l^4}\left(g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}\right)y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} + y\left(\frac{y_{\lambda\mu}}{l^4} - 4\frac{y_\lambda}{l^6}y^\epsilon y_{\epsilon\mu}\right)\left(-2g^{\lambda\alpha}\epsilon^{\mu\rho}\epsilon^{\beta\sigma} + g^{\lambda\mu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right)y_{\alpha\beta}y_{\rho\sigma} = \\
=& 5\frac{\Delta}{l^2} - \frac{4}{l^4}\left(g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}\right)y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} + \frac{y}{l^6}\left(y_\epsilon y_\nu y_{\lambda\mu} - 4y_\lambda y_\nu y_{\epsilon\mu}\right)\left(-2g^{\lambda\alpha}g^{\nu\epsilon}\epsilon^{\mu\rho}\epsilon^{\beta\sigma} + g^{\lambda\mu}g^{\nu\epsilon}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right)y_{\alpha\beta}y_{\rho\sigma} = \\
=& 5\frac{\Delta}{l^2} - \frac{4}{l^4}\left(g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}\right)y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} + \frac{y}{l^6}\left(-2g^{\lambda\alpha}g^{\nu\mu}\epsilon^{\epsilon\rho}\epsilon^{\beta\sigma} + g^{\lambda\epsilon}g^{\nu\mu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma} + 8g^{\mu\alpha}g^{\nu\epsilon}\epsilon^{\lambda\rho}\epsilon^{\beta\sigma} - 4g^{\mu\lambda}g^{\nu\epsilon}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right)y_\mu y_\nu y_{\epsilon\lambda}y_{\alpha\beta}y_{\rho\sigma}
\end{aligned} \tag{1.10.13}$$

One can show that

$$\left(g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}\right)y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = l^2\Delta \qquad \left(-2g^{\lambda\alpha}g^{\nu\mu}\epsilon^{\epsilon\rho}\epsilon^{\beta\sigma} + g^{\lambda\epsilon}g^{\nu\mu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma} + 8g^{\mu\alpha}g^{\nu\epsilon}\epsilon^{\lambda\rho}\epsilon^{\beta\sigma} - 4g^{\mu\lambda}g^{\nu\epsilon}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right)y_\mu y_\nu y_{\epsilon\lambda}y_{\alpha\beta}y_{\rho\sigma} = 0 \tag{1.10.14}$$

In fact, for the first identity one has

$$g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = \epsilon^{\alpha\lambda}\epsilon^{\mu}_{.\lambda}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = \left(g^{\mu\nu}\delta^{\sigma}_{\lambda} - g^{\mu\sigma}\delta^{\nu}_{\lambda}\right)\epsilon^{\alpha\lambda}\epsilon^{\beta\rho}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} =$$

$$= g^{\mu\nu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} - g^{\mu\alpha}\epsilon^{\sigma\nu}\epsilon^{\rho\beta}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = g^{\mu\nu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} - g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} \qquad \Rightarrow \tag{1.10.15}$$

$$\Rightarrow \quad g^{\alpha\mu}\epsilon^{\beta\rho}\epsilon^{\nu\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = \tfrac{1}{2}g^{\mu\nu}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}y_\mu y_\nu y_{\alpha\beta}y_{\rho\sigma} = l^2\Delta$$

so that the first of (1.10.14) holds true. For the second identity, one has

$$\left[-2g^{\nu\mu}\left(g^{\lambda\alpha}\epsilon^{\epsilon\rho}\epsilon^{\beta\sigma} - \tfrac{1}{2}g^{\lambda\epsilon}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right) + 8g^{\nu\epsilon}\left(g^{\mu\alpha}\epsilon^{\lambda\rho}\epsilon^{\beta\sigma} - \tfrac{1}{2}g^{\mu\lambda}\epsilon^{\alpha\rho}\epsilon^{\beta\sigma}\right)\right]y_\mu y_\nu y_{\epsilon\lambda}y_{\alpha\beta}y_{\rho\sigma} = 0 \tag{1.10.16}$$

by applying two times the identity (1.10.15).

Accordingly, field equation is

$$\mathbb{E} = \frac{\Delta}{l^2} = 0 \tag{1.10.17}$$

which is second order (and not fourth order as expected) and not quasi-linear (but quadratic in the second derivatives contained in Δ).

Real Klein–Gordon field

Let us consider a spacetime M equipped with a fixed Lorentzian metric g. We want to describe a scalar field $\varphi : M \to \mathbb{R}$ on the Lorentzian manifold (M, g). The first step is to address kinematics. When one considers two nearby charts (laboratories), by definition a *scalar field* φ transforms as

$$\varphi'(x') = \varphi(x) \tag{1.10.18}$$

This means that the value of the field φ does not depend on the observer. Of course, the local expression does.

Let us consider a simple special case $M = S^1$ with an angular coordinate θ. A function $\varphi_\theta : M \to \mathbb{R} : \theta \mapsto \cos(\theta)$. One can also consider two local coordinate systems $x = \cos(\theta)$ (defined on the open set $U_x := \{\theta \in (0, \pi)\}$) and $y = \sin(\theta)$ (defined on the open set $U_y := \{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$). The two observers are both defined on the overlap $U_{xy} := \{\theta \in (0, \frac{\pi}{2})\}$.

The function above is expressed with respect to the observer which uses x as $\varphi_x : U_x \to \mathbb{R} : x \mapsto x$. The same function is expressed with respect to the observer which uses y as $\varphi_y : U_y \to \mathbb{R} : y \mapsto \sqrt{1 - y^2}$.

As one can see the same function is locally expressed by different functions $(\varphi_\theta, \varphi_x, \varphi_y)$ and, in the overlaps, these local expressions are related by (1.10.18). For example, one has $x = \cos(\theta)$ and

$$\varphi_x(x) = \varphi_\theta(\theta) \tag{1.10.19}$$

The configuration bundle $\mathcal{C}$ is defined in terms of transition maps (1.10.18) which are the identities. Transformation rules (1.10.18) prescribe that the new fields are obtained replacing the new coordinates into the old local expression.

Then the configuration bundle is trivial, i.e. $K = M \times \mathbb{R}$ with fibered coordinates (x^μ, φ). Since we shall need the metric as well, let us consider the bundle $\mathrm{Lor}(M)$ of Lorentzian metrics and consider the configuration bundle $\mathcal{C} = K \times_M \mathrm{Lor}(M)$ with coordinates $(x^\mu, \varphi, g_{\mu\nu})$. Let us also denote by $(x^\mu, \varphi, \varphi_\mu, \ldots, g_{\mu\nu}, \{g\}^\alpha_{\mu\nu})$ fibered coordinates on the jet prolongations (we shall not need here derivatives of the metric higher than first order).

The dynamics is described by the Lagrangian

$$\boldsymbol{L}_{KG} = -\frac{\sqrt{g}}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi + \mu^2\varphi^2\right)\,d\sigma \tag{1.10.20}$$

where ∇_μ is the covariant derivative with respect to g and we denote it by $\overset{*}{\nabla}_\mu$ to emphasise the cases when it does not depend on the connection and it reduces to ordinary partial derivatives.

In general, one cannot assume that coordinates are lengths as in Cartesian coordinates. Even in relatively simple examples as polar coordinates, some of the coordinates are length, while the angles are adimensional. As a consequence, some of the coefficients of the metric, which are adimensional in Cartesian coordinates, retain dimensions so that the tensor is $[g] = L^2$.

On an m dimensional spacetime we have $[\sqrt{g}\,d\sigma] = L^m$, and set $[\varphi]$ for the unknown dimension of the field, so that $[g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi] = [\varphi]^2 L^{-2}$. We want, of course, that the action functional is an action, i.e. $[\boldsymbol{L}] = ML^2T^{-1}$, so that $[\varphi]^2 L^{m-2} = ML^2T^{-1}$ and then $[\varphi] = M^{1/2}L^{(4-m)/2}T^{-1/2}$. Finally, for dimensional coherence, we need to have $[\mu] = L^{-1}$.

Thus it seems difficult to consider μ as a mass! However, let us consider the constant $[\mathrm{c}/\hbar] = M^{-1}L^{-1}$ so that the constant

$$\hat\mu := \frac{\hbar\mu}{\mathrm{c}} \qquad \Rightarrow [\hat\mu] = M \tag{1.10.21}$$

is, in fact, a mass. Accordingly, we could (and should) write the Lagrangian density as

$$L_{KG} = -\frac{\sqrt{g}}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi + \frac{\mathrm{c}^2\hat\mu^2}{\hbar^2}\varphi^2\right) \tag{1.10.22}$$

so that $\hat\mu$ is a proper mass parameter.

The variation of the Lagrangian is

$$\begin{aligned}
\delta\boldsymbol{L}_{KG} =& \sqrt{g}\left(\delta g^{\alpha\beta}\left(\tfrac{1}{4}g_{\alpha\beta}g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi + \tfrac{1}{4}g_{\alpha\beta}\mu^2\varphi^2 - \tfrac{1}{2}\overset{*}{\nabla}_\alpha\varphi\overset{*}{\nabla}_\beta\varphi\right) - g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\delta\varphi - \mu^2\varphi\delta\varphi\right)\,d\sigma = \\
=& \sqrt{g}\left(-\tfrac{1}{2}T_{\alpha\beta}\delta g^{\alpha\beta} - \overset{g}{\nabla}_\nu\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\delta\varphi\right) + \left(\Box\varphi - \mu^2\varphi\right)\delta\varphi\right)\,d\sigma
\end{aligned} \tag{1.10.23}$$

$[T_{\mu\nu}] = ML^{2-m}T^{-1}$

where we set $T_{\alpha\beta} := \overset{*}{\nabla}_\alpha\varphi\overset{*}{\nabla}_\beta\varphi - \frac{1}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi + \mu^2\varphi^2\right)g_{\alpha\beta}$ for the so–called *energy-momentum stress tensor* of real Klein–Gordon field and $\Box\varphi := g^{\mu\nu}\overset{g}{\nabla}_\nu\overset{*}{\nabla}_\mu\varphi$ for the Dalambertian with respect to the metric g.

At this point, we have two possible paths to follow: either we regard the metric g as a *fixed* structure over spacetime or we consider it as a dynamical field. In the first case, it is not varied (as we did in the previous example) so we set $\delta g^{\alpha\beta} = 0$ and the variation of the Lagrangian simplifies accordingly and the real Klein–Gordon field equations are then

$$\Box\varphi - \mu^2\varphi = 0 \tag{1.10.24}$$

In the second case, also the metric is varied and field equations are

$$\begin{cases} \Box\varphi - \mu^2\varphi = 0 \\ T_{\alpha\beta} = 0 \end{cases}$$

(1.10.25)

In both cases, the Liouville part of $\boldsymbol{L}_{KG}$ reads as

$$\mathbb{F}(\boldsymbol{L}_{KG}) = -\sqrt{g}\, g^{\mu\nu} \overset{*}{\nabla}_{\mu}\varphi \,(\, d\varphi - \varphi_\lambda \, dx^\lambda)\wedge\, d\sigma_\nu$$

(1.10.26)

The first attitude of considering the metric g fixed is typical of (even though not unique to) SR. The metric g is considered physical but not dynamical. It is given together the specification of the system and it is known *a priori*. We shall point out in a number of instances that the fact that a field is not dynamical (i.e. it is given and not determined through some field equations) leave some spurious terms which must be dealt with with extra care.

The second attitude is typical of relativistic theories. Part of the definition of a relativistic theory is that any field *is dynamical* and no exception is allowed. In other words, spacetime is given without any field on it. Any physical structure defined on spacetime must be determined by field equations.

In this particular case, the field equations $T_{\alpha\beta} = 0$ are too strict to make physical sense. If they have to be obeyed, then they can be traced to obtain

$$T := g^{\alpha\beta} \overset{*}{\nabla}_{\alpha}\varphi \overset{*}{\nabla}_{\beta}\varphi - \tfrac{m}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_{\mu}\varphi\overset{*}{\nabla}_{\nu}\varphi + \mu^2\varphi^2\right) = 0 \qquad \Rightarrow (2-m)g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\varphi\overset{*}{\nabla}_{\beta}\varphi - m\mu^2\varphi^2 = 0$$

(1.10.27)

Let us first consider the case $\mu = 0$. In dimension other than $m = 2$, this equation implies $g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\varphi\overset{*}{\nabla}_{\beta}\varphi = 0$, which replaced back into $T_{\alpha\beta}$ gives

$$\overset{*}{\nabla}_{\alpha}\varphi\overset{*}{\nabla}_{\beta}\varphi = 0$$

(1.10.28)

This implies $\overset{*}{\nabla}_{\alpha}\varphi = 0$, which means that φ is constant. In dimension $m = 2$ and $\mu \neq 0$, (1.10.27) implies directly $\varphi = 0$. In the generic case $m \neq 2$ and $\mu \neq 0$, (1.10.27) can be plugged back in the field equation

$$T_{\alpha\beta} = \overset{*}{\nabla}_{\alpha}\varphi\overset{*}{\nabla}_{\beta}\varphi - \tfrac{1}{m}g^{\mu\nu}\overset{*}{\nabla}_{\mu}\varphi\overset{*}{\nabla}_{\nu}\varphi g_{\alpha\beta} = 0 \qquad T_{\alpha\beta} = -\tfrac{1}{m}\left(g^{\mu\nu}g_{\alpha\beta} - m\delta^\mu_{(\alpha}\delta^\nu_{\beta)}\right)\overset{*}{\nabla}_{\mu}\varphi\overset{*}{\nabla}_{\nu}\varphi = 0$$

(1.10.29)

This is a linear system given by an endomorphism $A^{\mu\nu}_{\alpha\beta} = g^{\mu\nu}g_{\alpha\beta} - m\delta^\mu_{(\alpha}\delta^\nu_{\beta)}$ acting on symmetric tensors $v_{\mu\nu}$.
In Minkowski space and Cartesian coordinates, this is

$$\begin{cases} \tfrac{3}{4}\overset{*}{\nabla}_{0}\varphi\overset{*}{\nabla}_{0}\varphi + \tfrac{1}{4}\overset{*}{\nabla}_{i}\varphi\overset{*}{\nabla}_{i}\varphi = 0 \\ \overset{*}{\nabla}_{0}\varphi\overset{*}{\nabla}_{i}\varphi = 0 \\ \overset{*}{\nabla}_{j}\varphi\overset{*}{\nabla}_{i}\varphi = 0 \qquad (i \neq j) \\ \tfrac{3}{4}\overset{*}{\nabla}_{i}\varphi\overset{*}{\nabla}_{i}\varphi = 0 \end{cases}$$

(1.10.30)

These altogether imply $\overset{*}{\nabla}_{\mu}\varphi\overset{*}{\nabla}_{\nu}\varphi = 0$, which with (1.10.27) implies $\varphi = 0$.

Anyway, in most cases, if one assumes $T_{\alpha\beta} = 0$ to be a serious field equation, then the Klein–Gordon field is constrained to be constant or zero, thence loosing most interesting solutions. By the way, this is one of the reasons why in Minkowski space one is *forced* to consider the metric fixed in order to get rid of this field equation.

If one wants to go by the second path and vary the metric as prescribed by relativistic theories, then the Klein–Gordon Lagrangian does not make much sense alone. But consider to add a Lagrangian for the metric only, a Lagrangian $\boldsymbol{L}_g$ which is varied to be $\delta \boldsymbol{L}_g = \frac{1}{2\kappa}\sqrt{g}\, G_{\mu\nu}\delta g^{\mu\nu}\, d\sigma + d\left(F^\mu\, d\sigma_\mu\right)$. Then the total Lagrangian $\boldsymbol{L} = \boldsymbol{L}_g + \boldsymbol{L}_{KG}$ leads to field equations in the form

$$\begin{cases} \Box\varphi - \mu^2\varphi = 0 \\ G_{\alpha\beta} = \kappa T_{\alpha\beta} \end{cases} \tag{1.10.31}$$

These field equations leave much more space for interesting solutions. The first equation $\Box\varphi - \mu^2\varphi = 0$ prescribes how the field φ evolves in presence of a metric g, while the second field equation $G_{\alpha\beta} = \kappa T_{\alpha\beta}$ prescribes how the metric g evolves in presence of a field φ. This is quite satisfactory from the physical viewpoint, though from the mathematical viewpoint this is a coupled non-linear system of partial differential equations (PDE) which is quite difficult to be analyzed. We shall see below solutions for these equations. Now we are just interested in pointing out how theories in which all fields are dynamical appear.

Another aspect of Klein–Gordon theory which is worth analyzing is conservation laws. The Klein–Gordon Lagrangian density is in fact a scalar density, namely $L_{KG} = -\frac{\sqrt{g}}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi + \mu^2\varphi^2\right)$; if one changes coordinates on spacetime $x'^\mu = x'^\mu(x)$ then the metric transforms as a tensor and the Klein–Gordon field as a scalar

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g'_{\mu\nu} = \bar{J}^\alpha_\mu g_{\alpha\beta}\bar{J}^\beta_\nu \\ \varphi' = \varphi \end{cases} \tag{1.10.32}$$

Then accordingly the Lagrangian density transforms as $L_{KG}(x', g', \varphi') = J L_{KG}(x, g, \varphi)$. This means that any spacetime diffeomorphism is a Lagrangian symmetry. Thus any spacetime vector field is an infinitesimal Lagrangian symmetry, as one can verify by using the covariance identity

$$\sqrt{g}\left(-\tfrac{1}{2}T_{\alpha\beta}\pounds_\xi g^{\alpha\beta} - \mu^2\varphi\pounds_\xi\varphi - \overset{*}{\nabla}^\mu\varphi\overset{*}{\nabla}_\mu\pounds_\xi\varphi\right) = d_\mu(\xi^\mu L_{KG}) \tag{1.10.33}$$

Just expand everything *without integrating by parts*. The Lie derivatives are

$$\pounds_\xi g^{\mu\nu} = -\overset{g}{\nabla}_\lambda\xi^\mu g^{\lambda\nu} - g^{\mu\lambda}\overset{g}{\nabla}_\lambda\xi^\nu \qquad \pounds_\xi\varphi = \xi^\mu\overset{*}{\nabla}_\mu\varphi \tag{1.10.34}$$

Thus expanding the left hand side of the covariance identity (1.10.33) one gets

$$\begin{aligned} &\sqrt{g}\left[-\left(\tfrac{1}{2}\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}_\nu\varphi - \tfrac{1}{4}\left(g^{\rho\sigma}\overset{*}{\nabla}_\rho\varphi\overset{*}{\nabla}_\sigma\varphi + \mu^2\varphi^2\right)g_{\mu\nu}\right)\left(-\overset{g}{\nabla}_\lambda\xi^\mu g^{\lambda\nu} - g^{\mu\lambda}\overset{g}{\nabla}_\lambda\xi^\nu\right) - \mu^2\varphi\xi^\mu\overset{*}{\nabla}_\mu\varphi - \overset{*}{\nabla}^\mu\varphi\overset{*}{\nabla}_\mu\left(\xi^\lambda\overset{*}{\nabla}_\lambda\varphi\right)\right] = \\ &= \sqrt{g}\left[\overset{*}{\nabla}_\mu\varphi\overset{*}{\nabla}^\lambda\varphi\overset{g}{\nabla}_\lambda\xi^\mu - \tfrac{1}{2}\left(g^{\rho\sigma}\overset{*}{\nabla}_\rho\varphi\overset{*}{\nabla}_\sigma\varphi + \mu^2\varphi^2\right)\overset{g}{\nabla}_\lambda\xi^\lambda - \mu^2\varphi\xi^\mu\overset{*}{\nabla}_\mu\varphi - \underline{\overset{*}{\nabla}^\mu\varphi\overset{*}{\nabla}_\mu\xi^\lambda\overset{*}{\nabla}_\lambda\varphi} - \overset{*}{\nabla}^\mu\varphi\xi^\lambda\overset{g}{\nabla}_{\mu\lambda}\varphi\right] = \\ &= \sqrt{g}\left[-\underline{\tfrac{1}{2}\overset{*}{\nabla}^\sigma\varphi\overset{*}{\nabla}_\sigma\varphi\overset{g}{\nabla}_\lambda\xi^\lambda} - \tfrac{1}{2}\mu^2\varphi^2\overset{g}{\nabla}_\lambda\xi^\lambda - \tfrac{1}{2}\mu^2\xi^\lambda\overset{*}{\nabla}_\lambda\varphi^2 - \tfrac{1}{2}\overset{g}{\nabla}_\lambda\left(\underline{\underline{\overset{*}{\nabla}^\mu\varphi\overset{g}{\nabla}_\mu\varphi}}\right)\xi^\lambda\right] = \frac{\sqrt{g}}{2}\left[-\overset{g}{\nabla}_\lambda\left(\overset{*}{\nabla}^\sigma\varphi\overset{*}{\nabla}_\sigma\varphi\xi^\lambda\right) - \overset{g}{\nabla}_\lambda\left(\mu^2\varphi^2\xi^\lambda\right)\right] = \\ &= \overset{*}{\nabla}_\lambda\left(-\frac{\sqrt{g}}{2}\left(\overset{*}{\nabla}^\sigma\varphi\overset{*}{\nabla}_\sigma\varphi + \mu^2\varphi^2\right)\xi^\lambda\right) = d_\mu(\xi^\mu L_{KG}) \end{aligned} \tag{1.10.35}$$

The corresponding Noether current is

$$\mathcal{E} = \left(-\sqrt{g}g^{\mu\nu}\overset{*}{\nabla}_\nu\varphi \pounds_\xi\varphi - L_{KG}\xi^\mu \right) \, d\sigma_\mu = \sqrt{g}\left(-g^{\mu\nu}\overset{*}{\nabla}_\nu\varphi\overset{*}{\nabla}_\lambda\varphi\,\xi^\lambda + \tfrac{1}{2}g^{\lambda\nu}\overset{*}{\nabla}_\lambda\varphi\overset{*}{\nabla}_\nu\varphi\xi^\mu + \tfrac{1}{2}\mu^2\varphi^2\xi^\mu \right)\, d\sigma_\mu =$$

$$= -\sqrt{g}g^{\mu\nu}\left(\overset{*}{\nabla}_\nu\varphi\overset{*}{\nabla}_\sigma\varphi - \tfrac{1}{2}\left(g^{\lambda\rho}\overset{*}{\nabla}_\lambda\varphi\overset{*}{\nabla}_\rho\varphi + \mu^2\varphi^2 \right)g_{\nu\sigma} \right)\xi^\sigma\,d\sigma_\mu = -\sqrt{g}g^{\mu\nu}T_{\nu\sigma}\xi^\sigma\,d\sigma_\mu \tag{1.10.36}$$

Notice that as long as we consider Klein–Gordon as a relativistic theory, thus g is a dynamical field and $T_{\mu\nu} = 0$ is a field equation, then one can set $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$ with $\mathcal{U} = 0$ and $\mathcal{E} = \tilde{\mathcal{E}}$ since $\mathcal{E}$ vanishes along solutions of field equations.

> In a more general perspective, the total Lagrangian will be $\boldsymbol{L} = \boldsymbol{L}_g + \boldsymbol{L}_{KG}$ and $\boldsymbol{L}_g$ will contribute to field equations as well as to Noether currents. The Noether currents of $\boldsymbol{L}_g$ will split to contribute both the reduced current and to superpotential. The contribution from $\boldsymbol{L}_g$ to the reduced current together with the contribution from $\boldsymbol{L}_{KG}$ will vanish along solutions as a consequence of the field equation $G_{\alpha\beta} = \kappa T_{\alpha\beta}$. The Klein–Gordon Lagrangian will not contribute to superpotential while the Lagrangian $\boldsymbol{L}_g$ will. We shall see this in detail later on.

If one considers Klein–Gordon in SR, thus g as a non-dynamical field, then Noether theorem will read as

$$d\mathcal{E} = \sqrt{g}\left(\tfrac{1}{2}T_{\alpha\beta}\pounds_\xi g^{\alpha\beta} - (\Box\varphi - \mu^2\varphi)\pounds_\xi\varphi \right)\,d\sigma \tag{1.10.37}$$

Klein–Gordon field equation $\Box\varphi - \mu^2\varphi = 0$ will vanish along solutions. As long as the first term is concerned, the energy-momentum stress tensor $T_{\alpha\beta}$ has no reason to vanish so in general $\mathcal{E}$ has no reason to be conserved (i.e. closed). On the contrary, in general along solutions, one has

$$d\left((j\sigma)^*\mathcal{E} \right) = (j\sigma)^*\left(\tfrac{1}{2}\sqrt{g}\,T_{\alpha\beta}\pounds_\xi g^{\alpha\beta} \right)\,d\sigma \tag{1.10.38}$$

That is why, in SR, one usually restricts to ξ being a Killing vector of the metric g so that $\pounds_\xi g^{\alpha\beta} = 0$ and $(j\sigma)^*\mathcal{E}$ is conserved anyway. In special ▷*Killing vectors* relativity Minkowski metric has $\frac{m}{2}(m+1)$ Killing vectors so that is not too restrictive and one still has $\frac{m}{2}(m+1)$ conserved currents (to accomodate for energy, momentum, angular momentum and three boosts as we shall see). However, let us notice that, on a generic spacetime, one has no Killing vectors at all. For this reason, probably it would be wise to learn to live without Killing vectors.

Maxwell electromagnetism

Electromagnetism is the theory of electric and magnetic field (E and B, respectively).

One should soon realise that the electromagnetic field is not a set of two (time-dependent) vector fields on space, but can be better described by a 2-form $\boldsymbol{F}$ on spacetime. In Cartesian coordinates on Minkowski spacetime the 2-form which represents the electromagnetic field (E, B) is

$$\boldsymbol{F} = -\mathrm{c}E_i\,dt \wedge\,dx^i + \tfrac{1}{2}\epsilon^k{}_{ij}B_k\,dx^i \wedge\,dx^j \tag{1.10.39}$$

> Let us remark that, in view of classical Lorentz force, one has

$$\vec{F} = q\left(\vec{E} + \tfrac{1}{c}\vec{v} \times \vec{B} \right)$$

then the electric and magnetic fields has the same dimensions $[E] = [B] = MLT^{-2}C^{-1}$. The factor c added in (1.10.39) is then there to restore dimensions, since $[L] = [ct]$. Of course, it is often custom to set $c = 1$. Then $[F] = ML^3T^{-2}C^{-1}$.

Notice that two spatial vector fields defined on a spacetime of dimension $m = 4$ are described by 6 functions (3 functions each). A 2-form on spacetime is described by $\frac{m}{2}(m-1)$, which is again 6 for a spacetime of dimension $m = 4$. Of course, by now, one should be convinced that objects on spacetime are better candidates to represent physical objects than objects on space.

However, one may argue why a 2-form and not a vector and 2 scalar fields, or something else?

For it, let us consider a transformation

$$\begin{cases} t' = \gamma\left(t + \frac{\beta}{c}x\right) \\ x' = \gamma\left(\beta c t + x\right) \\ y' = y \\ z' = z \end{cases} \Rightarrow \begin{cases} dt' = \gamma\left(dt + \frac{\beta}{c}\,dx\right) \\ dx' = \gamma\left(\beta c\,dt + dx\right) \end{cases} \Rightarrow dt' \wedge dx' = \gamma^2\left(1 - \beta^2\right)\,dt \wedge dx \tag{1.10.40}$$

where we set $\gamma^2(1 - \beta^2) = 1$. This transformation acts on the 2-form $\boldsymbol{F}$ as $F'_{\mu\nu} = \bar{J}^\alpha_\mu \bar{J}^\beta_\nu F_{\alpha\beta}$ which in turn induces a transformation on E and B, precisely

$$\begin{aligned} \boldsymbol{F}' &= -cE'_i\,dt' \wedge dx'^i + \tfrac{1}{2}\epsilon^k{}_{ij}B'_k\,dx'^i \wedge dx'^j = \\ &= -cE'_1\,dt \wedge dx - \gamma c E'_2\left(dt + \tfrac{\beta}{c}\,dx\right) \wedge dy - \gamma c E'_3\left(dt + \tfrac{\beta}{c}\,dx\right) \wedge dz + B'_1\,dy \wedge dz + \gamma B'_2\,dz \wedge (\beta c\,dt + dx) + \gamma B'_3(\beta c\,dt + dx) \wedge dy = \\ &= -cE'_1\,dt \wedge dx - c\gamma\left(E'_2 - \beta B'_3\right)\,dt \wedge dy - c\gamma\left(E'_3 + \beta B'_2\right)\,dt \wedge dz + \gamma\left(-\beta E'_2 + B'_3\right)\,dx \wedge dy - \gamma\left(\beta E'_3 + B'_2\right)\,dx \wedge dz + B'_1\,dy \wedge dz \end{aligned} \tag{1.10.41}$$

that is

$$\begin{cases} E_1 = E'_1 \\ E_2 = \gamma\left(E'_2 - \beta B'_3\right) \\ E_3 = \gamma\left(E'_3 + \beta B'_2\right) \end{cases} \qquad \begin{cases} B_1 = B'_1 \\ B_2 = \gamma\left(B'_2 + \beta E'_3\right) \\ B_3 = \gamma\left(B'_3 - \beta E'_2\right) \end{cases} \tag{1.10.42}$$

The transformation (1.10.40) will be seen to represent in SR a transformation between two inertial labs in relative motion by a constant velocity $\vec{v} = c\beta\,\vec{\imath}$ in the x-direction. For example, for a velocity small compared to c ($\beta c \to v$, $\beta \to \frac{v}{c}$, $\frac{\beta}{c} \to 0$, $\gamma \to 1$) it reduces to

$$\begin{cases} t' = t \\ x' = vt + x \\ y' = y \\ z' = z \end{cases} \tag{1.10.43}$$

i.e. to a Galileo boost. Under the same limit, the transformation laws of electric and magnetic field reads

$$\begin{cases} E_1 = E'_1 \\ E_2 = E'_2 - \beta B'_3 \\ E_3 = E'_3 + \beta B'_2 \end{cases} \qquad \begin{cases} B_1 = B'_1 \\ B_2 = B'_2 + \beta E'_3 \\ B_3 = B'_3 - \beta E'_2 \end{cases} \Rightarrow \begin{cases} \vec{E} = \vec{E}' + \frac{1}{c}\vec{v} \times \vec{B}' \\ \vec{B} = \vec{B}' - \frac{1}{c}\vec{v} \times \vec{E}' \end{cases} \tag{1.10.44}$$

This represents how an electromagnetic field (E', B') which is seen by an observer is seen by a different observer in motion with constant velocity with respect to the first observer. The second of (1.10.44) is recognized to be Biot-Savart rule for the magnetic field generated by a charge moving at speed $\vec{v}$ (and, in view of Galileo relativity principle, also the rule for the magnetic field seen by an observer in motion at speed $-\vec{v}$).

Accordingly, if one assumes the electromagnetic field to be a 2-form on spacetime, from the transformation laws one obtains as a consequence Biot-Savart law. Of course, one also obtains many more effects (e.g. the factors γ before performing the limit) that one can test. Transformations (1.10.42) are now very well tested and can be accepted as a matter of fact (at least in the Minkowskian regime).

We can also check transformation rules by spatial transformations which leave the time unchanged. Let us consider a transformation

$$\begin{cases} t' = t \\ x'^i = R^i_j x^j \end{cases} \Rightarrow \begin{cases} dt' = dt \\ dx'^i = R^i_j \, dx^j \end{cases} \tag{1.10.45}$$

For this transformation, one has

$$\begin{aligned} \boldsymbol{F}' &= -\mathrm{c} E'_i \, dt' \wedge \, dx'^i + \tfrac{1}{2}\epsilon^k{}_{ij} B'_k \, dx'^i \wedge \, dx'^j = \\ &= -\mathrm{c} E'_i R^i_j \, dt \wedge \, dx^j + \tfrac{1}{2}\epsilon^k{}_{ij} B'_k R^i_m R^j_n \, dx^m \wedge \, dx^n \end{aligned} \tag{1.10.46}$$

so that one has the new electric and magnetic field expressed as

$$\begin{cases} E_j = E'_i R^i_j \\ \epsilon^l_{mn} B_l = \epsilon^k{}_{ij} B'_k R^i_m R^j_n = \det(R) R^k_l \epsilon^l{}_{mn} B'_k \end{cases} \Rightarrow \begin{cases} E'_i = E_j \bar{R}^j_i \\ B'_l = \det(\bar{R}) \bar{R}^k_l B_k \end{cases} \tag{1.10.47}$$

that means that under spatial transformations the electric field is a (co)vector, while the magnetic field is a (co)vector density of weight 1. In particular, if one restricts to spatial rotations, both the magnetic and electric fields behaves like (co)vectors.

Once one accepts that the electromagnetic field must be represented by a spacetime 2-form $\boldsymbol{F}$, then Maxwell equations can be written as

$$\begin{cases} d\boldsymbol{F} = 0 \\ d * \boldsymbol{F} = *\boldsymbol{J} \end{cases} \tag{1.10.48}$$

where $\boldsymbol{J} = \frac{4\pi}{\mathrm{c}} \left(-\mathrm{c}^2 \rho \, dt + j_i \, dx^i \right)$ is a 1-form describing sources through the charge density ρ and the current densities j_i.

In fact, one has

$$\begin{aligned} d\boldsymbol{F} &= \left(-\mathrm{c}\partial_j E_i + \tfrac{1}{2}\epsilon^k{}_{ij}\partial_t B_k \right) dt \wedge \, dx^i \wedge \, dx^j + \tfrac{1}{2}\partial_l B^k \epsilon_{kij}\epsilon^{lij} \, d\sigma_0 = \tfrac{\mathrm{c}}{2}\epsilon_{kij}\left(e^{klm}\partial_l E_m + \tfrac{1}{\mathrm{c}}\partial_t B^k \right) dt \wedge \, dx^i \wedge \, dx^j + \partial_k B^k \, d\sigma_0 = \\ &= -\mathrm{c}\left(\mathrm{curl}(\vec{E}) + \tfrac{1}{\mathrm{c}}\partial_t \vec{B} \right)^k \, d\sigma_k + \mathrm{div}(\vec{B}) \, d\sigma_0 \end{aligned} \tag{1.10.49}$$

Thus $d\boldsymbol{F} = 0$ iff the two homogeneous Maxwell equations hold true, namely

$$\begin{cases} \mathrm{curl}(\vec{E}) + \tfrac{1}{\mathrm{c}} d_t \vec{B} = 0 \\ \mathrm{div}(\vec{B}) = 0 \end{cases} \tag{1.10.50}$$

For the non-homogeneous Maxwell equations, one must start computing the Hodge duality

$$\begin{aligned} *\boldsymbol{F} &= -\mathrm{c} E_i * \left(dt \wedge \, dx^i \right) + \tfrac{1}{2}\epsilon^k{}_{ij} B_k * \left(dx^i \wedge \, dx^j \right) = \sqrt{g}\left(\tfrac{1}{\mathrm{c}} E^i \, d\sigma_{0i} + \tfrac{1}{2}\epsilon^{kij} B_k \, d\sigma_{ij} \right) = \sqrt{g}\left(\tfrac{1}{2\mathrm{c}} E^i \epsilon_{0ijk} \, dx^j \wedge \, dx^k + \tfrac{\mathrm{c}}{2}\epsilon^{kij}\epsilon_{lij} B_k \, dt \wedge \, dx^l \right) = \\ &= \tfrac{1}{\mathrm{c}}\sqrt{g}\left(\tfrac{1}{2} E_k \epsilon^k{}_{ij} \, dx^i \wedge \, dx^j + \mathrm{c} B_k \, dt \wedge \, dx^k \right) \end{aligned} \tag{1.10.51}$$

and

$$*\boldsymbol{J} = \tfrac{4\pi}{\mathrm{c}}\left(-\mathrm{c}^2 \rho * \left(dt \right) + j_i * \left(dx^i \right) \right) = \tfrac{4\pi}{\mathrm{c}}\sqrt{g}\left(\rho \, d\sigma_0 + j^k \, d\sigma_k \right) \tag{1.10.52}$$

As it appears from the expression of $*\boldsymbol{F}$, the Hodge duality on 2-forms amounts to the substitutions

$$\boldsymbol{F} \mapsto *\boldsymbol{F} \qquad \begin{cases} E \mapsto -\frac{\sqrt{g}}{c}B \\ B \mapsto \frac{\sqrt{g}}{c}E \end{cases} \qquad \Rightarrow d\boldsymbol{F} \mapsto d*\boldsymbol{F} = -\left(-\mathrm{curl}\left(\sqrt{g}\vec{B}\right) + \frac{1}{c}d_t\left(\sqrt{g}\vec{E}\right)\right)^k d\sigma_k + \frac{1}{c}\,\mathrm{div}\left(\sqrt{g}\vec{E}\right)\,d\sigma_0 \tag{1.10.53}$$

The equation $d*\boldsymbol{F} = *\boldsymbol{J}$ then reads as

$$\begin{cases} \mathrm{curl}\left(\sqrt{g}\vec{B}\right) - \frac{1}{c}d_t\left(\sqrt{g}\vec{E}\right) = \frac{4\pi}{c}\sqrt{g}j^k \\ \frac{1}{c}\mathrm{div}\left(\sqrt{g}\vec{E}\right) = \frac{4\pi}{c}\sqrt{g}\rho \end{cases} \qquad \begin{cases} \mathrm{curl}\left(\vec{B}\right) - \frac{1}{c}d_t\vec{E} = \frac{4\pi}{c}j^k \\ \mathrm{div}\left(\vec{E}\right) = 4\pi\rho \end{cases} \tag{1.10.54}$$

which accounts for the non-homogeneous Maxwell equations. Moreover, from the non-homogeneous Maxwell equation it follows that

$$d*\boldsymbol{J} = 0 \tag{1.10.55}$$

which accounts for continuity equation which expresses conservation of electric charge.

The homogeneous field equation $d\boldsymbol{F} = 0$ just states that the electromagnetic field is a closed 2-form, i.e. $\boldsymbol{F}$ allows a local potential $\boldsymbol{A}$ such that $d\boldsymbol{A} = \boldsymbol{F}$. $\quad [\boldsymbol{A}] = ML^4T^{-2}C^{-1}$
The local potential $\boldsymbol{A} = A_\mu\,dx^\mu$ is a 1-form. Let us denote $\boldsymbol{A} = \mathrm{c}\phi\,dt + A_i\,dx^i$. Then we have:

$$d\boldsymbol{A} = -\mathrm{c}\left(\partial_i\phi - \frac{1}{c}\partial_t A_i\right)\,dt \wedge dx^i + \frac{1}{2}\epsilon_{klm}\partial_j A_i \epsilon^{kji}\,dx^l \wedge dx^m \qquad \Rightarrow \begin{cases} \vec{E} = \mathrm{grad}(\phi) - \frac{1}{c}\partial_t\vec{A} \\ \vec{B} = \mathrm{curl}(\vec{A}) \end{cases} \tag{1.10.56}$$

Comparing with classical formula, we can identity ϕ to be the *potential* of electric field and $\vec{A}$ to be the *vector potential* of magnetic field.

The local potential $\boldsymbol{A}$ is not uniquely defined. If one considers the local potential $\boldsymbol{A}' = \boldsymbol{A} + d\alpha$ for any 0-form α, then $d\boldsymbol{A}' = d\boldsymbol{A}$, i.e. both potentials $\boldsymbol{A}'$ and $\boldsymbol{A}$ defines the same electromagnetic field. If we also change chart on spacetime, the two local potentials are related by

$$A'_\mu = \bar{J}^\rho_\mu\left(A_\rho + \partial_\rho\alpha\right) \tag{1.10.57}$$

which are called *gauge transformations*.

With gauge transformations, one should feel that something strange is going on. We set up a global formalism, we have equations which are potentially global; however, we eventually introduced a *local* potential $\boldsymbol{A}$ and now, even if the potential were global, we can consider a local function α and spoil it forcing locality into our description.

The traditional explanation is that the potential is not observable and just a tool to describe physical fields which are contained in $\boldsymbol{F}$, not in $\boldsymbol{A}$. That is, of course, true even though one should discuss observability and derive it from the theory rather than proceeding the other way. Moreover, the potential $\boldsymbol{A}$ will be soon be selected as a fundamental field for a Lagrangian description of Maxwell electromagnetism and the variational framework we defined is not well suited to deal with discontinuous fields.

This is our first experience with a situation which will appear many times in different contexts. We often enforce a geometric meaning on objects (as we did with $\boldsymbol{A}$ when we said it was a 1-form so that its differential can be a 2-form and represent $\boldsymbol{F}$) just to discover that it is then discontinuous in

some situations when we push some transformation to the limit. Notice that *if* $\boldsymbol{A}$ were a 1-form *then* its differential $d\boldsymbol{A}$ certainly is a 2-form. However, there may be other objects which naturally produce a 2-form $\boldsymbol{F}$ though not as the differential of a 1-form.

Often, the reason of this awkward situation is that the original geometric meaning we imposed was incorrect and there is an *objectively better* geometric meaning for the object which preserves its continuity. The trick here is genuinely geometric: we have to consider that the fields (e.g. A_μ) are not a geometric object, but they are the *components* of a geometric object. As it happens, for example, to vector fields, the components can be discontinuous even just because coordinates are local and *they* (the coordinates, not the field) fail to extend.

In this case of the electromagnetic field, we can consider gauge transformations as a fundamental feature rather than a sort of side note and try to solder it right into the definition of the object rather than superimpose it onto a predetermined object (in this case a 1-form). Hence the issue is to see if there exists a geometric object which transforms canonically by means of gauge transformations. If we succeed, then usually what we obtain is a better understanding of the formalism in which the field $\boldsymbol{A}$ is continuous and an observer needs extra conventions to describe it in terms of local components. Gauge transformations then arise as changes of lab conventions and discontinuities are produced when the lab conventions fail to be global.

There is a different object ω to which a 2-form $\boldsymbol{F}$ is associated; namely, a connection on a principal $U(1)$-bundle P. We started by showing that the electromagnetic field is a spacetime 2-forms $\boldsymbol{F}$. However, $\boldsymbol{F}$ can be represented by the differential $\boldsymbol{F} = d\boldsymbol{A}$ of a 1-form $\boldsymbol{A}$ as well as the curvature of a $U(1)$-connection ω. We have to decide first which representation is the best for the potential. This is not something which can be decided locally; in fact a $U(1)$-connection of P is in the form $\quad\triangleright$*Principal connections*

$$\omega = dx^\mu \otimes \left(\partial_\mu - \hat{A}_\mu(x)\rho \right) \tag{1.10.58}$$

where ρ is a vertical right-invariant non-vanishing vector on P. Thus a $U(1)$-connection and a 1-form are both represented by the same type of functions. If one representation is to be preferred, it must be related to transformation rules. The coefficients of 1-forms transform under spacetime diffeomorphisms in the following way

$$\boldsymbol{A}' = A'_\mu \, dx'^\mu = A'_\mu J^\mu_\rho \, dx^\rho \qquad \Rightarrow A'_\mu = \bar{J}^\rho_\mu A_\rho \tag{1.10.59}$$

A $U(1)$-connection ω transforms under automorphism of the bundle P. An automorphism $\Phi : P \to P$ is locally expressed as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e^{i\theta'} = e^{i(\alpha+\theta)} \end{cases} \tag{1.10.60}$$

and the connection transforms as

$$\hat{A}'_\mu = \bar{J}^\rho_\mu \left(\hat{A}_\rho - i\partial_\rho\alpha \right) \qquad A'_\mu = \bar{J}^\rho_\mu \left(A_\rho + \partial_\rho\alpha \right) \tag{1.10.61}$$

where we set $\hat{A}_\mu := -iA_\mu$. (Notice that $T = -i$ is in fact a basis for the Lie algebra $\mathfrak{u}(1)$.) Thence the coefficients of a $U(1)$-connection transform by gauge transformations which are identified with automorphisms of P. However, there is also another more direct and physical motivation to prefer a representation of the electromagnetic field in terms of a $U(1)$-connection.

Let us think about *what* is an electromagnetic field and *which* potential A_μ can describe it. Let us fix an open set U and a 1-form $\boldsymbol{A} = A_\mu(x) \, dx^\mu$ defined on it. Any 1-form A defines a 2-form $\boldsymbol{F} = d\boldsymbol{A}$ which is exact and then closed. Accordingly, $\boldsymbol{F}$ satisfies the two homogeneous Maxwell equations.

The other two (non-homogeneous) equations can be considered as a definition of the source form $\boldsymbol{J} = (-1)^{m-1+s} * *\boldsymbol{J} = (-1)^{m-1+s} * d * \boldsymbol{F}$ (in dimension $m = 4$ and Lorentzian signature $(r, s) = (3, 1)$ this reads as $\boldsymbol{J} := *d * \boldsymbol{F}$) which is needed to produce that electromagnetic field $\boldsymbol{F}$.

> As we shall see, the matter current tensor $\boldsymbol{J}$ must be conserved which is hence a condition on $\boldsymbol{F}$. However, we shall see that the condition on $\boldsymbol{F}$ is automatically satisfied in view of the so-called *Bianchi identities*.

Hence any local potential A_μ is a candidate to locally describe an electromagnetic potential. Thus an electromagnetic field can always be described by a family of local potentials $A_\mu^{(\alpha)}$ defined on an open covering $U_{(\alpha)}$ of spacetime. However, the issue is what happens in the overlaps $U_{(\alpha\beta)} = U_{(\alpha)} \cap U_{(\beta)}$.

Since all observable quantities of the electromagnetic field are contained in $\boldsymbol{F}$, while the potential $\boldsymbol{A}$ is just a way to define observable quantities using 4 functions instead of 6 (which are constrained by the 4 homogeneous Maxwell equations). The potential A_μ is not itself observable. On the contrary, it is not uniquely defined being defined modulo gauge transformations. An observer, not only needs to set conventions to determine position on spacetime, but also needs to choose a representative in any gauge class, $[\boldsymbol{A}] = \{\boldsymbol{A}' = \boldsymbol{A} + d\alpha\}$. This is usually called a *gauge fixing* which is completely unrelated to a spacetime chart.

> As a matter of fact, a gauge fixing is related to choosing the function α which is an extra number at each spacetime point.

Hence two nearby observers on $U_{(\alpha)}$ and $U_{(\beta)}$ define local potentials $A_\mu^{(\alpha)}$ and $A_\mu^{(\beta)}$. In the overlap $U_{(\alpha\beta)}$ the two potentials have to agree, modulo a gauge transformation, i.e.

$$A_\mu^{(\beta)} = \bar{J}_\mu^\rho \left(A_\rho^{(\alpha)} + \partial_\rho \alpha \right) \tag{1.10.62}$$

for some function $\alpha(x)$. Accordingly, the potential cannot in general be described by a global 1-form, which would amount to set $\alpha = 0$. While sometimes one can set $\alpha = 0$ on $U_{(\alpha\beta)}$ there are situations in which this cannot be done on every overlaps at the same time.

Notice that $\varphi : U_{(\alpha\beta)} \to U(1) : x \mapsto e^{i\alpha(x)}$ are the transition maps of the bundle P on which the $U(1)$-connection ω is defined. The transition maps can be set all to the identity iff the bundle P is trivial. As long as one regards closed but not exact 2-forms $\boldsymbol{F}$ as legitimate electromagnetic fields, then they must be represented either by a family of local 1-forms which do not form global section of $A_1(M)$, or as a global $U(1)$-connection on a non-trivial bundle P. Since we based all variational calculus on global sections, we do not really have opinions and we are forced to opt for a description in terms of $U(1)$-connections.

Dirac monopole

We can here give a direct example of how a specific solution of Maxwell equations does determine the connection *and* the $U(1)$-bundle on which it lives. Let us consider Minkowski spacetime in Cartesian coordinates $x^\mu = (t, x^i) = (t, x, y, z)$. Let us consider the purely magnetic field

$$\vec{B} = \frac{1}{2r^3} x^i \partial_i \qquad \left(\vec{E} = 0 \right) \tag{1.10.63}$$

where we set $r^2 = x^2 + y^2 + z^2$ and which corresponds to the electromagnetic field

$$\boldsymbol{F} = \frac{1}{4r^3} \epsilon_{ijk} x^i \, dx^j \wedge \, dx^k \qquad \Rightarrow *\boldsymbol{F} = \frac{1}{4r^3} x_i \, dt \wedge \, dx^i \tag{1.10.64}$$

One can prove directly that such a 2-form $\boldsymbol{F}$ is a solution of vacuum Maxwell equations on $\mathbb{R} \times \left(\mathbb{R}^3 - \{0\}\right)$. In fact

$$d\boldsymbol{F} = -3\frac{1}{4r^5}\epsilon_{ijk}x^i x_l \, dx^l \wedge \, dx^j \wedge \, dx^k + \frac{1}{4r^3}\epsilon_{ijk} \, dx^i \wedge \, dx^j \wedge \, dx^k = -3\frac{1}{4r^5}\epsilon_{ijk}x^i x_l \epsilon^{ljk} \, d^3\sigma + \frac{1}{4r^3}\epsilon_{ijk}\epsilon^{ijk} \, d^3\sigma = -6\frac{1}{4r^3} \, d^3\sigma + 3!\frac{1}{4r^3} \, d^3\sigma = 0$$

$$d * \boldsymbol{F} = -3\frac{1}{4r^5}x_i x_l \, dx^l \wedge \, dt \wedge \, dx^i + \boxed{\frac{1}{4r^3} \, dx_i \wedge \, dt \wedge \, dx^i} = 3\frac{1}{4r^5}x_i x_l \, dt \wedge \, dx^l \wedge \, dx^i = 0$$

(1.10.65)

One can consider a quotient with respect to the equivalence relation

$$x^\mu \sim x'^\mu \quad \Longleftrightarrow \quad \exists \lambda \in \mathbb{R}^+ : x'^\mu = \lambda x^\mu$$

(1.10.66)

The quotient space is a sphere S_2 and the electromagnetic field (being homogeneous) projects to such a sphere.

One can show that there are two local potentials for such an electromagnetic field

$$\boldsymbol{A} = \frac{1}{2r}\frac{x \, dy - y \, dx}{z - r} \qquad\qquad \boldsymbol{A}' = \frac{1}{2r}\frac{x \, dy - y \, dx}{z + r}$$

(1.10.67)

Of course, at this stage we are not able to exclude that there exists another potential for the electromagnetic field $\boldsymbol{F}$ which is global on the whole sphere.

One can check that both $d\boldsymbol{A} = d\boldsymbol{A}' = \boldsymbol{F}$ and while $\boldsymbol{A}$ is defined on the sphere without the north pole, $\boldsymbol{A}'$ is defined on the sphere without the south pole.

Check first that

$$d\left(\frac{x \, dy - y \, dx}{r^2 - z^2}\right) = \frac{(\, dx \wedge \, dy - \, dy \wedge \, dx)(r^2 - z^2) - (x \, dy - y \, dx) \wedge 2(x \, dx + y \, dy)}{(r^2 - z^2)^2} = 2\left(r^2 - z^2 - (x^2 + y^2)\right)\frac{dx \wedge \, dy}{(r^2 - z^2)^2} = 0$$

(1.10.68)

and write the potential as

$$\boldsymbol{A} = -\frac{1}{2}\left(\frac{r + z}{r}\right)\left(\frac{x \, dy - y \, dx}{r^2 - z^2}\right)$$

(1.10.69)

Then one can compute

$$d\boldsymbol{A} = -\frac{1}{2}\frac{(\, dr + \, dz)r - (r + z) \, dr}{r^2} \wedge \frac{x \, dy - y \, dx}{r^2 - z^2} = -\frac{1}{2}\frac{r \, dz - \frac{z}{r}(x \, dx + y \, dy + z \, dz)}{r^2} \wedge \frac{x \, dy - y \, dx}{r^2 - z^2} = -\frac{1}{2r^3(r^2 - z^2)}\left(-zx \, dx - zy \, dy + (r^2 - z^2) \, dz\right) \wedge (x \, dy - y \, dx) =$$

$$= -\frac{1}{2r^3(r^2 - z^2)}\left(-z(x^2 + y^2) \, dx \wedge \, dy + (r^2 - z^2)x \, dz \wedge \, dy - (r^2 - z^2)y \, dz \wedge \, dx\right) = \frac{1}{2r^3}\left(z \, dx \wedge \, dy + x \, dy \wedge \, dz + y \, dz \wedge \, dx\right) = \boldsymbol{F}$$

(1.10.70)

One can also show that on the sphere without the poles the two potentials differ by a gauge transformation

$$\boldsymbol{A}' - \boldsymbol{A} = \frac{x \, dy - y \, dx}{(x^2 + y^2)} = d(\phi)$$

(1.10.71)

where ϕ is the longitude about the z-axis of the sphere S_2.

Repeat the same computation using spherical spatial coordinates (t, r, θ, ϕ).

The potential hence is not a global 1-form on the sphere. However, one can define a global $U(1)$-connection on a principal bundle P by setting

$$\omega = dx^\mu \otimes \left(\partial_\mu - A_\mu \rho\right) = dx^\mu \otimes \left(\partial_\mu - A'_\mu \rho\right) \tag{1.10.72}$$

provided the transition functions on P are given exactly by

$$\begin{cases} x' = x \\ e^{i\alpha'} = e^{i(\alpha+\phi)} \end{cases} \quad \Rightarrow \hat{g}_{(NS)} : S_2 - \{N, S\} \to U(1) : q \mapsto e^{i\phi} \tag{1.10.73}$$

where α are coordinate along the fibers.

Now we can show that in fact there exists a bundle P over the sphere S_2 which has precisely these transition functions.

Let us consider a 3-sphere S_3 embedded in $\mathbb{C}^2$ by the equation $|w_1|^2 + |w_2|^2 = 1$, and a 2-sphere S_2 embedded in $\mathbb{C} \times \mathbb{R}$ by the equation $|z_0|^2 + z^2 = 1$. Let us consider the map

$$\hat{\pi} : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R} : (w_1, w_2) \mapsto (2w_1^\dagger w_2, |w_1|^2 - |w_2|^2) \tag{1.10.74}$$

where $w_1^\dagger$ denotes the complex conjugate of $w_1 \in \mathbb{C}$. The map $\hat{\pi}$ restricts to a map $\pi : S_3 \to S_2$, because of the constraint $|w_1|^2 + |w_2|^2 = 1$ which ensures that $\hat{\pi}$ maps the 3-sphere S_3 onto the 2-sphere S_2.

Let us now fix a point $q = (z_0, z) \in S_2$; because of the constraint $|z_0|^2 + z^2 = 1$ defining S_2 we have $z_0 = \sqrt{1 - z^2}\, e^{i\phi}$ and $-1 \leq z \leq 1$. Notice that the north pole is defined for $z = 1$, while the south pole is for $z = -1$. The phase ϕ is the longitude of the point $q \in S_2$ and it is uniquely defined for $-1 < z < 1$ while it is undetermined when $z = \pm 1$, i.e. at the poles. A point (w_1, w_2) is in the fiber $\pi^{-1}(q)$ if following equations hold true

$$\begin{cases} |w_1|^2 + |w_2|^2 = 1 \\ |w_1|^2 - |w_2|^2 = z \\ 2w_1^\dagger w_2 = z_0 \end{cases} \tag{1.10.75}$$

Then all points $(w_1, w_2) \in \pi^{-1}(q)$ are in the form

$$\left(\sqrt{\frac{1+z}{2}}\, e^{i\alpha}, \sqrt{\frac{1-z}{2}}\, e^{i\phi} e^{i\alpha}\right) \tag{1.10.76}$$

for some $\alpha \in \mathbb{R}$. One readily recognises that each fiber is isomorphic to $U(1) \simeq S_1$, which is thence chosen as standard fiber. Let us now fix an open covering of the base manifold S_2 made of the open sets $(S_2)_N = \{q \in S_2 : q \neq S = (0, -1)\}$ and $(S_2)_S = \{q \in S_2 : q \neq N = (0, 1)\}$. We can define the following mappings:

$$\begin{cases} t_{(N)} : \pi^{-1}(S_2)_N \to (S_2)_N \times U(1) : \left(\sqrt{\dfrac{1+z}{2}}\, e^{i\alpha}, \sqrt{\dfrac{1-z}{2}}\, e^{i\phi} e^{i\alpha}\right) \mapsto (q, e^{i\alpha}) \\[3ex] t_{(S)} : \pi^{-1}(S_2)_S \to (S_2)_S \times U(1) : \left(\sqrt{\dfrac{1+z}{2}}\, e^{-i\phi} e^{i\alpha'}, \sqrt{\dfrac{1-z}{2}}\, e^{i\alpha'}\right) \mapsto (q, e^{i\alpha'}) \end{cases} \tag{1.10.77}$$

where we set $\alpha' = \phi + \alpha$.

These define a trivialisation $\{((S_2)_N, t_{(N)}), ((S_2)_S, t_{(S)})\}$ for the Hopf bundle $\mathcal{H} = (S_3, S_2, \pi; U(1))$. The transition maps $\hat{g}_{(NS)} : (S_2)_{NS} \to \mathrm{Diff}(U(1))$ is given by

$$\hat{g}_{(NS)} : (S_2)_{NS} \to U(1) : q \mapsto e^{i\phi} \qquad \hat{g}_{(NS)}(q) : U(1) \to U(1) : e^{i\alpha} \mapsto e^{i\phi} e^{i\alpha} \tag{1.10.78}$$

where we recall $q = (z_0, z)$ and $z_0 = \sqrt{1-z^2}e^{i\phi}$.

The trivialisation (1.10.77) allows us to define two local sections in the Hopf bundle:

$$\sigma_{(N)} : q \mapsto \left(\sqrt{\frac{1+z}{2}}, \sqrt{\frac{1-z}{2}}e^{i\phi} \right) \qquad \sigma_{(S)} : q \mapsto \left(\sqrt{\frac{1+z}{2}}e^{-i\phi}, \sqrt{\frac{1-z}{2}} \right) \tag{1.10.79}$$

defined on $(S_2)_N$ and $(S_2)_S$, respectively. They cannot patch together. In fact, at $N = (0,1)$, we have $z = 1$ and $z_0 = 0$ so that ϕ is here undetermined. Nevertheless, $\sigma_{(N)}(N) = (1,0)$ is well-defined, while it fails at $S = (0,-1)$.

Analogously, at $S = (0,-1)$, we have $z = -1$ and $z_0 = 0$ so that ϕ is again undetermined. Nevertheless, $\sigma_{(S)}(S) = (0,1)$ is well-defined, while it fails at $N = (0,1)$.

Accordingly, the local section $\sigma_{(N)}$, as well as $\sigma_{(S)}$, cannot be extended to a global section of the Hopf bundle.

The Hopf bundle $(S_3, S_2, \pi, U(1))$ is a principal bundle since trivialisations are built out of the local sections $\sigma_{(N)}$ and $\sigma_{(S)}$, as it always happens on principal bundles.

One can show that the Hopf bundle is not trivial (othewise one should have a diffeomorphism $S_3 \simeq S_2 \times S_1$ which is not the case). At this point, we are able to claim that there is no global representative for a connection (which would resort to a global gauge which is available only on trivial bundles).

Lagrangian formulation of Maxwell electromagnetism

An electromagnetic field is described by a global section of the bundle $\mathrm{Con}(P)$ of principal connections of a $U(1)$-principal bundle P. The bundle $\mathrm{Con}(P)$ has fibered coordinates (x^μ, A_μ) and its sections are in one-to-one correspondence with connections

$$\omega = dx^\mu \otimes \left(\partial_\mu - \hat{A}_\mu(x)\rho \right) \tag{1.10.80}$$

An automorphism of P is locally expressed by

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e^{i\theta'} = e^{i(\alpha+\theta)} \end{cases} \tag{1.10.81}$$

and it acts on $\mathrm{Con}(P)$ by

$$\begin{cases} x'^\mu = x'^\mu(x) \\ A'_\mu = \bar{J}^\rho_\mu \left(A_\rho + \partial_\rho \alpha \right) \end{cases} \tag{1.10.82}$$

Thus the group $\mathrm{Aut}(P)$ of automorphisms of P is identified with gauge transformations.

Let us remark that usually in literature *gauge transformations* is reserved to *vertical* automorphisms of P, which we denote by $\mathrm{Aut}_V(P)$ and we call *vertical gauge transformations*.

For us *gauge transformations* entangles in a non-trivial way a vertical gauge transformation and a spacetime diffeomorphism. The group structure is captured by the short exact sequence of group homomorphisms

$$\mathbb{I} \longrightarrow \mathrm{Aut}_V(P) \xrightarrow{\ i\ } \mathrm{Aut}(P) \xrightarrow{\ p\ } \mathrm{Diff}(M) \longrightarrow \mathbb{I} \tag{1.10.83}$$

From this sequence, it appears that vertical gauge transformations are a subgroup of gauge transformations. On the contrary, spacetime diffeomorphisms are not a subgroup of gauge transformations. Gauge transformations *project* onto spacetime diffeomorphisms but in general no *embedding* is defined.

Since we assume that gauge transformations do act on fields, if there were a group embedding $j : \mathrm{Diff}(M) \to \mathrm{Aut}(P)$ then spacetime diffeomorphisms would act on fields as well. Without this embedding, there is no action of diffeomorphisms on fields.

Let us consider $C = \mathrm{Con}(P) \times_M \mathrm{Lor}(M)$ as a configuration bundle with coordinates $(x^\mu, A_\mu, g_{\mu\nu})$. The curvature of a $U(1)$-connection is defined as

$$F_{\mu\nu} = d_\mu A_\nu - d_\nu A_\mu \tag{1.10.84}$$

▷*Curvature of a principal connections*

which is covariant with respect to gauge transformations.

Usually the curvature of a principal connection transforms in the ad-representation of the gauge group algebra. However, in this case $G = U(1)$ the gauge group is commutative, then the ad action is trivial and the curvature is a true 2-form on spacetime.

Let us consider the dynamics given by the Maxwell Lagrangian

$$\boldsymbol{L}_M = -\tfrac{1}{4\mathrm{g}} \sqrt{g} F_{\mu\nu} F^{\mu\nu}_{\cdot\cdot} \ d\sigma \tag{1.10.85}$$

where $\mathrm{g} = \mathrm{c}^3 \mu_0 = \frac{\mathrm{c}}{\epsilon_0}$ where μ_0 ($[\mu_0] = MLC^{-2}$) is the *vacuum magnetic permeability* and ϵ_0 is the *vacuum permittivity* ($[\epsilon_0] = M^{-1}L^{-3}T^2C^2$ as it should be since $\epsilon_0 \mu_0 = \mathrm{c}^{-2}$).

By dimensional analysis, one has $[\boldsymbol{L}] = M^2 L^2 T^{-4} C^{-2} [\mathrm{g}]^{-1} L^4 = ML^2 T^{-1}$ then $[\mathrm{g}] = MLC^{-2}L^3 T^{-3} = [\mathrm{c}^3 \mu_0]$ as in fact is.

As we did for Klein–Gordon equation, let us be agnostic for a while about whether the metric must be considered fixed or dynamical. The variation of the Lagrangian $\boldsymbol{L}_M$ is given by

$$\begin{aligned}
\delta \boldsymbol{L}_M &= \left[-\tfrac{1}{2\mathrm{g}} \sqrt{g} \left(F_{\mu\alpha} F^\mu_{\cdot\beta} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}_{\cdot\cdot} g_{\alpha\beta} \right) \delta g^{\alpha\beta} - \tfrac{1}{2\mathrm{g}} \sqrt{g} F^{\mu\nu}_{\cdot\cdot} \delta F_{\mu\nu} \right] \ d\sigma = \left[-\tfrac{1}{2} \sqrt{g} \, T_{\alpha\beta} \delta g^{\alpha\beta} - \tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\cdot\cdot} d_\mu \delta A_\nu \right] \ d\sigma = \\
&= \left[-\tfrac{1}{2} \sqrt{g} \, T_{\alpha\beta} \delta g^{\alpha\beta} - d_\mu \left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\cdot\cdot} \delta A_\nu \right) + d_\mu \left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\cdot\cdot} \right) \delta A_\nu \right] \ d\sigma
\end{aligned} \tag{1.10.86}$$

where we set $T_{\alpha\beta} := \tfrac{1}{\mathrm{g}} \left(F_{\mu\alpha} F^\mu_{\cdot\beta} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}_{\cdot\cdot} g_{\alpha\beta} \right)$ for the so-called *energy-momentum stress tensor* of electromagnetic field. We obtain the Euler–Lagrange part and Liouville part for the Maxwell Lagrangian to be

$$\mathbb{E}(\boldsymbol{L}_M) = \left[-\tfrac{1}{2} \sqrt{g} \, T_{\alpha\beta} \omega^{\alpha\beta} + d_\mu \left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\cdot\cdot} \right) \omega_\nu \right] \wedge \ d\sigma \qquad \mathbb{F}(\boldsymbol{L}_M) = \tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\cdot\cdot} \omega_\nu \wedge \ d\sigma_\mu \tag{1.10.87}$$

where we set $\omega^{\alpha\beta} = dg^{\alpha\beta} - d_\lambda g^{\alpha\beta} \, dx^\lambda$ and $\omega_\mu = dA_\mu - d_\lambda A_\mu \, dx^\lambda$ for the relevant contact forms.

Euler–Lagrange part induces the Maxwell equations in vacuum in covariant form

$$\begin{aligned}
d * \boldsymbol{F} = 0 \quad &\Rightarrow \tfrac{1}{2\mathrm{g}} d \left(\sqrt{g} F^{\mu\nu}_{\cdot\cdot} \, d\sigma_{\mu\nu} \right) = 0 \quad \Rightarrow \tfrac{1}{2\mathrm{g}} d_\lambda \left(\sqrt{g} F^{\mu\nu}_{\cdot\cdot} \right) dx^\lambda \wedge \ d\sigma_{\mu\nu} = 0 \quad \Rightarrow \\
&\Rightarrow \tfrac{1}{\mathrm{g}} d_\nu \left(\sqrt{g} F^{\mu\nu}_{\cdot\cdot} \right) \, d\sigma_\mu = 0 \quad \Rightarrow \tfrac{1}{\mathrm{g}} d_\nu \left(\sqrt{g} F^{\mu\nu}_{\cdot\cdot} \right) = 0 \quad \Rightarrow d_\nu \left(\sqrt{g} F^{\mu\nu}_{\cdot\cdot} \right) = 0
\end{aligned} \tag{1.10.88}$$

Since we assumed that gauge transformations reflect laboratory conventions, one should expect the dynamics to be unaffected by gauge transformations, i.e. one should expect gauge transformations to be symmetry of the Maxwell Lagrangian. This is in fact the case.

An automorphism of P is locally expressed as

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ e^{i\theta'} = e^{i(\alpha+\theta)} \end{cases} \tag{1.10.89}$$

and it acts on the configuration bundle $C = \mathrm{Con}(P) \times_M \mathrm{Lor}(M)$ as

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ A'_{\mu} = \bar{J}^{\rho}_{\mu}(A_{\rho} + \partial_{\rho}\alpha) \\ g'_{\mu\nu} = \bar{J}^{\alpha}_{\mu} g_{\alpha\beta} \bar{J}^{\beta}_{\nu} \end{cases} \tag{1.10.90}$$

The corresponding generator is $\Xi = \xi^{\mu}\partial_{\mu} + i\partial_{\mu}\alpha\rho$ and the Lie derivatives are

$$\begin{cases} \pounds_{\Xi}A_{\mu} = \xi^{\lambda}d_{\lambda}A_{\mu} - \left(-d_{\mu}\xi^{\lambda}A_{\lambda} + d_{\mu}\xi\right) = \xi^{\lambda}F_{\lambda\mu} + d_{\mu}\left(\xi^{\lambda}A_{\lambda} - \xi\right) = \xi^{\lambda}F_{\lambda\mu} - d_{\mu}\left(\xi_V\right) \\ \pounds_{\Xi}g_{\mu\nu} = \overset{g}{\nabla}_{\mu}\xi^{\lambda}g_{\lambda\nu} + g_{\mu\lambda}\overset{g}{\nabla}_{\nu}\xi^{\lambda} \quad \Rightarrow \pounds_{\Xi}g^{\mu\nu} = -\overset{g}{\nabla}_{\lambda}\xi^{\mu}g^{\lambda\nu} - g^{\mu\lambda}\overset{g}{\nabla}_{\lambda}\xi^{\nu} \end{cases} \tag{1.10.91}$$

where we set $\xi_V := \xi - A_{\mu}\xi^{\mu}$ for the vertical part of the gauge generator Ξ.
The covariance identity reads as

$$-\tfrac{1}{2}\sqrt{g}\,T_{\alpha\beta}\pounds_{\Xi}g^{\alpha\beta} - \sqrt{g}\,F^{\mu\nu}_{..}d_{\mu}\pounds_{\Xi}A_{\nu} = d_{\lambda}\left(\xi^{\lambda}L_M\right) \tag{1.10.92}$$

and it holds true as one can easily verify

$$\begin{aligned}
-\tfrac{1}{2}\sqrt{g}T_{\alpha\beta}\pounds_{\Xi}g^{\alpha\beta} - \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}d_{\mu}\pounds_{\Xi}A_{\nu} &= \tfrac{\sqrt{g}}{\mathrm{g}}\left(F_{\mu\alpha}F^{\mu}_{.\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}_{..}g_{\alpha\beta}\right)\overset{g}{\nabla}^{\alpha}\xi^{\beta} - \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}d_{\mu}\left(\xi^{\lambda}F_{\lambda\nu} - \boxed{d_{\nu}\left(\xi_V\right)}\right) = \\
&= \tfrac{\sqrt{g}}{\mathrm{g}}\left(F_{\mu\alpha}F^{\mu}_{.\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}_{..}g_{\alpha\beta}\right)\overset{g}{\nabla}^{\alpha}\xi^{\beta} - \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}\left(\xi^{\lambda}F_{\lambda\nu}\right) = \tfrac{\sqrt{g}}{\mathrm{g}}\left({\color{red}F_{\mu\alpha}F^{\mu}_{.\beta}} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}_{..}g_{\alpha\beta}\right)\overset{g}{\nabla}^{\alpha}\xi^{\beta} - {\color{red}\tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}\xi^{\lambda}F_{\lambda\nu}} - \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}\xi^{\lambda}\overset{g}{\nabla}_{\mu}F_{\lambda\nu} = \\
&= -\tfrac{\sqrt{g}}{4\mathrm{g}}F_{\mu\nu}F^{\mu\nu}_{..}g_{\alpha\beta}\overset{g}{\nabla}^{\alpha}\xi^{\beta} - \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}F_{\lambda\nu}\xi^{\lambda} = \\
&= -\tfrac{\sqrt{g}}{4\mathrm{g}}F_{\mu\nu}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}\xi^{\lambda} - \tfrac{\sqrt{g}}{2\mathrm{g}}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}F_{\mu\nu}\xi^{\lambda} = -\tfrac{1}{4\mathrm{g}}\sqrt{g}F_{\mu\nu}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}\xi^{\lambda} - \tfrac{1}{4\mathrm{g}}\overset{g}{\nabla}_{\lambda}\left(\sqrt{g}\,F^{\mu\nu}_{..}F_{\mu\nu}\right)\xi^{\lambda} = \overset{*}{\nabla}_{\lambda}\left(-\tfrac{1}{4\mathrm{g}}\sqrt{g}F_{\mu\nu}F^{\mu\nu}_{..}\xi^{\lambda}\right) = d_{\lambda}\left(\xi^{\lambda}L_M\right)
\end{aligned} \tag{1.10.93}$$

where we used the identity

$$F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}F_{\lambda\nu} = -F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}F_{\nu\mu} - F^{\mu\nu}_{..}\overset{g}{\nabla}_{\nu}F_{\mu\lambda} = F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}F_{\mu\nu} - F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}F_{\lambda\nu} \qquad \Rightarrow F^{\mu\nu}_{..}\overset{g}{\nabla}_{\mu}F_{\lambda\nu} = \tfrac{1}{2}F^{\mu\nu}_{..}\overset{g}{\nabla}_{\lambda}F_{\mu\nu} \tag{1.10.94}$$

Then any gauge transformation is a symmetry and the corresponding Noether current

$$\begin{aligned}
\mathcal{E} &= \left(-\tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}\pounds_{\Xi}A_{\nu} - \xi^{\mu}L_M\right)d\sigma_{\mu} = \tfrac{\sqrt{g}}{\mathrm{g}}\left(-F^{\mu\nu}_{..}(\xi^{\lambda}F_{\lambda\nu} - d_{\nu}\xi_V) + \tfrac{1}{4}\xi^{\mu}F^{\alpha\beta}_{..}F_{\alpha\beta}\right)d\sigma_{\mu} = \\
&= \tfrac{\sqrt{g}}{\mathrm{g}}\left(-\left(F^{\mu\nu}_{..}F_{\lambda\nu} - \tfrac{1}{4}F^{\alpha\beta}_{..}F_{\alpha\beta}\delta^{\mu}_{\lambda}\right)\xi^{\lambda} + F^{\mu\nu}_{..}d_{\nu}\xi_V\right)d\sigma_{\mu} = \left(-\sqrt{g}\,T^{\mu}_{.\lambda}\xi^{\lambda} + \tfrac{\sqrt{g}}{\mathrm{g}}F^{\mu\nu}_{..}d_{\nu}\xi_V\right)d\sigma_{\mu}
\end{aligned} \tag{1.10.95}$$

is conserved on shell (i.e. along solutions).

This Noether current also can be expressed as a superpotential since

$$
\mathcal{E} = \left(-\sqrt{g}\, T^{\mu}_{\ \cdot \lambda}\xi^{\lambda} + \tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot} d_{\nu}\xi_{V} \right) d\sigma_{\mu} = \left(-\sqrt{g}\, T^{\mu}_{\ \cdot \lambda}\xi^{\lambda} - d_{\mu}\left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot} \right)\xi_{V} + d_{\nu}\left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot}\xi_{V} \right) \right) d\sigma_{\mu} =
$$

$$
= \left(-\sqrt{g}\, T^{\mu}_{\ \cdot \lambda}\xi^{\lambda} - d_{\mu}\left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot} \right)\xi_{V} \right) d\sigma_{\mu} + d\left(\tfrac{\sqrt{g}}{2\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot}\xi_{V}\, d\sigma_{\mu\nu} \right) \tag{1.10.96}
$$

Then one can define

$$
\tilde{\mathcal{E}} = -\left(\sqrt{g}\, T^{\mu}_{\ \cdot \lambda}\xi^{\lambda} + d_{\mu}\left(\tfrac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot} \right)\xi_{V} \right) d\sigma_{\mu} \qquad \mathcal{U} = \tfrac{\sqrt{g}}{2\mathrm{g}} F^{\mu\nu}_{\ \cdot \cdot}\xi_{V}\, d\sigma_{\mu\nu} \tag{1.10.97}
$$

where the reduced current $\tilde{\mathcal{E}}$ vanishes on shell for any symmetry generator Ξ.

Thus for any solution and any $(m-2)$ closed surface Ω contained in a space slice Σ_{t_0} in spacetime determined as $t = t_0$, one can compute the integral

$$
Q = \tfrac{1}{2\mathrm{g}} \int_{\Omega} \sqrt{g}\, F^{\mu\nu}_{\ \cdot \cdot}\, d\sigma_{\mu\nu} \tag{1.10.98}
$$

which is conserved since it corresponds to the integral of the superpotential for $\xi_V = 1$.

In particular, in adapted coordinates (t, x^i), one has $d\sigma_{ij} = 0$ on $\Omega \subset \Sigma_{t_0}$. Thus the only contributions to the integral comes from $d\sigma_{0i}$ and $d\sigma_{i0}$

$$
Q = \tfrac{1}{\mathrm{g}} \int_{\Omega} \sqrt{g}\, F^{0i}_{\ \cdot \cdot}\, d\sigma_{0i} = \tfrac{\mathrm{c}^2}{\mathrm{g}} \int_{\Omega} \sqrt{g}\, E^i\, d\sigma_{0i} = \tfrac{1}{2}\tfrac{\mathrm{c}^2}{\mathrm{g}} \int_{\Omega} \sqrt{g}\, E^i \epsilon_{ijk}\, dx^j \wedge dx^k \tag{1.10.99}
$$

which coincides with the flow of the electric field through the surface Ω, which we know to be the *electric charge* contained within the boundary Ω, which we know to be conserved.

That is, as far vacuum electromagnetism is concerned. If we want to consider electromagnetism coupled to a charged field we need a matter field which produces an electric current density, i.e. a term factorising δA_μ in the variation of the matter Lagrangian. Accordingly, the matter Lagrangian needs to depend on the potential.

That is normally obtained by considering a field which transforms with respect to some $U(1)$ group action, so that gauge covariant derivative in fact depends on A_μ. The prototype of this construction, in its simplest form, is coupling to a scalar field, which must be complex to support the $U(1)$ group action.

Maxwell electromagnetism with a charged field

Let us now consider a theory for Maxwell electromagnetism together with complex Klein–Gordon field φ which is a section of the associated bundle $P \times_\lambda \mathbb{C}$ for the action

$$
\lambda : U(1) \times \mathbb{C} \to \mathbb{C} : (e^{i\alpha}, \varphi) \mapsto e^{iq\alpha} \cdot \varphi \tag{1.10.100}
$$

Usually, this is also called a complex *scalar* field. However, the value of the field φ at a point does not depend only on the spacetime point x but also on the gauge on the $U(1)$-bundle P. Accordingly, the name *scalar* is an abuse, since it transforms as

$$
\varphi'(x') = e^{iq\alpha(x)} \cdot \varphi(x) \tag{1.10.101}
$$

where $e^{i\alpha}$ are the transition maps on P and not simply as a scalar field $\varphi'(x') = \varphi(x)$.

Let us also set $D_\mu\varphi := \overset{A}{\nabla}_\mu\varphi = d_\mu\varphi - iqA_\mu\varphi$ for the gauge covariant derivative.

The name *gauge covariant derivative* comes with the following property:

$$
\begin{aligned}
D'_\mu\varphi' &= d_\mu\varphi' - iqA'_\mu\varphi' = \bar{J}^\nu_\mu d_\nu(e^{iq\alpha}\cdot\varphi) - iq\bar{J}^\nu_\mu(A_\nu + d_\nu\alpha)e^{iq\alpha}\cdot\varphi = {\color{red}e^{iq\alpha}\cdot iq\bar{J}^\nu_\mu d_\nu\alpha\varphi} + e^{iq\alpha}\cdot\bar{J}^\nu_\mu d_\nu\varphi - iq\bar{J}^\nu_\mu A_\nu e^{iq\alpha}\cdot\varphi - {\color{red}iq\bar{J}^\nu_\mu d_\nu\alpha e^{iq\alpha}\cdot\varphi} = \\
&= e^{iq\alpha}\cdot(d_\nu\varphi - iqA_\nu\varphi)\,\bar{J}^\nu_\mu = e^{iq\alpha}\cdot D_\nu\varphi\bar{J}^\nu_\mu
\end{aligned}
\tag{1.10.102}
$$

A connection ω on P (i.e. a potential for the electromagnetic field) induces a connection on the associate bundle $P\times_\lambda\mathbb{C}$ which in turn induces a covariant derivative of sections, which is precisely $D_\mu\varphi$.

Thus the configuration bundle is $C = \mathrm{Con}(P)\times_M \mathrm{Lor}(M)\times_M(P\times_\lambda\mathbb{C})$ with coordinates $(x^\mu, A_\mu, g_{\mu\nu}, \varphi, \varphi^\dagger)$ where $(\varphi)^\dagger$ denotes the complex conjugation of the complex field φ.

The gauge covariant derivative of the complex conjugate field is defined to be

$$
D_\mu\varphi^\dagger = d_\mu\varphi^\dagger + iqA_\mu\varphi^\dagger
\tag{1.10.103}
$$

The variation of the gauge covariant derivative is given by

$$
\delta D_\mu\varphi = d_\mu\delta\varphi - iqA_\mu\delta\varphi - iq\delta A_\mu\varphi = D_\mu\delta\varphi - iq\delta A_\mu\varphi \qquad \delta D_\mu\varphi^\dagger = d_\mu\delta\varphi^\dagger + iqA_\mu\delta\varphi^\dagger + iq\delta A_\mu\varphi^\dagger = D_\mu\delta\varphi^\dagger + iq\delta A_\mu\varphi^\dagger
\tag{1.10.104}
$$

Then we can choose a Lagrangian for the system

$$
\boldsymbol{L} = \boldsymbol{L}_M - \tfrac{1}{2}\sqrt{g}\left(D_\mu\varphi^\dagger D^\mu_{\cdot}\varphi + \mu^2\varphi^\dagger\varphi\right)\,d\sigma
\tag{1.10.105}
$$

where we understood what the gauge covariant derivative depends on. The field φ is accordingly called a *complex Klein–Gordon* field. We already computed the variation of the Maxwell Lagrangian $\boldsymbol{L}_M$ to be

$$
\delta\boldsymbol{L}_M = \left[-\tfrac{1}{2}\sqrt{g}\,T^{(M)}_{\alpha\beta}\delta g^{\alpha\beta} - d_\mu\left(\tfrac{\sqrt{g}}{\mathrm{g}}\,F^{\mu\nu}_{\cdot\cdot}\delta A_\nu\right) - d_\nu\left(\tfrac{\sqrt{g}}{\mathrm{g}}\,F^{\mu\nu}_{\cdot\cdot}\right)\delta A_\mu\right]\,d\sigma
\tag{1.10.106}
$$

The variation of the complex Klein–Gordon Lagrangian $\boldsymbol{L}_{KG}$ is easily computed as

$$
\begin{aligned}
\delta L_{KG} =&\,\tfrac{1}{2}\sqrt{g}\left(-D_\alpha\varphi^\dagger D_\beta\varphi + \tfrac{1}{2}\left(D_\mu\varphi^\dagger D^\mu_{\cdot}\varphi + \mu^2\varphi^\dagger\varphi\right)g_{\alpha\beta}\right)\delta g^{\alpha\beta} - \tfrac{1}{2}\sqrt{g}\left(\delta D_\mu\varphi^\dagger D^\mu_{\cdot}\varphi + \mu^2\delta\varphi^\dagger\varphi\right) - \tfrac{1}{2}\sqrt{g}\left(D_\mu\varphi^\dagger\delta D^\mu_{\cdot}\varphi + \mu^2\varphi^\dagger\delta\varphi\right) = \\
=&-\tfrac{1}{2}\sqrt{g}\,T^{(KG)}_{\alpha\beta}\delta g^{\alpha\beta} - \tfrac{1}{2}\sqrt{g}\left(D_\mu\delta\varphi^\dagger D^\mu_{\cdot}\varphi + iq\varphi^\dagger D^\mu_{\cdot}\varphi\delta A_\mu + \mu^2\delta\varphi^\dagger\varphi\right) - \tfrac{1}{2}\sqrt{g}\left(D^\mu_{\cdot}\varphi^\dagger D_\mu\delta\varphi - iqD^\mu_{\cdot}\varphi^\dagger\varphi\delta A_\mu + \mu^2\varphi^\dagger\delta\varphi\right) = \\
=&-\tfrac{1}{2}\sqrt{g}\,T^{(KG)}_{\alpha\beta}\delta g^{\alpha\beta} - \tfrac{1}{2}\left(D_\mu\left(\sqrt{g}\,\delta\varphi^\dagger D^\mu_{\cdot}\varphi\right) - \delta\varphi^\dagger\left(D_\mu\left(\sqrt{g}\,D^\mu_{\cdot}\varphi\right) - \mu^2\sqrt{g}\,\varphi\right) + iq\sqrt{g}\,\varphi^\dagger D^\mu_{\cdot}\varphi\delta A_\mu\right) + \\
&-\tfrac{1}{2}\left(D_\mu\left(\sqrt{g}\,D^\mu_{\cdot}\varphi^\dagger\delta\varphi\right) - \left(D_\mu(\sqrt{g}\,D^\mu_{\cdot}\varphi^\dagger) - \mu^2\sqrt{g}\,\varphi^\dagger\right)\delta\varphi - iq\sqrt{g}\,D^\mu_{\cdot}\varphi^\dagger\varphi\delta A_\mu\right) = \\
=&-\tfrac{1}{2}\sqrt{g}\,T^{(KG)}_{\alpha\beta}\delta g^{\alpha\beta} - \tfrac{1}{2}\left(-\delta\varphi^\dagger\left(D_\mu\left(\sqrt{g}\,D^\mu\varphi\right) - \mu^2\sqrt{g}\,\varphi\right) - \left(D_\mu(\sqrt{g}\,D^\mu_{\cdot}\varphi^\dagger) - \mu^2\sqrt{g}\,\varphi^\dagger\right)\delta\varphi\right) + \\
&-\tfrac{1}{2}\left(D_\mu\left(\sqrt{g}\left(\delta\varphi^\dagger D^\mu_{\cdot}\varphi + D^\mu_{\cdot}\varphi^\dagger\delta\varphi\right)\right) + iq\sqrt{g}\left(\varphi^\dagger D^\mu_{\cdot}\varphi - D^\mu_{\cdot}\varphi^\dagger\varphi\right)\delta A_\mu\right) = \\
=&-\tfrac{1}{2}\sqrt{g}\,T^{(KG)}_{\alpha\beta}\delta g^{\alpha\beta} - \tfrac{1}{2}\left(-\delta\varphi^\dagger\left(D_\mu\left(\sqrt{g}\,D^\mu\varphi\right) - \mu^2\sqrt{g}\,\varphi\right) - \left(D_\mu(\sqrt{g}\,D^\mu_{\cdot}\varphi^\dagger) - \mu^2\sqrt{g}\,\varphi^\dagger\right)\delta\varphi\right) + \\
&+\left(-\tfrac{1}{2}D_\mu\left(\sqrt{g}\left(\delta\varphi^\dagger D^\mu_{\cdot}\varphi + D^\mu_{\cdot}\varphi^\dagger\delta\varphi\right)\right) - \sqrt{g}J^\mu\delta A_\mu\right)
\end{aligned}
\tag{1.10.107}
$$

where we set $T^{(KG)}_{\alpha\beta} := D_{(\alpha}\varphi^\dagger D_{\beta)}\varphi - \frac{1}{2}\left(D_\mu\varphi^\dagger D^\mu_{\,\cdot}\varphi + \mu^2\varphi^\dagger\varphi\right)g_{\alpha\beta}$ for the matter energy-momentum stress tensor of the Klein–Gordon field and $J^\mu :=$ $\frac{iq}{2}\left(D^\mu_{\,\cdot}\varphi^\dagger\varphi - \varphi^\dagger D^\mu_{\,\cdot}\varphi\right)$ for the matter charge current and we considered φ and $\varphi^\dagger$ as two independent real fields.

Let us set $\Box\varphi := \frac{1}{\sqrt{g}}D_\mu\left(\sqrt{g}\,g^{\mu\nu}D_\nu\varphi\right)$ for the gauge covariant Dalambertian.

Let us stress that the gauge covariant derivatives D_μ are here defined with respect to a connection $\Gamma^\alpha_{\beta\mu}$ on the spacetime manifold (say $\Gamma = \{g\}$) and a $U(1)$-connection A_μ. There is no way to define a covariant derivative with respect to Γ only since φ is a section of a bundle associated to P. Thus it would not ambiguous to denote it by $\nabla_\mu\varphi$, since one has no other possible interpretation for the symbol rather than $D_\mu\varphi = \nabla^{(A)}_\mu\varphi$.

When it is applied to Klein–Gordon field, it actually depends on A only, being $\nabla_\mu\varphi = d_\mu\varphi - ieA_\mu\varphi$, while when it acts on an object with world indices, e.g. $v_\mu = \nabla_\mu\varphi$, it also depends on Γ, being $\nabla_\lambda v_\mu = d_\lambda v_\mu - iqA_\lambda v_\mu - \Gamma^\alpha_{\mu\lambda}v_\alpha$. This is needed to ensure the Leibniz rule holds true.

However, when acting on a vector *density* of weight 1, e.g. $v^\mu = \sqrt{g}\,g^{\mu\nu}\nabla_\nu\varphi$, it reads as

$$\nabla_\mu v^\nu = d_\mu v^\nu - iqA_\mu v^\nu + \Gamma^\nu_{\alpha\mu}v^\alpha - \Gamma^\alpha_{\alpha\mu}v^\nu \tag{1.10.108}$$

with a further extra term for being a density of weight 1. Thus the gauge covariant divergence

$$\nabla_\mu v^\mu = d_\mu v^\mu - iqA_\mu v^\mu + \Gamma^\mu_{\alpha\mu}v^\alpha - \Gamma^\alpha_{\alpha\mu}v^\mu = d_\mu v^\mu - iqA_\mu v^\mu \tag{1.10.109}$$

is independent of the spacetime connection Γ.

If we fix $\Gamma = \{g\}$ we also have

$$\Box\varphi = \frac{1}{\sqrt{g}}\nabla_\mu\left(\sqrt{g}\,g^{\mu\nu}\nabla_\nu\varphi\right) = \frac{1}{\sqrt{g}}\sqrt{g}\,\nabla_\mu\left(g^{\mu\nu}\nabla_\nu\varphi\right) = \nabla_\mu\left(\nabla^\mu\varphi\right) \tag{1.10.110}$$

Here

$$\nabla_\mu\left(\nabla^\mu\varphi\right) = d_\mu\nabla^\mu\varphi - iqA_\mu\nabla^\mu\varphi + \{g\}^\mu_{\alpha\mu}\nabla^\alpha\varphi$$

as prescribed for the covariant derivative of a vector. All these dirty tricks and the abuses of language amounting to denote all gauge derivatives by a single symbol ∇_μ show how important it is to have a general theory supporting us.

The two matter field equations $\Box\varphi - \mu^2\varphi = 0$ and $\Box\varphi^\dagger - \mu^2\varphi^\dagger = 0$ are one the complex conjugate of the other; they are then a unique complex equation.

The Euler–Lagrange part and Liouville part are then

$$\begin{cases} \mathbb{E}(\boldsymbol{L}) = \sqrt{g}\left(-\frac{1}{2}(T^{(M)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta})\omega^{\alpha\beta} - \frac{1}{g}\left(\frac{1}{\sqrt{g}}d_\nu\left(\sqrt{g}\,F^{\nu\mu}_{\,\cdot\cdot}\right) - \mathrm{g}J^\mu\right)\omega_\mu + \frac{1}{2}\omega^\dagger(\Box\varphi - \mu^2\varphi) + \frac{1}{2}(\Box\varphi^\dagger - \mu^2\varphi^\dagger)\boldsymbol{\omega}\right) \wedge \ d\sigma \\ \mathbb{F}(\boldsymbol{L}) = -\frac{1}{2}\sqrt{g}\left(\omega^\dagger\nabla^\mu_{\,\cdot}\varphi + \nabla^\mu_{\,\cdot}\varphi^\dagger\boldsymbol{\omega}\right) \wedge \ d\sigma_\mu \end{cases} \tag{1.10.111}$$

where we set $\omega^{\alpha\beta} = dg^{\alpha\beta} - d_\lambda g^{\alpha\beta}\,dx^\lambda$, $\omega_\mu = dA_\mu - d_\lambda A_\mu\,dx^\lambda$, $\boldsymbol{\omega} = d\boldsymbol{\varphi} - d_\lambda\varphi\,dx^\lambda$, $\omega^\dagger = d\boldsymbol{\varphi}^\dagger - d_\lambda\varphi^\dagger\,dx^\lambda$ for the relevant contact forms.

For field equations, as we pointed out above, one should decide which fields are fixed and which are dynamical. If we consider all fields dynamical, we have

$$T^{(M)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta} = 0 \qquad d_\nu\left(\sqrt{g}\,F^{\nu\mu}_{\,\cdot\cdot}\right) = \mathrm{g}\sqrt{g}\,J^\mu \qquad \Box\varphi - \mu^2\varphi = 0 \tag{1.10.112}$$

Let us ignore the first field equation (as we discussed for the Klein–Gordon case, it is often unreasonably strict and one should add a further Lagrangian $\boldsymbol{L}_g$ for the metric g to make it physically sound).

The second equation determines the electromagnetic field $\boldsymbol{A}$ generated by the sources $\boldsymbol{J} = \sqrt{g}J^{\mu}\,d\sigma_{\mu}$ which is now determined as a function of the Klein–Gordon field φ (and its first order derivatives) as well as from the electromagnetic field itself (which is hidden within the gauge covariant derivatives).

The third field equation determines the Klein–Gordon field under an electromagnetic field $\boldsymbol{A}$ (which means that the field φ does interact with electromagnetic field, i.e. it carries an electric charge).

Before focusing on conservation laws, let us remark what would happen considering the Lagrangian for the complex Klein–Gordon field alone

$$\boldsymbol{L}_{KG} = -\tfrac{\sqrt{g}}{2}\left(\nabla_{\mu}\varphi^{\dagger}\nabla^{\mu}_{\cdot}\varphi + \mu^{2}\varphi^{\dagger}\varphi\right)d\sigma \tag{1.10.113}$$

Its variation would be

$$\delta L_{KG} = -\tfrac{\sqrt{g}}{2}\,T^{(KG)}_{\alpha\beta}\delta g^{\alpha\beta} - \tfrac{\sqrt{g}}{2}\left(-\delta\varphi^{\dagger}\left(\Box\varphi - \mu^{2}\varphi\right) - \left(\Box\varphi^{\dagger} - \mu^{2}\varphi^{\dagger}\right)\delta\varphi\right) - \tfrac{\sqrt{g}}{2}J^{\mu}\delta A_{\mu} - \nabla_{\mu}\left(\tfrac{\sqrt{g}}{2}\left(\delta\varphi^{\dagger}\nabla^{\mu}_{\cdot}\varphi + \nabla^{\mu}_{\cdot}\varphi^{\dagger}\delta\varphi\right)\right) \tag{1.10.114}$$

which, even considering the metric fixed, would lead to field equations

$$\Box\varphi - \mu^{2}\varphi = 0 \qquad\qquad J^{\mu} = 0 \qquad\qquad T^{(KG)}_{\alpha\beta} = 0 \tag{1.10.115}$$

Thus if one does not consider A_{μ} fixed as well, one gets one field equation $J^{\mu} = 0$ which is, in most cases, too strict to make sense. On the contrary, if A is fixed there is only one field equation, namely $\Box\varphi - \mu^{2}\varphi = 0$ which describes the evolution of the matter field in a given electromagnetic field $\boldsymbol{A}$ (and a given metric g).

The quantity J^{μ} is always the matter charge current, but its meaning becomes manifest only when one adds the Maxwell Lagrangian. Similarly, the meaning of stress tensors $T^{(M)}_{\alpha\beta}$ and $T^{(KG)}_{\alpha\beta}$ will become manifest only when coupling to a dynamical metric g.

We can now turn to conservation laws. Gauge transformations

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ e^{i\theta'} = e^{i(\alpha+\theta)} \end{cases} \tag{1.10.116}$$

act on fields as

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ A'_{\mu} = \bar{J}^{\rho}_{\mu}\left(A_{\rho} + \partial_{\rho}\alpha\right) \\ g'_{\mu\nu} = \bar{J}^{\alpha}_{\mu}g_{\alpha\beta}\bar{J}^{\beta}_{\nu} \\ \varphi' = e^{iq\alpha}\cdot\varphi \end{cases} \tag{1.10.117}$$

and under these field transformations the Lagrangian $\boldsymbol{L} = \boldsymbol{L}_{M} + \boldsymbol{L}_{KG}$ is invariant, i.e. gauge transformations are symmetries. We already checked it for $\boldsymbol{L}_{M}$. We have to check covariance identity for the Lagrangian $\boldsymbol{L}_{KG}$.

Lie derivatives are

$$\begin{cases} \pounds_{\Xi}A_{\mu} = \xi^{\lambda}d_{\lambda}A_{\mu} - \left(-d_{\mu}\xi^{\lambda}A_{\lambda} + d_{\mu}\xi\right) = \xi^{\lambda}F_{\lambda\mu} + d_{\mu}\left(\xi^{\lambda}A_{\lambda} - \xi\right) = \xi^{\lambda}F_{\lambda\mu} - d_{\mu}\xi_{V} \\ \pounds_{\Xi}g_{\mu\nu} = \overset{g}{\nabla}_{\mu}\xi^{\lambda}g_{\lambda\nu} + g_{\mu\lambda}\overset{g}{\nabla}_{\nu}\xi^{\lambda} \quad \Rightarrow \pounds_{\Xi}g^{\mu\nu} = -\overset{g}{\nabla}_{\lambda}\xi^{\mu}g^{\lambda\nu} - g^{\mu\lambda}\overset{g}{\nabla}_{\lambda}\xi^{\nu} \\ \pounds_{\Xi}\varphi = \xi^{\mu}d_{\mu}\varphi - iq\xi\varphi = \xi^{\mu}\nabla_{\mu}\varphi - iq\xi_{V}\varphi \quad \Rightarrow \pounds_{\Xi}\varphi^{\dagger} = \xi^{\mu}\nabla_{\mu}\varphi^{\dagger} + iq\xi_{V}\varphi^{\dagger} \end{cases} \tag{1.10.118}$$

Then the covariance identity for the Lagrangian $\boldsymbol{L}_{KG}$ reads as

$$-\tfrac{1}{2}\sqrt{g}\,T_{\alpha\beta}\pounds_\Xi g^{\alpha\beta} - \tfrac{1}{2}\sqrt{g}\,\left(\pounds_\Xi\nabla_\mu\varphi^\dagger\nabla^\mu_{\cdot\cdot}\varphi + \mu^2\pounds_\Xi\varphi^\dagger\varphi\right) - \tfrac{1}{2}\sqrt{g}\,\left(\nabla_\mu\varphi^\dagger\pounds_\Xi\nabla^\mu_{\cdot\cdot}\varphi + \mu^2\varphi^\dagger\pounds_\Xi\varphi\right)\sigma = d_\mu\left(\xi^\mu L_{KG}\right) \tag{1.10.119}$$

Just expand it and verify.

Then once again, any gauge transformation is a symmetry of the Lagrangian $\boldsymbol{L}$ and Noether theorem applies. Noether currents of the Lagrangian $\boldsymbol{L}_{KG}$ are in the form

$$\begin{aligned}
\mathcal{E}_{KG} &= -\tfrac{1}{2}\sqrt{g}\,\left(\pounds_\Xi\varphi^\dagger\nabla^\mu_{\cdot\cdot}\varphi + \nabla^\mu_{\cdot\cdot}\varphi^\dagger\pounds_\Xi\varphi\right)\,d\sigma_\mu - \xi^\mu L_{KG}\,d\sigma_\mu = \\
&= -\tfrac{1}{2}\sqrt{g}\,\left((\xi^\nu\nabla_\nu\varphi^\dagger + iq\xi_V\varphi^\dagger)\nabla^\mu_{\cdot\cdot}\varphi + \nabla^\mu_{\cdot\cdot}\varphi^\dagger(\xi^\nu\nabla_\nu\varphi - iq\xi_V\varphi) - \xi^\mu(\nabla_\nu\varphi^\dagger\nabla^\nu_{\cdot\cdot}\varphi + \mu^2\varphi^\dagger\varphi)\right)\,d\sigma_\mu = \\
&= -\tfrac{1}{2}\sqrt{g}\,\left(\xi^\nu(\nabla_\nu\varphi^\dagger\nabla^\mu_{\cdot\cdot}\varphi + \nabla^\mu_{\cdot\cdot}\varphi^\dagger\nabla_\nu\varphi - \delta^\mu_\nu(\nabla_\lambda\varphi^\dagger\nabla^\lambda_{\cdot\cdot}\varphi + \mu^2\varphi^\dagger\varphi)) + iq(\varphi^\dagger\nabla^\mu_{\cdot\cdot}\varphi - \nabla^\mu_{\cdot\cdot}\varphi^\dagger\varphi)\xi_V\right)\,d\sigma_\mu = \\
&= -\sqrt{g}\,\left((\tfrac{1}{2}(\nabla_\nu\varphi^\dagger\nabla^\mu_{\cdot\cdot}\varphi + \nabla^\mu_{\cdot\cdot}\varphi^\dagger\nabla_\nu\varphi) - \tfrac{1}{2}(\nabla_\lambda\varphi^\dagger\nabla^\lambda_{\cdot\cdot}\varphi + \mu^2\varphi^\dagger\varphi)\delta^\mu_\nu)\xi^\nu + J^\mu\xi_V\right)\,d\sigma_\mu = \\
&= \sqrt{g}\,\left(-T^\mu_{\cdot\cdot\nu}\xi^\nu - J^\mu\xi_V\right)\,d\sigma_\mu
\end{aligned} \tag{1.10.120}$$

The form $\mathcal{E}_{KG}$ is calculated by the formula (1.4.12) and as such it is called *Noether current* and it obeys Noether theorem, i.e. $d\mathcal{E}_{KG} = \mathcal{W}(\boldsymbol{L}_{KG})$.

But once evaluated along a solution of $\boldsymbol{L}$, is it a closed form?

Here again everything goes back to what we consider field equations to see if the work current $\mathcal{W}(\boldsymbol{L}_{KG})$ vanishes along solutions of $\boldsymbol{L}$. In a purely relativistic perspective, all fields are dynamical and field equations are then (1.10.112). The work current of $\boldsymbol{L}_{KG}$ reads as

$$\mathcal{W}(\boldsymbol{L}_{KG}) = -\left(\pounds_\Xi\right)\lrcorner\,\mathbb{E}(\boldsymbol{L}_{KG}) = \sqrt{g}\Big(\tfrac{1}{2}T^{(KG)}_{\alpha\beta}\pounds_\Xi g^{\alpha\beta} + J^\mu\pounds_\Xi A_\mu - \tfrac{1}{2}\pounds_\Xi\varphi^\dagger(\Box\varphi - \mu^2\varphi) - \tfrac{1}{2}(\Box\varphi^\dagger - \mu^2\varphi^\dagger)\pounds_\Xi\varphi\Big)\wedge\,d\sigma \tag{1.10.121}$$

which along solutions of field equations (1.10.112) reduces to

$$\begin{aligned}
\mathcal{W}(\boldsymbol{L}_{KG}) &= \sqrt{g}\Big(-\tfrac{1}{2}\left(T^{(M)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta}\right)\pounds_\Xi g^{\alpha\beta} + \tfrac{1}{2}T^{(M)}_{\alpha\beta}\pounds_\Xi g^{\alpha\beta} - \tfrac{1}{\mathrm{g}}\left(\tfrac{1}{\sqrt{g}}d_\nu\left(\sqrt{g}\,F^{\nu\mu}_{\cdot\cdot}\right) - \mathrm{g}J^\mu\right)\pounds_\Xi A_\mu + \tfrac{1}{\mathrm{g}\sqrt{g}}d_\nu\left(\sqrt{g}\,F^{\nu\mu}_{\cdot\cdot}\pounds_\Xi A_\mu\right)\Big)\,d\sigma = \\
&= \sqrt{g}\Big(\tfrac{1}{2}T^{(M)}_{\alpha\beta}\pounds_\Xi g^{\alpha\beta} + \tfrac{1}{\mathrm{g}\sqrt{g}}d_\nu\left(\sqrt{g}\,F^{\nu\mu}_{\cdot\cdot}\pounds_\Xi A_\mu\right)\Big)\,d\sigma
\end{aligned} \tag{1.10.122}$$

which is non-zero in general.

This is because Noether theorem simply states that $d\mathcal{E} = \mathcal{W}$, not that $\mathcal{E}$ is closed on shell. The current $\mathcal{E}$ is closed on shell only when $\mathcal{W}$ vanishes on shell and this holds true only for the work current of the *total Lagrangian*, in this example for $\boldsymbol{L} = \boldsymbol{L}_M + \boldsymbol{L}_{KG}$. Both the Noether current and work current for partial Lagrangians (in this case $\boldsymbol{L}_{KG}$) can be easily computed (the expressions (1.4.12) are linear in the Lagrangian), they obey Noether theorem (1.4.9), but this implies that Noether current is closed on shell only for the Noether current of the *total Lagrangian*.

Let us stress that, also when considering the total Lagrangian $\boldsymbol{L}$, one can compute $\mathcal{E}(\boldsymbol{L})$ and $\mathcal{W}(\boldsymbol{L})$ and, by Noether theorem, one has $d\mathcal{E}(\boldsymbol{L}) = \mathcal{W}(\boldsymbol{L})$. In this case we have

$$\mathcal{W}(\boldsymbol{L}) = -\,(\pounds_{\Xi})\,\lrcorner\,\mathbb{E}(\boldsymbol{L}) = \sqrt{g}\Big(\tfrac{1}{2}\left(T^{(M)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta}\right) \pounds_{\Xi} g^{\alpha\beta} - \tfrac{1}{\mathrm{g}}\left(\tfrac{1}{\sqrt{g}}d_{\nu}\left(\sqrt{g}\,F^{\nu\mu}_{\cdot\cdot}\right) - \mathrm{g}J^{\mu}\right) \pounds_{\Xi} A_{\mu} - \tfrac{1}{2}\pounds_{\Xi}\varphi^{\dagger}(\Box\varphi - \mu^{2}\varphi) - \tfrac{1}{2}(\Box\varphi^{\dagger} - \mu^{2}\varphi^{\dagger})\pounds_{\Xi}\varphi\Big)\,d\sigma \tag{1.10.123}$$

If we had to ask whether the Noether current $\mathcal{E}(\boldsymbol{L})$ is closed along solutions of field equations, we still have to be clear on what we consider to be field equations.

In a purely relativistic attitude all fields are dynamical, field equations are given by (1.10.112), $\mathcal{W}(\boldsymbol{L}) = 0$, then the Noether current $\mathcal{E}(\boldsymbol{L})$ is closed.

If the metric g is considered fixed instead of dynamical though, then one fixed $\delta g^{\alpha\beta} = 0$ in the variation and field equations are just

$$\tfrac{1}{\sqrt{g}}d_{\nu}\left(\sqrt{g}\,F^{\nu\mu}_{\cdot\cdot}\right) = J^{\mu} \qquad\qquad \Box\varphi - \mu^{2}\varphi = 0 \tag{1.10.124}$$

Under these field equations, the work current reduces to

$$\mathcal{W}(\boldsymbol{L}) = -\,(\pounds_{\Xi})\,\lrcorner\,\mathbb{E}(\boldsymbol{L}) = \tfrac{1}{2}\sqrt{g}\left(T^{(M)}_{\alpha\beta} + T^{(KG)}_{\alpha\beta}\right)\pounds_{\Xi} g^{\alpha\beta}\,d\sigma \tag{1.10.125}$$

which still has no reason to vanish in general (in Minkowski spacetime one can gives examples of solutions $(A_{\mu}(x), \varphi(x))$ of field equations (1.10.124) for which the energy-momentum stress tensors do not vanish).

To get conservation for the Noether current $\mathcal{E}(\boldsymbol{L})$, one is forced to get rid of the terms surviving in the work current. This can be done by restricting gauge transformations and requiring that $\pounds_{\Xi} g^{\alpha\beta} = -2\overset{g}{\nabla}{}^{(\alpha}\xi^{\beta)} = 0$, i.e. by restricting to gauge transformations which project over spacetime isometries for g, i.e. requiring that ξ is a Killing vector for g.

In Minkowski spacetime, we have 10 Killing vectors to do the trick, but on generic spacetimes there are no Killing vectors at all. In many intermediate situations, one can do something. It is fortunate that in a purely relativistic framework, when one can encounter any sort of spacetimes (generically with no Killing vector) one does not need Killing vectors.

Finally, let us remark that the total Noether current reads as

$$\mathcal{E}(\boldsymbol{L}) = \mathcal{E}(\boldsymbol{L}_M) + \mathcal{E}(\boldsymbol{L}_{KG}) = \sqrt{g}\left(-\left(T^{(M)}\right)^{\mu}{}_{\cdot\lambda}\xi^{\lambda} - \tfrac{1}{\mathrm{g}\sqrt{g}}d_{\mu}\left(\sqrt{g}\,F^{\mu\nu}_{\cdot\cdot}\right)\xi_V\right)\,d\sigma_{\mu} + d\left(\tfrac{\sqrt{g}}{2\mathrm{g}}\,F^{\mu\nu}_{\cdot\cdot}\xi_V\,d\sigma_{\mu\nu}\right) + \sqrt{g}\left(-\left(T^{(KG)}\right)^{\mu}{}_{\cdot\nu}\xi^{\nu} + J^{\mu}\xi_V\right)\,d\sigma_{\mu} =$$
$$= \sqrt{g}\left(-\left(\left(T^{(M)}\right)^{\mu}{}_{\cdot\lambda} + \left(T^{KG}\right)^{\mu}{}_{\cdot\lambda}\right)\xi^{\lambda} - \tfrac{1}{\mathrm{g}}\left(\tfrac{1}{\sqrt{g}}d_{\mu}\left(\sqrt{g}\,F^{\mu\nu}_{\cdot\cdot}\right) - \mathrm{g}J^{\mu}\right)\xi_V\right)\,d\sigma_{\mu} + d\left(\tfrac{\sqrt{g}}{2\mathrm{g}}\,F^{\mu\nu}_{\cdot\cdot}\xi_V\,d\sigma_{\mu\nu}\right) \tag{1.10.126}$$

Thus one can define the reduced current

$$\tilde{\mathcal{E}}(\boldsymbol{L}) = \sqrt{g}\left(-\left(\left(T^{(M)}\right)^{\mu}{}_{\cdot\lambda} + \left(T^{(KG)}\right)^{\mu}{}_{\cdot\lambda}\right)\xi^{\lambda} - \tfrac{1}{\mathrm{g}}\left(\tfrac{1}{\sqrt{g}}d_{\mu}\left(\sqrt{g}\,F^{\mu\nu}_{\cdot\cdot}\right) - \mathrm{g}J^{\mu}\right)\xi_V\right)\,d\sigma_{\mu} \tag{1.10.127}$$

and the superpotential

$$\mathcal{U}(\boldsymbol{L}) = \tfrac{\sqrt{g}}{2\mathrm{g}}\,F^{\mu\nu}_{\cdot\cdot}\xi_V\,d\sigma_{\mu\nu} \tag{1.10.128}$$

so that one has

$$\mathcal{E}(\boldsymbol{L}) = \tilde{\mathcal{E}}(\boldsymbol{L}) + d\mathcal{U}(\boldsymbol{L}) \tag{1.10.129}$$

Since the reduced current is vanishing along solutions of field equations (at least in a purely relativistic attitude) then once again the model allows superpotential.

Notice that the Lagrangian $\boldsymbol{L}_{KG}$ does not contribute to the superpotential. This does not mean that the *field* φ does not contribute to the electric charge. In fact, it does (and that is what we mean by saying that it is a charged field). Actually, when defining the electric charge, one needs to integrate the superpotential on shell and the flow of the electric field remembers of the field φ, since one is considering a non-vacuum solutions of Maxwell equations.

Maxwell electromagnetism in dimension 2+1

Let us here make a point. It is not irrelevant what do you consider the best physical description of fields. Suppose we wish describe electromagnetism in a $2 + 1$ world. If we do not take the description of 2 vector fields on space seriously, and we believe instead in a description in terms of a 2-form on spacetime we still can introduce a potential $\boldsymbol{A}$ which has now 3 components.

Also in dimension 3 one needs to decide how seriously one should consider gauge transformations. Once we assume the potential $\boldsymbol{F} = d\boldsymbol{A}$ of course there are gauge transformations mapping potentials which produce the same $\boldsymbol{F}$.

When we accept gauge transformations we have a better description of potentials as connections of a $U(1)$-principal bundle $[P \to M]$. The electromagnetic field $\boldsymbol{F}$ is hence defined as the curvature of the gauge connection ω. Then we have the Maxwell Lagrangian as above and the same field equations $d_\mu(\sqrt{g}F^{\mu\nu}) = 0$ in vacuum.

Even in Minkowski spacetime now, $\boldsymbol{F}$ can be decomposed in a laboratory (t, x, y) as

$$\boldsymbol{F} = -\mathsf{c}E_i dt \wedge dx^i + Bdx \wedge dy \tag{1.10.130}$$

Accordingly, in dimension $2 + 1$ the electromagnetic field is described by a covector $\mathbf{E}$ and a scalar density $\mathbf{B}$.

> The argument we did for rotations still apply and it shows that $\mathbf{B}$ behaves as a scalar density, i.e. it is 2-form on space, not a scalar field.

Let us finally look for solutions on $M = \left(\mathbb{R}^2 - \{0\}\right) \times \mathbb{R}$ with $B = 0$, namely with the *ansatz* $\boldsymbol{A} = \mathsf{c}\phi(r)dt$, hence $\boldsymbol{F} = -\mathsf{c}\frac{d\phi}{dr}dt \wedge dr$. That corresponds to a static, spherically symmetric electric field, possibly singular at the origin. We can further specify the *ansatz* to $\frac{d\phi}{dr}(r) = r^\alpha$ and let us use field equations to determine α which in fact describes the asymptotic behaviour on Coulomb interaction.

The metric is $g = -dt^2 + dr^2 + r^2 d\theta^2$, which is the Minkowski metric just in spatially polar coordinates. Then the Maxwell equations are

$$0 = d_\mu(\sqrt{g}F_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}) = d_0\left(\sqrt{g}F_{01}g^{00}g^{1\nu}\right) + d_1\left(\sqrt{g}F_{10}g^{11}g^{0\nu}\right) = \mathsf{c}\left(d_0\left(r\frac{d\phi}{dr}g^{1\nu}\right) - d_1\left(r\frac{d\phi}{dr}g^{0\nu}\right)\right) = -\mathsf{c}\frac{d}{dr}\left(r^{\alpha+1}g^{0\nu}\right) \tag{1.10.131}$$

Splitting the equations, we get

$$\frac{d}{dr}\left(r^{\alpha+1}\right) = (\alpha+1)r^\alpha = 0 \qquad \frac{d}{dr}\left(r^{\alpha+1}g^{01}\right) = 0 \qquad \frac{d}{dr}\left(r^{\alpha+1}g^{02}\right) = 0 \tag{1.10.132}$$

The last two are identically satisfied since $g^{0i} = 0$. The only solution of the first is obtained for $\alpha = -1$. Then we have

$$\phi = k \ln(r) \qquad\qquad F = -\mathsf{c}\frac{k}{r}dt \wedge dr \qquad\qquad E_r = \frac{k}{r} \qquad\qquad (1.10.133)$$

Accordingly, in dimension $2 + 1$, the Coulomb force $F = eE$ goes as r^{-1} with the distance, instead of r^{-2} as in dimension 3+1.

Interlude

By now, following these examples, some general features about field theories are emerging. They are worth noticing now.

First of all, when one wants to define a field theory, it is important to discuss which are the fundamental fields and which are the allowed transformations among laboratories of observers. This is equivalent to discuss in which bundle C fields are globally defined as global sections.

Then a Lagrangian must be chosen depending on fields and their derivatives up to some finite order k (usually $k = 1$ or $k = 2$).

> Unlike in mechanics, where there is a universal prescription for dynamics, namely second law of dynamics $\vec{F} = m\vec{a}$ which leads to define Lagrangians in the form $L = T - U$, in field theory, one basically has to guess the Lagrangian so that it produces the *"observed"* field equations.

The Lagrangian should be checked to be global (so that field equations are automatically global). A case in which globality is easily checked is when the transformations among observers (the transition maps of the configuration bundle) are symmetries. If they are, not only one has

$$JL'(x'^{\mu}, y'^{i}, y'^{i}_{\mu}, \ldots) = L(x^{\mu}, y^{i}, y^{i}_{\mu}, \ldots) \qquad\qquad (1.10.134)$$

as needed for globality but one has in fact

$$JL(x'^{\mu}, y'^{i}, y'^{i}_{\mu}, \ldots) = L(x^{\mu}, y^{i}, y^{i}_{\mu}, \ldots) \qquad\qquad (1.10.135)$$

which is what is needed for covariance with respect to general change of lab.

In this case, which will be typical in relativistic theories, not only each observer defines a local Lagrangian and different local Lagrangians glue together in the intersections, but each observer uses the *same* function as local Lagrangian density, just each as a function of the fields of that observer's laboratory. Notice that covariance with respect to general change of lab implies globality, while of course not vice versa.

Then, by variation and integration by parts, one finds field equations and the Liouville part. This is quite an algorithmic computation and nothing particularly theoretical is really needed to do it. Just as in sports, knowing how it works is better, although knowledge is not really what one uses to do it.

Then fields transformation rules define Lie derivative of fields and one can write covariance identity. If the Lagrangian was covariant then one expects to have many symmetries, namely any one-parameter family of change of laboratories. The covariance identity must be verified without performing integrations by parts. One just has to expand and collect.

If symmetries are found then, for each symmetry, one can define Noether and work currents just by applying the general formula. We still need some theory and algorithm to split Noether currents into a reduced current and a superpotential. We saw few cases in which we were able to find a superpotential by a suitable integration by parts but we still need to know when superpotentials exist and a general strategy to find them. Analysing

the example above, one could see that starting from covariance identity where Lie derivatives appear under derivation the Noether theorem is obtained by integrating by parts derivatives without expanding Lie derivatives.

Then one can expand Lie derivatives in the Noether current so to obtain linear combinations of the symmetry generators (ξ^μ and ξ_V in the examples above) and their derivative up to some finite order (which depend on the fields under consideration). The superpotential is obtained by a suitable integration by parts of these derivatives.

As a general rule, covariant derivatives have to be preferred over ordinary derivatives, so that one can easily control globality of terms and objects.

Chern–Simons theory

For simplicity, let us consider the group $G = \mathrm{SO}(3)$ and denote its elements by S^i_j; it is understood that these parameters, which are actually coordinates on $\mathrm{GL}(3)$, are constrained by the relations ${}^tSS = \mathbb{I}$ and $\det S = 1$. These constraints identify $\mathrm{SO}(3)$ as a sub-manifold of $\mathrm{GL}(3)$.

Let P be a principal G-bundle over M; a right-invariant pointwise basis for vertical vectors is given by $\sigma_{ij} := \rho_{[ij]}$ where we set

$$\rho_{ij} := \delta_{hi} S^h_k \frac{\partial}{\partial S^j_k} \tag{1.10.136}$$

We remark that Lie algebra indices are raised and lowered by δ_{ij}. A principal connection is given by

$$\omega = dx^\mu \otimes (\partial_\mu - A^{ij}_\mu(x)\sigma_{ij}) \tag{1.10.137}$$

Let us also set $A^i_\mu := \frac{1}{2}\epsilon^i{}_{jk}A^{jk}_\mu$ and

$$F^i_{\alpha\beta} := \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha + \epsilon^i{}_{jk}A^j_\alpha A^k_\beta \qquad \boldsymbol{F} = \tfrac{1}{2}F^i_{\alpha\beta}dx^\alpha \wedge dx^\beta \otimes T_i \tag{1.10.138}$$

for a suitable basis $T_i = \frac{1}{2}\epsilon_{ij}{}^k\partial^j_k$ in the Lie algebra. The field $\boldsymbol{F}$ is called *curvature* in geometry, *field strength* in physics.

If we consider a gauge transformation on P locally expressed as

$$x'^\mu = x'^\mu(x) \qquad\qquad S' = \varphi(x) \cdot S \tag{1.10.139}$$

then the coefficients of the connection transforms as

$$A'^i_\mu = \bar{J}^\sigma_\mu \left(\varphi^i_k A^k_\sigma + \tfrac{1}{2}\epsilon^{ij}{}_k\varphi^k_l d_\sigma\bar{\varphi}^i_j\right) \tag{1.10.140}$$

As a consequence, the field strength transforms as

$$F'^i_{\mu\nu} = \bar{J}^\alpha_\mu \bar{J}^\beta_\nu \varphi^i_j F^j_{\alpha\beta} \tag{1.10.141}$$

However, transformation rules are irrelevant for field equations and pop in when discussing conservation laws.

Chern–Simons Lagrangian(s) is (a family of) local Lagrangians; on a 3-dimensional base M, and with obvious notation, one sets

$$\boldsymbol{L}_{CS}(j^1 A) = \epsilon^{\alpha\beta\lambda}\left(\delta_{ij}F^i{}_{\alpha\beta}A^j{}_\lambda - \tfrac{1}{3}\epsilon_{ijk}A^i{}_\alpha A^j{}_\beta A^k{}_\lambda\right)\,d\sigma \tag{1.10.142}$$

The variation of the Lagrangian is

$$\delta \boldsymbol{L}_{CS} = \epsilon^{\alpha\beta\lambda} \left(\delta_{ij} \delta F^i{}_{\alpha\beta} A^j{}_\lambda + \left(\delta_{ik} F^i{}_{\alpha\beta} - \epsilon_{ijk} A^i{}_\alpha A^j{}_\beta \right) \delta A^k{}_\lambda \right) \, d\sigma \tag{1.10.143}$$

The variation of the field strength is given by

$$\delta F^i_{\alpha\beta} = d_\alpha \delta A^i_\beta - d_\beta \delta A^i_\alpha + 2\epsilon^i{}_{jk} A^j_\alpha \delta A^k_\beta \tag{1.10.144}$$

so that the variation of the Lagrangian can be recast as

$$
\begin{aligned}
\delta \boldsymbol{L}_{CS} =& \epsilon^{\alpha\beta\lambda} \left(2\delta_{ij} \left(d_\alpha \delta A^i_\beta + \epsilon^i{}_{lk} A^l_\alpha \delta A^k_\beta \right) A^j{}_\lambda + \left(\delta_{ik} F^i{}_{\alpha\beta} - \epsilon_{ijk} A^i{}_\alpha A^j{}_\beta \right) \delta A^k{}_\lambda \right) \, d\sigma = \\
=& \epsilon^{\alpha\beta\lambda} \left(2\delta_{ij} d_\alpha \delta A^i_\beta A^j{}_\lambda + \underline{2\epsilon_{ijk} A^i{}_\alpha A^j_\beta \delta A^k_\lambda} + \left(\delta_{ik} F^i{}_{\alpha\beta} - \underline{\epsilon_{ijk} A^i{}_\alpha A^j{}_\beta} \right) \delta A^k{}_\lambda \right) \, d\sigma = \\
=& \left(d_\alpha \left(2\delta_{ij} \epsilon^{\alpha\beta\lambda} \delta A^i_\beta A^j{}_\lambda \right) - 2\delta_{ij} \epsilon^{\alpha\beta\lambda} d_\alpha A^j{}_\lambda \delta A^i_\beta + \epsilon^{\alpha\beta\lambda} \left(\delta_{ik} F^i{}_{\alpha\beta} \delta A^k{}_\lambda + \epsilon_{ijk} A^i{}_\alpha A^j{}_\beta \delta A^k{}_\lambda \right) \right) \, d\sigma = \\
=& \left(d_\alpha \left(2\delta_{ij} \epsilon^{\alpha\beta\lambda} \delta A^i_\beta A^j{}_\lambda \right) + \epsilon^{\alpha\beta\lambda} \delta_{lk} \left(d_\alpha A^l{}_\beta - d_\beta A^l{}_\alpha + \epsilon^l{}_{ij} A^i{}_\alpha A^j{}_\beta \right) \delta A^k{}_\lambda + \epsilon^{\alpha\beta\lambda} \delta_{lk} F^l{}_{\alpha\beta} \delta A^k{}_\lambda \right) \, d\sigma = \\
=& \left(d_\alpha \left(2\delta_{ij} \epsilon^{\alpha\beta\lambda} \delta A^i_\beta A^j{}_\lambda \right) + 2\epsilon^{\alpha\beta\lambda} \delta_{lk} F^l{}_{\alpha\beta} \delta A^k{}_\lambda \right) \, d\sigma
\end{aligned}
\tag{1.10.145}
$$

The Euler–Lagrange part of the Lagrangian is

$$\mathbb{E}(\boldsymbol{L}_{CS}) = 2\delta_{lk} \epsilon^{\alpha\beta\lambda} F^l{}_{\alpha\beta} \omega^k{}_\lambda \wedge \, d\sigma \tag{1.10.146}$$

where we set $\omega^k_\lambda := dA^k_\lambda - d_\alpha A^k_\lambda dx^\alpha$ for the contact form. This is manifestly a global form. For the Liouville part, one should set

$$< \mathbb{F} \mid X > = 2\delta_{ij} \epsilon^{\alpha\beta\lambda} A^j{}_\lambda \delta A^i{}_\beta \, d\sigma_\alpha \tag{1.10.147}$$

which, however, is not a global form (since the $A^j{}_\lambda$ does not transform covariantly). It is defined on a trivialisation patch though different patches do not glue together. In other words, one still has a family of Liouville parts, one for each trivialisation patch, which differ by gauge transformations on the overlaps and fail to glue together.

> In this situation one cannot do much to discuss global conservation laws. The Noether current would be in fact local and global conserved quantities cannot be defined since integrals are defined only on forms.
>
> However, if one has another connection, we shall see that one can define a global Lagrangian depending on both connections, which reduces to $\boldsymbol{L}_{CS}$ if the second connection is set to zero, which however makes sense only locally in a trivialisation domain (in fact, when a gauge transformation acts on a zero connection, the transformed connection is not zero any longer).

▷*Noether currents for CS theories*

Hilbert–Einstein theory

Let us consider $\mathrm{Lor}(M)$ as a configuration bundle. Fibered coordinates are $(x^\mu, g_{\mu\nu})$ and sections are in one-to-one correspondence with global Lorentzian metrics on the spacetime M. One can use alternatively $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$ (as well as the densitised inverse $\hat{g}^{\mu\nu} := \sqrt{g}\, g^{\mu\nu}$) as coordinates along the standard fiber since they are in one-to-one correspondence. On the first jet, one can use the Christoffel symbols

$$\{g\}^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\epsilon} \left(-d_\epsilon g_{\mu\nu} + d_\mu g_{\nu\epsilon} + d_\nu g_{\epsilon\mu} \right) \tag{1.10.148}$$

as coordinates, since they are in one-to-one correspondence with first derivatives of the metric.

One can also define curvatures on the second jet bundle $J^2\mathrm{Lor}(M)$ ▷*Curvature tensors*

$$R^{\alpha}{}_{\beta\mu\nu} = d_{\mu}\{g\}^{\alpha}_{\beta\nu} + \{g\}^{\alpha}_{\epsilon\mu}\{g\}^{\epsilon}_{\beta\nu} - [\mu\nu] \qquad R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu} \qquad R = g^{\mu\nu}R_{\mu\nu} \tag{1.10.149}$$

Let us choose the second order Lagrangian

$$\boldsymbol{L}_H = \tfrac{\sqrt{g}}{2\kappa}\,(R - 2\Lambda)\ d\sigma \tag{1.10.150}$$

where R is scalar curvature defined by the metric and Λ a constant, called the *cosmological constant*. This Lagrangian is called the *Hilbert–Einstein Lagrangian* (sometimes setting $\Lambda = 0$) and its field theory is called the *(vacuum) standard (purely metric) GR*.

The variation of the Lagrangian reads as

$$\delta\boldsymbol{L}_H = \tfrac{\sqrt{g}}{2\kappa}\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\,d\sigma + d\left(\tfrac{\sqrt{g}}{2\kappa}g^{\mu\nu}\delta u^{\lambda}_{\mu\nu}\,d\sigma_{\lambda}\right) \tag{1.10.151}$$

where, setting $u^{\lambda}_{\mu\nu} := \{g\}^{\lambda}_{\mu\nu} - \delta^{\lambda}_{(\mu}\{g\}^{\epsilon}_{\nu)\epsilon}$, we used the identity $\delta R_{\mu\nu} = \nabla_{\lambda}\delta u^{\lambda}_{\mu\nu}$. Thus Euler–Lagrange part and Liouville part are

$$\mathbb{E}(L_H) = \tfrac{\sqrt{g}}{2\kappa}\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\omega^{\mu\nu}\wedge\ d\sigma \qquad \mathbb{F}(L_H) = \tfrac{\sqrt{g}}{2\kappa}g^{\mu\nu}\omega^{\lambda}_{\mu\nu}\wedge\ d\sigma_{\lambda} \tag{1.10.152}$$

where we define the contact forms

$$\omega^{\mu\nu} =\ dg^{\mu\nu} - d_{\alpha}g^{\mu\nu}\,dx^{\alpha} \qquad \omega^{\lambda}_{\mu\nu} =\ du^{\lambda}_{\mu\nu} - d_{\alpha}u^{\lambda}_{\mu\nu}\,dx^{\alpha} \tag{1.10.153}$$

The tensor $G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}$ is a symmetric tensor, called the *Einstein tensor* for the metric g. In dimension $m = 4$, it has 10 components, as the metric tensor. Field equations are in the form

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{1.10.154}$$

For $\Lambda = 0$ these are called (vacuum) *Einstein equations*. They are 10 non-linear PDE. For example, Minkowski metric, being flat, is a solution for them.

The Lagrangian is generally covariant, meaning that any spacetime diffeomorphism is a symmetry. In fact, the Lagrangian density $\sqrt{g}R$ is a scalar density of weight 1 and, for any diffeomorphism, one has

$$\bar{J}\sqrt{g'}R' = \sqrt{g}R \tag{1.10.155}$$

which directly proves .

One can also prove directly the covariance identity with respect to any spacetime vectorfield ξ.

$$\tfrac{\sqrt{g}}{2\kappa}\left((G_{\mu\nu} + \Lambda g_{\mu\nu})\,\pounds_{\xi}g^{\mu\nu} + g^{\mu\nu}\pounds_{\xi}R_{\mu\nu}\right) = \tfrac{\sqrt{g}}{2\kappa}\left(-2R_{\mu\nu}\nabla^{\mu}\xi^{\nu} + R\nabla_{\lambda}\xi^{\lambda} + \xi^{\lambda}\nabla_{\lambda}R + 2\nabla^{\mu}\xi^{\nu}R_{\mu\nu} - 2\Lambda\nabla_{\mu}\xi^{\mu}\right) = \tfrac{\sqrt{g}}{2\kappa}\nabla_{\lambda}\left((R - 2\Lambda)\,\xi^{\lambda}\right) = \overset{*}{\nabla}_{\lambda}\left(\tfrac{\sqrt{g}}{2\kappa}\,(R - 2\Lambda)\,\xi^{\lambda}\right) \tag{1.10.156}$$

where we used the Lie derivatives

$$\pounds_{\xi}g^{\mu\nu} = -\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu} \qquad \pounds_{\xi}R_{\mu\nu} = \xi^{\lambda}\nabla_{\lambda}R_{\mu\nu} + \nabla_{\mu}\xi^{\lambda}R_{\lambda\nu} + \nabla_{\nu}\xi^{\lambda}R_{\mu\lambda} \tag{1.10.157}$$

and all covariant derivatives are with respect to the metric g.

Being the Lagrangian generally covariant, if g is a solution of Einstein equations, then any $g' = \Phi^* g$ is a solution as well. This does not rely in the specific form of Einstein equations, it just depends on the fact that they are expressed through a tensor.

Any equation in the form $t_{\mu\nu\ldots} = 0$ for any tensor $t_{\mu\nu\ldots}(x, y)$ has the same property. In fact for any diffeomorphism $\Phi : x \mapsto x'$ which acts on fields by $y' = \Phi_* y$ one has

$$t'_{\mu\nu\ldots}(x', y') = \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \ldots t_{\rho\sigma\ldots}(x, y) \tag{1.10.158}$$

so that one has that

$$t(x, y) = 0 \quad \Longleftrightarrow \quad t'(x', y') = 0 \tag{1.10.159}$$

Accordingly, the section $y(x)$ is a solution iff $y'(x')$ is a solution as well.

Of course, a similar argument holds true for equations which are expressed by vanishing of tensor densities, while, for example, the equation $\Gamma^\alpha_{\beta\mu} = 0$ has not the same property since connections transform affinely.

Let us stress that invariance with respect to spacetime diffeomorphisms will have a special meaning to us, since it means solutions are invariant with respect to changes of laboratories. A physical law expressed by vanishing of tensors (or tensor densities) is independent of the observer or of the laboratory, i.e. *absolute*.

▷ *General covariance*

Noether currents are in the form

$$
\begin{aligned}
\mathcal{E}(L_H, \xi) &= \tfrac{\sqrt{g}}{2\kappa} \left(g^{\mu\nu} \pounds_\xi u^\lambda_{\mu\nu} - \xi^\lambda (R - 2\Lambda) \right) d\sigma_\lambda = \tfrac{\sqrt{g}}{2\kappa} \left(\left(g^{\mu\nu} \delta^\lambda_\epsilon - g^{\lambda\nu} \delta^\mu_\epsilon \right) \pounds_\xi \{g\}^\epsilon_{\mu\nu} - \xi^\lambda R + 2\Lambda \xi^\lambda \right) d\sigma_\lambda = \\
&= \tfrac{\sqrt{g}}{2\kappa} \left(\left(-\underline{g^{\mu\nu} g^{\lambda\sigma}} + \underline{g^{\lambda\nu} g^{\mu\sigma}} + {\color{red}g^{\nu\sigma} g^{\lambda\mu}} - \underline{g^{\lambda\sigma} g^{\nu\mu}} + \underline{g^{\sigma\mu} g^{\lambda\nu}} - {\color{red}g^{\lambda\mu} g^{\sigma\nu}} \right) \nabla_\sigma \nabla_\mu \xi_\nu - \xi^\lambda R + 2\Lambda \xi^\lambda \right) d\sigma_\lambda = \\
&= \tfrac{\sqrt{g}}{2\kappa} \left(\left(2 g^{\lambda\nu} g^{\mu\sigma} - 2 g^{\mu\nu} g^{\lambda\sigma} \right) \nabla_\sigma \nabla_\mu \xi_\nu - R \xi^\lambda + 2\Lambda \xi^\lambda \right) d\sigma_\lambda = \\
&= \tfrac{\sqrt{g}}{2\kappa} \left(\left(\delta^\lambda_\nu g^{\mu\sigma} - \delta^\mu_\nu g^{\lambda\sigma} \right) \nabla_{\sigma\mu} \xi^\nu + \tfrac{1}{2} \left(\boxed{g^{\lambda\nu} g^{\mu\sigma}} - g^{\mu\nu} g^{\lambda\sigma} \right) R_{\nu\epsilon\sigma\mu} \xi^\epsilon - R \xi^\lambda + 2\Lambda \xi^\lambda \right) d\sigma_\lambda = \\
&= \tfrac{\sqrt{g}}{2\kappa} \left(\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\rho} \delta^\sigma_\epsilon \right) \nabla_{\rho\sigma} \xi^\epsilon + \left(\tfrac{1}{2} R^\lambda_{\ \epsilon} - R \delta^\lambda_\epsilon + 2\Lambda \delta^\lambda_\epsilon \right) \xi^\epsilon \right) d\sigma_\lambda
\end{aligned}
\tag{1.10.160}
$$

where we used the identities

$$\pounds_\xi u^\lambda_{\mu\nu} = \pounds_\xi \{g\}^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \pounds_\xi \{g\}^\epsilon_{\nu)\epsilon} \qquad \pounds_\xi \{g\}^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\epsilon} \left(-\overset{g}{\nabla}_\epsilon \pounds_\xi g_{\mu\nu} + \overset{g}{\nabla}_\mu \pounds_\xi g_{\nu\epsilon} + \overset{g}{\nabla}_\nu \pounds_\xi g_{\epsilon\mu} \right) \tag{1.10.161}$$

Eventually, we can write Lie derivatives as (tensorial) linear combinations of the symmetry generators ξ^ϵ together with their (symmetrised covariant) derivatives up to some finite order k (in this case $k = 2$).

For any fixed generator ξ^ϵ and any solution of field equations g, one obtains a $(m-1)$-form $\mathcal{E} = (j^2 g)^* \mathcal{E}(L_H, \xi)$ on spacetime which turns out to be closed as a consequence of Noether theorem. The Noether current can be written as

$$\mathcal{E}(L_H, \xi) = \tfrac{\sqrt{g}}{2\kappa} \left(\mathcal{E}^\lambda_\epsilon \xi^\epsilon + \mathcal{E}^{\lambda\rho}_\epsilon \nabla_\rho \xi^\epsilon + \mathcal{E}^{\lambda\rho\sigma}_\epsilon \nabla_{\rho\sigma} \xi^\epsilon \right) d\sigma_\lambda \tag{1.10.162}$$

where we set

$$\mathcal{E}^\lambda_\epsilon := \tfrac{3}{2} R^\lambda_{\ \epsilon} - R \delta^\lambda_\epsilon + 2\Lambda \delta^\lambda_\epsilon \qquad\qquad \mathcal{E}^{\lambda\rho}_\epsilon := 0 \qquad\qquad \mathcal{E}^{\lambda\rho\sigma}_\epsilon := \delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda(\rho} \delta^{\sigma)}_\epsilon \tag{1.10.163}$$

These are called the *canonical tensors* of the theory. In the Noether current can take the second order term and integrate it by parts as

$$
\begin{aligned}
\tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\sigma}\delta^\rho_\epsilon\right)\nabla_{\rho\sigma}\xi^\epsilon &= \tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\sigma}\delta^\rho_\epsilon\right)\left(\nabla_\rho\nabla_\sigma\xi^\epsilon - \tfrac{1}{2}R^\epsilon_{\ \alpha\rho\sigma}\xi^\alpha\right) = \\
&= \tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\sigma}\delta^\rho_\epsilon\right)\nabla_\rho\nabla_\sigma\xi^\epsilon - \tfrac{1}{2}\tfrac{\sqrt{g}}{2\kappa}\left(\boxed{\delta^\lambda_\epsilon g^{\rho\sigma}} - g^{\lambda\sigma}\delta^\rho_\epsilon\right)R^\epsilon_{\ \alpha\rho\sigma}\xi^\alpha = \\
&= \nabla_\rho\left(\tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\sigma}\delta^\rho_\epsilon\right)\nabla_\sigma\xi^\epsilon\right) + \tfrac{1}{2}\tfrac{\sqrt{g}}{2\kappa}R^\lambda_{\ \epsilon}\xi^\epsilon = \nabla_\rho\left(2\tfrac{\sqrt{g}}{2\kappa}\nabla^{[\rho}\xi^{\lambda]}\right) + \tfrac{1}{2}\tfrac{\sqrt{g}}{2\kappa}R^\lambda_{\ \epsilon}\xi^\epsilon
\end{aligned}
\tag{1.10.164}
$$

The Noether current is then

$$
\mathcal{E}(L_H,\xi) = \left(\nabla_\rho\left(2\tfrac{\sqrt{g}}{2\kappa}\nabla^{[\rho}\xi^{\lambda]}\right) + 2\tfrac{\sqrt{g}}{2\kappa}\left(R^\lambda_{\ \epsilon} - \tfrac{1}{2}R\delta^\lambda_\epsilon + \Lambda\delta^\lambda_\epsilon\right)\xi^\epsilon\right)\,d\sigma_\lambda
\tag{1.10.165}
$$

Then we can define the *reduced current* as

$$
\tilde{\mathcal{E}}(L_H,\xi) = 2\tfrac{\sqrt{g}}{2\kappa}\left(R^\lambda_{\ \epsilon} - \tfrac{1}{2}R\delta^\lambda_\epsilon + \Lambda\delta^\lambda_\epsilon\right)\xi^\epsilon\,d\sigma_\lambda
\tag{1.10.166}
$$

which gracefully vanishes along solutions, and the *superpotential*

$$
\mathcal{U}(L_H,\xi) = \tfrac{\sqrt{g}}{2\kappa}\nabla^{[\rho}\xi^{\lambda]}\,d\sigma_{\lambda\rho} \qquad \Rightarrow d\mathcal{U}(L_H,\xi) = \nabla_\alpha\left(\tfrac{\sqrt{g}}{2\kappa}\nabla^{[\rho}\xi^{\lambda]}\right)\,dx^\alpha \wedge\,d\sigma_{\lambda\rho} = \nabla_\rho\left(2\tfrac{\sqrt{g}}{2\kappa}\nabla^{[\rho}\xi^{\lambda]}\right)\,d\sigma_\lambda
\tag{1.10.167}
$$

Please notice how the integration by parts is really critical here. If one just started from $\tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\rho}\delta^\sigma_\epsilon\right)\nabla_{\rho\sigma}\xi^\epsilon$ one would have got

$$
\tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\rho}\delta^\sigma_\epsilon\right)\nabla_{\rho\sigma}\xi^\epsilon = \nabla_\rho\left(\tfrac{\sqrt{g}}{2\kappa}\left(\delta^\lambda_\epsilon g^{\rho\sigma} - g^{\lambda\rho}\delta^\sigma_\epsilon\right)\nabla_\sigma\xi^\epsilon\right) - \tfrac{1}{2}\tfrac{\sqrt{g}}{2\kappa}R^\lambda_{\ \epsilon}\xi^\epsilon = \nabla_\rho\left(\tfrac{\sqrt{g}}{2\kappa}\nabla^\rho\xi^\lambda - g^{\lambda\rho}\nabla_\epsilon\xi^\epsilon\right) - \tfrac{1}{2}\tfrac{\sqrt{g}}{2\kappa}R^\lambda_{\ \epsilon}\xi^\epsilon
\tag{1.10.168}
$$

which is also true, though useless to our purposes (the volume term would not vanish on shell and the boundary term is not antisymmetric).

Then any Noether current in Hilbert theory allows a superpotential. This sounds now as an odd coincidence. We shall see below that something similar can be done in any generally covariant theory. In any general covariant theory, one has a suitable covariant integration by part which allows us to define superpotentials.

This is particularly strange at this stage under the following viewpoint. One can regard Noether theorem as a way to produce closed forms on the spacetime; in fact, given a solution σ and a spacetime vector field ξ, one can define a closed form $\mathcal{E}(L_H,\xi,\sigma) = (j^2\sigma)^*\mathcal{E}(L_H,\xi)$. Closed forms are also locally exact, but when superpotentials exist, they are *globally* exact.

If spacetime is contractible (or, more generally, if $H^{m-1}_{dR}(M) = 0$) then all closed $(m-1)$-forms on M are also exact. However, we shall see that in gravitational theories spacetime is not expected to be globally diffeomorphic to $\mathbb{R}^m$ and its cohomology group can be as non-trivial as we wish. In general then on reasonable spacetimes there are plenty of closed forms which are not exact.

In view of this fact, one could expect that, in general, not all Noether currents admitted a superpotential.

We shall show that this is not the case. No matter how complicated the cohomology of spacetime is, in any relativistic theory the Noether theorem will always single out infinitely many exact forms as Noether currents.

We shall prove it in general in Chapter 3.

Purely metric $f(R)$-theories

Let us consider a spacetime M (connected, paracompact, which allows global Lorentzian metrics), a Lorentzian metric field $g_{\mu\nu}$ and a matter field v^i on it. The metric field is a section of the bundle $\mathrm{Lor}(M)$ while the vector field v^i is a section of some natural vector bundle E defined on M, e.g. the bundle for some tensor density. The configuration bundle $C = \mathrm{Lor}(M) \times_M E$ has fibered coordinates $(x^\mu, g^{\mu\nu}, v^i)$. Let us consider the Lagrangian

$$\boldsymbol{L} = \left[\frac{\sqrt{g}}{2\kappa} f(R) + L_m(g_{\mu\nu}, v^i, v^i_\mu) \right] d\sigma \tag{1.10.169}$$

which is a horizontal form on $J^2\mathrm{Lor}(M) \times_M J^1 E$, we set $v^i_\mu := d_\mu v^i + Z^{i\beta}_\alpha(v)\{g\}^\alpha_{\beta\mu}$ for the covariant derivative of v^i. One can specify $Z^{i\beta}_\alpha(v)$ to represent any specific tensor density. The function $f(R)$ is a *regular function* of the scalar curvature R of the metric g. This is called a *purely metric $f(R)$-theory*. Field equations are easily obtained by varying the Lagrangian with respect to the fundamental fields $(g^{\mu\nu}, v^i)$.

$$\begin{cases} f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} + \nabla_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R) \\ E_i = 0 \end{cases} \tag{1.10.170}$$

where we set

$$\begin{aligned} T_{\mu\nu} &:= {}^m H_{\alpha\beta} + \nabla_\epsilon \, {}^m H^\epsilon_{\alpha\beta} & H^\rho_{\alpha\beta} &:= {}^g H^\rho_{\alpha\beta} + {}^m H^\rho_{\alpha\beta} \\ {}^g H_{\alpha\beta} &:= \nabla_\epsilon \, {}^g H^\epsilon_{\alpha\beta} = -\left(\nabla_{\alpha\beta}f'(R) - g_{\alpha\beta}\Box f'(R)\right) & {}^g H^{\rho\epsilon}_{\alpha\beta} &:= \left(g_{\alpha\beta}g^{\rho\epsilon} - \delta^\rho_{(\alpha}\delta^\epsilon_{\beta)}\right) f'(R) \\ {}^g H^\epsilon_{\alpha\beta} &:= \nabla_\rho \, {}^g H^{\epsilon\rho}_{\alpha\beta} = \left(g_{\alpha\beta}g^{\rho\epsilon} - \delta^\rho_{(\alpha}\delta^\epsilon_{\beta)}\right) \nabla_\rho f'(R) & \sqrt{g}E_i &:= p_i - \nabla_\mu p^\mu_i \\ \tfrac{1}{2}\sqrt{g}\, {}^m H^\epsilon_{\beta\mu} &:= -\tfrac{1}{2}p^\rho_i Z^{i\sigma}_\alpha \left(g_{\mu\sigma}\delta^\epsilon_\rho\delta^\alpha_\beta + g_{\beta\rho}\delta^\epsilon_\sigma\delta^\alpha_\mu - g_{\mu\rho}g_{\beta\sigma}g^{\alpha\epsilon}\right) & -\tfrac{1}{2}\sqrt{g}\, {}^m H_{\alpha\beta} &:= \frac{\partial L_m}{\partial g^{\alpha\beta}} \end{aligned} \tag{1.10.171}$$

For the variation one has

$$\begin{aligned} \delta L ={} & \frac{\sqrt{g}}{2\kappa}\left(f'(R)R_{\alpha\beta} - \tfrac{1}{2}f(R)g_{\alpha\beta} - \kappa\, {}^m H_{\alpha\beta}\right)\delta g^{\alpha\beta} + \left(-\tfrac{1}{2}p^\mu_i Z^{i\beta}_\alpha g^{\alpha\epsilon} + \tfrac{1}{2}p^\epsilon_i Z^{i\mu}_\alpha g^{\alpha\beta} + \tfrac{1}{2}p^\beta_i Z^{i\epsilon}_\alpha g^{\alpha\mu}\right)\nabla_\epsilon \delta g_{\beta\mu} + (p_i - \nabla_\mu p^\mu_i)\delta v^i + \\ & + \frac{\sqrt{g}}{2\kappa}f'(R)g^{\beta\mu}\nabla_\alpha\left(\delta\{g\}^\alpha_{\beta\mu} - \delta^\alpha_\mu\delta^\sigma_\rho\delta\{g\}^\rho_{\sigma\beta}\right) + \nabla_\lambda\left(p^\lambda_i\delta v^i\right) = \\ ={} & \frac{\sqrt{g}}{2\kappa}\left(f'(R)R_{\alpha\beta} - \tfrac{1}{2}f(R)g_{\alpha\beta} - \kappa\, {}^m H_{\alpha\beta}\right)\delta g^{\alpha\beta} + \frac{\sqrt{g}}{2}\, {}^m H^\epsilon_{\alpha\beta}\nabla_\epsilon\delta g^{\alpha\beta} + \sqrt{g}E_i\delta v^i + \\ & + \tfrac{1}{2}\frac{\sqrt{g}}{2\kappa}f'(R)\left(-\underline{g^{\beta\sigma}g^{\alpha\epsilon}} + \underline{g^{\beta\alpha}g^{\sigma\epsilon}} + \underline{g^{\epsilon\beta}g^{\alpha\sigma}} - \underline{g^{\epsilon\alpha}g^{\beta\sigma}} + \underline{g^{\sigma\epsilon}g^{\alpha\beta}} - \underline{g^{\sigma\alpha}g^{\epsilon\beta}}\right)\nabla_\alpha\nabla_\epsilon\delta g_{\sigma\beta} + \nabla_\lambda\left(p^\lambda_i\delta v^i\right) = \\ ={} & \frac{\sqrt{g}}{2\kappa}\left(f'(R)R_{\alpha\beta} - \tfrac{1}{2}f(R)g_{\alpha\beta} - \kappa\, {}^m H_{\alpha\beta}\right)\delta g^{\alpha\beta} + \frac{\sqrt{g}}{2}\, {}^m H^\epsilon_{\alpha\beta}\nabla_\epsilon\delta g^{\alpha\beta} + \sqrt{g}E_i\delta v^i + \frac{\sqrt{g}}{2\kappa}\, {}^g H^{\rho\epsilon}_{\alpha\beta}\nabla_\rho\nabla_\epsilon\delta g^{\alpha\beta} + \nabla_\lambda\left(p^\lambda_i\delta v^i\right) = \\ ={} & \frac{\sqrt{g}}{2\kappa}\left(f'(R)R_{\alpha\beta} - \tfrac{1}{2}f(R)g_{\alpha\beta} - \kappa\left({}^m H_{\alpha\beta} + \nabla_\epsilon\, {}^m H^\epsilon_{\alpha\beta}\right) + {}^g H_{\alpha\beta}\right)\delta g^{\alpha\beta} + \sqrt{g}E_i\delta v^i + \nabla_\lambda\left(p^\lambda_i\delta v^i + \frac{\sqrt{g}}{2}\, {}^m H^\lambda_{\alpha\beta}\delta g^{\alpha\beta} + \frac{\sqrt{g}}{2\kappa}\, {}^g H^{\lambda\epsilon}_{\alpha\beta}\nabla_\epsilon\delta g^{\alpha\beta} - \frac{\sqrt{g}}{2\kappa}\, {}^g H^\lambda_{\alpha\beta}\delta g^{\alpha\beta}\right) \end{aligned} \tag{1.10.172}$$

We can trace the first field equation by $g^{\mu\nu}$ to obtain the so called *master equation*

$$f'(R)R - \tfrac{m}{2}f(R) = \kappa T - (m-1)\Box f'(R) \qquad \Rightarrow f(R) = \tfrac{2}{m}\left(f'(R)R - \kappa T + (m-1)\Box f'(R)\right) \tag{1.10.173}$$

where we set $T := g^{\mu\nu}T_{\mu\nu}$ for the trace of the energy-momentum stress tensor.

The first field equation can also be put in a form

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \kappa \left(f'(R)\right)^{-1} \left(T_{\mu\nu} + \tfrac{1}{\kappa}\left[\nabla_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R) - \tfrac{1}{2}\left(f'(R)R - f(R)\right)g_{\mu\nu}\right]\right) =: \kappa\hat{T}_{\mu\nu} \tag{1.10.174}$$

which seems an Einstein equation though with a modified energy-momentum stress tensor

$$\hat{T}_{\mu\nu} := \frac{1}{f'(R)}\left(T_{\mu\nu} - \tfrac{1}{m}T + \tfrac{1}{\kappa}\left[\nabla_{\mu\nu}f'(R) - \left(\tfrac{m-2}{2m}f'(R)R + \tfrac{1}{m}\Box f'(R)\right)g_{\mu\nu}\right]\right) \tag{1.10.175}$$

Brans–Dicke theories with a potential

Let us consider a metric g, a scalar field φ and a generic set of matter fields v^i. A *purely metric Brans–Dicke theory* (in dimension m) is described by a Lagrangian of the form

$$\boldsymbol{L}_{BD} = \left[\frac{\sqrt{g}}{2\kappa}\left(\varphi^\alpha R - \frac{\omega}{\varphi^\beta}\nabla_\mu\varphi\nabla^\mu\varphi + U(\varphi)\right) + \mathcal{L}_m(g_{\mu\nu}, v^i, v^i_\mu)\right] d\sigma \tag{1.10.176}$$

where R is, of course, the scalar curvature of g, α, β, ω are constants and $U(\varphi)$ a potential describing self interactions of the scalar field φ. Let us stress that the matter fields v couple to g, not to φ.

By variation of the Lagrangian, we obtain

$$
\begin{aligned}
\delta L_{BD} =& \tfrac{\sqrt{g}}{2\kappa}\left(\varphi^\alpha\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right) - \tfrac{\omega}{\varphi^\beta}\nabla_\mu\varphi\nabla_\nu\varphi - \kappa T_{\mu\nu} + \tfrac{1}{2}\left(\tfrac{\omega}{\varphi^\beta}\nabla_\epsilon\varphi\nabla^\epsilon\varphi - U(\varphi)\right)g_{\mu\nu}\right)\delta g^{\mu\nu} - \tfrac{\sqrt{g}}{2\kappa}\alpha\varphi^{\alpha-1}\nabla_\lambda\varphi\left(g^{\sigma\nu}\delta^\lambda_\rho - g^{\lambda\nu}\delta^\sigma_\rho\right)\delta\{g\}^\rho_{\sigma\nu} + \\
&+ \tfrac{\sqrt{g}}{2\kappa}\left(\alpha R\varphi^{\alpha-1} + \beta\tfrac{\omega}{\varphi^{\beta+1}}\nabla_\mu\varphi\nabla^\mu\varphi + U'(\varphi)\right)\delta\varphi - 2\tfrac{\sqrt{g}}{2\kappa}\tfrac{\omega}{\varphi^\beta}\nabla^\mu\varphi\nabla_\mu\delta\varphi + (p_i - \nabla_\mu p_i^\mu)\delta v^i + \nabla_\lambda\left(p_i^\lambda\delta v^i + \tfrac{\sqrt{g}}{2\kappa}\varphi^\alpha g^{\mu\nu}\delta u^\lambda_{\mu\nu}\right) = \\
=& \tfrac{\sqrt{g}}{2\kappa}\left(\varphi^\alpha G_{\mu\nu} - \tfrac{\omega}{\varphi^\beta}\nabla_\mu\varphi\nabla_\nu\varphi - \kappa T_{\mu\nu} + \tfrac{1}{2}\left(\tfrac{\omega}{\varphi^\beta}\nabla_\epsilon\varphi\nabla^\epsilon\varphi - U(\varphi)\right)g_{\mu\nu}\right)\delta g^{\mu\nu} - \tfrac{\sqrt{g}}{2\kappa}\tfrac{\alpha}{2}\varphi^{\alpha-1}\nabla_\lambda\varphi\left(\underline{-g^{\sigma\nu}g^{\lambda\epsilon}} + \underline{g^{\lambda\nu}g^{\sigma\epsilon}} + \underline{g^{\epsilon\sigma}g^{\lambda\nu}} - \textcolor{red}{g^{\lambda\sigma}g^{\epsilon\nu}} + \textcolor{red}{g^{\nu\epsilon}g^{\lambda\sigma}} - \underline{g^{\lambda\epsilon}g^{\nu\sigma}}\right)\nabla_\epsilon\delta g_{\sigma\nu} + \\
&+ \tfrac{\sqrt{g}}{2\kappa}\left(\alpha R\varphi^{\alpha-1} + \underline{\underline{\beta\tfrac{\omega}{\varphi^{\beta+1}}\nabla_\mu\varphi\nabla^\mu\varphi}} + U'(\varphi) - \underline{\underline{2\beta\tfrac{\omega}{\varphi^{\beta+1}}\nabla_\mu\varphi\nabla^\mu\varphi}} + 2\tfrac{\omega}{\varphi^\beta}\Box\varphi\right)\delta\varphi + E_i\delta v^i + \nabla_\lambda\left(p_i^\lambda\delta v^i + \tfrac{\sqrt{g}}{2\kappa}\varphi^\alpha g^{\mu\nu}\delta u^\lambda_{\mu\nu} - \tfrac{\sqrt{g}}{\kappa}\tfrac{\omega}{\varphi^\beta}\nabla^\lambda\varphi\delta\varphi\right) = \\
=& \tfrac{\sqrt{g}}{2\kappa}\left(\varphi^\alpha G_{\mu\nu} - \tfrac{\omega}{\varphi^\beta}\nabla_\mu\varphi\nabla_\nu\varphi - \kappa T_{\mu\nu} + \tfrac{1}{2}\left(\tfrac{\omega}{\varphi^\beta}\nabla_\epsilon\varphi\nabla^\epsilon\varphi - U(\varphi)\right)g_{\mu\nu}\right)\delta g^{\mu\nu} - \tfrac{\sqrt{g}}{2\kappa}\alpha\varphi^{\alpha-1}\nabla_\lambda\varphi\left(g^{\lambda\epsilon}g_{\sigma\nu} - \delta^\lambda_\nu\delta^\epsilon_\sigma\right)\nabla_\epsilon\delta g^{\sigma\nu} + \\
&+ \tfrac{\sqrt{g}}{2\kappa}\left(\alpha R\varphi^{\alpha-1} + U'(\varphi) - \beta\tfrac{\omega}{\varphi^{\beta+1}}\nabla_\mu\varphi\nabla^\mu\varphi + 2\tfrac{\omega}{\varphi^\beta}\Box\varphi\right)\delta\varphi + E_i\delta v^i + \nabla_\lambda\left(p_i^\lambda\delta v^i + \tfrac{\sqrt{g}}{2\kappa}\varphi^\alpha g^{\mu\nu}\delta u^\lambda_{\mu\nu} - \tfrac{\sqrt{g}}{\kappa}\tfrac{\omega}{\varphi^\beta}\nabla^\lambda\varphi\delta\varphi\right) = \\
=& \tfrac{\sqrt{g}}{2\kappa}\left(\varphi^\alpha G_{\mu\nu} - \kappa T_{\mu\nu} - \alpha\varphi^{\alpha-1}\left(\nabla_{\mu\nu}\varphi - \Box\varphi g_{\mu\nu}\right) - \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{\omega}{\varphi^\beta}\right)\nabla_\mu\varphi\nabla_\nu\varphi + \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{1}{2}\tfrac{\omega}{\varphi^\beta}\right)\nabla_\epsilon\varphi\nabla^\epsilon\varphi g_{\mu\nu} - \tfrac{1}{2}U(\varphi)g_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&+ \tfrac{\sqrt{g}}{2\kappa\varphi}\left(\alpha R\varphi^\alpha + \varphi U'(\varphi) + \tfrac{2\omega}{\varphi^\beta}\left(\varphi\Box\varphi - \tfrac{\beta}{2}\nabla_\mu\varphi\nabla^\mu\varphi\right)\right)\delta\varphi + E_i\delta v^i + \\
&+ \nabla_\lambda\left(p_i^\lambda\delta v^i + \tfrac{\sqrt{g}}{2\kappa}\varphi^\alpha g^{\mu\nu}\delta u^\lambda_{\mu\nu} - \tfrac{\sqrt{g}}{\kappa}\tfrac{\omega}{\varphi^\beta}\nabla^\lambda\varphi\delta\varphi - \tfrac{\sqrt{g}}{2\kappa}\alpha\varphi^{\alpha-1}\nabla_\epsilon\varphi\left(g^{\lambda\epsilon}g_{\mu\nu} - \delta^\lambda_\nu\delta^\epsilon_\mu\right)\delta g^{\mu\nu}\right)
\end{aligned}
\tag{1.10.177}
$$

where we set $E_i := p_i - \nabla_\mu p_i^\mu$ for the matter field equations, $G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}$ for the Einstein tensor of g.

Then equations for the Brans–Dicke Lagrangian are

$$\begin{cases} \varphi^\alpha G_{\alpha\beta} = \kappa T_{\alpha\beta} + \alpha\varphi^{\alpha-1}\left(\nabla_{\alpha\beta}\varphi - \Box\varphi g_{\alpha\beta}\right) + \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \frac{\omega}{\varphi^\beta}\right)\nabla_\alpha\varphi\nabla_\beta\varphi - \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \frac12\frac{\omega}{\varphi^\beta}\right)\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} + \frac12 U g_{\alpha\beta} \\[2mm] \alpha\varphi^\alpha R + \varphi U' + \frac{2\omega}{\varphi^\beta}\left(\varphi\Box\varphi - \frac{\beta}{2}\nabla_\lambda\varphi\nabla^\lambda\varphi\right) = 0 \\[2mm] E_i = 0 \end{cases} \tag{1.10.178}$$

By tracing the first equation by $g^{\alpha\beta}$, one obtains

$$\tfrac12\varphi^\alpha R = \tfrac{1}{2-m}\left(\kappa T - \alpha\varphi^{\alpha-1}(m-1)\Box\varphi - \left((m-1)\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{m-2}{2}\tfrac{\omega}{\varphi^\beta}\right)\nabla_\lambda\varphi\nabla^\lambda\varphi + \tfrac{m}{2}U\right) \tag{1.10.179}$$

which can be used to eliminate the Ricci scalar from the first equation

$$\begin{aligned} \varphi^\alpha R_{\alpha\beta} =&\kappa\left(T_{\alpha\beta} - \tfrac{1}{m-2}T g_{\alpha\beta}\right) + \alpha\varphi^{\alpha-1}\nabla_{\alpha\beta}\varphi + \tfrac{\alpha}{m-2}\varphi^{\alpha-1}\Box\varphi g_{\alpha\beta} + \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{\omega}{\varphi^\beta}\right)\nabla_\alpha\varphi\nabla_\beta\varphi + \\ &+ \tfrac{\alpha(\alpha-1)}{m-2}\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} - \tfrac{1}{m-2}U g_{\alpha\beta} \end{aligned} \tag{1.10.180}$$

Then we can eliminate the Ricci curvature from the second field equation and (1.10.179)

$$\alpha\varphi^{\alpha-1}(m-1)\Box\varphi + \left((m-1)\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{m-2}{2}\tfrac{(\alpha-\beta)\omega}{\alpha}\tfrac{1}{\varphi^\beta}\right)\nabla_\lambda\varphi\nabla^\lambda\varphi + \tfrac{m-2}{\alpha}\tfrac{\omega}{\varphi^\beta}\varphi\Box\varphi - \tfrac{m}{2}U + \tfrac{m-2}{2\alpha}\varphi U' = \kappa T \tag{1.10.181}$$

Thus field equations can be recast as

$$\begin{cases} \varphi^\alpha R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \tfrac{1}{m-2}T g_{\alpha\beta}\right) + \alpha\varphi^{\alpha-1}\nabla_{\alpha\beta}\varphi + \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{\omega}{\varphi^\beta}\right)\nabla_\alpha\varphi\nabla_\beta\varphi + \tfrac{1}{m-2}\left(\alpha\varphi^{\alpha-1}\Box\varphi + \alpha(\alpha-1)\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi - U\right)g_{\alpha\beta} \\[2mm] \left(\alpha\varphi^{\alpha-1}(m-1) + \tfrac{m-2}{\alpha}\tfrac{\omega}{\varphi^\beta}\varphi\right)\Box\varphi + \left((m-1)\alpha(\alpha-1)\varphi^{\alpha-2} + \tfrac{m-2}{2\alpha}\tfrac{(\alpha-\beta)\omega}{\varphi^\beta}\right)\nabla_\lambda\varphi\nabla^\lambda\varphi + \left(\tfrac{m-2}{2\alpha}\varphi U' - \tfrac{m}{2}U\right) = \kappa T \\[2mm] E_i = 0 \end{cases} \tag{1.10.182}$$

11. Lagrangian mechanics

Mechanics is formally a special field theory in which the base manifold M is reduced to $M = \mathbb{R}$. The "fields" are identified with Lagrangian coordinates q^i and one wants to determine them as functions of the coordinate t in M. Solutions will be locally expressed by $q^i(t)$. Since the base manifold $\mathbb{R}$ is contractible then any bundle over it is trivial, thus one can choose $C = \mathbb{R} \times Q$ with no loss of generality.

A general vertical map on C is in the form

$$t' = t \qquad\qquad q' = q'(t, q) \tag{1.11.1}$$

and it does not preserve the product $C = \mathbb{R} \times Q$. An example is provided by Galilei boosts for a free particle. Thus different (Galilean inertial) observers see the same configuration bundle, though different products. They see the same synchronous surfaces $t = const$ though they define different lines $q = const$. The congruence of these lines defines what an observer regards as rest motions, which is, of course, observer dependent.

Once again the bundle framework is *absolute* while the coordinate (or product) structures are *relative* to the observer.

The first jet bundle J^1C is modelled on the vector bundle $\mathbb{R} \times TQ$ (as a bundle on C).

For any first order Lagrangian $\boldsymbol{L} = L(t, q, u)\, dt$, one has standard Euler–Lagrange equations

$$\mathbb{E}(\boldsymbol{L}) = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial u^i} \right) \right) (\, dq^i - u^i\, dt) \wedge\, dt = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial u^i} \right) \right)\, dq^i \wedge\, dt \tag{1.11.2}$$

If one restricts to vertical symmetries $X = X^i \partial_i$ on C, so that $\pounds_X q^i = -X^i$, the covariance identity reads as

$$\frac{\partial L}{\partial q^i} \pounds_X q^i + \frac{\partial L}{\partial u^i} \frac{d}{dt} \pounds_X q^i = 0 \qquad \Longleftrightarrow \qquad \hat{X}(L) = 0 \tag{1.11.3}$$

where $\hat{X}$ denotes the prologation to the tangent bundle $\mathbb{R} \times TQ$ of the symmetry field.

A Noether current is a 0-form, i.e. a function. Being conserved, i.e. closed, for a 0-form means to be constant along solutions of equations of motion. Thus a Noether current is a *first integral of the system*. Its expression for a vertical symmetry is the standard

$$F := \frac{\partial L}{\partial u^i} \pounds_X q^i = -\pi_i X^i(t, q) \tag{1.11.4}$$

which is of course linear in the momenta π_i.

If one considers the vector field $\xi = \partial_t$ over C (which is, in fact, global being $C = \mathbb{R} \times Q$). The Lie derivatives of Lagrangian coordinates are

$$\pounds_\xi q^i = u^i \tag{1.11.5}$$

The covariance identity for that is

$$\frac{\partial L}{\partial q^i} \pounds_\xi q^i + \frac{\partial L}{\partial u^i} \frac{d}{dt} \pounds_\xi q^i = \frac{d}{dt}(L) \qquad \Rightarrow \frac{\partial L}{\partial q^i} u^i + \frac{\partial L}{\partial u^i} \dot{u}^i = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^i} u^i + \frac{\partial L}{\partial u^i} \dot{u}^i \qquad \Rightarrow \frac{\partial L}{\partial t} = 0 \tag{1.11.6}$$

Accordingly, the vector field $\xi = \partial_t$ is a Lagrangian symmetry only for time-independent Lagrangians. The corresponding Noether first integral

$$\mathcal{H} = \frac{\partial L}{\partial u^i} \pounds_\xi q^i - L = \pi_i u^i - L \tag{1.11.7}$$

which is called the *total energy*.

Of course, in mechanics, superpotentials do not make much sense. Noether currents are in fact 0-forms and the superpotentials would be of degree -1.

Oddly enough, this could be given a meaning in the so-called *augmented de Rham sequence*

$$0 \to \mathbb{R} \hookrightarrow \Omega^0(M) \to \Omega^1(M) \to \ldots \to \Omega^m(M) \to 0 \tag{1.11.8}$$

where one adds constants in the first position of the standard de Rham sequence (constants can be identified with constant functions in $\Omega^0(M)$). Then superpotentials live in $\mathbb{R}$ and state that first integrals are *constant* along solutions.

However, this is not particularly useful (and in fact one does not usually spend time to do it) because the augmented de Rham sequence is always, by construction, exact at the level of 0-forms, at least if M is connected. If the manifold M is connected, a 0-form is closed if and only if it is constant. Thus any Noether current in $\Omega^0(M)$ allows a superpotential which selects its value along solutions.

Homogeneous formalism for mechanics

In the framework for mechanics above, *time* has a special role over positions. Although C encompasses time and position together as spacetime, then one assumes *absolute time* which corresponds to the fibration $\pi : C \to \mathbb{R}$. Moreover, time also plays a special role being the parameter along which motions are described, i.e. it is the independent variable of the formalism.

Both these aspects can be overcome in general showing a homogeneous formulation for classical (non-relativistic) mechanics which relies on a bigger space $C' = \mathbb{R} \times C$ with coordinates (s, t, x) in which (t, x) enter on equal footing.

The core idea is introducing an extra parameter s to describe trajectories on C. The parameter s is somehow unphysical since the parameterisation of trajectories of C does not contain information about the physical motion. Accordingly, equations of motion must select a curve together with all its reparameterisations, so that it, in fact, singles out *trajectories* which are associated to physical motions. The information about velocities and accelerations is encoded into the slope and curvature of trajectories in C, not in tangent vectors to C which instead depend on the parameterisation.

Of course, by doing that, we introduce an enormous redundancy in the description of the mechanical systems. Above the history of a system was encoded in a parameterised curve $x(t)$. One model for one system; unfortunately different observers give different representations of the same system. Now the history of a system is described by infinitely many parameterised curves $(t(s), x(s))$, namely all reparameterisations of the same trajectory.

It is precisely by this redundancy that the description becomes absolute. This is not the only case in which one trades redundancy for an absolute description. This principle could be taken as a core idea of relativity and gauge theories and we shall see many examples in detail.

Thus we are trying a description of mechanical systems on the bundle $[C' \to \mathbb{R}]$ which is necessarily trivial so that one can take $C' = \mathbb{R} \times C$ with coordinates (s, t, x^i). In order to specify a dynamics for such a model we have to choose a Lagrangian on $J^1C' \simeq \mathbb{R} \times TC \simeq \mathbb{R} \times T(\mathbb{R} \times Q)$ with coordinates $(s, x^0 = t, x^i, u^0, u^i)$.

The tangent vectors (u^0, u^i) in $T(\mathbb{R} \times Q)$ have not to be confused with velocities $\dot{x}^i$ of motion, which are in fact related to

$$\dot{x}^i = \frac{u^i}{u^0} \tag{1.11.9}$$

being it the slope of the trajectory in $\mathbb{R} \times Q$. Tangent vectors $u = u^\mu \partial_\mu$ have no physical meaning while velocities, of course, have.

It turns out that one can construct a Lagrangian for each system with a classical dynamics. Let us consider for simplicity a standard holonomic system described by a Lagrangian on TQ of the form

$$\boldsymbol{L} = \left(\tfrac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j - U(x) \right) \, dt \tag{1.11.10}$$

To this Lagrangian we can associate a Lagrangian on TC in the form

$$\boldsymbol{L}_h = \left(\tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - U(x)\right) dt = \left(\tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - U(x)\right) u^0 \, ds \qquad (1.11.11)$$

Before discussing equations of motion, let us first notice that the Lagrangian $\boldsymbol{L}_h$ is covariant with respect to arbitrary reparameterisations.

If we consider a section $\gamma : s \mapsto (s, t(s), x(s))$ and a reparameterisation $\phi : \mathbb{R} \to \mathbb{R} : s' \mapsto s(s')$ (from which $ds = \phi' \, ds'$ follows) of it given by

$$\gamma' : s' \mapsto (s', t'(s'), x'(s')) \qquad \begin{cases} t'(s') := t(s(s')) \\ x'(s') := x(s(s')) \end{cases} \Rightarrow \begin{cases} u'^0 = \phi' u^0 \\ u'^i = \phi' u^i \end{cases} \qquad (1.11.12)$$

then the Lagrangian transforms under reparameterisations as

$$\boldsymbol{L}'_h = \left(\tfrac{1}{2}g_{ij}(x')\tfrac{u'^i}{u'^0}\tfrac{u'^j}{u'^0} - U(x')\right) u'^0 \, ds' = \left(\tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - U(x)\right) \phi' u^0 \, ds' = \left(\tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - U(x)\right) u^0 \, ds = \boldsymbol{L}_h \qquad (1.11.13)$$

i.e. arbitrary reparameterisations are Lagrangian symmetries.

The Lagrangian $\boldsymbol{L}_h$ is varied in the standard way with respect to (t, x^i).

The variation of the Lagrangian $\boldsymbol{L}_h$ reads as

$$\delta\boldsymbol{L}_h = \left(g_{ij}\tfrac{u^j}{u^0}\delta u^i + \tfrac{1}{2}\partial_k g_{ij}u^i\tfrac{u^j}{u^0}\delta x^k - u^0\partial_k U\delta x^k - \left(g_{ij}\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - \tfrac{1}{u^0}L_h\right)\delta u^0\right) ds = \left(-g_{kj}\left(\tfrac{d}{ds}\tfrac{u^j}{u^0} + \{g\}^j_{il}\tfrac{u^i}{u^0}u^l + u^0 g^{jl}\partial_l U\right)\delta x^k + \tfrac{d}{ds}\mathcal{H}\delta t + \tfrac{d}{ds}\left(g_{ij}\tfrac{u^j}{u^0}\delta x^i - \mathcal{H}\delta t\right)\right) ds \qquad (1.11.14)$$

where we set $\mathcal{H} := g_{ij}\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} - \tfrac{1}{u^0}L_h = \tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} + U(x)$ which in fact coincides with the classical total energy of the system (which is conserved iff the Lagrangian L does not explicitly depend on time t, i.e. if the metric g and the potential U do not).

Notice that as the standard velocity is $\dot{x}^i = \tfrac{u^i}{u^0}$, the standard acceleration is in the form

$$\ddot{x}^i = \tfrac{d\dot{x}^i}{dt} = \tfrac{1}{u^0}\tfrac{d}{ds}\left(\tfrac{u^i}{u^0}\right) = \tfrac{1}{(u^0)^2}\left(\tfrac{du^i}{ds} - \tfrac{u^i}{u^0}\tfrac{du^0}{ds}\right) \qquad (1.11.15)$$

Equations of motion for the Lagrangian L_h read as

$$\begin{cases} \tfrac{d}{ds}\left(\tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} + U(x)\right) = 0 & \Rightarrow \quad \tfrac{1}{2}g_{ij}(x)\tfrac{u^i}{u^0}\tfrac{u^j}{u^0} + U(x) = E \\ \ddot{x}^i + \{g\}^i_{jk}\tfrac{u^j}{u^0}\tfrac{u^k}{u^0} = -g^{ik}\partial_k U & \Rightarrow \quad \tfrac{du^i}{ds} + \{g\}^i_{jk}u^j u^k = -(u^0)^2 g^{ik}\partial_k U + u^i\tfrac{d}{ds}\ln u^0 \end{cases} \qquad (1.11.16)$$

The first equation is satisfied in view of conservation of energy (and it is, in fact, implied by the second). The second equation is equivalent to Newton equations for the original system (as it is manifest by the first form of the equation $\ddot{x}^i + \{g\}^i_{jk}\tfrac{u^j}{u^0}\tfrac{u^k}{u^0} = -g^{ik}\partial_k U$). The second form $\tfrac{du^i}{ds} + \{g\}^i_{jk}u^j u^k = -(u^0)^2 g^{ik}\partial_k U + u^i\tfrac{d}{ds}\ln u^0$ is the normal form of the same equation in terms of variables on C.

The Euler–Lagrange part of the Lagrangian $\boldsymbol{L}_h$ is

$$\mathbb{E}(L_h) = -g_{kj}\left(\tfrac{d}{ds}\left(\tfrac{u^j}{u^0}\right) + \{g\}^j_{il}\tfrac{u^i}{u^0}u^l + u^0 g^{jl}\partial_l U\right)\boldsymbol{\omega}^k \wedge ds + \tfrac{d}{ds}\left(\mathcal{H}\right)\boldsymbol{\omega} \wedge ds \qquad (1.11.17)$$

where we set $\boldsymbol{\omega}^k = dx^k - u^k\,ds$ and $\boldsymbol{\omega} = dt - u^0\,ds$ for the relevant contact forms. The Liouville part is

$$\mathbb{F}(L_h) = g_{ij}\frac{u^j}{u^0}\boldsymbol{\omega}^i - \mathcal{H}\,\boldsymbol{\omega} \tag{1.11.18}$$

The equivalence between standard Newtonian equations of motion and the homogeneous ones is worth to be discussed in detail. We already proved that a solution of Newtonian equations is a solution of homogeneous equations. Of course, solutions of Newtonian equations are curves in Q, while solutions of homogeneous equations are curves in $C = \mathbb{R} \times Q$. Thence by *equivalence* we mean that a solution $\gamma : t \mapsto x(t)$ of Newtonian equations induces a curve $\hat{\gamma} : s \mapsto (t = s, x(s))$ on C which is also a solution of homogeneous equations.

However the correspondence is not one-to-one. If $\hat{\gamma} : s \mapsto (t(s), x(s))$ is a solution of homogeneous equations, the curve $\gamma : s \mapsto x(s)$ is not necessarily a solution of Newtonian equations. What is true, is that there always exists a reparameterisation of γ (namely, the one obtained by using $t = t(s)$ as a parameter, i.e. $\hat{\gamma}' : s \mapsto x(s(t))$) which is a solution of Newtonian equations. In other words, to a single solution of Newtonian equations, there correspond infinitely many solutions of homogeneous equations. A parameterised curve γ on Q uniquely selects a trajectory on C and if γ is a solution of Newtonian equations then any parameterisation of such a trajectory is a solution of homogeneous equations.

It is worth showing here that if $\hat{\gamma}$ is a solution of homogeneous equations then also any of its reparameterisations is a solution as well.

Let $\gamma : s \mapsto (t(s, x^i(s)))$ be a solution and $\phi : s' \mapsto s = s(s')$ a reparameterisation. Then one can define a new curve $\gamma' : s' \mapsto (t(s(s')), x^i(s(s')))$ by reparameterising γ. One has

$$u'^0 = \phi' u^0 \qquad u'^i = \phi' u^i \qquad \Rightarrow v'^i = \frac{u'^i}{u'^0} = \frac{u^i}{u^0} = v^i \quad a'^i = \frac{dv'^i}{dt'} = \frac{dv^i}{dt} = a^i \tag{1.11.19}$$

By replacing these into (1.11.16), one sees immediately that γ' is a solution as well.

From the point of view of conserved quantities, the homogeneous formulation is equivalent to the standard Newtonian one as well. The original Lagrangian was covariant with respect to time translations (as far as L does not depend explicitly on time). In the homogeneous formalism, t is promoted to be a Lagrangian coordinate and it is a *cyclic* coordinate for the Lagrangian L_h. The total energy $\mathcal{H}$ is (minus) the conjugate momentum to t and, since t is cyclic, $\mathcal{H}$ is conserved.

The homogeneous Lagrangian L_h does not explicitly depend on s either, thus one expects another first integral, namely

$$F = \frac{\partial L_h}{\partial u^\mu}u^\mu - L_h = -\mathcal{H}u^0 + \frac{\partial L_h}{\partial u^i}u^i - L_h = u^0\left(-\mathcal{H} + g_{ij}\frac{u^i}{u^0}\frac{u^j}{u^0} - \frac{1}{u^0}L_h\right) \equiv 0 \tag{1.11.20}$$

Thus as expected F is a first integral though of a trivial kind since it is constant along solutions just because it is constant along any curve. In other words, F does not add any further non-trivial conservation laws to the original system.

The Lagrangian $\boldsymbol{L}_h$ can also be rewritten as

$$\boldsymbol{L}_h = \tfrac{1}{2}\left(g_{ij}(x)u^iu^j - 2U(x)u^0u^0\right)\frac{1}{u^0}\,ds = \frac{1}{2u^0}\hat{g}_{\mu\nu}(x)u^\mu u^\mu\,ds \tag{1.11.21}$$

where we defined the bilinear form $\hat{g}$ on C as

$$|\hat{g}_{\mu\nu}| = \begin{pmatrix} -2U & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{1.11.22}$$

This is degenerate for the free particle (i.e. $U(x) = 0$) and it captures the geometric structure of Newton-Galilei absolute space and time.

Let us stress that in the expression (1.11.21) time still appears to be a special coordinate due to the u^0 left in the Lagrangian L_h as a global factor. As a matter of fact, time *is* (and it remains) a special coordinate in classical mechanics, whether it is written in homogeneous formalism or not, even though at least we now moved its special role from kinematics to dynamics. Time directions are also singled out on C as the directions in which the bilinear form $\hat{g}$ is degenerate.

None of what we saw here in this Section has anything to do with SR or GR. The dynamics of the system we considered is *exactly* equivalent to standard Newtonian mechanics and no relativistic effect whatsoever is considered.

This is to show that homogeneous formalism is not relativity. One can have homogeneous formalism and spacetime for Newtonian mechanics. Relativity is a specific homogeneous dynamics which predicts effects that slightly (or dramatically, depending on the velocity) modify Newtonian mechanics.

The phase space

Let us now go back to the Newtonian Lagrangian and study a different aspect: initial condition problem. Equations of motions are in the form (1.11.2). We say that the Lagrangian is *non-degenerate* iff the Hessian with respect to the velocities is invertible, i.e.

$$\det\left(\frac{\partial L}{\partial u^i \partial u^j}\right) \neq 0 \tag{1.11.23}$$

For Lagrangians in the form (1.11.10), this amounts to require g_{ij} to be non-degenerate. Let us denote by g^{ij} the inverse of such a Hessian. In general g^{ij} depends on (q, u), while for Lagrangians in the form (1.11.10), i.e. quadratic in the velocities, it depends on q, only.

For non-degenerate Lagrangians, the equations of motion are expanded to

$$\frac{\partial L}{\partial u^i \partial u^k}\dot{u}^k + \frac{\partial L}{\partial u^i \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^i} = 0 \tag{1.11.24}$$

and they can be set in normal form, namely

$$\ddot{q}^i = g^{ij}\left(\frac{\partial L}{\partial q^j} - \frac{\partial L}{\partial u^j \partial q^k}\dot{q}^k\right) =: X_L^i(q, \dot{q}) \tag{1.11.25}$$

This is a second order ODE in normal form. As it always happens, it can be recast as a system of first order ODE in the unknown (q^i, u^i), namely

$$\begin{cases} \dot{q}^i = u^i \\ \dot{u}^i = X_L^i(q, u) \end{cases} \tag{1.11.26}$$

In turn, these can be seen as the equations for integral curves of a vector field on TQ

$$X_L = u^i \frac{\partial}{\partial q^i} + X_L^i(q, u)\frac{\partial}{\partial u^i} \tag{1.11.27}$$

This vector field captures the whole dynamics for the Lagrangian system; it is equivalent to giving the Lagrangian itself and solutions of Lagrangian equations are obtained as its integral curves.

Since integral curves always exist (of course, provided the vector field is sufficiently regular, but, remember, for us all objects are smooth by default) that means that, for any initial condition $(q_0, u_0) \in TQ$, one can determine a unique integral curve $\gamma_{(q_0, u_0)}$ such that $\gamma_{(q_0, u_0)}(0) = (q_0, u_0)$.

Taking inspiration from this example, in general we define the *state* of a system as the minimal information needed to determine uniquely the evolution of the system. The space of the states of the systems is called the *phase space*, in this case let us call it the *Lagrangian phase space*. The points of the phase space of a system are in one-to-one correspondence with solutions.

Accordingly, TQ is the (Lagrangian) phase space of the non-degenerate Lagrangian system.

> Notice that being TQ a manifold, we are implicitly giving a differential structure to the space of solutions, which is a space of curves.

Let us finally stress that this discussion is based on the assumption that the Lagrangian is non-degenerate. If the Lagrangian is degenerate, usually that means that there are constraints on initial conditions and at the same time that some subset of allowed initial conditions in fact determine the same evolution of the system. In these cases the phase space of the system is not TQ but something else.

We shall see examples of this below. By now let us just say that there is nothing *magic* about the phase space (or about Hamiltonian formalism which usually goes with it). The phase space is just studying the space of solutions and providing it with a differential structure. Everything which goes with it are just tools to achieve that goal. Usually, in mechanics, this is done canonically for non-degenerate Lagrangian systems by introducing Hamiltonian formalism. Unfortunately, going to field theories, Hamiltonian formalism breaks covariance and people invented a number of covariant Hamiltonian formalisms to keep doing what Hamilton does for mechanics. This is a way of doing it, but for us the *Hamiltonian viewpoint* will be anything which makes transparent that the initial condition problem is well-posed, i.e. that some sort of Cauchy theorem holds true. Accordingly, the vector field X_L for us is already a form of Hamiltonian viewpoint somehow, at least for non-degenerate, first order Lagrangian mechanical systems.

Hamiltonian formalism

If the Lagrangian is global, it means that one can cover TQ with coordinate patches (q^i, u^i), (q'^i, u'^i), … and give a Lagrangian on each patch $L(q, u)$, $L'(q', u')$, … such that in the overlaps

$$L'(q', u') = L(q, u) \tag{1.11.28}$$

(Here we assumed we keep time fixed on all patches, i.e. vertical transformations.) If the coordinate patches are related by

$$q'^i = q^i(q) \qquad u'^i = J^i_j(q)\, u^j \tag{1.11.29}$$

and $J^i_j(q)$ denotes the Jacobian of q' with respect to q, then one has $L'(q'(q), u'(q, u)) = L(q, u)$ and can consider its derivatives with respect to u^k

$$\frac{\partial L'}{\partial u'^i} J^i_k = \frac{\partial L}{\partial u^k} \qquad\qquad \Rightarrow \pi'_i J^i_k = \pi_k \qquad \Rightarrow \pi'_i = \bar{J}^k_i \pi_k \tag{1.11.30}$$

which can be interpreted as transformation rules for the conjugate momenta π_i.

One can easily check that such transformation laws are the ones for the components of covectors, i.e. points in T^*Q. Accordingly, setting natural coordinates (q^i, p_i) on T^*Q, any global Lagrangian L gives a global map

$$\Phi_L : TQ \to T^*Q : (q, u) \mapsto (q^i, p_i = \pi_i(q, u)) \tag{1.11.31}$$

which is called the *Legendre map* of L. The Lagrangian L is *regular* iff such a map is a diffeomorphism.

A necessary condition to be regular is that the Jacobian of the map

$$\begin{pmatrix} \mathbb{I} & 0 \\ \frac{\partial \pi_i}{\partial q^j} & \frac{\partial \pi_i}{\partial u^j} \end{pmatrix} \tag{1.11.32}$$

is non-degenerate, i.e. the Lagrangian L is non-degenerate.

One can push forward the vector field X_L along the Legendre map to obtain a vector field $X_H = (\Phi_L)_* X_L$ on T^*Q.

Let us compute X_H. One has $F(q,u) = F'(q, \pi(q,u))$ for any function $F : TQ \to \mathbb{R}$ and, by taking the derivatives of it, one gets

$$\begin{cases} (\Phi_L)_* \left(\frac{\partial}{\partial q^i} \right) = \frac{\partial}{\partial q^i} + \frac{\partial \pi_k}{\partial q^i} \frac{\partial}{\partial p_k} \\ (\Phi_L)_* \left(\frac{\partial}{\partial u^i} \right) = \frac{\partial \pi_k}{\partial u^i} \frac{\partial}{\partial p_k} \end{cases} \tag{1.11.33}$$

$$X_H = (\Phi_L)_* X_L = u^i \left(\frac{\partial}{\partial q^i} + \frac{\partial \pi_k}{\partial q^i} \frac{\partial}{\partial p_k} \right) + X_L^i(q,u) \frac{\partial \pi_k}{\partial u^i} \frac{\partial}{\partial p_k} = u^i \frac{\partial}{\partial q^i} + \left(u^i \frac{\partial \pi_k}{\partial q^i} + X_L^i(q,u) \frac{\partial \pi_k}{\partial u^i} \right) \frac{\partial}{\partial p_k} \tag{1.11.34}$$

where u^i is considered to be expressed as a function of (q,p) by inverting $p_i = \pi_i(q,u)$, which is again possible if the Lagrangian L is non-degenerate.

The form of X_H can be simplified by remembering that $g^{ij} \frac{\partial \pi_k}{\partial u^i} = \delta_k^j$ so that

$$X_H = u^i \frac{\partial}{\partial q^i} + \left(u^i \frac{\partial \pi_k}{\partial q^i} + \frac{\partial L}{\partial q^k} - \frac{\partial L}{\partial u^k \partial q^l} u^l \right) \frac{\partial}{\partial p_k} = u^i \frac{\partial}{\partial q^i} + \frac{\partial L}{\partial q^k} \frac{\partial}{\partial p_k} \tag{1.11.35}$$

Before simplifying further the expression for X_H, we already know a great deal of things about it. In particular, its integral curves are, by construction and definition of push-forward, the Legendre transform of solutions of Euler–Lagrange equations. They are solutions of a system of first order normal ODE, called the *Hamilton equations*.

Recalling that T^*Q is a symplectic manifold for the symplectic form $\boldsymbol{\omega} = dp_i \wedge dq^i$, X_H is a locally Hamiltonian vector field, i.e. the symplectic dual of the vector field X_H is a closed 1-form. In fact, we have since

$$di_{X_H}\boldsymbol{\omega} = -\frac{\partial u^i}{\partial q^k}\, dq^k \wedge dp_i - \frac{\partial u^i}{\partial p_k}\, dp_k \wedge dp_i + \frac{\partial L}{\partial q^i \partial q^k}\, dq^i \wedge dq^k + \frac{\partial L}{\partial u^i \partial q^k}\frac{\partial u^i}{\partial q^j}\, dq^j \wedge dq^k + \frac{\partial L}{\partial u^i \partial q^k}\frac{\partial u^i}{\partial p_j}\, dp_j \wedge dq^k =$$
$$= \left(\frac{\partial L}{\partial q^j \partial q^k} + \frac{\partial L}{\partial u^i \partial q^k}\frac{\partial u^i}{\partial q^j} \right) dq^j \wedge dq^k + \left(\frac{\partial u^j}{\partial q^k} + \frac{\partial L}{\partial u^i \partial q^k}\frac{\partial u^i}{\partial p_j} \right) dp_j \wedge dq^k - \frac{\partial u^i}{\partial p_k}\, dp_k \wedge dp_i \tag{1.11.36}$$

where the functions $u^i(q,p)$ are obtained by inverting $p_i = \pi_i(q,u)$, i.e. the identity $p_i = \frac{\partial L}{\partial u^i}(q, u(q,p))$ holds true.

From these identities, it follows by differentiation that

$$0 = \frac{\partial L}{\partial u^i \partial q^k}(q, u(q,p)) + \frac{\partial L}{\partial u^i \partial u^j}(q, u(q,p))\frac{\partial u^j}{\partial q^k} \qquad \delta_j^k = \frac{\partial L}{\partial u^i \partial u^j}(q, u(q,p))\frac{\partial u^j}{\partial p_k} \tag{1.11.37}$$

From these identities, it follows that $\frac{\partial u^j}{\partial p_k}$ is the inverse of the Hessian $\frac{\partial L}{\partial u^i \partial u^j}$; since the Hessian is symmetric its inverse in symmetric as well. Thus the term $\frac{\partial u^i}{\partial p_k} dp_k \wedge dp_i$ vanishes.

The coefficient of the term $\left(\frac{\partial L}{\partial q^j \partial q^k} + \frac{\partial L}{\partial u^i \partial q^k} \frac{\partial u^i}{\partial q^j} \right) dq^j \wedge dq^k$ can be recast as

$$\frac{\partial L}{\partial q^j \partial q^k} - \frac{\partial L}{\partial u^i \partial u^l}(q, u(q,p)) \frac{\partial u^l}{\partial q^k} \frac{\partial u^i}{\partial q^j} \tag{1.11.38}$$

which is manifestly symmetric as well. Finally, the coefficient of the term $\left(\frac{\partial u^j}{\partial q^k} + \frac{\partial L}{\partial u^i \partial q^k} \frac{\partial u^i}{\partial p_j} \right) dp_j \wedge dq^k$ can be recast as

$$\frac{\partial u^j}{\partial q^k} + \frac{\partial L}{\partial u^i \partial q^k} \frac{\partial u^i}{\partial p_j} = \frac{\partial u^j}{\partial q^k} - \frac{\partial L}{\partial u^i \partial u^l} \frac{\partial u^l}{\partial q^k} \frac{\partial u^i}{\partial p_j} = \frac{\partial u^j}{\partial q^k} - \delta^j_l \frac{\partial u^l}{\partial q^k} = 0 \tag{1.11.39}$$

Accordingly, we have that in general one has $d i_{X_H} \boldsymbol{\omega} = 0$ and the vector field X_H is locally Hamiltonian.   ▷*Locally Hamiltonian fields*

This is a first interesting result. The flow of X_H, which is by construction the evolution of the system, is a canonical flow. Thus the evolution of any regular Lagrangian system written on T^*Q, is a canonical flow.

Since the vector field X_H is locally Hamiltonian, one can check whether it is also globally Hamiltonian. In fact, it is. One can check that

$$i_{X_H} \boldsymbol{\omega} = -u^i \, dp_i + \frac{\partial L}{\partial q^k} \, dq^k = -d \left(p_i u^i(q,p) - L(q, u(q,p)) \right) \tag{1.11.40}$$

We have that

$$i_{X_H} \boldsymbol{\omega} = \boldsymbol{\omega}(X_H, \cdot) = dp_i(X_H)dq^i - dq^i(X_H)dp_i = \frac{\partial L}{\partial q^i} dq^i - u^i dp_i \tag{1.11.41}$$

Let us define the *Hamiltonian* $H(q,p) = p_i u^i(q,p) - L(q, u(q,p))$ and compute

$$\frac{\partial H}{\partial q^k} = p_i \frac{\partial u^i}{\partial q^k} - \frac{\partial L}{\partial q^k} - \frac{\partial L}{\partial u^i} \frac{\partial u^i}{\partial q^k} = -\frac{\partial L}{\partial q^k} \qquad \frac{\partial H}{\partial p_k} = u^k(q,p) + p_i \frac{\partial u^i}{\partial p_k} - \frac{\partial L}{\partial u^i} \frac{\partial u^i}{\partial p_k} = u^k \tag{1.11.42}$$

Thus $i_{X_H} \boldsymbol{\omega}$ can be written as

$$i_{X_H} \boldsymbol{\omega} = -\frac{\partial H}{\partial p_i} \, dp_i - \frac{\partial H}{\partial q^k} \, dq^k = -dH \tag{1.11.43}$$

which shows that H is a global potential for $i_{X_H} \boldsymbol{\omega}$ and thus a generating function for the field X_H.

The function $H(q,p) = p_i u^i(q,p) - L(q, u(q,p))$ on T^*Q is called the *Hamiltonian* of the system and the vector field X_H can be written as

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^k} \frac{\partial}{\partial p_k} \tag{1.11.44}$$

The integral curves of X_H obey the equations

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\[2mm] \dot{p}_k = -\dfrac{\partial H}{\partial q^k} \end{cases} \tag{1.11.45}$$

which, in the regular case, are called *Hamilton equations*.

Hamilton equations are, by construction, equivalent to Euler–Lagrange equations. They are first order normal ODE, thus Cauchy theorem applies to them. Hence given any point $(q_0, p^0) \in T^*Q$ there is one and only one solution $\gamma_{(q,p)} : t \mapsto (q(t), p(t))$ such that $\gamma_{(q,p)}(0) = (q_0, p^0)$. The space T^*Q is also a phase space for the system. Let us call it the *(Hamiltonian) phase space*.

Hamilton equations can be also written intrinsically. A curve $\gamma : \mathbb{R} \to T^*Q$ is a solution of Hamilton equations iff

$$i_{\dot\gamma}\boldsymbol{\omega} = -dH \qquad \Longleftrightarrow \qquad \dot\gamma = X_H \tag{1.11.46}$$

Along solutions of Hamilton equations, we have

$$i_{\dot\gamma}\boldsymbol{\omega} = dp_i(\dot\gamma)dq^i - dq^i(\dot\gamma)dp_i = \dot p_i dq^i - \dot q^i dp_i = -\frac{\partial H}{\partial q^i}dq^i - \frac{\partial H}{\partial p_i}dp_i = -dH \tag{1.11.47}$$

Given a Hamiltonian system on T^*Q which is described by the Hamiltonian H, one can define the inverse Legendre transform $\Phi_L^{-1} : T^*Q \to TQ$ by

$$\dot q^i = \frac{\partial H}{\partial p_i}(q, p) \tag{1.11.48}$$

The Hamiltonian is *regular* if such a map is a diffeomorphism and it is non-degenerate iff the Hessian with respect to momenta is invertible

$$\det\left(\frac{\partial H}{\partial p_\mu \partial p_\nu}\right) \neq 0 \tag{1.11.49}$$

One can hence associate to the Hamiltonian system the Lagrangian system described by the Lagrangian

$$L(q, u) = p_i(q, u)u^i - H(q, p(q, u)) \tag{1.11.50}$$

where the functions $p_i(q, u)$ are obtained by solving inverse Legendre transform with respect to the velocities, which is possible for non-degenerate Hamiltonians. Hence there is a complete symmetry between the Lagrangian and Hamiltonian side of the (non-degenerate) mechanics.

Let us also define the *Poisson bracket* of two functions F and G on T^*Q to be the function

$$\{F, G\} = X_F(G) = -X_G(F) \tag{1.11.51}$$

Poisson brackets are obviously bilinear and antisymmetric. They also obeys the following properties:

(a) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \qquad$ (Jacobi identities)

(b) $\forall c \in \mathbb{R} : \quad \{F, c\} = 0$

(c) $\{F, GH\} = \{F, G\}H + G\{F, H\} \qquad$ (Liebniz rule)

(d) $[X_F, X_G] = X_{\{F, G\}}$

Property (a) tells us that functions on T^*Q forms a Lie algebra.

Property (b) tells us that Poisson brackets can be defined on the quotient algebra of functions on T^*Q with respect to constants.

Property (c) tells us that Poisson brackets are a derivative of the algebra of functions on T^*Q.

Property (d) tells us that the map $X : \mathcal{F}(T^*Q) \to \mathfrak{X}(T^*Q) : F \mapsto X_F$ is a morphism of Lie algebras. Constants are in the kernel of such a map and can be quotiented away.

Let us prove property (c).

$$\{F, GH\} = X_F(GH) = X_F(G)H + GX_F(H) = \{F,G\}H + G\{F,H\} \tag{1.11.52}$$

Let us prove property (d) by applying the property (16.A.1).

$$i_{[X_F, X_G]}\boldsymbol{\omega} = \pounds_{X_F} i_{X_G}\boldsymbol{\omega} - i_{X_G}\pounds_{X_F}\boldsymbol{\omega} = \pounds_{X_F} dG - i_{X_G} di_{X_F}\boldsymbol{\omega} = di_{X_F} dG - \boxed{i_{X_G} ddF} = dX_F(G) = d\{F,G\} \qquad \Rightarrow X_{\{F,G\}} = [X_F, X_G] \tag{1.11.53}$$

Let us finally prove property (a).

$$\begin{aligned}\{F, \{G,H\}\} =& -X_{\{G,H\}}(F) = -[X_G, X_H](F) = -X_G(X_H(F)) + X_H(X_G(F)) = -\{G, \{H,F\}\} + \{H, \{G,F\}\} = \\ =& -\{G, \{H,F\}\} - \{H, \{F,G\}\}\end{aligned} \tag{1.11.54}$$

As it rarely happens, everybody more or less agree on what is to be intended by Hamiltonian formulation of mechanical non-degenerate systems. Different approaches differ mainly in notation. The discussion then develops essentially on how to extend this to field theories or to the degenerate cases.

Field theories are different from mechanics essentially for two reasons: there are many independent variables and consequently ODE becomes PDE as well as one can define many conjugate momenta instead of one for each position coordinate. Moreover, most of Lagrangians in field theories are degenerate.

One can decide of solving the degeneracy problem by discussing Hamiltonian formalism for degenerate mechanical system first. This approach was first embraced by Dirac, Bergmann and others. Some other approaches of this type go for identifying minimal structures, weaker than the symplectic structure, which still allow to treat mechanics, hoping that these less strict structures are then better generalised to field theory. These attempts split essentially between the *presymplectic world* (relaxing the request that the symplectic form is invertible, keeping a bilinear closed 2-form) and the *Poisson world* (in which one assumes that the tensor defining Poisson brackets exists, though it can be degenerate).

On the other hand, one can face extension to field theory first. One way is to regard Lagrangian field theory as a mechanical system with infinitely many degrees of freedom, basically treating a section of configuration bundle as a parameterised curve $\sigma : t \mapsto y^i(t, x^i)$ as if $y^i(\cdot, x^i)$ were all distinct Lagrangian coordinates. Unfortunately, this requires the choice of a preferred time breaking general covariance if the theory has it. Although this is the most-followed approach, a minority goes for preserving covariance as long as possible.

Standard *ADM* and canonical approaches are of the first type. See[5][6].

The situation is so intricate because different techniques are better suited than the others to solve particular issues or in particular situations. As a matter of fact one should master all these approaches and their mutual relations. And at the moment this is far beyond the scope of this book. As long as we are concerned we already did our choice. We already declared that for us *Hamiltonian viewpoint* means studying when the initial condition

problem is well-posed and we shall follow that line trying to keep covariance as long as it is reasonable, since we are going exactly to study covariant theories.

Hereafter we shall consider a couple of slightly different formulations for Hamiltonian systems which will be then friendly to extensions to field theory.

Before proceeding, let us spend few words about why one should care to study Hamiltonian formalism, in the first place. The question is not trivial, since we already noticed that Euler–Lagrange equations and Hamilton equations are in fact *equivalent!* There are many different answers, each bringing a piece of truth with it.

First, the Hamiltonian world is a symplectic world and symplectic geometry comes with a huge apparatus of tools which helps analysing pathological situations. In the symplectic world you have Poisson brackets and canonical transformations. The first turns out to be useful to disentangle the mess of constraints in degenerate mechanics, the second is essential to take the observation that the evolution of systems is a canonical flow and use it to characterise solutions.

The Lagrange world is more natural, though, in view of the Legendre transform, it carries a symplectic structure which is not independent of the dynamics as it happens for Hamiltonian systems. One can consider two Hamiltonian systems on the same phase space and hence define two different inverse Legendre transforms. Then the same symplectic form $\boldsymbol{\omega}$ on the phase space T^*Q defines two different symplectic structures $(\Phi_H^{-1})^*\boldsymbol{\omega}$ and $(\Phi_{H'}^{-1})^*\boldsymbol{\omega}$ on TQ. Of course, having a canonical symplectic structure greatly simplifies the analysis of systems.

This is essentially why Lagrangian systems are preserved by fibered transformations, while Hamiltonian systems are preserved by canonical transformations. Being the group of canonical transformations way bigger than fibered transformations, one has more freedom to find a specific transformation that better simplifies the equations of motion and allow to go for solutions.

Second, Hamiltonian formalism comes with a mythology about what is to be considered a physical degree of freedom. We call it *mythology* not because it is not true, but because the tools that come with Hamiltonian analysis of constraints are often confused with something more fundamental. The phase space is T^*Q not because one defines Hamiltonian formalism on cotangent bundle, but because (and as long as) one is free to take any point in cotangent bundles as a legitimate initial condition for the system. It is *since* Hamilton equations are in normal forms, that Cauchy theorem holds and points of T^*Q can be taken as (uncostrained) initial conditions. We shall see below that the analysis of initial condition problem in field theory can be carried over in the Lagrangian framework as well. Often people argue about which is the most fundamental approach to physical system, whether the Lagrangian approach or the Hamiltonian framework. This argument is particularly amusing when one remembers that they are discussing about two *equivalent* frameworks.

Third, Poisson brackets play a prominent role in quantisation, at least in some of the approaches to quantisation. Unfortunately, once again when one leaves the comfortable sunny beaches of systems which everybody agree on how should be quantised, the quantisation techniques are at stake once again. Also in this case, again because we are interested in covariant field theories (for which, despite some partially successful attempts, nobody agrees on what quantisation means) we feel it is particularly dangerous to buy one side of the world and rejecting the other.

It is certainly true that the tools developed in the last century are adapted to Hamiltonian formalism and for that it is often easier to use Hamiltonian formalism. It is also true that since Lagrangian framework is certainly more natural, Hamiltonian formalism itself is at stake when one tries applications outside the accepted scope (e.g. to covariant field theories). For these reasons, we believe it would be wiser to keep an agnostic attitude about this issue, using both formalisms keeping in mind what Hamiltonian framework has proved to do better and keep stuck to the fundamental issues instead of embracing a framework over the other.

Helmholtz Lagrangian

Let us first notice that one can obtain Hamilton equations from a Lagrangian on TT^*Q. Let us fix coordinates $(q, p, \dot q, \dot p)$ on TT^*Q and consider the Lagrangian

$$\boldsymbol{L}_{Hel} = \left(p_i \dot q^i - H(q, p) \right) \, dt \tag{1.11.55}$$

which is called the *Helmholtz Lagrangian*. By variation of it, one gets

$$\delta \boldsymbol{L}_{Hel} = \left(\delta p_k \dot{q}^k + p_k \delta \dot{q}^k - \frac{\partial H}{\partial q^k} \delta q^k - \frac{\partial H}{\partial p_k} \delta p_k \right) \, dt = \left(\left(\dot{q}^k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q^k} \right) \delta q^k \right) dt + \frac{d}{dt} \left(p_k \delta q^k \right) \, dt \tag{1.11.56}$$

Thus one can associate to the Helmholtz Lagrangian its Euler–Lagrange and Liouville parts as

$$\mathbb{E}(L_{Hel}) = \left(\dot{q}^k - \frac{\partial H}{\partial p_k} \right) \omega_k \wedge \, dt - \left(\dot{p}_k + \frac{\partial H}{\partial q^k} \right) \omega^k \wedge \, dt \qquad \mathbb{F}(L_{Hel}) = p_k \omega^k \tag{1.11.57}$$

where we set $\omega_k = \, dp_k - \dot{p}_k \, dt$ and $\omega^k = \, dq^k - \dot{q}^k \, dt$ for the relevant contact forms.

One can see directly that Hamilton equations follow by requiring criticality with respect to independent variation of δq^k and δp_k, with the variation of the positions δq^k vanishing at the boundary $\{t_0, t_1\}$.

> Notice in fact that the Helmholtz Lagrangian is first order in the position but zero order in the momenta. Thus no integration by parts of variations of momenta. Thus the Liouville part has no component in the momenta directions. Thus one does not need to impose that $\delta p_k = 0$ at the boundary to ensure that the boundary term vanishes.
>
> Moreover, let us remark that there is nothing magic about Helmholtz Lagrangian. The action defined by the Helmholtz Lagrangian is the same defined by the origianal Lagrangian, since, in view of the inverse Legendre transform, one has
>
> $$\boldsymbol{L} = \left(p_i \dot{q}^i - H(q, p) \right) \, dt \tag{1.11.58}$$
>
> Still it is geometricly interesting that the same system can be described as a Lagrangian system on Q and on T^*Q. In the first case, one varies curves in Q, in the second, curves on T^*Q. This is another characteristic of Hamiltonian viewpoint. Lagrangian viewpoint means to vary curves on configuration space, Hamiltonian viewpoint means to vary curves in the phase space.

The Helmholtz Lagrangian is linear in the velocities, thus it is degenerate (thus it does not induce directly a Hamiltonian on T^*T^*Q).

We also need to remark that there is also *another* Helmholtz Lagrangian, namely

$$\boldsymbol{L}'_{Hel} = \left(q^i \dot{p}_i + H(q, p) \right) \, dt \tag{1.11.59}$$

which still induced Hamilton equations, though being 0-order in q and first order in p. Its variation gives

$$\delta L'_{Hel} = \left(\dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i - \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i + \frac{d}{dt} \left(q^i \delta p_i \right) \tag{1.11.60}$$

which, in fact, still implies Hamilton equations.

As we shall discuss in Chapter 5, these two Helmholtz Lagrangians are related by a Legendre transforms. In fact one has

$$\delta L_{Hel} = \delta \left(p_i \dot{q}^i \right) - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i = -\dot{p}_i \delta q^i - q^i \delta \dot{p}_i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i + \delta(p_i \dot{q}^i + q^i \dot{p}_i) = -\delta L'_{Hel} + \delta(p_i \dot{q}^i + q^i \dot{p}_i) \tag{1.11.61}$$

from which one gets

$$L_{Hel} = -L'_{Hel} + p_i\dot{q}^i + q^i\dot{p}_i = -L'_{Hel} + \frac{d}{dt}\left(p_i q^i\right) \tag{1.11.62}$$

That directly shows that the two Lagrangians L_{Hel} and L'_{Hel} are dynamically equivalent, since they differ by a total time derivative.

Time-dependent mechanics

Time-dependent mechanics seems to be of very limited interest for covariant field theories. We shall see that there is no generally covariant Lagrangian which explicitly depends on the independent variables. However, the natural arena for time-dependent mechanics is $\mathbb{R} \times TQ$ which means also considering general transformations

$$\begin{cases} t' = t'(t) \\ q'^i = q'^i(t, q) \end{cases} \qquad \Rightarrow u'^i = \bar{J}\left(J^i + J^i_j u^j\right) \tag{1.11.63}$$

even when the Lagrangian does not depend on time. And this is exactly what happens in covariant theories in which one should be free to change independent variables at will.

On the other hand, allowing arbitrary changes of time (or any change but $t' = t - t_0$) in Newtonian mechanics seems useless since such transformations in some sense do not preserve uniform absolute time. It would be as looking for a formulation in which one can use arbitrarily non-uniform times. Maybe it could be of interest from a mathematical viewpoint though our uniform time seems well rooted in our Newtonian description of the world.

As a curiosity, and in view of our interest for covariant field theories, this is exactly what we shall do here. We shall search for a formulation of mechanics which is friendly to transformations (1.11.63), sometimes restricting to $\dot{t}' = 1$. What we have done until now was restricting to T^*Q *and* to changes of coordinates in the form $q'^i = q'^i(q)$.

If the Lagrangian depends on time for example the vector field X_L is not a vector field on TQ since its component depends explicitly on t. And in general the form of vector fields X_L is not preserved by transformations (1.11.63).

The vector field X_L should be regarded as an object on $\mathbb{R} \times TQ$.

However, it cannot be seen as a vertical vector field on the bundle $\mathbb{R} \times TQ \to \mathbb{R}$. Transformations (1.11.63) imply, by considering $F(t, q, u) = F'(t', q', u')$, on the natural basis of vectors

$$\begin{cases} \frac{\partial}{\partial t} = J\frac{\partial}{\partial t'} + J^i\frac{\partial}{\partial q'^i} + \partial_0\left(\bar{J}\left(J^j + J^j_k u^k\right)\right)\frac{\partial}{\partial u'^j} \\ \frac{\partial}{\partial q^i} = J^j_i\frac{\partial}{\partial q'^j} + \bar{J}(d_i J^j + J^j_{ki} u^k)\frac{\partial}{\partial u'^j} \\ \frac{\partial}{\partial u^i} = \bar{J}J^j_i\frac{\partial}{\partial u'^j} \end{cases} \tag{1.11.64}$$

From this, one can obtain transformation rules for the components of a vector field on $R \times TQ$.

$$\xi\left(J\frac{\partial}{\partial t'} + J^i\frac{\partial}{\partial q'^i} + \partial_0\left(\bar{J}\left(J^j + J^j_k u^k\right)\right)\frac{\partial}{\partial u'^j}\right) + \xi^i\left(J^j_i\frac{\partial}{\partial q'^j} + \bar{J}(d_i J^j + J^j_{ki} u^k)\frac{\partial}{\partial u'^j}\right) + \dot{\xi}^i\left(\bar{J}J^j_i\frac{\partial}{\partial u'^j}\right) =$$
$$= \xi J\frac{\partial}{\partial t'} + \left(\xi^i J^j_i + \xi J^j\right)\frac{\partial}{\partial q'^j} + \left(\xi\partial_0\left(\bar{J}\left(J^j + J^j_k u^k\right)\right) + \xi^i\bar{J}(d_i J^j + J^j_{ki} u^k) + \dot{\xi}^i\bar{J}J^j_i\right)\frac{\partial}{\partial u'^j} \tag{1.11.65}$$

Thus components of vector fields transform as

$$
\begin{cases}
\xi' = J\xi \\
\xi'^j = \xi^i J_i^j + \xi J^j \\
\dot{\xi}'^j = \xi \partial_0 \left(\bar{J} \left(J^j + J_k^j u^k \right) \right) + \xi^i \bar{J}(d_i J^j + J_{ki}^j u^k) + \dot{\xi}^i \bar{J} J_i^j
\end{cases}
\tag{1.11.66}
$$

Even restricting to transformations with $J = 1$ the vector field X_L cannot be vertical (i.e. with $\xi = 0$). In fact one would have $\xi'^j = \xi^i J_i^j$ which is incompatible with posing $\xi^i = u^i$ (unless one further restricts to time-independent transformations as done above).

By keeping stuck to the position $\xi^i = u^i$, it suggests to pose $\xi = 1$, which is then violated by general transformations with $J \neq 1$. In order to have everything working in full generality, we have to define X_L to be a vector density of weight 1, i.e. to be in the form

$$
X_L = \ dt \otimes \left(\tfrac{\partial}{\partial t} + u^i \tfrac{\partial}{\partial q^i} + X_L^i(t,q,u) \tfrac{\partial}{\partial u^i} \right)
\tag{1.11.67}
$$

We already know objects in this form. What we are saying is that X_L is a connection on the bundle $\mathbb{R} \times TQ \to \mathbb{R}$. In general the coefficients of an   ▷*Connections*
object $\ dt \otimes \left(\xi \partial_t + \xi^i \tfrac{\partial}{\partial q^i} + \dot{\xi}^i \tfrac{\partial}{\partial u^i} \right)$ transforms as

$$
\begin{cases}
\xi' = \xi \\
\xi'^j = \bar{J} \left(\xi^i J_i^j + \xi J^j \right) \\
\dot{\xi}'^j = \bar{J} \left(\xi \partial_0 \left(\bar{J} \left(J^j + J_k^j u^k \right) \right) + \xi^i \bar{J}(d_i J^j + J_{ki}^j u^k) + \dot{\xi}^i \bar{J} J_i^j \right)
\end{cases}
\tag{1.11.68}
$$

These are compatible with setting $\xi = 1$ and $\xi^i = u^i$. We have to check if it is compatible with setting

$$
\dot{\xi}^i = g^{ij} \left(\frac{\partial L}{\partial q^j} - \frac{\partial L}{\partial u^j \partial q^k} u^k - \frac{\partial L}{\partial u^j \partial t} \right)
\tag{1.11.69}
$$

The Lagrangian is a scalar density, i.e. it transforms as

$$
J L'(t',q',u') = L(t,q,u)
\tag{1.11.70}
$$

Then one has

$$
\begin{cases}
\dfrac{\partial L}{\partial q^i} = J J_i^j \dfrac{\partial L'}{\partial q'^j} + \left(d_i J^j + J_{ik}^j u^k \right) \dfrac{\partial L'}{\partial u'^j} \\
\dfrac{\partial L}{\partial u^i} = J_i^j \dfrac{\partial L'}{\partial u'^j} \quad \Rightarrow \dfrac{\partial L}{\partial u^k \partial u^i} = \bar{J} J_i^j J_k^l \dfrac{\partial L'}{\partial u'^l \partial u'^j} \quad \Rightarrow g'^{ij} = \bar{J} J_k^i J_l^j g^{kl}
\end{cases}
\tag{1.11.71}
$$

which, in turn, imply

$$
\begin{cases}
\dfrac{\partial L}{\partial u^i \partial q^k} = J_{ik}^j \dfrac{\partial L'}{\partial u'^j} + J_i^j J_k^l \dfrac{\partial L'}{\partial u'^j \partial q'^l} + \bar{J} J_i^j \dfrac{\partial L'}{\partial u'^j \partial u'^l} \left(d_k J^l + J_{kn}^l u^n \right) \\
\dfrac{\partial L}{\partial u^i \partial t} = d_i J^j \dfrac{\partial L'}{\partial u'^j} + J J_i^j \dfrac{\partial L'}{\partial u'^j \partial t'} + J_i^j J^k \dfrac{\partial L'}{\partial u'^j \partial q'^k} + J_i^j \dfrac{\partial L'}{\partial u'^j \partial u'^k} \partial_0 \left(\bar{J}(J^k + u^l J_l^k) \right)
\end{cases}
\tag{1.11.72}
$$

Thus one has

$$
\dot\xi^i = g^{ij}\left(\frac{\partial L}{\partial q^j} - \frac{\partial L}{\partial u^j \partial q^k}u^k - \frac{\partial L}{\partial u^j \partial t}\right) = g^{ij}\left(JJ_j^l\frac{\partial L'}{\partial q'^l} + \left(d_j J^l + J_{jk}^l u^k\right)\frac{\partial L'}{\partial u'^l} - \left(J_{jk}^l\frac{\partial L'}{\partial u'^l} + J_j^n J_k^l\frac{\partial L'}{\partial u'^n \partial q'^l} + \bar{J}J_j^n\frac{\partial L'}{\partial u'^n \partial u'^l}\left(d_k J^l + J_{km}^l u^m\right)\right)u^k + \right.
$$
$$
\left. - d_i J^j\frac{\partial L'}{\partial u'^j} - JJ_j^l\frac{\partial L'}{\partial u'^l \partial t'} - J_j^l J^k\frac{\partial L'}{\partial u'^l \partial q'^k} - J_j^l\frac{\partial L'}{\partial u'^l \partial u'^k}\partial_0\left(\bar{J}(J^k + u^l J_l^k)\right)\right)
\tag{1.11.73}
$$

This in principle gives the transformation rules for $\dot\xi^i$ which replaced into (1.11.68) should tell us if position (1.11.69) is compatible.

$$
\dot\xi'^j = \bar{J}\left(\partial_0\left(\bar{J}\left(J^j + J_k^j u^k\right)\right) + u^i \bar{J}(d_i J^j + J_{ki}^j u^k) + \bar{J}J_i^j g^{im}\left[JJ_m^n\frac{\partial L'}{\partial q'^n} + \left(d_m J^l + J_{mk}^l u^k\right)\frac{\partial L'}{\partial u'^l} + \right.\right.
$$
$$
\left. - \left(J_{mk}^l\frac{\partial L'}{\partial u'^l} + J_m^n J_k^l\frac{\partial L'}{\partial u'^n \partial q'^l} + \bar{J}J_m^n\frac{\partial L'}{\partial u'^n \partial u'^l}\left(d_k J^l + J_{kp}^l u^p\right)\right)u^k - d_m J^k\frac{\partial L'}{\partial u'^k} - JJ_m^n\frac{\partial L'}{\partial u'^n \partial t'} - J_m^n J^l\frac{\partial L'}{\partial u'^n \partial q'^l} - J_m^l\frac{\partial L'}{\partial u'^l \partial u'^k}\partial_0\left(\bar{J}(J^k + u^l J_l^k)\right)\right]\right) =
\tag{1.11.74}
$$
$$
= \bar{J}g'^{jn}\left[J\frac{\partial L'}{\partial q'^n} - \left(J_k^l u^k + J^l\right)\frac{\partial L'}{\partial u'^n \partial q'^l} - J\frac{\partial L'}{\partial u'^n \partial t'}\right] = g'^{jn}\left[\frac{\partial L'}{\partial q'^n} - u'^l\frac{\partial L'}{\partial u'^n \partial q'^l} - \frac{\partial L'}{\partial u'^n \partial t'}\right]
$$

That in fact proves that the position (1.11.69) is compatible with transformation laws.

Thus dynamics in time-dependent mechanics is expressed by a connection X_L on the bundle $\mathbb{R} \times TQ \to \mathbb{R}$ in the form (1.11.67). Once such a connection is given, it determines the horizontal distribution, which in mechanics is first order and hence integrable. That defines horizontal curves $\gamma : \mathbb{R} \to \mathbb{R} \times TQ : s \mapsto (t(s), q(s), u(s))$ such that

$$
\begin{cases}
\dot t = 1 \\
\dot q^i = u^i \\
\dot u^i = X_L^i(t, q, u)
\end{cases}
\tag{1.11.75}
$$

which are equivalent to Euler–Lagrange equations.

Let us stress that it is not by a *choice* of ours that we defined the dynamics in terms of a connection X_L. Once we agree that the Lagrangian is global and that the most general transformations allowed are fibered transformations (1.11.63) then transformation laws for $X_L^i(t, q, u)$ follows and they are such that they enter as components of a connection and not as components of a vector field. That is a fact independent of our wishes.

In other words, the character of an object is not something we decide; it is already written there and we discover it, if we are patient enough to work it out.

Let us also remark that once we accept that dynamics should be written in term of a connection, then that is true even by restricting to time-independent Lagrangian and even restricting to time-independent transformations of coordinates. Simply by these restrictions one can define a one-to-one correspondence between *any connection* and vector fields. If we know we shall never consider time-dependent transformations, then we are free to represent dynamics by vector fields (as we did above) but still with no real advantage except simplicity (which, by the way, is based only on the fact that somehow we regard vector fields as simpler objects than connections, when, of course, there is no real support to such attitude). On the other hand, if we assume dynamics to be described by a vector field and then we allow time-dependent transformations then we are easily lead to erroneous conclusions.

For these reasons, we should be patient enough to discover the true character of the objects we use and then act accordingly.

Let us finally stress that we implicitly discovered another thing. If the Lagragian is a scalar density, then the Hamiltonian cannot be a function on T^*Q or on $\mathbb{R} \times T^*Q$. In fact, the transformation laws of velocities $u'^i = \bar{J}\left(J^i + J^i_j u^j\right)$ are such that the term $p_i u^i$ does not transform properly.

$$H' = p'_i u'^i - L' = p_k \bar{J}^k_i \bar{J}\left(J^i + J^i_j u^j\right) - \bar{J}L = \bar{J}J^i\bar{J}^k_i p_k + \bar{J}\left(p_k u^k - L\right) = \bar{J}J^i\bar{J}^k_i p_k + H \tag{1.11.76}$$

Also the Hamiltonian is a function only with respect to time-independent transformations.

Poincaré–Cartan form

Notice that the action of the Helmholtz Lagrangian can be written as

$$A_D[\sigma] = \int_D \sigma^* \boldsymbol{L}_{Hel} = \int_D \left(p_i \dot{q}^i - H(q,p)\right) dt = \int_D \sigma^* \left(p_i\, dq^i - H(q,p)\, dt\right) = \int_D \sigma^* \boldsymbol{\Theta}_H \tag{1.11.77}$$

This suggests to define

$$\boldsymbol{\Theta}_H := -H\, dt + p_i\, dq^i \tag{1.11.78}$$

which is called the *Poincaré–Cartan form*, which is another equivalent way to describe the dynamics of the system. It is a 1-form on $\mathbb{R} \times T^*Q$.

Let us compute how the time component of a 1-form over $\mathbb{R} \times T^*Q$ transforms with respect to transformations

$$\begin{cases} t' = t'(t) & \Rightarrow dt' = J dt \\ q'^i = q'^i(t,q) & \Rightarrow dq'^i = J^i\, dt + J^i_k\, dq^k \\ p'_i = \bar{J}^k_i p_k \end{cases} \tag{1.11.79}$$

Then one has

$$-h'\, dt' + \alpha'_i\, dq'^i = -h'J\, dt + \alpha'_i\left(J^i\, dt + J^i_k\, dq^k\right) = -\left(h'J - \alpha'_i J^i\right)\, dt + \alpha'_i J^i_k\, dq^k \tag{1.11.80}$$

Then the transformation laws of the coefficients α_i match with the ones of momenta (1.11.79). At the same time, the transformation law of the coefficient h matches the one expected for the Hamiltonian, namely (1.11.76). In fact one gets

$$h = h'J - \alpha'_i J^i \qquad \Rightarrow h' = h + \alpha_k \bar{J}^k_i J^i \tag{1.11.81}$$

Accordingly, the Hamiltonian is not a function and Hamiltonian dynamics is described by the Poincaré–Cartan form in its full generality.

This is an example of the dangerous route one takes by assuming the wrong character for objects.

In standard Hamiltonian mechanics, one shows that flows of symplectic transformations are related to locally Hamiltonian vector fields. Locally Hamiltonian vector fields are locally generated by a potential, which is called the *generating function* of the flow.

Then one can locally associate a flow of symplectic transformations to a generating function $F(q,u)$. This is done in standard analytic mechanics, maybe with a different notation. In any event, one uses it, for example, to associate flows to first integrals and eventually rewrite Noether theorem in Hamiltonian formalism.

Then going to Hamilton-Jacobi (i.e. studying the canonical flow of a Hamiltonian system) one defines *another theory* for generating functions of canonical flows which are time-dependent canonical transformations. This new theory of generating functions is obtained by starting from Poincaré–Cartan form (which must be preserved

modulo the differential of the generating functions). It sometimes remains obscure (to be kind) why does one need a new theory for generating functions and what is the difference between flows of symplectic transformations and canonical flows.

Sometimes one is told that symplectic transformations are kinemanical while canonical flows are dynamical, which is true, though it does not clarify why does one need a different theory of generating functions.

Now it is clear that canonical flows are time-dependent transformations and they hence need the true and full story about Poincaré–Cartan forms, while the first theory of symplectic transformations was developed within a *simplified* framework in which time was frozen.

The dynamics can be described in terms of Poincaré–Cartan form. One can in fact see that a section $\sigma : \mathbb{R} \to \mathbb{R} \times T^*Q$ is a solution of Hamilton equations iff one has

$$(\sigma)^* i_X d\boldsymbol{\Theta}_H = 0 \tag{1.11.82}$$

for all vertical vector fields of $X = \delta q^i \frac{\partial}{\partial q^i} + \delta p_i \frac{\partial}{\partial p_i}$.

$$
\begin{aligned}
d\boldsymbol{\Theta}_H &= -\tfrac{\partial H}{\partial q^k}\, dq^k \wedge\, dt - \tfrac{\partial H}{\partial p_k}\, dp_k \wedge\, dt +\, dp_k \wedge\, dq^k \\
i_X d\boldsymbol{\Theta}_H &= -\left(\tfrac{\partial H}{\partial q^k}\delta q^k + \tfrac{\partial H}{\partial p_k}\delta p_k\right)\, dt + \delta p_k\, dq^k - \delta q^k\, dp_k \\
(\sigma)^* i_X d\boldsymbol{\Theta}_H &= \left(-\left(\tfrac{\partial H}{\partial q^k} + \dot p_k\right)\delta q^k + \left(-\tfrac{\partial H}{\partial p_k} + \dot q^k\right)\delta p_k\right)\, dt
\end{aligned}
\tag{1.11.83}
$$

Thus $(\sigma)^* i_X d\boldsymbol{\Theta}_H = 0$ is equivalent to Hamilton equations.

The same result can be derived by variational calculus. The variation of the action is

$$\delta_X A_D[\sigma] = \int_D \sigma^* \pounds_X \boldsymbol{\Theta}_H = \int_D \sigma^* i_X d\boldsymbol{\Theta}_H + \int_{\partial D} \sigma^* i_X \boldsymbol{\Theta}_H \tag{1.11.84}$$

The boundary term vanishes as long as one has $\delta q^k = 0$ on the boundary ∂D. Then field equations are $\sigma^* i_X d\boldsymbol{\Theta}_H = 0$ (for all deformations X).

If we consider an arbitrary (projectable) vector field $X = \delta t(t)\frac{\partial}{\partial t} + \delta q^i(t,q,p)\frac{\partial}{\partial q^i} + \delta p_i(t,q,p)\frac{\partial}{\partial p_i}$ and compute again $(\sigma)^* i_X d\boldsymbol{\Theta}_H$, we get

$$(\sigma)^* i_X d\boldsymbol{\Theta}_H = \left(-\left(\tfrac{\partial H}{\partial q^k} + \dot p_k\right)\delta q^k + \left(-\tfrac{\partial H}{\partial p_k} + \dot q^k\right)\delta p_k + \left(\tfrac{\partial H}{\partial q^k}\dot q^k + \tfrac{\partial H}{\partial p_k}\dot p_k\right)\delta t\right)\, dt \tag{1.11.85}$$

which vanishes for solutions of Hamilton equations. Thus $(\sigma)^* i_X d\boldsymbol{\Theta}_H = 0$ for a general X.

The Legendre map can be easily extended to $\Phi_L : \mathbb{R} \times TQ \to \mathbb{R} \times T^*Q : (t,q,u) \mapsto (t,q,p_k = \frac{\partial L}{\partial u^k})$. We can then define by pull-back the counterpart of the Poincaré–Cartan form on the Lagrangian side. It is still called the *Poincaré–Cartan form* and denoted by

$$\boldsymbol{\Theta}_L = L\, dt + \pi_k \omega^k \tag{1.11.86}$$

where $\omega^k =\, dq^k - u^k\, dt$ is the usual contact form.

One has the pull-back as

$$\boldsymbol{\Theta}_L = (\Phi_L^{-1})^* \boldsymbol{\Theta}_H = -(\pi_i u^i - L)\, dt + \pi_i\, dq^i = L\, dt + \pi_i\left(\, dq^i - u^i\, dt\right) = L\, dt + \pi_i \omega^i \tag{1.11.87}$$

For the expression of Θ_L, one sees directly that the action can be expressed as

$$\delta_X A_D[\sigma] = \int_D (j^1\sigma)^* \boldsymbol{\Theta}_L \tag{1.11.88}$$

since this amounts to add a contact form which vanishes when pulled back along the section $j^1\sigma$ (which clarifies the relation with the Helmholtz Lagrangian as well).

Also Euler–Lagrange dynamics can be written entirely in terms of the Poincaré–Cartan form $\boldsymbol{\Theta}_L$.

Given a section $\sigma : \mathbb{R} \to \mathbb{R} \times Q$ and its prologation $j^1\sigma : \mathbb{R} \to \mathbb{R} \times TQ$, the section σ is a solution of Euler–Lagrange equations iff, for all vertical vector fields $X = \delta q^k \frac{\partial}{\partial q^k}$, one has

$$(j^1\sigma)^* i_{J^1 X} d\boldsymbol{\Theta}_L = 0 \tag{1.11.89}$$

Let us consider a (projectable) vector field $X = \delta t \frac{\partial}{\partial t} + \delta q^k \frac{\partial}{\partial q^k} + \delta u^k \frac{\partial}{\partial u^k}$ on $\mathbb{R} \times TQ$ and a section $\sigma : \mathbb{R} \to \mathbb{R} \times TQ : t \mapsto (t, q(t), u(t))$. Then we have

$$
\begin{aligned}
d\boldsymbol{\Theta}_L =& dL \wedge dt + d\pi_k \wedge \omega^k - \pi_k \, du^k \wedge \, dt \\
i_X d\boldsymbol{\Theta}_L =& X(L) \, dt - \delta t dL + X(\pi_k)\omega^k - \delta q^k d\pi_k + u^k \delta t d\pi_k - \pi_k \delta u^k \, dt + \pi_k \, du^k \delta t \\
(\sigma)^* i_X d\boldsymbol{\Theta}_L =& \left(\left(\tfrac{\partial L}{\partial q^k} - \tfrac{d\pi_k}{dt} \right) \delta q^k + \left(\tfrac{\partial L}{\partial u^k} - \pi_k \right) \delta u^k - \left(\tfrac{\partial L}{\partial q^k} \dot{q}^k + \tfrac{\partial L}{\partial u^k} \dot{u}^k - \tfrac{d\pi_k}{dt} u^k - \pi_k \dot{u}^k \right) \delta t \right) \, dt = \left(\left(\tfrac{\partial L}{\partial q^k} - \tfrac{d\pi_k}{dt} \right) \delta q^k - \left(\tfrac{\partial L}{\partial q^k} \left(\dot{q}^k - u^k \right) + \left(\tfrac{\partial L}{\partial q^k} - \tfrac{d\pi_k}{dt} \right) u^k \right) \delta t \right) \, dt = 0
\end{aligned} \tag{1.11.90}
$$

If one considers a projectable vector field and a generic section on $\mathbb{R} \times TQ$ then Euler–Lagrange equations hold true and, for a generic Lagrangian, also the holonomic conditions for the section. However, if the Lagrangian has cyclic coordinates the corresponding holonomic condition has to be provided by hand.

If one starts from a holonomic section $\sigma = j^1\rho$ then for that Euler–Lagrange equations hold true.

In order to obtain Euler–Lagrange equations, one can restrict to vertical fields, or even to prolongations of vertical vector fields on $\mathbb{R} \times Q$ and still Euler–Lagrange equations follow.

Constrained systems

Let us consider two examples of constrained systems.

Consider a material point moving on a horizontal plane. The free system is described by Cartesian coordinates (x, y) and the dynamics is given by

$$\boldsymbol{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{1.11.91}$$

Then we impose the constraint $x^2 + y^2 = r^2$. There are no free motions on this constraint. However, we can derive constrained solutions as configurations of the free action critical *on the constraint*, i.e. configurations on the constraint which are critical points of the action functional only with respect to deformations compatible with the constraint. We reduce the deformations, we obtain more general critical configurations.

We know we can consider the variation of the free Lagrangian along deformations tangent to the constraint or, equivalently, consider general deformations of a Lagrangian modified by Lagrangian multipliers

$$\hat{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \alpha(x^2 + y^2 - r^2) \tag{1.11.92}$$

where α will be eventually interpreted as the force needed to keep the point on the constraint. The constrained Lagrangian L^* defines the momenta

$$p_x = \dot{x} \qquad\qquad p_y = \dot{y} \qquad\qquad p_\alpha = 0 \tag{1.11.93}$$

and we get the equations of motion

$$\ddot{x} = -2\alpha x \qquad\qquad \ddot{y} = -2\alpha y \qquad\qquad x^2 + y^2 = r^2 \tag{1.11.94}$$

By the way, we know the general solution to these equations, as

$$x(t) = x_0 \cos(\omega t) + \tfrac{\dot{x}_0}{\omega} \sin(\omega t) \qquad\qquad y(t) = y_0 \cos(\omega t) + \tfrac{\dot{y}_0}{\omega} \sin(\omega t) \tag{1.11.95}$$

where we set $2\alpha = \omega^2$ and initial conditions $(x_0, y_0, \dot{x}_0, \dot{y}_0)$. However, if we impose the constraint on initial conditions $x_0^2 + y_0^2 = r^2$, the material point does not stay on the constraint. In fact,

$$\begin{aligned} x^2 + y^2 - r^2 &= \left(x_0 \cos(\omega t) + \tfrac{\dot{x}_0}{2}\sin(\omega t)\right)^2 + \left(y_0 \cos(\omega t) + \tfrac{\dot{y}_0}{\omega}\sin(\omega t)\right)^2 - r^2 = (x_0^2 + y_0^2 - r^2)\cos^2(\omega t) + \tfrac{\dot{x}_0^2 + \dot{y}_0^2 - \omega^2 r^2}{\omega^2}\sin^2(\omega t) + 2\tfrac{x_0\dot{x}_0 + y_0\dot{y}_0}{\omega}\cos(\omega t)\sin(\omega t) = \\ &= \left(\tfrac{\dot{x}_0^2 + \dot{y}_0^2 - \omega^2 r^2}{\omega^2}\sin(\omega t) + 2\tfrac{x_0\dot{x}_0 + y_0\dot{y}_0}{\omega}\cos(\omega t)\right)\sin(\omega t) = 0 \end{aligned} \tag{1.11.96}$$

The material point stays on the constraint iff, beside starting on the constraint, one has $x_0\dot{x}_0 + y_0\dot{y}_0 = 0$, i.e. the initial velocity is orthogonal to the position, and $\dot{x}_0^2 + \dot{y}_0^2 = \omega^2 r^2$ which constrains the initial speed.

Once we impose these further constraints, they also need to be maintained by the dynamics. For,

$$\begin{aligned} x\dot{x} + y\dot{y} &= (x_0\cos(\omega t) + \tfrac{\dot{x}_0}{\omega}\sin(\omega t))(\dot{x}_0\cos(\omega t) - x_0\omega\sin(\omega t)) + (y_0\cos(\omega t) + \tfrac{\dot{y}_0}{\omega}\sin(\omega t))(\dot{y}_0\cos(\omega t) - y_0\omega\sin(\omega t)) = \\ &= (x_0\dot{x}_0 + y_0\dot{y}_0)\cos^2(\omega t) + \left(-(x_0^2 + y_0^2)\omega + \tfrac{\dot{x}_0^2 + \dot{y}_0^2}{\omega}\right)\cos(\omega t)\sin(\omega t) - (x_0\dot{x}_0 + y_0\dot{y}_0)\sin^2(\omega t) = \\ &= \left(\tfrac{\dot{x}_0^2 + \dot{y}_0^2 - \omega^2 r^2}{\omega} - (x_0^2 + y_0^2 - r^2)\omega\right)\cos(\omega t)\sin(\omega t) = 0 \end{aligned} \tag{1.11.97}$$

and

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 - \omega^2 r^2 &= (\dot{x}_0\cos(\omega t) - x_0\omega\sin(\omega t))^2 + (\dot{y}_0\cos(\omega t) - y_0\omega\sin(\omega t))^2 - \omega^2 r^2 = \\ &= (\dot{x}_0^2 + \dot{y}_0^2 - \omega^2 r^2)\cos^2(\omega t) + (x_0^2 + y_0^2 - r^2)\omega^2\sin^2(\omega t) - 2(x_0\dot{x}_0 + y_0\dot{y}_0)\cos(\omega t)\sin(\omega t) = 0 \end{aligned} \tag{1.11.98}$$

Also we have

$$\dot{x}^2 + \dot{y}^2 = 2\alpha r^2 \quad\Rightarrow\quad \alpha = \tfrac{1}{2}\frac{\dot{x}_0^2 + \dot{y}_0^2}{x_0^2 + y_0^2} = \tfrac{1}{2r^2}|v|^2 = \tfrac{1}{2}\omega^2 \tag{1.11.99}$$

That is a constant of motion, hence also ω is constant, and $F = -r\alpha u_r$ is the centripetal force needed to keep the material point on the constraint.

Let us define Q as the plane (x, y) and $\hat{Q} = Q \times \mathbb{R}$ as the extended plane $(x, y; \alpha)$.

For this system we can define the Poincaré-Cartan form on $TQ \times \mathbb{R}$ with coordinates $(q, u; \alpha)$

$$\boldsymbol{\Theta}_{\hat{L}} = \hat{L}\,dt + p(dq - u\,dt) = -\hat{\mathcal{H}}\,dt + p\,dq \tag{1.11.100}$$

where $\hat{\mathcal{H}} = \tfrac{1}{2}(u_x^2 + u_y^2) + \alpha(x^2 + y^2 - r^2)$ is the *total energy*. The Poincaré-Cartan form defines equivalent equations, it defines the Helmholtz Lagrangian on $T(TQ \times \mathbb{R})$ (on which we are using the coordinates $(q, u, \alpha, \dot{q}, \ddot{q}, \dot{\alpha})$ as an abuse of notation)

$$\hat{L}_H = \left(-\hat{\mathcal{H}}(q, u) + p(q, u)\dot{q}\right)dt = \left(-\tfrac{1}{2}(u_x^2 + u_y^2) - \alpha(x^2 + y^2 - r^2) + u_x\dot{x} + u_y\dot{y}\right)dt \tag{1.11.101}$$

If we repeat the construction for the free Lagrangian (which can be obtained simply by setting $\alpha = 0$) we obtain the Helmholtz Lagrangian as

$$L_H = \left(-\tfrac{1}{2}(u_x^2 + u_y^2) + u_x\dot{x} + u_y\dot{y}\right) \, dt \tag{1.11.102}$$

The corresponding free variation on TQ is

$$\delta L_H = \left(-(u_x\delta u_x + u_y\delta u_y) + \delta u_x\dot{x} + \delta u_y\dot{y} + u_x\delta\dot{x} + u_y\delta\dot{y}\right) \, dt = \left((-u_x + \dot{x})\delta u_x + (-u_y + \dot{y})\delta u_y - \dot{u}_x\delta x - \dot{u}_y\delta y\right) \, dt + d\left(u_x\delta x + u_y\delta y\right) \tag{1.11.103}$$

The equations endowed by independent variations of $(\delta q, \delta u)$ are

$$\dot{x} = u_x \qquad \dot{y} = u_y \qquad\qquad \dot{u}_x = 0 \qquad \dot{u}_y = 0 \tag{1.11.104}$$

which are, in fact, the first order form of the free equations, as we see by eliminating (u_x, u_y).

If we restrict to the variations tangent to the constraint $x^2 + y^2 = r^2$, we have instead that δL is orthogonal to the constraint, possibly non-zero.

$$\dot{x} = u_x \qquad \dot{y} = u_y \qquad\qquad \dot{u}_x = -2\alpha x \qquad \dot{u}_y = -2\alpha y \tag{1.11.105}$$

where α is the ratio between δL and the canonical normal covector $d(x^2 + y^2 - r^2)$. These, in fact, produce again the constrained equations.

Check that the free variation of (1.11.101)

$$\delta\hat{L}_H = \left((-u_x + \dot{x})\delta u_x + (\dot{y} - u_y)\delta u_y - (\dot{u}_x + 2\alpha x)\delta x - (\dot{u}_y + 2\alpha y)\delta y - \delta\alpha(x^2 + y^2 - r^2)\right) \, dt + d(u_x x + u_y y) \tag{1.11.106}$$

produces the same equations

$$\dot{x} = u_x \qquad \dot{y} = u_y \qquad\qquad \dot{u}_x = -2\alpha x \qquad \dot{u}_y = -2\alpha y \qquad\qquad x^2 + y^2 = r^2 \tag{1.11.107}$$

The Lagrangian $\hat{L}$ defines a degenerate Lagendre map since $p_\alpha = 0$. That defines the primary constraint S_1 as the surface $p_\alpha = 0$. One can however define the Hamiltonian on (and only on) the primary constraint $\phi^1 := p_\alpha = 0$ as

$$\hat{H} = \tfrac{1}{2}(p_x^2 + p_y^2) + \alpha(x^2 + y^2 - r^2) \tag{1.11.108}$$

If we start on the primary constraint S_1 we need to impose the solution stays on the constraint. That is equivalent to ask that X_H is tangent to S_1, which is equivalent to ask that

$$\{\phi^1, \hat{H}\} = -(x^2 + y^2 - r^2) =: -\phi^2 \tag{1.11.109}$$

on the constraint S_1. We see that starting on S_1 the solution is not going to stay on S_1 unless we add a further constraint $\phi^2 = 0$ that defines a new constraint $S_2 = \{\phi^1 = \phi^2 = 0\}$, which is called the *secondary constraint*.

Of course, now we start on S_2 and we want to know if the solution will stay on $S_2 \subset S_1$. For, we already know $\{\phi^1, H\} = 0$ and we need to check

$$\{\phi^2, \hat{H}\} = 4(xp_x + yp_y) =: 4\phi^3 \tag{1.11.110}$$

That defines the tertiary constraint S_3 by adding the condition $\phi^3 = 0$. Now we need to to check

$$\{\phi^3, \hat{H}\} = (p_x^2 + p_y^2 - 2\alpha r^2) - 2\alpha\phi^2 =: \phi^4 - 2\alpha\phi^2 \tag{1.11.111}$$

The we set $\phi^4 = 0$ to (finally) define S_4. The last check is

$$\{\phi^4, \hat{H}\} = -4\alpha\phi^3 = 0 \tag{1.11.112}$$

which is now satisfied on the constraint S_4 which is hence called the *final constraint* denoted by $S = S_4$. A solution with initial conditions on S stays on S.

The Hamiltonian $\hat{H}$ is originally defined on S_1, while the final dynamics is restricted to $S \subset S_1$. We can try to find out whether we can use a different Hamiltonian to describe the same dynamics on S. Such may be Hamiltonian H needs to coincide with $\hat{H}$ on S (hence $H = \hat{H} + u_a\phi^a$) and it still defines the same dynamics on S. Let us remark, in principle, $H = \hat{H}$ on S however they can differ on S_1, e.g. if $H = \hat{H} + u_2\phi^2 + u_3\phi^3 + u_4\phi^4$

The evolution of the system on $S \subset S_1$ just depends on

$$\{\phi^b, \hat{H}\} \approx \{\phi^b, H\} = \{\phi^b, \hat{H}\} + \{\phi^b, u_a\}\phi^a + u_a\{\phi^b, \phi^a\} \approx \{\phi^b, \hat{H}\} + u_a\{\phi^b, \phi^a\} \approx 0 \tag{1.11.113}$$

where $\approx$ stands for equality on constraint S, i.e. for equality modulo linear combinations of the constraints.

For the candidate equivalent Hamiltonian $H = \hat{H} + u_a\phi^a$ on the constraint S to be equivalent to $\hat{H}$ we must have

$$
\begin{aligned}
\{\hat{H}, \phi^1\} + u_a\{\phi^a, \phi^1\} &\approx -2r^2 u_4 \approx 0 && \Rightarrow && u_4 = 0 \\
\{\hat{H}, \phi^2\} + u_a\{\phi^a, \phi^2\} &\approx -2r^2 u_3 \approx 0 && \Rightarrow && u_3 = 0 \\
\{\hat{H}, \phi^3\} + u_a\{\phi^a, \phi^3\} &\approx 2r^2 u_2 - 4\alpha r^2 u_4 \approx 0 && \Rightarrow && u_2 = 0 \\
\{\hat{H}, \phi^4\} + u_a\{\phi^a, \phi^4\} &\approx 2r^2 u_1 + 4\alpha r^2 u_3 \approx 0 && \Rightarrow && u_1 = 0
\end{aligned}
\tag{1.11.114}
$$

Let us compute on the side

$$\{\phi^1, \phi^2\} = 0 \quad \{\phi^1, \phi^3\} = 0 \quad \{\phi^1, \phi^4\} = 2r^2 \quad \{\phi^2, \phi^3\} = 2\phi^2 + 2r^2 \quad \{\phi^2, \phi^4\} = 4\phi^3 \quad \{\phi^3, \phi^4\} = 2\phi^4 + 4\alpha r^2 \tag{1.11.115}$$

That gives a unique Hamiltonian

$$H = \hat{H} = \tfrac{1}{2}(p_x^2 + p_y^2) + \alpha(x^2 + y^2 - r^2) \tag{1.11.116}$$

In this case we completely determine the coefficients u_a (which depends on the non-degeneracy of the matrix $M^{ab} = \{\phi^a, \phi^b\}$) and the corresponding Hamiltonian on the final constraint S.

The second example we want to discuss is from the degenerate Lagrangian

$$\boldsymbol{L} = \tfrac{1}{2}(\dot{x} + \dot{y})^2 + V(x, y) \tag{1.11.117}$$

which defines the momenta

$$p_x = \dot{x} + \dot{y} = p_y \tag{1.11.118}$$

If we want to define a Hamiltonian formalism for this system, we have a primary constraint $\phi^1 = p_x - p_x$.

The total energy is

$$\mathcal{H} = \tfrac{1}{2}(\dot{x} + \dot{y})^2 - V(x, y) \tag{1.11.119}$$

and it defines a Hamiltonian on S_1

$$H = \tfrac{1}{2}p_x^2 - V(x, y) \tag{1.11.120}$$

This Hamiltonian is not well defined on $T^*\mathbb{R}^2$ since we could have equally well written it as

$$H_1 = \tfrac{1}{2}p_y^2 - V(x, y) = H - \tfrac{1}{2}\left(\phi^1\right)^2 \qquad\qquad H_2 = \tfrac{1}{2}p_x p_y - V(x, y) = H - \tfrac{1}{2}p_x\phi^1 \tag{1.11.121}$$

There are infinitely many expressions for the Hamiltonian $\hat{H} = H + u(q, p)\phi^1$ on $T^*\mathbb{R}^2$, all of them agree on S_1.

We need to enforce compatibility between the constraint and the dynamics

$$\{\phi^1, H\} = \partial_x V - \partial_y V \tag{1.11.122}$$

If the potential is in the form $V(x, y) := U(x + y)$, then it is called a *compatible potential* and one does not need any further compatibility conditions since $\{\phi^1, H\} = 0$.

If the potential is not compatible, we have different situations depending on the specific potential. For example, if $V = ax + by$ we have $\{\phi^1, H\} = a - b$ which is constant, non-zero iff $a \neq b$ and we cannot add any further constraint to obtain compatibility.

If we have $H = x^2 + y^2$ one has $\{\phi^1, H\} = 2(x + y)$ and we can enforce compatibility on the secondary constraint $\phi^2 = x + y = 0$. Then we have $\{\phi^2, H\} = 2p_x$, which enforces the tertiary constraint $\phi^3 = p_x$ which also replaces ϕ^1 with $\tilde{\phi}^1 = p_y$. Finally, we have

$$\{\phi^3, H\} = -2x \tag{1.11.123}$$

which enforces the final constraint $\phi^4 = x$, which also simplifies the second constraint to $\tilde{\phi}^2 = y$.

The final constraint is then zero-dimensional, which is not very interesting. The same thing happens with different, non-compatible, potentials.

Let us consider a compatible potential $V(x, y) := U(x + y)$, the primary constraint $\phi^1 = p_x - p_y$ is also the final constraint $S = S_1$ and Hamilton equations are

$$\dot{x} = p_x + \alpha \qquad \dot{y} = -\alpha \qquad\qquad \dot{p}_x = U' \qquad \dot{p}_y = U' \qquad\qquad \Rightarrow \qquad \ddot{x} = \dot{p}_x + \dot{\alpha} = U' + \dot{\alpha} \qquad \dot{y} = -\alpha \qquad \dot{p}_y = U' \tag{1.11.124}$$

We have one solution for each initial condition $(t_0, x_0, y_0, p_x^0, p_y^0)$ and for any function $\alpha(t)$ which is left completely undetermined.

Let us consider initial conditions $(x_0, y_0, \dot{x}_0)$ and for any function $\alpha(t)$ we can compute $y(t) = y_0 - \int_0^t \alpha(t)\, dt$ and replace it into the potential, which then depends on (t, x) only. The other equations read as

$$\dot{s} = t \qquad \dot{x} = p_x \qquad \dot{p}_x = U'(x + y(t)) \qquad\qquad \dot{p}_y = U'(x + y(t)) \tag{1.11.125}$$

That defines a flow on the space with coordinates (t, x, p_x, p_y). At least locally, it defines integral curves for any initial position (t_0, x_0, p_x^0, p_y^0).

We see that these equations are underdetermined, the solution depends on an arbitrary function $\alpha(t)$ which is not determined by the dynamics. As a direct consequence, one has to forget Cauchy theorem and determinism, or at least one needs to fight to have it back.

If we fix initial conditions $(t_0, x_0, y_0, p_x^0, p_y^0)$ and a function $\alpha(t)$ we have a solution. Then we can fix a different function $\alpha'(t) = \alpha(t) + \epsilon(t)$ to obtain a different solution. We have the same initial positions, if we have $\epsilon(0) = 0$ we even have the same initial velocities, still two different solutions.

This is a typical feature of these kind of systems in which evolution is not entirely determined by dynamics, we shall see similar examples even in field theory. If we want to enforce a deterministic of the physical state of the system, we have only one way out: the state of the system is not faithfully described by (x, y). There are many configurations which represent the same *physical state* of the system so that even if the evolution of the configuration is not uniquely determined, the evolution of the physical state still can be defined uniquely.

In other words, we define equivalent 2 configurations which describe the same physical state. Then physical state are points in the quotient space, i.e. classes of configurations. The equations for configurations need to be compatible with the quotient, i.e. they need to be covariant with respect to changes of representatives within the classes of equivalent configurations, so that they induce equations for the classes themselves. And, finally, we have that equations in the quotient obeys Cauchy theorem, they determine uniquely the evolution of the physical state for each initial physical state.

That structure is pretty common and we shall find it in different contexts, from mechanics to field theories. In the example we are considering here, the situation is a bit awkward. We are declaring that curves $(x(t), y(t))$ over-parameterize the physical state of the system we are considering, still we work hard to determine the evolution of the configuration. We introduce $\alpha(t)$ exactly to be able to determine the evolution of the configuration, at the same time we are arguing that $(x(t), y(t))$ cannot be given a physical meaning, only the physical state can. Since observing is a physical process and the result can classically depend only on the physical state, we are claiming that $(x(t), y(t))$ cannot be even observed, just quantities depending on the physical state can.

In this case the situation is pretty clear. Although $(x(t), y(t))$ cannot be observed, the quantity $z = x + y$ can. We can change coordinates to $z = x + y$ and $u = x - y$ and write equations in terms of the new coordinates (z, u).

Equations read as

$$\dot{z} = p \qquad \dot{u} = p + 2\alpha \qquad \dot{p} = U'(z) \tag{1.11.126}$$

from which we see that the evolution of z is uniquely determined by the equation $\ddot{z} = U'(z)$, while the evolution of u is left completely undetermined.

In this system, the physical state is described by z and u is called a *gauge* degrees of freedom.

The parameter α defines a flow generated by $X = \alpha(\partial_x - \partial_y)$, corresponding to

$$x' = x + \epsilon \qquad\qquad y' = y - \epsilon \tag{1.11.127}$$

This flow leaves the physical state unchanged and spans the gauge degree of freedom.

Of course, here one could ask why one should not simply write the system in terms of z and throw away u altogether. And, in fact, one can, in this case one should. The system can be written in terms of z simply by using the Lagrangian

$$\boldsymbol{L} = \left(\tfrac{1}{2} p^2 - U(z) \right) \, dt \tag{1.11.128}$$

Let us stress that that is because we assume (x, y) are global Cartesian coordinates on a plane. The definition of the physical state is then invariant with respect to changes of Cartesian coordinates.

However, if, for some reason, we consider more general changes of coordinates (e.g. because (x, y) are more general coordinates, maybe in a more general manifold, maybe *local* coordinates), the definition of physical state becomes observer-dependent and we trade off covariance with using physical quantities only.

This is more or less the approach to gauge systems proposed originally by Dirac (also, mainly, in view of quantization). One can also assume a different standpoint. Instead of describing constraints as level sets, we can use embeddings and adapted coordinates. Let us sketch here this strategy in this simple case.

The (primary, as well as final) constraint is parameterised by adapted coordinates $(x, y, z; \chi)$ in which the constraint is $\chi = p_x + p_y = 0$. The constraint is, in fact, a sub-manifold $S \subset T^*Q$ on which we choose coordinates (x, y, z) and we describe the embedding $\iota : S \to T^*Q : (x, y, z) \mapsto (x, y, p_x, p_y)$ given by $p_x = z = p_y$, so that $\chi \circ \iota = 0$, identically.

In view of this constraint, we have in general

$$\frac{\partial \chi}{\partial q} + \frac{\partial \chi}{\partial p_i} \frac{\partial p_i}{\partial q} = 0 \qquad \frac{\partial \chi}{\partial p_i} \frac{\partial p_i}{\partial z} = 0 \tag{1.11.129}$$

We can define the Poincaré-Cartan form on S directly as

$$\Theta_H = -H(x, y, z)dt + zdx + zdy \tag{1.11.130}$$

for the Hamiltonian $H = \frac{1}{2}z^2 - U(x + y)$ defined on S.

That defines directly Hamilton equations as $(\hat{\sigma})^* X \lrcorner d\Theta_H = 0$.

We have

$$dH = -dH \wedge dt + dz \wedge dx + dz \wedge dy \tag{1.11.131}$$

We consider a (vertical on $S \times \mathbb{R}$) vector field $X = \delta x \partial_x + \delta y \partial_y + \delta z \partial_z$ on S and we compute

$$X \lrcorner dH = -(\tfrac{\partial H}{\partial x}\delta x + \tfrac{\partial H}{\partial y}\delta y + \tfrac{\partial H}{\partial z}\delta z)dt + \delta z dx - \delta x dz + \delta z dy - \delta y dz \tag{1.11.132}$$

Now we can fix a configuration $\hat{\sigma}$ locally given by $(x(t), y(t), z(t))$ and compute

$$(\hat{\sigma})^* X \lrcorner dH = \left(\left(-\tfrac{\partial H}{\partial x} - \dot{z} \right) \delta x - \left(\tfrac{\partial H}{\partial y} + \dot{z} \right) \delta y \left(-\tfrac{\partial H}{\partial z} + \dot{x} + \dot{y} \right) \delta z \right) dt \tag{1.11.133}$$

Accordingly, we have Hamilton equations in the form

$$\dot{z} = -\tfrac{\partial H}{\partial x} = -\tfrac{\partial H}{\partial y} = U'(x + y) \qquad \dot{x} + \dot{y} = \tfrac{\partial H}{\partial z} = z \tag{1.11.134}$$

These are clearly implied by the equation obtained above. These do not introduce the extra gauge degree of freedom α. We can further improve this form by quotienting out the gauge flow.

Any observable $F : T^*Q \to R$ can be restricted to the final constraint $f = F|_S : S \to \mathbb{R}$. Dirac defines F to be *first class* when $\{\phi^a, F\} \approx 0$, *second class* otherwise.

In the current example the constraint is first class, as any constraint of codimension 1. In the example above, each ϕ^a is second class.

As any function on T^*Q, a constraint ϕ defines a (Hamiltonian) flow X_ϕ, which happens to be tangent to the final constraint S iff ϕ is first class. Accordingly, first class constraints defines flows on the constraint S and since

$$X_H(\phi) = \{H, \phi\} = -\{\phi, H\} = -X_\phi(H) \approx 0 \tag{1.11.135}$$

these flows are symmetries of the dynamics, they send one solution into solutions.

In terms of symplectic geometry, the final constraint is coisotropic iff all constraints ϕ^a are first class. In this case, X_ϕ is not only tangent to S, it is in fact characteristic, since ϕ is not only constant along solutions, they are constant on the whole final constraint S. (In physical terms, they are not only first integrals, they are constants on the phase space S, off-shell).

Given a first class constraint ϕ one has $\{\phi, \phi^a\}$, i.e. $\omega(d\phi, d\phi^a) = 0$, i.e. $\omega((d\phi)^\sharp, v) = 0$ for all $v \in TS$, which is the definition of $(d\phi)^\sharp$ being a characteristic vector.

As a matter of fact, $(d\phi^a)^\sharp$ span the characteristic distribution and their flows are locally transitive on the characteristic surfaces. When we quotient out the characteristic distribution we perform a symplectic reductions are we are left with a Hamiltonian system dealing with physical degrees of freedom only. Accordingly, points on one characteristic surface are different configuration representatives of the same physical state, which is identified with the quotient physical phase space.

In the example we are considering, we have the final constraint S which is a coisotropic surface in T^*Q. A vector tangent to the constraint is in the form

$$v = v_x \partial_x + v_y \partial_y + v \partial_{p_x} + v \partial_{p_y} \tag{1.11.136}$$

In fact we have

$$d\phi^1(v) = (dp_x - dp_y)(v) = v - v = 0 \tag{1.11.137}$$

A characteristic vector $u = u_x \partial_x + u_y \partial_y + u_{p_x} \partial_{p_x} + u_{p_y} \partial_{p_y}$ is such that

$$0 = \omega(u, v) = v \lrcorner (u_{p_x} dx - u_x dp_x + u_{p_y} dy - u_y dp_y) = u_{p_x} v_x + u_{p_y} v_y + (u_x - u_y)v \quad \Rightarrow \quad u \propto \partial_x - \partial_y \tag{1.11.138}$$

which is also tangent to S in view of the fact S is coisotropic.

The characteristic curves are $x = -y + c$. The characteristic vector $u = (d\phi^1)^\sharp$ is tangent to the characteristic curves and it drags representatives of a physical phase space. We see that $(x, y, z) \sim (x', y', z')$ iff they are on the same characteristic curve $x + y = c = x' + y'$.

Then (c, z) are coordinates on the quotient physical space. The induced symplectic form is $\tilde{\omega} = dz \wedge dc$. The Hamiltonian on the physical phase space is

$$\tilde{H} = \tfrac{1}{2}z^2 - U(c) \tag{1.11.139}$$

That is a description of the dynamics in terms of the physical degrees of freedom only.

These two example are not at all exhaustive. We just consider two very different examples of constrained system. One with second class constraint in which the constraint is imposed just in place of a holonomic constraint. As a consequence, some of the degrees of freedom are determined by the others and the constraint, not by the dynamics. The degrees of freedom which are not determined directly by the dynamics are however observables so that it makes sense to write them as part of the solution of the system.

In the other examples, we have a first class constraint. That corresponds to a gauge degree of freedom, which in this case we can also quotient out explicitly.

In practice the situation is much more complicated and these two situations mix together in all possible inconvenient ways. However, they provide a scheme to analyze complex situations.

Bisogna ricontrollare i segni e le convenzioni sulla forma simplettica e le parentesi di Poisson

Hamilton equations for constrained mechanical systems

We are now ready to discuss Hamilton equations for first order constrained mechanical systems in general and compare the different forms from describing constraints as level surface $\chi^a = 0$ or by choosing a parameterization.

Let us start from a degenerate Lagrangian $\boldsymbol{L} = L(t, q, u)\, dt$ which defines a primary constraint $\Lambda_1 \subset \mathbb{R} \times T^*Q$. We have a Legendre transform $\hat{\Phi}_L : \mathbb{R} \times TQ \to \mathbb{R} \times T^*Q : (t, q, u) \mapsto \left(t, q, p = \frac{\partial L}{\partial u}\right)$ which is not invertible, just as the Lagrangian is degenerate. We suppose the image to be a sub-manifold, which is the primary constraint.

The primary constraint Λ_1 can be described as a level set $\Lambda_1 = \{\chi^a(t, q, p) = 0\}$, as well as a sub-manifold $\Lambda_1 \subset \mathbb{R} \times T^*Q$, the embedding being $\iota : \Lambda_1 \to \mathbb{R} \times T^*Q : (t, q, z) \mapsto (t, q, p(t, q, z))$. We can redefine the Lagendre map as $\Phi_L : \mathbb{R} \times TQ \to \Lambda_1$ such that $\hat{\Phi}_L = \iota \circ \Phi_L$.

Let us also consider $T^*(\mathbb{R} \times Q)$ which is directly a symplectic manifold (and, in particular, a cotangent bundle). As such it carries the canonical Liouville form $\boldsymbol{\Theta} = p\, dt + \pi\, dq$ as well the canoinical symplectic form $\boldsymbol{\Omega} = d\boldsymbol{\Theta}$. We have a fibration $[T^*(\mathbb{R} \times Q) \to \mathbb{R} \times T^*Q]$.

We can define the (global) map $\psi_L : \mathbb{R} \times TQ \to T^*(\mathbb{R} \times Q) : (t, q, u) \mapsto (t, q, p, \pi)$ defined as

$$p = -\mathcal{H}(t, q, u) \qquad\qquad \pi = \frac{\partial L}{\partial u}(t, q, u) \tag{1.11.140}$$

We already discussed transformation laws which make this map global.

The Poincaré-Cartan form $\boldsymbol{\Theta}_L$ can be defined as a pull-back $\boldsymbol{\Theta}_L = (\psi_L)^* \boldsymbol{\Theta}$, which produces, equivalently, Euler-Lagrange equations. The map ψ_L also factorizes through the Legendre transform, as $\psi_H : \Lambda_1 \to T^*(\mathbb{R} \times Q)$ such that $\psi_L = \psi_H \circ \Phi_L$.

Consider a curve $\gamma : \mathbb{R} \to \mathbb{R} \times TQ : s \mapsto (t(s), q(s), u(s))$ and its image $\Phi_L \circ \gamma : \mathbb{R} \to \Lambda_1$.
That defines a single point on the primary constraint iff

$$\dot{t} = 0 \qquad\qquad \dot{q} = 0 \qquad\qquad \dot{\pi} = \frac{\partial L}{\partial t \partial u}\dot{t} + \frac{\partial L}{\partial q \partial u}\dot{q} + \frac{\partial L}{\partial u \partial u}\dot{u} = \frac{\partial L}{\partial u \partial u}\dot{u} = 0 \tag{1.11.141}$$

If the Lagrangian were non-degenerate that would imply $\dot{u} = 0$, but it is degenerate. Accordingly, $\dot{u}$ is in the kernel of the bilinear form $\frac{\partial L}{\partial u \partial u}$. The directions $\dot{u}$ are vertical with respect to the fibration $[\mathbb{R} \times TQ \to \Lambda_1]$ defined by the Legendre transform.
On the other hand, we have a curve $\psi_L \circ \gamma : \mathbb{R} \times T^*(\mathbb{R} \times Q)$ which defines a single point on $T*(\mathbb{R} \times Q)$ iff

$$\dot{\pi} = \frac{\partial L}{\partial u \partial u}\dot{u} = 0 \qquad\qquad \dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial t}\dot{t} + \frac{\partial \mathcal{H}}{\partial q}\dot{q} + \frac{\partial \mathcal{H}}{\partial u}\dot{u} = \frac{\partial L}{\partial u \partial u}u\dot{u} + \pi - \textcolor{red}{\frac{\partial L}{\partial u}} = 0 \tag{1.11.142}$$

In other words, the vertical directions along which Φ_L is constant are the same directions along which ψ_L is constant, i.e. the map ψ_L is compatible with the quotient to Λ where it defines the map $\psi_H : \Lambda_1 \to T^*(\mathbb{R} \times Q)$. This quotient map ψ_H defines, in particular, the Hamiltonian H on Λ_1

We can define the *Hamiltonian Poincaré-Cartan form* $\boldsymbol{\Theta}_H = (\psi_H)^* \boldsymbol{\Theta}$ and use it to define Hamilton equations. As we did on the Lagrangian framework we can describe equations intrinsically as $\hat{\gamma} : \mathbb{R} \times \Lambda_1$ is a Hamiltonian critical curve iff for all (vertical) vectors X on Λ we have

$$\hat{\gamma}^* X \lrcorner d\boldsymbol{\Theta}_H = 0 \tag{1.11.143}$$

For describing this condition in coordinates we can go through the Helmholtz Lagrangian $L_{Hel} = (-H(t,q,z) + p\dot{q})ds$ and obtain Hamilton equations as Euler Lagrange equations of the Helmholtz Lagrangian. The form we obtain depends on whether we describe the primary constraint as $\Lambda_1 = \{\chi^a(q,p) = 0\}$ or by parameterizing it by coordinates (t,q,z).

If we use coordinates (t,q,z) on Λ_1 we have

$$L_{Hel} = (-H(t,q,z) + p(t,q,z)\dot{q})\, ds \qquad \Rightarrow \qquad \delta L_{Hel} = \left(\left(-\frac{\partial H}{\partial q} + \frac{\partial p}{\partial q}\dot{q} - \dot{p} \right) \delta q + \left(-\frac{\partial H}{\partial z} + \frac{\partial p}{\partial z}\dot{q} \right) \delta z \right) ds + d(p\delta q) \tag{1.11.144}$$

Being variations along the constraint obtained for independent $(\delta q, \delta z)$, we have Hamilton equations in the form

$$\dot{p} = -\frac{\partial H}{\partial q} + \frac{\partial p}{\partial q}\dot{q} \qquad\qquad \frac{\partial p}{\partial z}\dot{q} = \frac{\partial H}{\partial z} \tag{1.11.145}$$

Equivalently, we can consider a Hamiltonian $\hat{H}(t,q,p)$ that extends $H(t,q,z)$ out of the primary constraint Λ_1, i.e. one has $\hat{H}(t,q,p(t,q,z)) = H(t,q,z)$, and we describe it as $\chi^a(t,q,p) = 0$, with of course $\chi^a(t,q,p(t,q,z)) = 0$. In this alternate setting, we have

$$L_{Hel} = \left(-\hat{H}(t,q,p) + p\dot{q} \right) ds \qquad \Rightarrow \qquad \delta L_{Hel} = \left(\left(-\frac{\partial \hat{H}}{\partial q} - \dot{p} \right) \delta q + \left(-\frac{\partial \hat{H}}{\partial p} + \dot{q} \right) \delta p \right) ds + d(p\delta q) \tag{1.11.146}$$

This time $(\delta q, \delta p)$ are not independent since variations are tangent to the constraint. We can consider the constraint $\chi^a(t,q,p) = 0$ and we have that

$$\dot{p} = -\frac{\partial \hat{H}}{\partial q} - \lambda_a \frac{\partial \chi^a}{\partial q} \qquad\qquad \dot{q} = \frac{\partial \hat{H}}{\partial p} + \lambda_a \frac{\partial \chi^a}{\partial p} \tag{1.11.147}$$

Although this form of Hamilton equations appears to be quite different from the one above, they are in fact equivalent.

In fact we have $\chi^a(t,q,p(t,q,z)) = 0$ from which we get the identities

$$\frac{\partial \chi^a}{\partial t} + \frac{\partial \chi^a}{\partial p}\frac{\partial p}{\partial t} = 0 \qquad\qquad \frac{\partial \chi^a}{\partial q} + \frac{\partial \chi^a}{\partial p}\frac{\partial p}{\partial q} = 0 \qquad\qquad \frac{\partial \chi^a}{\partial p}\frac{\partial p}{\partial z} = 0 \tag{1.11.148}$$

and we have $\hat{H}(t,q,p(t,q,z)) = H(t,q,z)$ from which we get the identities

$$\frac{\partial \hat{H}}{\partial t} + \frac{\partial \hat{H}}{\partial p}\frac{\partial p}{\partial t} = \frac{\partial H}{\partial t} \qquad\qquad \frac{\partial \hat{H}}{\partial q} + \frac{\partial \hat{H}}{\partial p}\frac{\partial p}{\partial q} = \frac{\partial H}{\partial q} \qquad\qquad \frac{\partial \hat{H}}{\partial p}\frac{\partial p}{\partial z} = \frac{\partial H}{\partial z} \tag{1.11.149}$$

We consider a solution $(q(t), z(t))$ of (1.11.145) and we show that $p(t) = p(t, q(t, z(t)))$ gives a solution of (1.11.147). In fact, for the first we have, we have

$$\dot{q} = \frac{\partial \hat{H}}{\partial p} + \lambda_a \frac{\partial \chi^a}{\partial p} \qquad \Rightarrow \qquad \frac{\partial p}{\partial z} \dot{q} = \frac{\partial \hat{H}}{\partial p} \frac{\partial p}{\partial z} + \lambda_a \frac{\partial \chi^a}{\partial p} \frac{\partial p}{\partial z} = \frac{\partial H}{\partial z} \tag{1.11.150}$$

while for the second we have

$$\dot{p} = -\frac{\partial H}{\partial q} + \frac{\partial p}{\partial q} \dot{q} = -\frac{\partial \hat{H}}{\partial q} - \frac{\partial \hat{H}}{\partial p} \frac{\partial p}{\partial q} + \frac{\partial p}{\partial q} \frac{\partial \hat{H}}{\partial p} + \lambda_a \frac{\partial \chi^a}{\partial p} \frac{\partial p}{\partial q} = -\frac{\partial \hat{H}}{\partial q} - \lambda_a \frac{\partial \chi^a}{\partial q} \tag{1.11.151}$$

The relativistic particle

Here we shall deal with the theory of a test particle moving in a spacetime M equipped with a (Lorentzian) metric g. We want to discuss the system both from the Lagrangian and Hamiltonian viewpoint. In fact, as a Hamiltonian system, it shows how to deal with constraints in mechanics and, in some sense, it is analogous to the a relativistic field theory, although in mechanics, since covariance with respect to reparameterizations mimics general covariance. This will be used below for a model of test particles in GR (and SR).

The relativistic material point is described by timelike trajectories on a Lorentzian spacetime (M, g). If one is doing special relativity (SR) then (M, g) is the Minkowski spacetime.

Since the state of the system is described by trajectories while Lagrangian formalism deals with parameterised curves $\gamma : \mathbb{R} \to M$, then reparameterisations must be symmetries of the system so that, as a consequence, if γ is a solution of equations of motion then any representative for the same trajectory $\gamma' \in [\gamma]$ is a solution as well. Accordingly, equations of motion are compatible with the quotient and they in fact define equations for trajectories.

The Lagrangian for the free point then cannot be simply

$$\hat{\boldsymbol{L}} = \tfrac{m}{2} g_{\mu\nu} u^\mu u^\nu \, ds \tag{1.11.152}$$

since this is not covariant with respect to arbitrary reparameterisations. One could use the Lagrangian

$$\tilde{\boldsymbol{L}} = \tfrac{m}{2} g_{\mu\nu} \frac{u^\mu}{u^0} u^\nu \, ds \tag{1.11.153}$$

which is covariant with respect to reparameterizations. However, the time coordinate is treated differently from other coordinates and we already know that this Lagrangian describes in a homogeneous framework the standard Newtonian particle. Let us consider instead the somewhat simpler Lagrangian

$$\boldsymbol{L} = m c \sqrt{-g_{\mu\nu} u^\mu u^\nu} \, ds \tag{1.11.154}$$

which is also covariant with respect to arbitrary reparameterizations.

Let us write the variation

$$\delta \boldsymbol{L} = \frac{-mc}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} u^\mu g_{\mu\sigma} \left(\delta u^\sigma + \{g\}^\sigma_{\nu\alpha} u^\nu \delta x^\alpha \right) \, ds = \frac{d}{ds} \left(\frac{-mc}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} u^\mu g_{\mu\alpha} \delta x^\alpha \right) \, ds + \left(\frac{d}{ds} \left(\frac{mc}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} u^\mu g_{\mu\alpha} \right) - \frac{mc}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} u^\mu g_{\mu\sigma} \{g\}^\sigma_{\nu\alpha} u^\nu \right) \delta x^\alpha \, ds \tag{1.11.155}$$

From the first variation formula we get

$$\mathbb{E} = \left(\frac{d}{ds} \left(\frac{m\mathbb{c}}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} u^\mu g_{\mu\alpha} \right) - \frac{m\mathbb{c}}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} u^\mu g_{\mu\sigma}\{g\}^\sigma_{\nu\alpha}u^\nu \right) \bar{d}x^\alpha \otimes \, ds \qquad \mathbb{F} = \frac{-m\mathbb{c}}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} u^\mu g_{\mu\alpha}\bar{d}x^\alpha \otimes \, ds \qquad (1.11.156)$$

The Euler–Lagrange equations for the Lagrangian (1.11.154) are

$$\ddot{x}^\lambda + \{g\}^\lambda_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \tfrac{d}{ds}\left(\ln(|\dot{x}|)\right)\dot{x}^\lambda \qquad (1.11.157)$$

which are invariant with respect to reparameterisations, meaning that if $x^\lambda(s)$ is a solution, then any $x'^\lambda(s') = x^\lambda(s(s'))$ is again a solution for any reparameterization $\phi : \mathbb{R} \to \mathbb{R} : s' \mapsto s(s')$.

The first and the second derivatives, namely $\dot{x}^\mu$ and $\ddot{x}^\lambda$, transform as

$$\dot{x}'^\lambda = \phi'\dot{x}^\lambda \qquad\qquad \ddot{x}'^\lambda = (\phi')^2\ddot{x}^\lambda + \phi''\dot{x}^\lambda \qquad (1.11.158)$$

The left hand side of the equations of motion is denoted by $a^\lambda = \ddot{x}^\lambda + \{g\}^\lambda_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ and it is called the *covariant acceleration*; it transforms as a tangent vector to M with respect to changes of coordinates

$$x'^\lambda(x) \qquad \dot{x}'^\lambda = J^\lambda_\alpha \dot{x}^\alpha \qquad \ddot{x}'^\lambda = J^\lambda_\alpha \ddot{x}^\alpha + J^\lambda_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \quad \Rightarrow$$
$$a'^\lambda = \ddot{x}'^\lambda + \{g\}'^\lambda_{\alpha\beta}\dot{x}'^\alpha\dot{x}'^\beta = J^\lambda_\alpha \ddot{x}^\alpha + J^\lambda_{\alpha\beta}\dot{x}^\alpha\dot{q}^\beta + J^\lambda_\mu\left(\{g\}^\mu_{\rho\sigma}\bar{J}^\rho_\alpha\bar{J}^\sigma_\beta + \bar{J}^\mu_{\alpha\beta}\right)J^\alpha_\gamma \dot{x}^\gamma J^\beta_\delta \dot{x}^\delta = J^\lambda_\alpha \ddot{x}^\alpha + \textcolor{red}{J^\lambda_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} + \left(J^\lambda_\alpha\{g\}^\alpha_{\gamma\delta} - \textcolor{red}{J^\lambda_{\gamma\delta}}\right)\dot{x}^\gamma\dot{x}^\delta = J^\lambda_\alpha\left(\ddot{x}^\alpha + \{g\}^\alpha_{\gamma\delta}\dot{x}^\gamma\dot{x}^\delta\right) = J^\lambda_\alpha a^\alpha \qquad (1.11.159)$$

while with respect to reparameterizations it transforms as

$$a'^\lambda = \ddot{x}'^\lambda + \{g\}^\lambda_{\alpha\beta}\dot{x}'^\alpha\dot{x}'^\beta = (\phi')^2\ddot{x}^\lambda + \phi''\dot{x}^\lambda + \{g\}^\lambda_{\alpha\beta}(\phi')^2\dot{x}^\alpha\dot{x}^\beta = (\phi')^2 a^\lambda + \phi''\dot{x}^\lambda \qquad (1.11.160)$$

The right hand side of the equations of motion transforms with respect to reparameterizations as

$$\tfrac{d}{ds'}\left(\ln(|\dot{x}'|)\right)\dot{x}'^\lambda = \tfrac{d}{ds'}\left(\ln(\phi'|\dot{x}|)\right)\phi'\dot{x}^\lambda = \tfrac{d}{ds}\left(\ln(|\dot{x}|)\right)(\phi')^2\dot{x}^\lambda + \phi''\dot{x}^\lambda \qquad (1.11.161)$$

Accordingly, the equations of motion transform as

$$a'^\lambda - \tfrac{d}{ds'}\left(\ln(|\dot{x}'|)\right)\dot{x}'^\lambda = (\phi')^2 a^\lambda + \textcolor{red}{\phi''\dot{x}^\lambda} - \tfrac{d}{ds}\left(\ln(|\dot{x}|)\right)(\phi')^2\dot{x}^\lambda - \textcolor{red}{\phi''\dot{x}^\lambda} = (\phi')^2\left(a^\lambda - \tfrac{d}{ds}\left(\ln(|\dot{x}|)\right)\dot{x}^\lambda\right) \qquad (1.11.162)$$

so that, the reparameterization of a solution is still a solution.

One can show that, for any solutions $x(s)$ of equations of motions, there is always a reparameterisation $x'(s')$ of it which is solution of

$$\ddot{x}'^\lambda + \{g\}^\lambda_{\alpha\beta}\dot{x}'^\alpha\dot{x}'^\beta = 0 \qquad (1.11.163)$$

and, on the other hand, for any $x'(s')$ solutions of this, any reparameterisation of it, namely $x(s)$, is a solution of (1.11.157).

The Euler-Lagrange equations are not in normal form, nor they are independent as we can see on Minkowski spacetime in Cartesian coordinates.

Equations are

$$\dot{u}^\lambda = \frac{d}{ds}\left(\ln\left(\sqrt{(u^0)^2 - \delta_{kh}u^k u^h}\right)\right)u^\lambda = \frac{u^0\dot{u}^0 - \delta_{ij}u^j\dot{u}^i}{(u^0)^2 - \delta_{kh}u^k u^h}u^\lambda \tag{1.11.164}$$

They can be written equivalently as

$$\left((u^0)^2 - \delta_{kh}u^k u^h\right)\dot{u}^\lambda = \left(u^0\dot{u}^0 - \delta_{ij}u^j\dot{u}^i\right)u^\lambda \tag{1.11.165}$$

which are clearly not in normal form.
We can split them as

$$(u^0)^2\dot{u}^0 - |\vec{u}|^2\dot{u}^0 = (u^0)^2\dot{u}^0 - u^0\vec{u}\cdot\dot{\vec{u}} \qquad\qquad (u^0)^2\dot{u}^i - |\vec{u}|^2\dot{u}^i = \left(u^0\dot{u}^0 - \vec{u}\cdot\dot{\vec{u}}\right)u^i \tag{1.11.166}$$

from which independence is not clear.
Let us introduce new coordinates $\left(N^2 = -|u|^2, v^i = \frac{u^i}{u^0}\right)$ to replace u^λ. This is simply a change of coordinates in $T_x M$, in fact, we can solve it

$$\begin{cases} v^i = \dfrac{u^i}{u^0} \\ N^2 = (u^0)^2\left(1 - |\vec{v}|^2\right) \end{cases} \qquad\qquad \begin{cases} u^i = \dfrac{Nv^i}{\sqrt{1 - |\vec{v}|^2}} \\ u^0 = \dfrac{N}{\sqrt{1 - |\vec{v}|^2}} \end{cases} \tag{1.11.167}$$

For the derivative (or equivalently, by a tangent prolongation) we have

$$N\dot{N} = -u^\alpha\eta_{\alpha\beta}\dot{u}^\beta = u^0\dot{u}^0 - \vec{u}\cdot\dot{\vec{u}} \qquad\qquad \dot{v}^i = \frac{u^0\dot{u}^i - u^i\dot{u}^0}{(u^0)^2} \tag{1.11.168}$$

Then we can use equations

$$N\dot{N} = \left((u^0)^2 - |\vec{u}|^2\right)\frac{\dot{u}^0}{u^0} = N^2\frac{\dot{u}^0}{u^0} \qquad \left((u^0)^2 - |\vec{u}|^2\right)\dot{u}^i = \left((u^0)^2 - |\vec{u}^0|^2\right)\frac{\dot{u}^0}{u^0}u^i \qquad \begin{cases} \dot{N} = N\dfrac{\dot{u}^0}{u^0} \\ \dot{u}^i = u^i\dfrac{\dot{u}^0}{u^0} \end{cases} \Rightarrow \dot{v}^i = \frac{u^i\dot{u}^0 - u^i\dot{u}^0}{(u^0)^2} = 0 \tag{1.11.169}$$

That forces v^i to be constant, $N = u^0\sqrt{1 - |\vec{v}|^2}$ (as well as u^0) are left unconstrained and they account for reparameterization freedom.
Once we know v^i are constants, then we can integrate the motion as

$$\begin{cases} x^i(s) = x_o^i + \dfrac{v^i s}{\sqrt{1 - |\vec{v}|^2}} \\ x^0(s) = x_o^0 + \dfrac{s}{\sqrt{1 - |\vec{v}|^2}} \end{cases} \tag{1.11.170}$$

where we set the parameterization as $s = \int_{s_o}^s N\,ds$ for an arbitrary function $N(s)$.
To summarize, Euler-Lagrange equations correspond to $(m-1)$-independent equations $\dot{v}^i = 0$. They do not determine a unique solution, the solution is determined up to an arbitrary function that represents the parameterization, instead.

Since equations (1.11.163) are Euler–Lagrange equations for the Lagrangian (1.11.152) this justifies using Lagrangian (1.11.152) in place of Lagrangian (1.11.154). Another reason for justifying Lagrangian (1.11.152) instead Lagrangian (1.11.154), although it does not implement the fundamental invariance with respect to reparameterisations, is that the invariant Lagrangian (1.11.154) is degenerate. In fact, momenta are

$$p_\mu = m\mathrm{c}\frac{-g_{\mu\nu}u^\nu}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} \tag{1.11.171}$$

This cannot be inverted with respect to velocity since the map $\hat{\Phi}_L : TM \to T^*M : (x, u) \mapsto (x, p)$ is not surjective since one has

$$|p|^2 := p_\mu g^{\mu\nu} p_\nu = m^2 c^2 \frac{g_{\mu\alpha} u^\alpha g^{\mu\nu} g_{\nu\beta} u^\beta}{-g_{\mu\nu} u^\mu u^\nu} = -m^2 c^2 \tag{1.11.172}$$

Thus one has a sub-manifold $\Lambda = \text{Im}(\hat{\Phi}_L) \subset T^*M$ given by

$$\Lambda := \{(x, p) : |p|^2 = -m^2 c^2\} \tag{1.11.173}$$

which is the level set of the constraint $\chi = p_\alpha g^{\alpha\beta} p_\beta + m^2 c^2$. That is called the *primary constraint*. We can define the map $\Phi_L : TM \to \Lambda$ which in fact is the projection of a fiber bundle.

A tangent vector $w = v^\mu \frac{\partial}{\partial x^\mu} + w^\mu \frac{\partial}{\partial u^\mu}$ to TM is vertical in the fibration $\hat{\Phi}_L : TM \to T^*M$ iff $T\hat{\Phi}_L(w) = 0$. The map tangent map $T\hat{\Phi}_L : TTM \to TT^*M$ is obtained by differentiating the equation $F(x, u) = F(x, p(x, u))$, to obtain

$$T\hat{\Phi}_L\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\mu} + \frac{\partial p_\alpha}{\partial x^\mu}\frac{\partial}{\partial p_\alpha} \qquad\qquad T\hat{\Phi}_L\left(\frac{\partial}{\partial u^\mu}\right) = \frac{\partial p_\alpha}{\partial u^\mu}\frac{\partial}{\partial p_\alpha} \tag{1.11.174}$$

In view of this, we have

$$T\hat{\Phi}_L(w) = v^\mu \frac{\partial}{\partial u^\mu} + \left(v^\mu \frac{\partial p_\alpha}{\partial x^\mu} + w^\mu \frac{\partial p_\alpha}{\partial u^\mu}\right)\frac{\partial}{\partial p_\alpha} = 0 \qquad \Rightarrow \qquad v^\mu = 0, \, w^\mu \frac{\partial L}{\partial u^\mu \partial u^\alpha} = 0 \tag{1.11.175}$$

Thus we see vertical vectors are related directly to the degeneracy of the Lagrangian. For the Lagrangian we are considering, we have

$$w^\mu \frac{\partial L}{\partial u^\mu \partial u^\nu} = \frac{mc}{(-g_{\alpha\beta} u^\alpha u^\beta)^{3/2}} w^\mu (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}) u^\rho u^\sigma = 0 \tag{1.11.176}$$

This condition is clearly satisfied by $w^\mu = u^\mu$ (just check that (1.11.176) becomes $u^\mu g_{\mu\nu} |u|^2 - u^\rho g_{\rho\nu} |u|^2 = 0$).

Moreover, when we fix a point $x \in \Lambda$ we can fix coordinates such that $g_x = \eta$ and we can use the residual Lorentzian freedom to set $u^\mu = \delta_0^\mu$ and $w^2 = w^3 = 0$. In this special coordinates we see that $w^\mu = u^\mu$ is the only solution at x which, of course, is independent of coordinates.

Accordingly, vertical vectors are in the form $v = \alpha u^\mu \frac{\partial}{\partial u^\mu}$. Those are radial vectors in the fibers of TM. A vertical vector field then has a flow that rescales (at any point) the velocity u which is something that does not affect the dynamics (which in fact depends only on the projection into T^*M).

In other words, we confirmed that the norm of u is not observable, only its direction is.

The vertical vectors define a distribution of rank 1, which is automatically involutive (being of rank 1) and the integral lines of it are the fibers of the fibration defined by $[\Phi_L : TM \to \Lambda]$.

Vectors in $T\Lambda$ are in the form $v = v^\mu \partial_\mu + v_\mu \partial^\mu$; if this is tangent to Λ, one has $v \in [\gamma = (x(s), p(s))]$, thus

$$p_\mu g^{\mu\nu} p_\nu = -m^2 c^2 \quad \Rightarrow 2v_\mu g^{\mu\nu} p_\nu + p_\mu p_\nu \partial_\alpha g^{\mu\nu} v^\alpha = 2v_\mu g^{\mu\nu} p_\nu + \partial_\alpha (p_\mu p_\nu g^{\mu\nu}) v^\alpha = 2v_\mu g^{\mu\nu} p_\nu + \partial_\alpha \left(-m^2 c^2\right) v^\alpha = 2v_\mu g^{\mu\nu} p_\nu = 0 \tag{1.11.177}$$

That defines a distribution Δ of rank $2m - 1$ on T^*M. Let us show it is involutive by considering two generators $X = X^\mu \partial_\mu + X_\mu \partial^\mu$, $Y = Y^\mu \partial_\mu + Y_\mu \partial^\mu$ (hence such that $X_\mu g^{\mu\nu} p_\nu = 0$ and $Y_\mu g^{\mu\nu} p_\nu = 0$) and consider their commutator

$$[X, Y] = (X^\alpha \partial_\alpha Y^\mu + X_\alpha \partial^\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu - Y_\alpha \partial^\alpha X^\mu)\partial_\mu + (X^\alpha \partial_\alpha Y_\mu + X_\alpha \partial^\alpha Y_\mu - Y^\alpha \partial_\alpha X_\mu - Y_\alpha \partial^\alpha X_\mu)\partial^\mu \tag{1.11.178}$$

We need to show $[X, Y]$ is, in fact, tangent to Λ namely that

$$
\begin{aligned}
[X,Y]_\mu g^{\mu\alpha} p_\alpha =& (X^\nu \partial_\nu Y_\mu + X_\nu \partial^\nu Y_\mu - Y^\nu \partial_\nu X_\mu - Y_\nu \partial^\nu X_\mu) g^{\mu\alpha} p_\alpha = X^\nu \partial_\nu Y_\mu g^{\mu\alpha} p_\alpha + X_\nu \partial^\nu Y_\mu g^{\mu\alpha} p_\alpha - Y^\nu \partial_\nu X_\mu g^{\mu\alpha} p_\alpha - Y_\nu \partial^\nu X_\mu g^{\mu\alpha} p_\alpha = \\
=& - \tfrac{1}{2} \left(X^\nu p_\rho p_\sigma \partial_\nu \left(\partial_\alpha g^{\rho\sigma} Y^\alpha \right) + X_\nu \partial_\alpha g^{\rho\sigma} \partial^\nu \left(p_\rho p_\sigma Y^\alpha \right) - Y^\nu p_\rho p_\sigma \partial_\nu \left(\partial_\alpha g^{\rho\sigma} X^\alpha \right) - Y_\nu \partial_\alpha g^{\rho\sigma} \partial^\nu \left(p_\rho p_\sigma X^\alpha \right) \right) + \\
& - {\color{red}X^\nu Y_\mu \partial_\nu g^{\mu\alpha} p_\alpha} - {\color{red}X_\nu Y_\mu g^{\mu\nu}} + {\color{blue}Y^\nu X_\mu \partial_\nu g^{\mu\alpha} p_\alpha} + {\color{blue}Y_\nu X_\mu g^{\mu\nu}} = \\
=& - \tfrac{1}{2} \left(X^\nu \partial_\nu Y^\alpha + X_\nu \partial^\nu Y^\alpha - Y^\nu \partial_\nu X^\alpha - Y_\nu \partial^\nu X^\alpha \right) \partial_\alpha g^{\rho\sigma} p_\rho p_\sigma - {\color{blue}X_\nu \partial_\alpha g^{\nu\sigma} p_\sigma Y^\alpha} + {\color{red}Y_\nu \partial_\alpha g^{\nu\sigma} p_\sigma X^\alpha} + \\
& - {\color{red}X^\nu Y_\mu \partial_\nu g^{\mu\alpha} p_\alpha} + {\color{blue}Y^\nu X_\mu \partial_\nu g^{\mu\alpha} p_\alpha} = -\tfrac{1}{2} [X,Y]^\alpha \partial_\alpha g^{\rho\sigma} p_\rho p_\sigma
\end{aligned}
\tag{1.11.179}
$$

Hence the distribution is integrable and, in fact, it is tangent to Λ which is an integral surface of that distribution.

The primary constraint Λ is a sub-manifold of codimension 1, thus it is coisotropic, hence we say that it is a *first class constraint*. Being coisotropic, it supports a characteristic distribution

$$
\Delta = \{ w \in TT^*M : \forall v \in T\Lambda : \boldsymbol{\omega}(w, v) = 0 \}
\tag{1.11.180}
$$

We have $w = w^\alpha \partial_\alpha + w_\alpha \partial^\alpha$ and $v = v^\alpha \partial_\alpha + v_\alpha \partial^\alpha$ which is tangent to Λ, i.e. $v_\mu g^{\mu\nu} p_\nu = 0$. Then one has

$$
0 = \boldsymbol{\omega}(w, v) = w_\alpha v^\alpha - w^\alpha v_\alpha \qquad \Rightarrow \qquad w_\alpha = 0, \ w^\alpha v_\alpha = w^0 v_0 + w^i v_i = 0
\tag{1.11.181}
$$

Since $p_\mu g^{\mu\nu} p_\nu = -m^2 c^2$, the covector p is timelike, then $p_0 \neq 0$. By the tangent condition $v_0 p^0 = -v_i g^{i\nu} p_\nu$, w is characteristic iff for all vectors v tangent to Λ $\left(w^0 g^{i\nu} p_\nu - w^i p^0 \right) v_i = 0$ iff $w^0 g^{i\nu} p_\nu = w^i p^0$.

Hence characteristic vectors are generated $w = g^{0\nu} p_\nu \partial_0 + g^{i\nu} p_\nu \partial_i = p_\nu g^{\mu\nu} \partial_\mu$. This is in fact tangent to Λ as

$$
w_\mu g^{\mu\nu} p_\nu = 0
\tag{1.11.182}
$$

The characteristic distribution is of rank 1, then involutive, then integrable. Then we can find characteristic curves tangent to it as

$$
\dot{x}^\mu = g^{\mu\alpha} p_\alpha \qquad\qquad \dot{p}_\alpha = 0
\tag{1.11.183}
$$
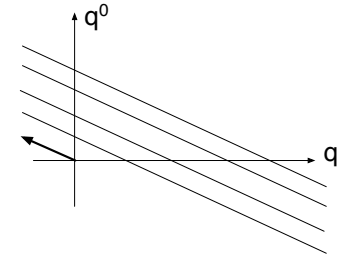
Fig 1.1: Characteristic distribution for relativistic point

In Minkowski spacetime $(p_0)^2 = m^2 c^2 + |\vec{p}|^2 \neq 0$, we have characteristics curves explicitly as constant momenta and

$$x^0 = \sqrt{m^2 c^2 + |\vec{p}|^2}\, s + x_0^0 \qquad\qquad x^i = \delta^{ik} p_k\, s + x_0^i \qquad\qquad (1.11.184)$$

These correspond to curves in Fig. 1.1. Since we are considering the case $p_0 \neq 0$ the lines are never horizontal.

Accordingly, the axis $x^0 = x_o^0$ is transverse to the characteristics.

Two points in Λ are equivalent iff they lie on the same characteristics and we define the quotient

$$S := \Lambda / \sim \qquad\qquad (1.11.185)$$

with coordinates (x^i, p_i) together with the embedding

$$\varphi : S \to T^* M : (x^i, p_i) \mapsto (x_o^0, x^i, \sqrt{m^2 c^2 + |\vec{p}|^2}, p_i) \qquad\qquad (1.11.186)$$

On Λ, we can define the symplectic form

$$\hat{\boldsymbol{\omega}} = \ dp_i \wedge \ dx^i = \varphi^* \boldsymbol{\omega} \qquad\qquad (1.11.187)$$

Notice that Λ is a cotangent bundle since we can define also a Liouville form

$$\hat{\boldsymbol{\theta}} = \varphi^* \boldsymbol{\theta} = p_i\, dx^i \qquad \hat{\boldsymbol{\omega}} = d\hat{\boldsymbol{\theta}} \qquad\qquad (1.11.188)$$

from which one gets that $\Lambda \simeq T^* \mathbb{R}^3$. The Poincaré–Cartan form induced by the Lagrangian (1.11.154) is

$$\boldsymbol{\Theta}_L = L\, ds + \frac{\partial L}{\partial u^\mu}\, (\ dx^\mu - u^\mu\, ds) \qquad\qquad (1.11.189)$$

We already showed that a section γ of $\mathbb{R} \times M$ is critical iff $\forall X \in \mathfrak{X}(\mathbb{R} \times TM)$

$$(\hat{\gamma})^* X \lrcorner d\boldsymbol{\Theta}_L = 0 \qquad\qquad (1.11.190)$$

where $\hat{\gamma}$ denotes the tangent lift of γ to $\mathbb{R} \times TM$.

The form $\boldsymbol{\Theta}_L$ induces a form $\boldsymbol{\Theta}_H$ on Λ such that $(\Phi_L)^* \boldsymbol{\Theta}_H = \boldsymbol{\Theta}_L$, namely

$$\boldsymbol{\Theta}_H = -\left(p_\mu u^\mu - L\right)\, ds + p_\mu\, dx^\mu = -H\, ds + p_\mu\, dx^\mu \qquad\qquad (1.11.191)$$

where for us one has

$$H = -\frac{mc g_{\mu\nu} u^\nu}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} u^\mu - mc\sqrt{-g_{\mu\nu} u^\mu u^\nu} = mc\sqrt{-g_{\mu\nu} u^\mu u^\nu} - mc\sqrt{-g_{\mu\nu} u^\mu u^\nu} = 0 \qquad\qquad (1.11.192)$$

so that

$$\boldsymbol{\Theta}_H = p_\mu \, dx^\mu = p_0 \, dx^0 + p_i \, dx^i \tag{1.11.193}$$

where p_0 solves the primary constraint $(p_0)^2 g^{00} + 2 p_0 g^{0i} p_i + p_i g^{ik} p_k = -m^2 \mathrm{c}^2$.

It also shows that, since $H = 0$ on the primary constraint Λ, there is no secondary constraint and Λ is also the final constraint. As a primary constraint, Λ is obviously first class.

For example, on Minkowski spacetime we have

$$\boldsymbol{\Theta}_H = \sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2} \, dx^0 + p_i \, dx^i \qquad (p_0)^2 - |\vec{p}|^2 = m^2 \mathrm{c}^2 \tag{1.11.194}$$

That is also a Poincaré-Cartan form on S for the Hamiltonian $h = -\sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2}$, which should describe the dynamics after we removed the constraints.

Also the form $\boldsymbol{\Theta}_H$ contains the information about dynamics. In fact, a section $\sigma : \mathbb{R} \to \mathbb{R} \times T^*M$ is critical iff $\forall X \in \mathfrak{X}(\Lambda)$ one has

$$(\sigma)^* X \lrcorner d\boldsymbol{\Theta}_H = 0 \tag{1.11.195}$$

One has $\sigma : \mathbb{R} \to \mathbb{R} \times T^*M : s \mapsto (x^\mu(s), p_\mu(s))$ and $X \in \mathfrak{X}(\Lambda)$ iff

$$X = X^\mu \partial_\mu + X_\mu \partial^\mu \tag{1.11.196}$$

where $X_\mu g^{\mu\nu} p_\nu = 0$, hence $X = X^\mu \partial_\mu + X_i \left(g^{0\nu} p_\nu \partial^i - g^{i\nu} p_\nu \partial^0 \right)$. In our case, we also have

$$d\boldsymbol{\Theta}_H = dp_\mu \wedge dx^\mu \quad \Rightarrow X \lrcorner d\boldsymbol{\Theta}_H = -X^\mu \, dp_\mu + X_i \left(g^{0\nu} p_\nu dx^i - g^{i\nu} p_\nu dx^0 \right) \tag{1.11.197}$$

Thus one has

$$(\sigma)^* X \lrcorner d\boldsymbol{\Theta}_H = -X^\mu \dot{p}_\mu \, ds + X_i \left(g^{0\nu} p_\nu \dot{x}^i - g^{i\nu} p_\nu \dot{x}^0 \right) \, ds \tag{1.11.198}$$

$$\dot{p}_\mu = 0 \qquad g^{0\nu} p_\nu \dot{x}^i = g^{i\nu} p_\nu \dot{x}^0 \quad \Rightarrow \quad \dot{x}^i = \alpha g^{i\nu} p_\nu, \, \dot{x}^0 = \alpha g^{0\nu} p_\nu \qquad \Rightarrow \quad \dot{x}^\mu = \alpha g^{\mu\nu} p_\nu \tag{1.11.199}$$

On Minkowski, that shows that momenta are constant and, because of the primary constraint, $p_0 = \sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2}$. Then p_i parameterized solutions and for each solution we have

$$\dot{x}^0 = -\alpha p_0 = -\alpha \sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2} \qquad \dot{x}^i = \alpha \delta^{ik} p_k = \alpha p^i \tag{1.11.200}$$

which can be integrated to

$$x^0(s') = -\sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2} \, \alpha s' + x_o \qquad x^i(s') = p^i \, \alpha s' + x_o^i \tag{1.11.201}$$

which agrees with (1.11.170) just by changing the parameterization to $s = -m\mathrm{c}\alpha s'$ and $p^i = -\frac{m\mathrm{c}\, v^i}{\sqrt{1 - |\vec{v}|^2}}$ and, consequently, $p_0 = m\mathrm{c}\sqrt{1 + \frac{|\vec{v}|^2}{1 - |\vec{v}|^2}} = \frac{m\mathrm{c}}{\sqrt{1 - |\vec{v}|^2}}$.

We can consider (x^0, x^i, p_i) as coordinates on S. The form $\boldsymbol{\Theta}_H = p_\mu dq^\mu$ on Λ can be pulled back on S as

$$\boldsymbol{\Theta}_h = \sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2} \, dq^0 + p_i dq^i \tag{1.11.202}$$

This produces Hamilton equations for the Hamiltonian h, namely

$$\dot{x}^i = \frac{\partial h}{\partial p_i} = -\frac{\delta^{ik} p_k}{\sqrt{m^2 \mathrm{c}^2 + |\vec{p}|^2}} = v^i \qquad \dot{p}_i = 0 \tag{1.11.203}$$

12. Epilogue

It is time to stop and look back on what we have discussed in this Chapter.

Essentially, we have learnt to take a dynamics given in terms of a Lagrangian and obtain field equations, to consider transformations and check whether they are Lagrangian symmetries and, if they are, compute Noether currents. In some cases, we learnt to build superpotentials for conservation laws. We also discussed a number of examples.

The beginner has to learn both the general ideas *and* learn to do computations. Here we *begun* to learn about computations and set up the basis for future discussions.

Most of the examples we considered were not discussed in their physical meaning. In Section 1.10, we also discussed Hilbert–Einstein gravity, though without discussing any meaning of the objects we introduced. Most of this book will be spent in discussing these examples in greater and greater detail and meaning. Here all the examples are introduced for giving examples of field equations, conservation laws and superpotentials in Lagrangian field theory and discussions are limited to these aspects leaving more physical discussion for later. What the reader should be able to grasp is the mathematical notation and techniques and, if needed, one should resort to Part IV for filling the mathematical gaps.

As we said, there are plenty of books introducing to GR and gravity on a physical stance, with somehow minimal mathematical background. This is not our way. Here we are booting up a (massive and highly correlated) mathematical framework which needs time, examples, detailed discussions to be glued together.

As we also remarked, for us, physical intuition has to be built on mathematics, so one needs to examine examples *before* having a physical grasp on what is going on. This is precisely what we have done in this Chapter; we examined a number of examples we do not yet really understand from a physical viewpoint. Physical understanding will come later, for now we are happy enough when we realise we are skilled enough the do our maths.

There will be time to come back on these examples with all we shall learn and eventually see them in their true colours.

References

Extensive and now standard background in Lagrangian formulation of mechanics can be found in Marsden[7], G. Sardanashvily[8].

About the formulations of field theories which relates to symplectic structures, (pre-symplectic, multi-symplectic, ...) can be found Souriou[9], Gimpsy[5]

A standard introduction to classical field theory Gelfand[2]

Some of the global aspects and higher order are in Fatibene–Francaviglia[3]

Classical primers about GR are Carroll[10], Gravitation[11]

Some of the more geometric aspects are Baez[12]

M.Henneaux, C.Teiteilboim, *Quantization of Gauge Systems*

H.J.Matschull, *Diracs Canonical Quantization Programme*, arXiv:quant-ph/9606031

13. Exercises

Exercise 1:

Chapter 2. General covariance

> *Can you help me?*
> *Hey you, don't tell me there's no hope at all*
> *Together we stand, divided we fall.*

(Pink Floyd, *Hey You*)

1. Introduction

This Chapter is devoted to introduce *relativistic theories* and discuss the role of *general covariance principle*.

There are two types of relativistic theories called *natural theories* and *gauge-natural theories*. Both natural and gauge-natural theories obey the same simple principles and the difference is, in a sense, just technical.

> Moreover, natural theories can be seen as special gauge-natural theories (of degree $(0, h)$ and with a trivial gauge group $G = \{e\}$). We keep them separated because historically natural theories are closer to what was originally considered a relativistic theory, often they are simpler to discuss, and they refer to better known objects, which are the same objects studied in standard differential geometry.
>
> However, sometimes considering gauge-natural theories as more fundamental gives us a better, more general, viewpoint. And it is necessary, for example, to deal with spinor fields.

Let us start by stating these principles in words. Precise technical formulation will follow. As usual, what matters are the technical precise formulation, while the naive statements are only used for heuristic purposes.

We can summarise the naive principles for relativistic field theories as follows:

(i) any physical law should be expressed in an absolute fashion;

(ii) all fields appearing in the Lagrangian are dynamical.

Of course, the focus goes on the definition of *absolute* and of *dynamical field*. *Absolute* means *independent of the observer*; it is a principle of relativity similar to the old principle of Galilei, though extended to *any* observer, not restricted to *inertial* observers. Of course, said that, the focus is shifted from *absolute* to the definition of *observer*.

We discussed examples of non-dynamical fields in the previous Chapter. A precise definition of *dynamical fields* is a core issue, unfortunately not completely addressed. It will be discussed hereafter. It is often very clear what is *not* a dynamical field (for example the Minkowski metric in SR) while sometimes non-dynamical structures are difficult to be pinpointed. What is a dynamical field is connected to discussing what is a *background*

and a *background-free field theory*. The definition of dynamical field is somehow easier in a Lagrangian framework: for us, *dynamical* will roughly mean that it is varied in the action and it obeys the endowed field equations.

As far as the observers are concerned, here we are using a rather general definition of *observer*. An observer is any collection of conventions which allows to report physical observations as a list of numbers. Here *observer* is identified with the laboratory it locally defines.

> There are other definitions of observer which usually are particular cases of our definition.
>
> In Newtonian mechanics, an observer is a reference frame (O, e_i) in the affine space $\mathbb{A}^3$. Of course, there is a Cartesian coordinate system associated to the frame (if $P \in \mathbb{A}^3$, then $P - O = x^i e_i$ and one can define a chart $r : \mathbb{A}^3 \to \mathbb{R}^3 : P \mapsto x^i$). By using these coordinates, one can specify the positions of material points as a list of coordinates.
>
> The same construction holds in Minkowski spacetime for special relativity.
>
> In relativity (both GR and SR), one sometimes uses a clock (a parameterised worldline) as an observer. We shall see how this will allow one to define a local coordinate system which allows to report events as m-tuples (t, x^i).
>
> Sometimes we want to keep distinct the chart (laboratory) and the pointlike observer (observer). However, an observer can define a laboratory e.g. by choosing synchronization conventions, thus sometimes we use observer for both the pointlike observer and its laboratory.

▷*Fermi coordinates*

In field theory, reporting observations means recording when, where and which field reading has been done. Since fields are represented by a section σ of a configuration bundle $\mathcal{C} = (C, M, \pi, F)$, observers (labs) are canonically identified with fibered coordinates on the configuration bundle. Once a fibered chart is given, the value of fields at a given spacetime point x^μ is described by the numbers (x^μ, y^i) which are the coordinates representing the point $\sigma(x) \in C$.

> We can consider a *laboratory* as log file of observation reports in the form (x^μ, y^i), to be understood as:
>
> *the fields have been measured at x^μ to have to values y^i.*
>
> This is all a laboratory does. Even if it wants to report its conventions, usually the convention can be expressed as measuring some conventional object. For example:
>
> *I read the temperature of boiling water to be 100*
>
> is a (partial) declaration of Celsius unit.
>
> Well, one has to also declare the pressure, that water was distilled, that the thermometer had a certain precision, that it was not illuminated by incoming radiation, and so on. All these other additional details is why we want to avoid declaring conventions and we prefer declaring a correspondence between the reports of two different laboratories (which is not something one laboratory does but something extra *we* do).

When one is able to report the observation of one laboratory, then one should be able to predict the list of numbers produced by another laboratory when the description of the first observer of the same physical situation is known. In other words, one should be able to map one set of fibered coordinates into the other, i.e. for a generic pair of fibered charts

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = y'^i(x, y) \end{cases} \tag{2.1.1}$$

As we discussed, this information is enough to define the configuration bundle (modulo isomorphisms) by glueing tubelike patches together.

> One naturally has an equivalence relation between local descriptions of physical observations. Two local descriptions (x^μ, y^i) and (x'^μ, y'^i) are equivalent if they can be produced by two different observers looking at the same physical situation; this corresponds to require that the two local descriptions are related by (2.1.1). One

can define an *absolute* physical reality to be the equivalence class of observations modulo this equivalence relation. This is absolute since of course it does not depend on the observer anymore (since it collects the descriptions by all possible observers).

Notice that there is a natural identification between absolute physical facts and points in configuration bundles (which are themselves defined to be the equivalence class of all local descriptions in terms of fibered coordinates). In other words, the configuration bundle is *by definition* an absolute description of the physical situation under consideration.

The most general change of observer lab is, in this general case, a map $\varphi : U_{(\alpha\beta)} \to \mathrm{Diff}(F)$ defined on the intersection $U_{(\alpha\beta)} \subset M$ of the two labs and valued in the diffeomorphisms of the standard fiber F.

This most general case is almost hopeless since the group $\mathrm{Diff}(F)$ is not a finite dimensional manifold. It is a fortunate event that we will not need to go that way. In most cases considered in current physical modelling, the configuration bundle C does not need such a generality. As a matter of fact, one can restrict allowed trivialisations so that one can always cover the bundle C just using these special trivialisations and all transition maps among these special trivialisations take the values in some finite dimensional subgroup $\varphi : U_{(\alpha\beta)} \to G \subset \mathrm{Diff}(F)$.

The simplest example of this situation is what happens with vector bundles and affine bundles, which alone cover all cases of interest for fundamental interactions and matter with the (not so irrelevant) exception of models with gravitation (for which a similar situation occurs though the relevant bundles are not vector or affine). Affine bundles have an affine space $\mathbb{A}$ as a standard fiber and *by definition* they allow trivialisations so that transition maps take values in the group of affine maps $\mathrm{GA}(\mathbb{A})$, so that one has $\varphi : U_{(\alpha\beta)} \to \mathrm{GA}(\mathbb{A}) \subset \mathrm{Diff}(\mathbb{A})$. Vector bundles have a vector space V as a standard fiber and *by definition* they allow trivialisations so that transition maps take values in the group of linear maps $\mathrm{GL}(V)$, so that one has $\varphi : U_{(\alpha\beta)} \to \mathrm{GL}(V) \subset \mathrm{Diff}(V)$.

The group embedding $i : G \hookrightarrow \mathrm{Diff}(F)$ is a (left) action of G on the manifold F, which is something quite familiar to work with. It can be represented by a map $\lambda : G \times F \to F$ which is an ordinary map between two ordinary finite dimensional manifolds.

In all these cases, one can define *group valued transition maps* $\hat{\varphi} : U_{(\alpha\beta)} \to G$ which induce then, by composition with the group embedding, the standard transition maps $\varphi = i \circ \hat{\varphi} : U_{(\alpha\beta)} \to \mathrm{Diff}(F)$. The group-valued transition maps $\hat{\varphi}$ form a cocycle, which defines a principal bundle P, modulo isomorphisms. Then the original configuration bundle C is isomorphic to the associated bundle $C \simeq P \times_\lambda F$.

Accordingly, in these cases one can restrict *without loss of generality* to bundles associated to a principal bundle P, which is called the *structure bundle*. On these bundles C, a fibered chart is induced by a fibered chart on P. Let us define *gauge-natural laboratory* to be a fibered chart on P. In this way, we also provide P with a role of describing observer laboratories.

This framework is particularly inspiring since automorphisms $\Phi \in \mathrm{Aut}(P)$ of P canonically act on associated bundles (and in particular on C) and define a special subgroup $\mathrm{Aut}(P) \subset \mathrm{Aut}(C)$ of transformations on C, which are called *gauge transformations*. If one requires these transformations to be Lagrangian symmetries, we shall see in detail that this implies that all gauge-natural observers are treated on equal footing and accordingly, the physical laws are independent of the gauge-natural laboratory. This is, modulo some technicality to be discussed in detail, the general scheme for so-called *gauge-natural theories*.

Sometimes the situation is even simpler than that! In many cases (among which all tensor bundles in which what follows specify to the case $h = 1$) each chart in the base manifold does induce a trivialisation on C, then transition maps are induced by transition maps in the base manifold and they depend on the Jacobians up to some finite order h.

As an example, let us consider the tangent bundle $T(M)$. Once a chart x^μ in the base M is chosen then spacetime coordinates induce a natural basis ∂_μ of tangent vectors and any tangent vector can be written in the form $v = v^\mu \partial_\mu$. This induces a trivialisation of the tangent bundle

$$t : \pi^{-1}(U) \to U \times \mathbb{R}^m : (x, v) \mapsto (x^\mu, v^\mu) \tag{2.1.2}$$

These special trivialisations of $T(M)$ which are induced by spacetime charts are called *natural trivialisations* (or *natural observers* in the physical jargon).
The tangent bundles can always be covered by natural trivialisations (just because the spacetime can always be covered by charts). The transition maps between natural trivialisations on $T(M)$ are in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^\mu = J^\mu_\nu(x) v^\nu \end{cases} \tag{2.1.3}$$

where, following the standard notation for Jacobians, J^μ_ν are the Jacobians of the spacetime transition maps $x'^\mu = x'^\mu(x)$. Hence the transition maps between natural trivialisations on $T(M)$ are completely determined by the spacetime atlas.
Of course, on $T(M)$ one could also use more general trivialisations associated to non-natural pointwise basis V_a. Once such a basis has been fixed, then any tangent vector v can be expanded as $v = v^a V_a$ and one can define a trivialisation

$\triangleright$*Standard Jacobians*

$$\hat{t} : \pi^{-1}(U) \to U \times \mathbb{R}^m : (x, v) \mapsto (x^\mu, v^a)$$

which is not natural but it is still a trivialisation of the bundle $T(M)$. Accordingly, the transition maps between two such trivialisations are in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^a = J^a_b(x) v^b \end{cases} \tag{2.1.4}$$

Now the matrix J^a_b is not a Jacobian any longer, it is instead related to the bases fixed, namely $V_a = J^b_a V'_b$. Thus one cannot obtain J^a_b from what is going on on spacetime (i.e. from $x'^\mu = x'^\mu(x)$).
A similar situation can be shown to hold on any tensor bundle defined over M. One can also define the bundle of Lorentzian metrics $\mathrm{Lor}(M)$. This has local fibered coordinates $(x^\mu, g_{\mu\nu})$ where the field *"coordinates"* $g_{\mu\nu}$ are coefficients of symmetric, non-degenerate, Lorentzian bilinear forms. Transition maps are fixed to represent the tensorial transformation rules, namely,

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g'_{\mu\nu} = \bar{J}^\alpha_\mu g_{\alpha\beta} \bar{J}^\beta_\nu \end{cases} \tag{2.1.5}$$

Also in this case, transition maps are uniquely determined in terms of spacetime charts.
Let us stress for future reference that transformation laws between natural observers contain as an essential part the transformation laws of fields, namely $g'_{\mu\nu} = \bar{J}^\alpha_\mu g_{\alpha\beta} \bar{J}^\beta_\nu$,

This is the general scheme for so-called *natural bundles*. On natural bundles, the group $\mathrm{Diff}(M)$ of base diffeomorphisms canonically acts on C and defines a subgroup $\mathrm{Diff}(M) \subset \mathrm{Aut}(C)$ of transformations which can be reasonably required to be symmetries for the system. This identifies a class of special observers (which are associated to charts in M), called *natural observers* or *natural laboratories*. We shall see in detail that this implies that all these natural observers are treated on equal footing and accordingly, the physical laws are independent of the natural observer. This is, modulo some technicality to be discussed in detail, the general scheme for a so-called *natural theories*.

Diffeomorphisms of M act by tangent maps on TM and can be required to be symmetries of the dynamics, obtaining a natural theory.

The tangent bundle TM is associated to the general frame bundles $L(M)$, which is thence a kind of structure bundle. If the dynamics were covariant with respect to the whole $\mathrm{Aut}(L(M))$ it would be a gauge-natural theory. As long as it is covariant only with respect to diffeomorphisms of M lifted on $L(M)$ by the natural lift and then represented on TM as on associated bundles, then the theory is a natural theory. In other words, in a natural theory on TM, one has the relevant group actions

$$\mathrm{Diff}(M) \subset \mathrm{Aut}(L(M)) \subset \mathrm{Aut}(TM) \qquad (2.1.6)$$

which are implemented as symmetries.

Currently, all physics of relativistic fundamental interactions and matter can be understood in terms of natural and gauge-natural theories. Let us summarising the situation to an intermediate level of precision.

One can think about the classification of observer labs in *natural* and *gauge-natural observers* is after all a classification as *inertial* and *non-inertial observers* in Newtonian mechanics. And that would be wrong.

The difference between natural and gauge-natural observers is dictated by the kind of physical phenomenology one is required to observe. It is not an attribute of the observer, it does not single out a class of well behaving observers in a more general class.

If we are observing a Newtonian system, we can do it by using an inertial frame or a rotating one (whatever that could mean from a fundamental viewpoint). Here the critical issue is that both inertial and non-inertial observers can report on the same system.

If we are describing electromagnetic field, instead, there is no choice and we are forced to use gauge-natural observers. As long as the electromagnetic field is modelled by a connection on a $U(1)$-principal bundle, it is a fact that it is a section of a gauge-natural bundle which, in general, is not natural. Whatever observer is used, it needs extra conventions to deal with gauge invariance.

It is a matter of fact that spacetime diffeomorphisms do not act (globally) on the electromagnetic potential A_μ while gauge transformations do. To deal with this situation one *needs* gauge-natural observers. There is no natural observer which has anything to say about this. On the other hand, when a natural observer is enough to do the job, any gauge-natural observer just has extra conventions which are not used and ignored. In those situations, the gauge-natural observers are *equivalent* to natural observer.

In other words, one could say that there is only one class of observers, which are gauge-natural observers with general enough conventions to deal with *any* physical situation. Natural observers just are those general observers which happen to keep some of their conventions silent. As we said, one could introduce gauge-natural theories only and then regard natural theories as specific simple cases of this general class. Here we decided not to do it, since this general approach would have been hard to follow in the beginning and to connect to historical standard examples.

Hereafter we shall give a definition of natural and gauge-natural field theories.

In Section 2.2 we shall discuss how (gauge) covariance constrains allowed dynamics in these theories. Section 2.3 will be devoted to discuss how evolution connects to the homogeneous formalism. In Section 2.4 we shall consider how general (covariance) affects the interpretation of a relativistic theory, in particular how it constrains the notion of physical state. Section 2.5 will be devoted to review some traditional critiques to general covariance with the aim of render them in modern language. In Section 2.6 we discuss when a field can be considered a background and when it is not, so sketching a discussion about what is a *background-free theory*. Finally, in Section 2.7 we collect some examples to show how one defines a relativistic theory.

Natural theories

Natural theories are the framework for situations in which the observers just need to know a local description of spacetime to report about the physical events happening in their neighbourhood. This is not always the case. In particular, it is not the case when, as we shall see, one has to take some

gauge symmetry into account. However, in some situations the transformation of fields $y'^i = y'^i(x, y)$ can be uniquely determined by the spacetime transformations $x'^\mu = x'^\mu(x)$. When this happens we say we are considering a *natural bundle*.

Accordingly, let us define a *natural bundle C* iff one can lift functorially any base map ϕ to a bundle endomorphism $\hat\phi$, namely

$$
\begin{array}{ccc}
C & \dashrightarrow{\hat\phi} & C \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi} & M
\end{array}
\tag{2.1.7}
$$

Let us stress that the prolongation must be functorial, i.e. the prolongation of the identity is the identity and the prolongation must preserve compositions, i.e.

$$
(\phi \circ \psi)^\wedge = \hat\phi \circ \hat\psi
\tag{2.1.8}
$$

For the tangent bundle TM the lift is provided by the tangent map $\hat\phi := T(\phi)$, which is, in fact, functorial.

Hence, on natural bundles, one has a canonical action of spacetime diffeomorphisms on the configuration bundle, i.e. a group homomorphism

$$
(\cdot)^\wedge : \mathrm{Diff}(M) \to \mathrm{Aut}(C) : \phi \mapsto \hat\phi
$$

which is called the *natural lift*.

The natural lift $(\cdot)^\wedge$ is a group homomorphism, thus it preserves compositions.

One can define a (covariant) functor from the category of manifolds to the category of bundles associating the bundle C to the base manifold M (remember that the cocycle of transition maps of the bundle C is constructed out of the cocycle of transition maps of an atlas of M) and a bundle morphism $\hat\phi$ on C to a base morphism ϕ. Accordingly, we could write $C = \hat M$ and $(\cdot)^\wedge : \mathfrak{Man} \to \mathfrak{FibB}$.

The situation can be summarized by the following diagram

$$
\begin{array}{ccc}
C & \hat\phi \curvearrowleft & \\
\downarrow \pi & \vdots & (\cdot)^\wedge \\
\phi \quad M \rightsquigarrow M & \phi &
\end{array}
\tag{2.1.9}
$$

In natural bundles, the natural lift is a canonical splitting of the short exact sequence of groups

$$
\mathbb{I} \to \mathrm{Aut}_V(C) \xrightarrow{i} \mathrm{Aut}(C) \xrightarrow{p} \mathrm{Diff}(M) \to \mathbb{I}
$$

$$
\underset{(\cdot)^\wedge}{\longleftarrow\,\dashleftarrow}
\tag{2.1.10}
$$

 Recognising and building natural bundles is quite important since they appear to be configuration bundles in natural theories. Luckily, a bundle is natural iff it is associated to some frame bundle $L^h(M)$.

 In this context, observers are identified with *natural observers*, i.e. spacetime charts and changes of (natural) observers are local spacetime diffeomorphisms. The canonical lift defines a subgroup of transformations $\text{Diff}(M) \subset \text{Aut}(C)$ and one can drag sections of C along spacetime diffeomorphisms

$$\hat{\sigma} := \hat{\phi} \circ \sigma \circ \phi^{-1} \tag{2.1.11}$$

This is an action of the group $\text{Diff}(M)$ on the set of sections of C, namely $\text{Sec}(C)$.

 Sections, as well as the group action of spacetime diffeomorphisms, are prolonged to jet bundles and one can require these transformations to leave the Lagrangian invariant. A Lagrangian which is covariant with respect to any spacetime diffeomorphism is called a *natural Lagrangian* or a *covariant Lagrangian*.

> Let us stress that if the configuration bundle is not a natural bundle then there is no canonical action of spacetime diffeomorphisms on the configuration bundle. Spacetime diffeomorphisms do not act on fields. In this context, asking the dynamics to be generally covariant does not make *any sense*.

 Moreover, in a natural theory, all fields appearing in the Lagrangian are dynamical. This means that one cannot fix any field structure on spacetime. Spacetime is given as a bare manifold and all additional structures defined on it are determined by field equations.

> Let us remark that that is not completely true. There are some structures (e.g. the Kronecker delta, or the Levi Civita tensor densities) which are *canonical* and are somehow in between constants and fields. Deciding if they need to be dynamical or can be used as fixed structures in relativistic theories, and how they relate to covariance is something to be discussed. We shall do it below.

 In other words, one considers a general variation of a section of the configuration bundle C and each field has to satisfy the field equations endowed by Hamilton principle.

Definition (2.1.12): *a* natural theory *is a field theory defined on a natural bundle C over a spacetime M by a global Lagrangian which is covariant with respect to any spacetime diffeomorphism. A section σ of the configuration bundle represents a set of fields, all to be considered dynamical.*

 We shall show below that the covariance imposed on dynamics does impose observer independence. The dynamics has to be defined by choosing some Lagrangian. We assume that the Lagrangian must be covariant with respect to all spacetime diffeomorphisms. Since a one-parameter subgroup of diffeomorphisms on M is generated by a spacetime vector field ξ, this is equivalent to require that the Lagrangian $\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu}, \ldots)\, d\sigma$ satisfies the covariance identity for *any* spacetime vector field ξ, i.e.

$$p_i \pounds_\xi y^i + p_i^\mu d_\mu \pounds_\xi y^i + p_i^{\mu\nu} d_{\mu\nu} \pounds_\xi y^i + \ldots = d_\mu (\xi^\mu L) \tag{2.1.13}$$

where the Lie derivative of a section with respect to ξ is defined to be $\pounds_\xi \sigma = \pounds_{\hat{\xi}} \sigma$, i.e. as the Lie derivative of the section with respect to the natural lift of the vector field to the configuration bundle (the natural lift which exists since the configuration bundle is assumed to be natural).

 The identities (2.1.13)—one for any spacetime vector field ξ—are a strong constraint on the functional form of the Lagrangian. We already noticed that, in order to be covariant, a Lagrangian must transform as a scalar density of weight 1 with respect to the *finite* symmetry transformation and

(2.1.13) was obtained as the infinitesimal form of this condition. Thus a Lagrangian is natural iff it transforms as a scalar density of weight 1 under any spacetime diffeomorphism (at least in the connected component of the identity).

Usually, one can use this covariance identity (2.1.13) to find out all possible natural Lagrangians depending on some set of fields and their derivatives up to some fixed order. This sort of results are called *Utiyama-like theorems*. Some examples will be considered below.

Gauge-natural theories

In some situations, natural theories are not enough. If we consider electromagnetism as an example, we decided (see above) to describe the electromagnetic field by a potential A_μ on spacetime. In fact, that, not $\boldsymbol{F}$, is the field with respect to which the Maxwell Lagrangian is varied.

However, as a matter of fact, one measures $\boldsymbol{F}$ (each observer through measuring E_i and B_i), not A_μ. And as we noticed, there are infinitely many local potentials A'_μ which correspond to the same $\boldsymbol{F}$. In fact, one has that if $\boldsymbol{F} = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\ dx^\mu \wedge\ dx^\nu$ then $\boldsymbol{F} = \frac{1}{2}\left(\partial'_\mu A'_\nu - \partial'_\nu A'_\mu\right)\ dx'^\mu \wedge\ dx'^\nu$ for any $A'_\mu(x') = \bar{J}^\nu_\mu\left(A_\nu(x) + \partial_\nu\alpha(x)\right)$, where α is a local function on M.

In order to describe the electromagnetic field $\boldsymbol{F}$, each laboratory needs to select a representative in each class of potentials which determine the same $\boldsymbol{F}$. That means that given two labs, one should know, beside how to transform spacetime coordinates from one lab to the other, which local function α one has to use in order to transform the potential A_μ of the first lab into the potential A'_μ of the second lab. Notice also that the function α is completely uncorrelated to the change of coordinates $x'^\mu = x'^\mu(x)$.

As a matter of fact, we already discussed it, though in the specific example of Dirac monopole in the previous Chapter. If a specific configuration of electromagnetic field $\boldsymbol{F}$ is observed then one is already able to connect global properties of it to a specific (class of isomorphism of) structure bundles. When the electromagnetic field $\boldsymbol{F}$ is closed but not exact, then one can find a family of local potentials which are connected in the overlaps by a 2-cocycle of vertical gauge transformations. Such a cocycle is not uniquely defined but two different representatives differ by a coboundary. Then the cocycle defines a $U(1)$-principal bundle P, and a different cohomology representative does in fact define a different representative of the same class of diffeomorphisms. (If $\boldsymbol{F}$ is exact a global potential exists and this leads to a a trivial $U(1)$-bundle.)

$\triangleright$*Dirac Monopole*

Let us in fact suppose we live on a spacetime M on which an electromagnetic field $\boldsymbol{F}$ is defined. Then we have a collection of local laboratories (each defined on an open set U_α) and each lab defines its potential $A_\mu^{(\alpha)}(x)$ on U_α. Since they do describe the same electromagnetic field F on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (whenever $U_{\alpha\beta} \neq \varnothing$) the local potentials are related by

$$A'^{(\beta)}_\mu(x') = \bar{J}^\nu_\mu\left(A^{(\alpha)}_\nu(x) + \partial_\nu\alpha_{(\beta\alpha)}(x)\right) \tag{2.1.14}$$

Of course, since *defining the same* $\boldsymbol{F}$ is an equivalence relation on the potentials, one has

$$\begin{cases} \alpha_{(\alpha\alpha)}(x) = 0 \\ \alpha_{(\beta\alpha)}(x) = -\alpha_{(\alpha\beta)}(x) \\ \alpha_{(\alpha\gamma)}(x) + \alpha_{(\gamma\beta)}(x) + \alpha_{(\beta\alpha)}(x) = 0 \end{cases} \tag{2.1.15}$$

Then we can associate a local map $\phi_{(\beta\alpha)} : U_{\alpha\beta} \to U(1) : x \mapsto e^{i\alpha_{(\beta\alpha)}(x)}$ to each non-empty overlap $U_{\alpha\beta} \neq \varnothing$ This is a cocycle since

$$\begin{cases} \phi_{(\alpha\alpha)}(x) = 1 \\ \phi_{(\beta\alpha)}(x) = e^{i\alpha_{(\beta\alpha)}(x)} = e^{-i\alpha_{(\alpha\beta)}(x)} = \left(\phi_{(\alpha\beta)}(x)\right)^{-1} \\ \phi_{(\alpha\gamma)}(x) \cdot \phi_{(\gamma\beta)}(x) \cdot \phi_{(\beta\alpha)}(x) = e^{i\alpha_{(\alpha\gamma)}(x)} \cdot e^{i\alpha_{(\gamma\beta)}(x)} \cdot e^{i\alpha_{(\beta\alpha)}(x)} = e^{i\left(\alpha_{(\alpha\gamma)}(x) + \alpha_{(\gamma\beta)}(x) + \alpha_{(\beta\alpha)}(x)\right)} = 1 \end{cases} \tag{2.1.16}$$

and it defines (uniquely, up to isomorphisms) a $U(1)$-principal bundle $\mathcal{P} = (P, M, \pi, U(1))$ as a structure bundle on which the local potentials define a global connection

$$\omega = dx^\mu \otimes (\partial_\mu - A_\mu(x)\rho) \tag{2.1.17}$$

Of course, if the same labs measure a different electromagnetic field $\boldsymbol{F}'$ they define different potentials, related by a different cocycle $(U_{\alpha\beta}, \phi'_{(\beta\alpha)})$. Whether the two cocycles define the same principal bundle (and, of course, we mean *modulo isomorphisms*), or two different bundles $\mathcal{P}$ and $\mathcal{P}'$ (we mean that they are not isomorphic) depends on the details encoded in the maps $\phi_{(\beta\alpha)}(x)$.

If the spacetime M is contractible then any bundle defined on it is trivial. Then any two $U(1)$-principal bundles defined on M are isomorphic. Then one has necessarily $\mathcal{P} \sim \mathcal{P}'$.

If M is not contractible than there exist $U(1)$-principal bundles $\mathcal{P}$ and $\mathcal{P}'$ which are not isomorphic. Then one can solve Maxwell equations on the two bundles obtaining solutions on $\mathcal{P}$ or on $\mathcal{P}'$.

On a given structure bundle $\mathcal{P}$ (or $\mathcal{P}'$) there exist infinitely many connections (which, in fact, form an affine space) and infinitely many electromagnetic fields $\boldsymbol{F}$ which lead precisely to that structure bundle $\mathcal{P}$ (or $\mathcal{P}'$).

In other words, *being a solution of Maxwell equations on the same structure bundle* is an equivalence relation on solutions of Maxwell equations. The space of solutions disconnects in families each associated to a particular structure bundle, each in view of their global properties.

Thus the general solution of Maxwell equations is obtained by solving Maxwell equations on each structure bundle and then taking the union of all solutions found.

There are two essential things to keep in mind. First, $\mathcal{P}$ contains extra information about the (gauge-natural) observers which is not contained in M. Second, the extra information is not contained in any particular laboratory, it is not something which can be measured by one observer. It is a collective property of all labs which although non-local and conventional is in principle observable analysing the labs conventions (or their reports).

Since one has a principal bundle $\mathcal{P}$ around, then fibered coordinates on $\mathcal{P}$ and associated bundles are not induced by an atlas on M. One needs to select a trivialisation on $\mathcal{P}$, for example by selecting a family of local sections in $\mathcal{P}$ each inducing a local trivialisation. Once a trivialisation on $\mathcal{P}$ is given then its transition functions $\phi_{(\beta\alpha)} : U_{\alpha\beta} \to G$ are determined.

Principal bundles are not suited for variational calculus. Since they support a bijection between local sections and local trivialisations, a principal bundle allows global sections iff it is trivial. Then, in general, there are no global sections on principal bundles. If one selects such a bundle with no global section as a configuration bundle for a field theory, the action (which is a functional on the set of global sections) is defined on the empty set. It is quite trivial and non-sensical to look for solutions of field equations (i.e. global sections which obey Euler–Lagrange equations) if there are no global section at all!

In gauge-natural theories, the configuration bundle C is a gauge-natural bundle; it is associated to the structure bundle P, which is a principal bundle for some Lie group G, which is called the *gauge group*.

Since the configuration bundle C is associated to the structure bundle P, then any automorphism $(\Phi, \varphi) \in \mathrm{Aut}(\mathcal{P})$ induces an automorphism $(\Phi, \varphi)^\wedge \in \mathrm{Aut}(C)$ which preserves compositions

$$(\mathrm{id}_P, \mathrm{id}_M)^\wedge = (\mathrm{id}_C, \mathrm{id}_M) \qquad (\Phi_2, \varphi_2)^\wedge \circ (\Phi_1, \varphi_1)^\wedge = (\Phi_2 \circ \Phi_1, \varphi_2 \circ \varphi_1)^\wedge \tag{2.1.18}$$

Accordingly, one has a canonical action of gauge transformations $\mathrm{Aut}(P)$ on the configuration bundle, i.e. a group homomorphism

$$(\cdot)^\wedge : \mathrm{Aut}(P) \to \mathrm{Aut}(C) : \phi \mapsto \hat{\phi}$$

which is called the *gauge-natural group action*.

The gauge-natural group action $(\cdot)^\wedge$ is a group homomorphism, thus it preserves compositions.

One can define a (covariant) functor from the category of principal bundles to the category of bundles, associating the bundle C to the principal bundle P (remember that the cocycle of transition maps of P induces the cocycle of transition maps of the bundle C) and a bundle morphism $\hat\phi$ on C to an automorfism $\phi \in \mathrm{Aut}(P)$. Accordingly, we could write $C = \hat P$ and $(\cdot)^\wedge : \mathfrak{P}(\mathfrak{M},\mathfrak{G}) \to \mathfrak{FibB}$.

The configuration bundle comes with a G-structure (trivialisations of $\mathcal{P}$ define trivialisations on $\mathcal{C}$) and with special transformations $\mathrm{Aut}(\mathcal{P}) \subset \mathrm{Aut}(\mathcal{C})$ which are called *gauge transformations*.

One can also define *vertical gauge transformations* to be vertical automorphisms of the structure bundle, namely $\mathrm{Aut}_V(P) \subset \mathrm{Aut}(P) \subset \mathrm{Aut}(C)$. The situation can be summarized by the following diagram

$$
\begin{array}{ccccc}
\Phi & P & & C & \hat\Phi \\
\vdots & \big\downarrow & \rightsquigarrow & \big\downarrow{\scriptstyle\pi} & \vdots \\
\phi & M & & M & \phi
\end{array}
\tag{2.1.19}
$$

The relevant transformation groups are described by the following exact sequence

$$
\mathbb{I} \longrightarrow \mathrm{Aut}_V(P) \xrightarrow{\ i\ } \mathrm{Aut}(P) \xrightarrow{\ p\ } \mathrm{Diff}(M) \longrightarrow \mathbb{I}
\tag{2.1.20}
$$

Notice that this exact sequence does not in general split (as it happened canonically for (2.1.10)). It is precisely because this exact sequence does not split (in general) that we can say that spacetime diffeomorphisms do not act (in general) on configuration bundle.

As it happened in natural theories on $L^h(M)$, it will be particularly important to be able to recognise and build gauge-natural bundles associated to a structure bundle $\mathcal{P}$; luckily, a gauge-natural bundle $\mathcal{C} = (C, M, \pi, F)$ is necessarily associated to $J^k P \times_M L^h(M)$ with $h \geq k$. The integer pair (k, h) is called the *degree* of the gauge-natural bundle $\mathcal{C}$.

In this context, laboratories are identified to *gauge-natural labs*, i.e. trivialisations of P and changes of (gauge-natural) laboratories are local gauge transformations. The canonical lift defines a subgroup of transformations $\mathrm{Aut}(P) \subset \mathrm{Aut}(C)$ and one can drag sections of C along gauge transformations

$$
\hat\sigma := \hat\phi \circ \sigma \circ \phi^{-1}
\tag{2.1.21}
$$

This is an action of the group $\mathrm{Aut}(P)$ on the set of sections $\mathrm{Sec}(C)$.

Sections, as well as the action of gauge transformations, are prolonged to jet bundles and one can require these transformations to leave the Lagrangian invariant. A Lagrangian which is covariant with respect to any automorphism of P is called a *gauge-natural Lagrangian* or a *gauge covariant Lagrangian*.

Moreover, in a gauge-natural theory, all fields appearing in Lagrangian are dynamical. This means that one cannot fix any field structure on spacetime or on P. Again the structure bundle P is given as a bare principal bundle and all additional structures defined on it are determined by field equations. In other words, one considers a general variation of a section of the configuration bundle C and each field obeys the field equations obtained by Hamilton principle.

Definition (2.1.22): *let us define a gauge-natural theory as a field theory defined on a gauge-natural bundle $\mathcal{C}$ associated to a structure bundle $\mathcal{P} = (P, M, \pi, G)$ and defined by a global Lagrangian $\boldsymbol{L}$ which is covariant with respect to all gauge transformations. A section σ of the configuration bundle represents a set of fields, all to be considered dynamical.*

Also in this more general case, we shall show below that the gauge covariance imposed on dynamics does implement observer independence.

Again, the covariance requests on the Lagrangian is a strong constraint on allowed dynamics and, also in this case, one can use Utiyama-like arguments to find all possible gauge-natural Lagrangians. These are the Lagrangians for which the following equation is identically satisfied

$$p_i \pounds_\Xi y^i + p_i^\mu d_\mu \pounds_\Xi y^i + p_i^{\mu\nu} d_{\mu\nu} \pounds_\Xi y^i + \ldots = d_\mu \left(\xi^\mu L \right) \tag{2.1.23}$$

for any generator of gauge transformations.

> Any vector field Ξ on $\mathcal{P}$ is right-invariant iff its flow is made of automorphisms. For this reason, right-invariant vector fields on $\mathcal{P}$ are also called *infinitesimal generators of automorphisms* or *generators of gauge transformations*.
>
> Then the flow (Φ_s, φ_s) of Ξ induces a flow $(\Phi_s, \varphi_s)^\wedge$ on $\mathcal{C}$ (which is a flow just because $(\cdot)^\wedge$ is a covariant functor). As any flow on $\mathcal{C}$, $(\Phi_s, \varphi_s)^\wedge$ on $\mathcal{C}$ has an infinitesimal generator denoted by $\hat{\Xi} \in \mathfrak{X}(C)$.
>
> The vector field $\hat{\Xi}$ is also called a *generator of gauge transformations* on $\mathcal{C}$ and it is induced canonically by Ξ.
>
> Then one can define the Lie derivative of sections of $\mathcal{C}$ with respect to $\hat{\Xi}$. However, that depends in fact just on Ξ (through $\hat{\Xi}$). For that reason, let us denote it as
>
> $$\pounds_\Xi \sigma := \pounds_{\hat{\Xi}} \sigma \tag{2.1.24}$$
>
> as one does for natural objects.
>
> Let us stress once again that since in general $\mathcal{C}$ is not a natural bundle, the sections of $\mathcal{C}$, i.e. the fields, do not support a group action of spacetime diffeomorphisms. One cannot define the transformation rules (or Lie derivatives) of fields with respect to spacetime diffeomorphisms but just with respect to gauge transformations.

Everything is in transformation rules

The situation described above may appear intricate at first. The requests about symmetries probably are acceptably clear though we refer to group actions which encode a lot of geometric information about the configuration bundle and its relation with spacetime or the structure bundle. Such relations are expressed by the request of being a natural or gauge-natural bundle which, in turn, understands a whole Chapter of bundle theory. *Golden rule*

It is important that eventually one sees the structure of a field theory in these general terms and we shall below present a number of examples to get used to this vewpoint. However, as a matter of fact and on a practical stance, we stress that:

> *everything is encoded into transformation laws of fields.*

If we are considering a field which we know as a natural object, that means we know how it transforms under spacetime diffeomorphisms. For example, if we are considering a metric field $g_{\mu\nu}$, then we know that under a spacetime diffeomorphism $\phi : M \to M : x^\mu \mapsto x'^\mu(x)$ it transforms as

$$g'_{\mu\nu}(x') = \bar{J}^\alpha_\mu \, g_{\alpha\beta}(x) \, \bar{J}^\beta_\nu \tag{2.1.25}$$

which can be recognised to come from a group action on the configuration bundle

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ g'_{\mu\nu} = \bar{J}^{\alpha}_{\mu} \, g_{\alpha\beta} \, \bar{J}^{\beta}_{\nu} \end{cases} \tag{2.1.26}$$

That is a group action $\lambda : \mathrm{GL}(m) \times L \to L$ where L denotes the set of all non-degenerate bilinear forms of a fixed signature (r, s), with $m = r + s$, for example Lorentzian, which is also the standard fiber of the configuration bundle.

Everything comes from this group action λ; the configuration bundle is associated to $L(M)$, which has $\mathrm{GL}(m)$ as structure group, by the group action λ. One can *define* the configuration bundle as being associated to $L(M)$, namely

$$\mathrm{Lor}(M) \simeq C := L(M) \times_{\lambda} L \tag{2.1.27}$$

Since we build it as an associated bundle to $L(M)$, this comes with the group action of $\mathrm{Diff}(M) \subset \mathrm{Aut}(C)$ canonically defined and it coincides by construction with (2.1.26). This canonical group action induces on sections, again by construction, the group action of diffeomorphisms on fields, i.e. the action (2.1.25).

Let us stress once again that there is no extra work in setting a global bundle framework. The bundle framework is exactly equivalent to the local setting enhanced with the declaration of transformation laws of fields.

If one has a group action written in terms of Jacobians, that can be written as an action of the group $\mathrm{GL}(m)$ and the configuration bundle is associated to $L(M)$. If the action depends on higher derivatives of Jacobians (Hessians, ...) (e.g. it depends on Hessians as it happens for connections $\Gamma^{\alpha}_{\beta\mu}$) then the group action can be written in terms of the group $\mathrm{GL}^h(m)$ (e.g. $\mathrm{GL}^2(m)$) and the configuration bundle is associated to $L^h(M)$ (e.g. $L^2(M)$).

For example, the bundle of connections can be defined to be

$$\mathrm{Con}(M) = L^2(M) \times_{\lambda} T^1_2(\mathbb{R}) \tag{2.1.28}$$

where one sets coordinates (J^a_b, J^a_{bc}) on the group $\mathrm{GL}^2(m)$ and defines the action

$$\Gamma'^a_{bc} = J^a_d \left(\Gamma^d_{ef} \bar{J}^e_b \bar{J}^f_d + \bar{J}^d_{ab} \right) \tag{2.1.29}$$

which can be shown to be a left action on the space $T^1_2(\mathbb{R})$ which has been given coordinates Γ^a_{bc}.
Then one can show that this construction is equivalent to more canonical ones, e.g.

$$L^2(M) \times_{\lambda} T^1_2(\mathbb{R}) \simeq J^1 L(M)/\mathrm{GL}(m) \tag{2.1.30}$$

and that (global) sections of $\mathrm{Con}(M)$ are in one-to-one correspondence with (global) connections on M.

If, then, transformation laws are written with another group G, then the configuration bundle is associated to a structure bundle P which is a G-principal bundle P

$$C = P \times_{\lambda} F \qquad\qquad \lambda : G \times F \to F \tag{2.1.31}$$

which could be called a *pure gauge theory*.

In the examples of the previous Chapter, a complex Klein–Gordon field is a pure gauge theory for the group $U(1)$.

If transformation laws depend algebraically on Jacobians and a group G, one can repeat the construction of the associated configuration bundle as

$$C = (P \times L(M)) \times_\lambda F \qquad\qquad \lambda : G \times \mathrm{GL}(m) \times F \to F \qquad\qquad (2.1.32)$$

The most general case is when transformation laws of fields depend on Jacobians and their derivatives up to order h together with the elements in group G together with their derivatives up to order k (and for technical reasons in those cases one must necessarily have $h \geq k$). In such cases, one has a group action

$$\lambda : J^k G \times \mathrm{GL}^h(m) \times F \to F \qquad\qquad (2.1.33)$$

and the configuration bundle is associated to

$$C \simeq J^k P \times_M L^h(M) \times_\lambda F \qquad\qquad (2.1.34)$$

The electromagnetic potential A_μ seen as a $U(1)$-connection is an example of such a general case with $k = h = 1$.

One can prove that the most general gauge-natural bundle is associated to the principal bundle

$$W^{(k,h)}P = J^k P \times L^h(M) \qquad\qquad (2.1.35)$$

for some (k, h) with $h \geq k$. In all gauge-natural theories a gauge transformation in $\phi \in \mathrm{Aut}(P)$ projects onto $\varphi \in \mathrm{Diff}(M)$ (since it is projectable) and it defines an automorphism of $W^{(k,h)}P$ by $(j^k\phi, \hat\varphi)$ which is then represented on the configuration bundle and its sections.

Natural and gauge-natural theories are structurally quite different. In order to allow to discuss them on an equal footing, one needs unified notation which specifies to different structures in the two cases. For example, we shall denote by $\mathcal{S}$ the group of symmetries, which specifies to spacetime diffeomorphims in natural theories or to generalised gauge transformations in gauge-natural theories. This unified notation will, of course, make discussion easier but it also encapsulate irrelevant details, which are essential to be recognised as irrelevant.

Let us stress it once again:

> *everything is encoded in transformation rules of fields.*

Bundle framework is just a mean to trace covariance of the transformation laws of fields, i.e. to trace covariance of gauge transformations. As long as you keep track of transformation laws in a field theory (unlike we did in first example about vibrating plate in Chapter 1, but as already done in the second example after one reformulates it in terms of tensor densities) you are in fact using bundle theory, or at least a formalism equivalent to it.

If one does not declare transformation laws for fields, then we have a local field theory, a local expression for dynamics which defines local field equations and local solutions. Unfortunately, these local solutions encode physical facts together with observer conventions but we do not know how to relate observations from different laboratories. Then we have no absolute description for physics and the theory tells us something, only if we support it with a detailed description of the conventions of the observers. For it to make a minimal sense, the description of the lab conventions relies on *accepted* physical assumptions (at least about the structure of spacetime and the nature of fields).

Here we are attempting to boot up a *fundamental* theory, which *defines* an absolute description of the world without relying on assumptions of the structure of spacetime. And in such a situation, the only option we have is to define *absolute knowledge* as a *intersubjective* knowledge. That is the only

known way to define what is absolute knowledge together with what is an observer, without having to describe an observer (i.e. its lab conventions) in an absolute way.

That is the best we can do. That is the best we can *currently* do. It is the only known escape to define absolute knowledge without an absolute benchmark, something that medieval philosophy was not able to even imagine. That is the most heuristically essential, precise definition of what is a *relativistic theory*. A *relativistic theory* is a field theory defining *absolute knowledge* as *general intersubjective knowledge*.

If this is not philosophy, I do not know what else it is.

2. Utiyama-like arguments

As a matter of fact, imposing general covariance with respect to spacetime diffeomorphisms in a natural field theory, or general covariance with respect to gauge transformations in a gauge-natural field theory does constrain the allowed Lagrangians severely.

When one is setting up a natural or a gauge-natural field theory, once fundamental fields have beed fixed and the order at which they appear in the Lagrangian has been decided, then just some Lagrangians are allowed. The Utiyama-like theorems explicit the conditions to be checked for being generally (gauge) covariant and they fix which combinations of fields can appear in (gauge) covariant Lagrangians.

The trick is to find combinations which *transform well*. These usually are rather complicated functions of fundamental fields and their derivatives; examples are (gauge) covariant derivatives which depend on the connection and the derivatives of fields, though they transforms better than partial derivatives. One trades better transformation rules with more complicated expression as functions of the fundamental fields.

Having simple transformation rules translate in having simple Lie derivatives, while being a complicate functions of fundamental fields translate in having complicated expression under variation. However, one can check that covariance identity just depends on Lie derivatives, not on variations. Accordingly, it is highly recommended to use combinations with simple transformation rules. It is a (another) fortunate event, that differential geometry spent over a century by now building geometric objects which in fact are exactly objects with a possibly quite intricate expression which though transform better than expected. For example, the curvature tensors are depending on second derivatives of the metric field, but they are tensors! They transforms with Jacobians only while second derivative of a tensor is in general expected to transform with third order Jacobians.

Hereafter we shall discuss examples of Utiyama-like arguments for natural and gauge-natural field theories. It will be important to keep two results from these examples: first, covariance do in fact constrains dynamics. Second, the examples discussed will be further analysed for their physical meaning in next Chapters. More examples will be discussed at the end of this Chapter.

Utiyama-like theorems for natural theories

Let us consider the bundle $\mathrm{Lor}(M)$. We already discussed that it is a natural bundle with coordinates $(x^\mu, g_{\mu\nu})$. Let us look for natural Lagrangians defined on $J^2\mathrm{Lor}(M)$. Locally we have

$$\boldsymbol{L} = \tilde{L}(x^\mu, g_{\mu\nu}, d_\alpha g_{\mu\nu}, d_{\alpha\beta} g_{\mu\nu})\, d\sigma \tag{2.2.1}$$

This example contains also natural Lagrangians on $J^1\mathrm{Lor}(M)$, which can be obtained by assuming that the Lagrangian does not in fact depend on $d_{\alpha\beta}g_{\mu\nu}$. This particular case will be discussed below.

First of all, let us notice that we can change fibered coordinates on $J^1\mathrm{Lor}(M)$ by setting

$$\begin{cases} x^\mu = x^\mu \\ g_{\mu\nu} = g_{\mu\nu} \\ \{g\}^\alpha_{\beta\mu} = \tfrac{1}{2}g^{\alpha\epsilon}\left(-d_\epsilon g_{\beta\mu} + d_\beta g_{\mu\epsilon} + d_\mu g_{\epsilon\beta}\right) \end{cases} \tag{2.2.2}$$

This is a change of fibered coordinates since the relation between $\{g\}$ and first derivatives is invertible.

One can permute indices

$$\begin{cases} 2g_{\epsilon\alpha}\{g\}^\alpha_{\beta\mu} = -d_\epsilon g_{\beta\mu} + d_\beta g_{\mu\epsilon} + d_\mu g_{\epsilon\beta} \\ 2g_{\beta\alpha}\{g\}^\alpha_{\mu\epsilon} = -d_\beta g_{\mu\epsilon} + d_\mu g_{\epsilon\beta} + d_\epsilon g_{\beta\epsilon} \end{cases} \tag{2.2.3}$$

and summing to obtain

$$2g_{\epsilon\alpha}\{g\}^\alpha_{\beta\mu} + 2g_{\beta\alpha}\{g\}^\alpha_{\mu\epsilon} = {\color{red}-d_\epsilon g_{\beta\mu}} + {\color{blue}d_\beta g_{\mu\epsilon}} + d_\mu g_{\epsilon\beta} - {\color{blue}d_\beta g_{\mu\epsilon}} + d_\mu g_{\epsilon\beta} + {\color{red}d_\epsilon g_{\beta\epsilon}} = 2d_\mu g_{\epsilon\beta} \tag{2.2.4}$$

from which one obtains

$$d_\mu g_{\epsilon\beta} = g_{\epsilon\alpha}\{g\}^\alpha_{\beta\mu} + g_{\beta\alpha}\{g\}^\alpha_{\mu\epsilon} \tag{2.2.5}$$

Let us also define new *"coordinates"* on $J^2\mathrm{Lor}(M)$

$$\begin{cases} x^\mu = x^\mu \\ g_{\mu\nu} = g_{\mu\nu} \\ \{g\}^\alpha_{\beta\mu} = \tfrac{1}{2}g^{\alpha\epsilon}\left(-d_\epsilon g_{\beta\mu} + d_\beta g_{\mu\epsilon} + d_\mu g_{\epsilon\beta}\right) \\ R^\alpha{}_{\beta\mu\nu} = d_\mu\{g\}^\alpha{}_{\beta\nu} - d_\nu\{g\}^\alpha{}_{\beta\mu} + \{g\}^\alpha_{\epsilon\mu}\{g\}^\epsilon_{\beta\nu} - \{g\}^\alpha_{\epsilon\nu}\{g\}^\epsilon_{\beta\mu} \\ S^\alpha{}_{\beta\mu\nu} = d_{(\mu}\{g\}^\alpha{}_{\beta\nu)} \end{cases} \tag{2.2.6}$$

which is also invertible for second derivatives, as one can show.

Of course, $R^\alpha{}_{\beta\mu\nu}$ (as well as $S^\alpha{}_{\beta\mu\nu}$) are not properly coordinates, since their components are not independent, since they obey symmetry properties. The curvature $R^\alpha{}_{\beta\mu\nu}$ is antisymmetric in $[\mu\nu]$ and it obeys first Bianchi identities. Moreover, since here it is a metric curvature, it is also antisymmetric in the first pair $[\alpha\beta]$ and symmetric with respect to pair swap $(\{\alpha\beta\}\{\mu\nu\})$, once α has been lowered by the metric g. Analogously, $S^\alpha{}_{\beta\mu\nu}$ is symmetric with respect to $(\beta\mu\nu)$.

One means that, from $R^\alpha{}_{\beta\mu\nu}$ and $S^\alpha{}_{\beta\mu\nu}$, a number of independent quantities (in this metric case $\frac{m^2}{12}(m^2-1)$ for $R^\alpha{}_{\beta\mu\nu}$ and $\frac{m^2}{6}(m+2)(m+1)$ for $S^\alpha{}_{\beta\mu\nu}$, thus $\frac{m^2}{4}(m+1)^2$ overall) are extracted and used as real coordinates. Thus they well account for the second derivatives of the metric, which are precisely $\frac{m^2}{4}(m+1)^2$.

By using more or less the same quantities, we shall also deal with Palatini formalism. In that case, the curvature has more independent components (since it obeys less symmetry identities) and, also in that case, they will precisely account for the first derivatives of the (torsionless) connection, which are $\frac{m^3}{2}(m+1)$ independent components, i.e. more than the second derivative of the metric.

One can check that

$$\begin{aligned} 3S^\alpha{}_{\beta\mu\nu} - 2R^\alpha{}_{(\beta\nu)\mu} = {}&3S^\alpha{}_{\beta\mu\nu} - R^\alpha{}_{\beta\nu\mu} - R^\alpha{}_{\nu\beta\mu} = \\ ={}&\underline{d_\mu\{g\}^\alpha{}_{\beta\nu}} + {\color{red}d_\nu\{g\}^\alpha{}_{\mu\beta}} + {\color{blue}d_\beta\{g\}^\alpha{}_{\nu\mu}} - {\color{red}d_\nu\{g\}^\alpha{}_{\beta\mu}} + \underline{d_\mu\{g\}^\alpha{}_{\beta\nu}} - {\color{blue}d_\beta\{g\}^\alpha{}_{\nu\mu}} + \underline{d_\mu\{g\}^\alpha{}_{\nu\beta}} + \ldots = \\ ={}&d_\mu\{g\}^\alpha{}_{\beta\nu} + \ldots \end{aligned} \tag{2.2.7}$$

where ... denotes terms which do not depend on second derivatives of the metric. That shows that the correspondence between $(R^\alpha{}_{\beta\mu\nu}, S^\alpha{}_{\beta\mu\nu})$ and $d_\mu\{g\}^\alpha{}_{\beta\nu}$ is one-to-one.

Of course, while $R^\alpha{}_{\beta\mu\nu}$ is tensorial, $S^\alpha{}_{\beta\mu\nu}$ is not.

The Lagrangian can be written in terms of the new coordinates

$$\boldsymbol{L} = \sqrt{g}\, l(x^\mu, g_{\mu\nu}, \{g\}^\alpha_{\beta\mu}, R^\alpha{}_{\beta\mu\nu}, S^\alpha{}_{\beta\mu\nu})\, d\sigma \tag{2.2.8}$$

We can also write Lie derivatives of the new coordinates as

$$\begin{cases} \pounds_\xi g_{\mu\nu} = 2\overset{g}{\nabla}_{(\mu}\xi_{\nu)} \\ \pounds_\xi \{g\}^\alpha_{\beta\mu} = \xi^\lambda R^\alpha{}_{\beta\lambda\mu} + \overset{g}{\nabla}_{\beta\mu}\xi^\alpha \end{cases} \quad \begin{cases} \pounds_\xi R^\alpha{}_{\beta\mu\nu} = \xi^\lambda\overset{g}{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} - \overset{g}{\nabla}_\lambda\xi^\alpha R^\lambda{}_{\beta\mu\nu} + \overset{g}{\nabla}_\beta\xi^\lambda R^\alpha{}_{\lambda\mu\nu} + \overset{g}{\nabla}_\mu\xi^\lambda R^\alpha{}_{\beta\lambda\nu} + \overset{g}{\nabla}_\nu\xi^\lambda R^\alpha{}_{\beta\mu\lambda} \\ \pounds_\xi S^\alpha{}_{\beta\mu\nu} = d_{(\mu}(\xi^\lambda R^\alpha{}_{\beta\lambda\nu)} + \overset{g}{\nabla}_{\beta\nu)}\xi^\alpha) \end{cases} \tag{2.2.9}$$

where $\overset{g}{\nabla}_{\beta\mu}$ denotes symmetrised second order covariant derivatives.

Let us also set

$$\begin{cases} p^{\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial g_{\mu\nu}} + \tfrac{1}{2}\sqrt{g}g^{\mu\nu}l = \sqrt{g}\hat{p}^{\mu\nu} + \tfrac{1}{2}\sqrt{g}g^{\mu\nu}l \\ p_\alpha^{\beta\mu} = \sqrt{g}\dfrac{\partial l}{\partial\{g\}^\alpha_{\beta\mu}} = \sqrt{g}\hat{p}_\alpha^{\beta\mu} \end{cases} \quad \begin{cases} p_\alpha{}^{\beta\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial R^\alpha{}_{\beta\mu\nu}} = \sqrt{g}\hat{p}_\alpha{}^{\beta\mu\nu} \\ P_\alpha{}^{\beta\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial S^\alpha{}_{\beta\mu\nu}} = \sqrt{g}\hat{P}_\alpha{}^{\beta\mu\nu} \end{cases} \tag{2.2.10}$$

for the naive momenta. Then the covariance identity reads as

$$\sqrt{g}\Bigg[(2\hat{p}^{\mu\nu} + {\color{red}g^{\mu\nu}l})\overset{g}{\nabla}_\mu\xi_\nu + \hat{p}_\alpha^{\beta\mu}\left(\xi^\lambda R^\alpha{}_{\beta\lambda\mu} + \overset{g}{\nabla}_{\beta\mu}\xi^\alpha\right) + \hat{p}_\alpha{}^{\beta\mu\nu}\left(\xi^\lambda\overset{g}{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} - \overset{g}{\nabla}_\lambda\xi^\alpha R^\lambda{}_{\beta\mu\nu} + \overset{g}{\nabla}_\beta\xi^\lambda R^\alpha{}_{\lambda\mu\nu} + 2\overset{g}{\nabla}_\mu\xi^\lambda R^\alpha{}_{\beta\lambda\nu}\right) + $$
$$+ 2\hat{P}_\alpha{}^{\beta\mu\nu}d_\mu\left(\xi^\lambda R^\alpha{}_{\beta\lambda\nu} + \overset{g}{\nabla}_{\beta\nu}\xi^\alpha\right)\Bigg] = \sqrt{g}\left[{\color{red}\overset{g}{\nabla}_\lambda\xi^\lambda l} + \xi^\lambda\overset{g}{\nabla}_\lambda l\right] = \overset{g}{\nabla}_\lambda\left(\sqrt{g}\, l\,\xi^\lambda\right) \tag{2.2.11}$$

Since this must hold for any ξ^λ and the values of $\xi^\lambda(x)$, $\overset{g}{\nabla}_\alpha\xi^\lambda(x)$, ... are pointwise independent, one has

$$\begin{cases} \boxed{\hat{p}_\alpha^{\beta\mu}R^\alpha{}_{\beta\lambda\mu}} + \hat{p}_\alpha{}^{\beta\mu\nu}\overset{g}{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} = \overset{g}{\nabla}_\lambda l \\ 2\hat{p}^{\mu\nu}g_{\mu\lambda} + \hat{p}_\alpha{}^{\beta\mu\epsilon}\left(-\delta^\alpha_\lambda R^\nu{}_{\beta\mu\epsilon} + \delta^\nu_\beta R^\alpha{}_{\lambda\mu\epsilon} + 2\delta^\nu_\mu R^\alpha{}_{\beta\lambda\epsilon}\right) = 0 \end{cases} \quad \begin{cases} \hat{p}_\alpha^{\beta\mu} = 0 \\ \hat{P}_\alpha{}^{\beta\mu\nu} = 0 \end{cases} \tag{2.2.12}$$

where some terms (in the first three equations) factorising $\hat{P}_\alpha{}^{\beta\mu\nu}$ have been neglected since, in view of the fourth equation, $\hat{P}_\alpha{}^{\beta\mu\nu} = 0$.

From the last two equations one gets that the Lagrangian (2.2.8) cannot depend explicitly on $S^\alpha{}_{\beta\mu\nu}$ and $\{g\}^\alpha_{\beta\mu}$. The condition for covariance then reduces to

$$\begin{cases} \hat{p}_\alpha{}^{\beta\mu\nu}\overset{g}{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} = \nabla_\lambda l = \partial_\lambda l + \hat{p}^{\alpha\beta}d_\lambda g_{\alpha\beta} + \hat{p}_\alpha{}^{\beta\mu\nu}d_\lambda R^\alpha{}_{\beta\mu\nu} \\ 2\hat{p}^{\mu\nu}g_{\mu\lambda} + \hat{p}^{\alpha\beta\mu\epsilon}\left(-g_{\alpha\lambda}R^\nu{}_{\beta\mu\epsilon} - \delta^\nu_\alpha R_{\beta\lambda\mu\epsilon} + 2\delta^\nu_\mu R_{\alpha\beta\lambda\epsilon}\right) = 0 \end{cases} \tag{2.2.13}$$

Then we can extract information from the first equation by restoring covariant derivatives with respect to g and using the second equation to obtain that $\partial_\lambda l = 0$.

One has

$$\hat{p}_\alpha{}^{\beta\mu\nu}\left(d_\lambda R^\alpha{}_{\beta\mu\nu} + \Gamma^\alpha_{\epsilon\lambda}R^\epsilon{}_{\beta\mu\nu} - \Gamma^\epsilon_{\beta\lambda}R^\alpha{}_{\epsilon\mu\nu} - 2\Gamma^\epsilon_{\mu\lambda}R^\alpha{}_{\beta\epsilon\nu}\right) = \partial_\lambda l + \hat{p}^{\alpha\beta}\left(\boxed{\nabla_\lambda g_{\alpha\beta}} + 2\Gamma^\epsilon_{\alpha\lambda}g_{\epsilon\beta}\right) + \hat{p}_\alpha{}^{\beta\mu\nu}d_\lambda R^\alpha{}_{\beta\mu\nu} \tag{2.2.14}$$

and

$$\partial_\lambda l + \left(2\hat{p}^{\theta\beta}g_{\epsilon\beta} + \hat{p}^{\alpha\beta\mu\nu}\left(-g_{\alpha\epsilon}R^\theta{}_{\beta\mu\nu} - \delta^\theta_\alpha R_{\beta\epsilon\mu\nu} + 2\delta^\theta_\mu R_{\alpha\beta\epsilon\nu}\right)\right)\Gamma^\epsilon_{\theta\lambda} = 0 \quad \Rightarrow \partial_\lambda l = 0 \tag{2.2.15}$$

Accordingly, the first condition implies that the Lagrangian (2.2.8) cannot depend explicitly on x. Then one is left with a Lagrangian

$$\boldsymbol{L} = \sqrt{g}\, l(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu})\, d\sigma \tag{2.2.16}$$

which obeys the second condition

$$2\hat{p}^{\mu\nu}g_{\mu\lambda} + \hat{p}^{\alpha\beta\mu\epsilon}\left(-g_{\alpha\lambda}R^\nu{}_{\beta\mu\epsilon} - \delta^\nu_\alpha R_{\beta\lambda\mu\epsilon} + 2\delta^\nu_\mu R_{\alpha\beta\lambda\epsilon}\right) = 0 \tag{2.2.17}$$

Let us start considering zero order covariant Lagrangians (so that $\boldsymbol{L} = \sqrt{g}l(g_{\mu\nu})\, d\sigma$ and $\hat{p}^{\alpha\beta\mu\epsilon} = 0$). Then one must have

$$\hat{p}^{\mu\nu} = 0 \quad \Rightarrow \boldsymbol{L} = -2\frac{\sqrt{g}}{2\kappa}\Lambda\, d\sigma \tag{2.2.18}$$

for some constant Λ which is usually called a *cosmological constant*. For historical and dimensional reasons, the constant $2\kappa = 16\pi \mathbf{G}\mathbf{c}^{-3}$ is fixed to $[\Lambda] = L^{-2}$
produce a Newtonian limit which agrees with standard units.

For first order Lagrangians, one still has $\hat{p}^{\alpha\beta\mu\epsilon} = 0$. Then there is no first order covariant Lagrangians (other than the Lagrangians which are actually zero order) which depend only on the metric and its first derivatives.

Second order covariant Lagrangians then are the first option if one wants a theory depending on the metric only.

Choosing covariant Lagrangians amounts to list scalar densities one can build with $(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu})$. First of all, any scalar density $L(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu})$ of weight 1 induces a scalar density of weight zero $l(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu}) = \frac{1}{\sqrt{g}}L(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu})$. Accordingly, all scalar densities of weight 1 can be written in the form

$$L(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu}) = \sqrt{g}\, l(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu}) \tag{2.2.19}$$

for some scalar $l(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu})$.

For example, one has a scalar $l(g_{\mu\nu}, R^\alpha{}_{\beta\mu\nu}) = R = g^{\beta\nu}\delta^\mu_\alpha R^\alpha{}_{\beta\mu\nu} = \frac{1}{2}\left(g^{\beta\nu}\delta^\mu_\alpha - g^{\beta\mu}\delta^\nu_\alpha\right)R^\alpha{}_{\beta\mu\nu}$ for which

$$p^{\mu\nu} = \sqrt{g}\left(\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) \qquad p_\alpha{}^{\beta\mu\epsilon} = \sqrt{g}g^{\beta[\epsilon}\delta^{\mu]}_\alpha = \frac{\sqrt{g}}{2}\left(g^{\beta\epsilon}\delta^\mu_\alpha - g^{\beta\mu}\delta^\epsilon_\alpha\right) \tag{2.2.20}$$

i.e.

$$\hat{p}^{\mu\nu} = \left(\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) - \tfrac{1}{2}g^{\mu\nu}R = -R^{\mu\nu} \qquad \hat{p}_\alpha{}^{\beta\mu\epsilon} = \tfrac{1}{2}\left(g^{\beta\epsilon}\delta^\mu_\alpha - g^{\beta\mu}\delta^\epsilon_\alpha\right) \tag{2.2.21}$$

This satisfies the condition (2.2.17); in fact

$$-2R^\nu{}_\lambda + \tfrac{1}{2}\left(g^{\beta\epsilon}\delta^\mu_\alpha - g^{\beta\mu}\delta^\epsilon_\alpha\right)\left(-\delta^\alpha_\lambda R^\nu{}_{\beta\mu\epsilon} - g^{\alpha\nu}R_{\beta\lambda\mu\epsilon} + 2\delta^\nu_\mu R^\alpha{}_{\beta\lambda\epsilon}\right) = -2R^\nu{}_\lambda - R^\nu{}_\lambda + R^\nu{}_\lambda + 2R^\nu{}_\lambda = 0 \tag{2.2.22}$$

Other examples are

$$L = \sqrt{g}\, Q = \sqrt{g}\, R_{\alpha\beta}R^{\alpha\beta} \qquad L = \sqrt{g}\, K = \sqrt{g}\, R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \tag{2.2.23}$$

Also any algebraic combination of the invariants (R, Q, K) gives examples of Lagrangians which are generally covariant, namely

$$L = \sqrt{g}\, f(R, Q, K) \tag{2.2.24}$$

for any sufficiently regular function f. Here *sufficiently regular* means that a number of its derivatives must exist in order to write down field equations and field equations must themselves be sufficiently regular so that they can be reasonably analysed looking for solutions.

As usual *smooth* is ok, though we shall show that some smooth functions f lead to degenerate dynamics and are usually excluded. Nevertheless a *generic* smooth function is ok, meaning that one has to rule out a finite number of small families which will be discussed below.

A more geometric account of gauge-natural bundles and Utiyama theorem can be found in[13][14].

Utiyama-like arguments for gauge-natural theories

As for natural theories, also gauge-natural theories have their Utiyama-like theorems. If one fixes fields and the order of the theory, then gauge covariance imposes strong constraints on the allowed gauge-natural Lagrangians.

For example, let us consider a Lagrangian for electromagnetism, which depends on $(g_{\mu\nu}, A_\mu)$, which is zero order in $g_{\mu\nu}$ and first order in A_μ. A general Lagrangian is in the form:

$$\boldsymbol{L} = \tilde{L}(x^\mu, g_{\mu\nu}, A_\mu, d_\mu A_\nu)\, d\sigma \tag{2.2.25}$$

One can define new fibered *"coordinates"* $(x^\mu, g_{\mu\nu}, F_{\mu\nu}, S_{\mu\nu})$ on the prolongation of the bundle of potentials A_μ:

$$F_{\mu\nu} = d_\mu A_\nu - d_\nu A_\mu \qquad\qquad S_{\mu\nu} = d_\mu A_\nu + d_\nu A_\mu \tag{2.2.26}$$

One can easily check that the first derivatives $d_\mu A_\nu$ can be expressed as a function of $F_{\mu\nu}$ and $S_{\mu\nu}$.

Of course, $F_{\mu\nu}$ is tensorial, $S_{\mu\nu}$ is not. Then one can equivalently consider a Lagrangian in the form:

$$\boldsymbol{L} = \sqrt{g}\, l(x^\mu, g^{\mu\nu}, A_\mu, F_{\mu\nu}, S_{\mu\nu})\, d\sigma \tag{2.2.27}$$

Let us now consider a symmetry generator, which is a right-invariant vector field $\Xi = \xi^\mu \partial_\mu + \zeta\rho$ on P. Lie derivatives of fields read as

$$
\begin{cases}
\pounds_\Xi g^{\mu\nu} = -2\overset{g}{\nabla}{}^{(\mu}\xi^{\nu)} \\
\pounds_\Xi A_\mu = \xi^\lambda F_{\lambda\mu} + D_\mu \zeta_V
\end{cases}
\qquad
\begin{cases}
\pounds_\Xi F_{\mu\nu} = \xi^\lambda \overset{g}{\nabla}_\lambda F_{\mu\nu} + \overset{g}{\nabla}_\mu \xi^\lambda F_{\lambda\nu} + \overset{g}{\nabla}_\nu \xi^\lambda F_{\mu\lambda} \\
\pounds_\Xi S_{\mu\nu} = d_\mu\left(\xi^\lambda F_{\lambda\nu} + \nabla_\nu \zeta_V\right) + d_\nu\left(\xi^\lambda F_{\lambda\mu} + \nabla_\mu \zeta_V\right) = 2\nabla_{\mu\nu}\zeta_V + \dots
\end{cases}
\tag{2.2.28}
$$

where we set $\zeta_V = \zeta + \xi^\mu A_\mu$ for the vertical part of the symmetry generator and $\nabla_\mu \zeta_V = \partial_\mu \zeta_V$ denotes the gauge covariant derivative (which, in this case, coincides with the ordinary partial derivative since the group is commutative).

Let us also define the naive momenta

$$\begin{cases} p^\mu = \sqrt{g}\dfrac{\partial l}{\partial A_\mu} = \sqrt{g}\hat{p}^\mu \\[2ex] p_{\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial g^{\mu\nu}} - \tfrac{1}{2}\sqrt{g}lg_{\mu\nu} \end{cases} \qquad \begin{cases} p^{\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial F_{\mu\nu}} = \sqrt{g}\hat{p}^{\mu\nu} \\[2ex] P^{\mu\nu} = \sqrt{g}\dfrac{\partial l}{\partial S_{\mu\nu}} = \sqrt{g}\hat{P}^{\mu\nu} \end{cases} \tag{2.2.29}$$

The covariance identity reads as

$$p_{\mu\nu}\pounds_\Xi g^{\mu\nu} + p^\mu \pounds_\Xi A_\mu + p^{\mu\nu}\pounds_\Xi F_{\mu\nu} + P^{\mu\nu}\pounds_\Xi S_{\mu\nu} = d_\lambda\left(\xi^\lambda L\right) \tag{2.2.30}$$

$$\begin{cases} \hat{p}^{\mu\nu}\overset{g}{\nabla}_\lambda F_{\mu\nu} = \nabla_\lambda l \\[2ex] -2\hat{p}_{\lambda\nu}g^{\mu\nu} + l\delta^\mu_\lambda + 2\hat{p}^{\mu\nu}F_{\lambda\nu} = l\delta^\mu_\lambda \end{cases} \Rightarrow \hat{p}_{\lambda\nu}g^{\mu\nu} = \hat{p}^{\mu\nu}F_{\lambda\nu} \qquad \begin{cases} p^\mu = 0 \\[2ex] P^{\mu\nu} = 0 \end{cases} \tag{2.2.31}$$

The third and fourth conditions force the Lagrangian to be independent of A_μ and $S_{\mu\nu}$. The first equation (using the second) implies that l does not depend explicitly on x.

$$\textcolor{red}{\hat{p}^{\mu\nu}\overset{g}{\nabla}_\lambda F_{\mu\nu}} = \partial_\lambda l + \boxed{\hat{p}_{\mu\nu}\nabla_\lambda g^{\mu\nu}} + \hat{p}^{\mu\nu}\nabla_\lambda F_{\mu\nu} \qquad \Rightarrow \partial_\lambda l = 0 \tag{2.2.32}$$

Then we have the most general covariant Lagrangian depending on $(x^\mu, g_{\mu\nu}, A_\mu, d_\mu A_\nu)$

$$\boldsymbol{L} = \sqrt{g}\, l(g^{\mu\nu}, F_{\mu\nu})\, d\sigma \tag{2.2.33}$$

which satisfies

$$\hat{p}_{\lambda\nu}g^{\mu\nu} = \hat{p}^{\mu\nu}F_{\lambda\nu} \tag{2.2.34}$$

This condition is satisfied, for example, by Maxwell Lagrangian (1.10.85).

In fact:

$$\boldsymbol{L} = -\tfrac{1}{4\mathrm{g}}\sqrt{g}F_{\alpha\beta}F^{\alpha\beta}_{..}\, d\sigma \qquad \Rightarrow \begin{cases} \hat{p}_{\lambda\nu} = -\tfrac{1}{2\mathrm{g}}F_{\lambda\alpha}F_\nu{}^\alpha_{.} \\[2ex] \hat{p}^{\mu\nu} = -\tfrac{1}{2\mathrm{g}}F^{\mu\nu}_{..} \end{cases} \Rightarrow \textcolor{red}{-\tfrac{1}{2\mathrm{g}}F_{\lambda\alpha}F_\nu{}^\alpha_{.}g^{\mu\nu}} = -\tfrac{1}{2\mathrm{g}}F^{\mu\nu}_{..}F_{\lambda\nu} \tag{2.2.35}$$

3. Initial value problems for covariant theories

Now that we defined natural and gauge-natural field theories and we showed how general (as well as gauge) covariance constrains allowed dynamics, we can go on analysing general features of relativistic theories. Let us here consider initial problem formulation for covariant theories.

Generally speaking, *initial value problem formulation* means regarding a covariant field σ on the spacetime M as a 1-parameter family of fields on a hypersurface $\Sigma \subset M$. Instead of having one covariant field which describes the whole history of the field on spacetime, one can think of selecting a

space manifold Σ and considering a 1-parameter family of fields each representing a configuration, which evolves as the parameter passes by. There is no *a priori* need in general to regard the parameter as a physical *time*; still often it is.

First ingredient for initial value formulation is a 1-parameter foliation of spacetime M. Each leaf Σ_s of such a foliation is a frame in the evolution of the fields. The covariant field σ on M induces fields on the hyperfurfaces Σ_s which represent the system at that value of the parameter s. This will be called an *ADM slicing* of spacetime or an *ADM foliation.*

Then the covariant field equations can be written in terms of the fields on Σ_s and they (may) become a Cauchy problem: one assigns the value of fields on Σ_s (and their derivatives up to some order) and field equations determine the evolution of fields as s passes by. As a result, one has a 1-parameter family of fields, one on each Σ_s.

Finally, one would like to build the covariant field σ out of the 1-parameter family of fields on Σ_s and be sure that σ is a solution of covariant field equations as a consequence of being built out of solutions of equations on Σ_s.

Let us remark that this scheme is quite simplistic and it almost never works this way. As a consequence we shall elaborate it as we proceed.

Let us stress that, by selecting a foliation, the general covariance is broken. The *ADM* slicing is a structure fixed on spacetime which is not (completely) determined by field equations. If one thinks to the parameter s as time, the surfaces Σ_s are equal time surfaces and they select preferred observers which agree that events on Σ_s are contemporary.

On the other hand, studying initial value problem clearly states what one is free to fix and what is determined by field equations. In other words, it is equivalent to Hamiltonian viewpoint and it defines the *physical state of the system.*

As we already said, covariant theories trade absolute description of fields with redundancy of description. One can think to a system as an absolute description of physical world, at the price of adding a lot of redundancy accounting for the observer freedom; this is the covariant formulation. Alternatively, one can describe the system with minimal data, which unfortunately usually depends on the observer; and this is the initial value formulation. Both viewpoints tell us a piece of the story and we have to be able to do both.

ADM space-times

Let us start by defining the foliation which is the starting point for initial value formulation.

Definition (2.3.1): *an ADM splitting of spacetime or a ADM foliation is a bundle $\mathcal{B} = (M, \mathbb{R}, t, \Sigma)$ which foliates the spacetime M in a 1-parameter family of $(m-1)$-manifolds isomorphic to Σ.*

This is a general result since any foliation in hypersurfaces can be considered as a bundle (provided the leaves are all diffeomorphic to a standard model Σ). Since the base $\mathbb{R}$ is contractible, then the bundle over it is necessarily trivial. Accordingly, for any *ADM* slicing the spacetime M is necessarily a product $M \simeq \mathbb{R} \times \Sigma$ for some abstract $(m-1)$-dimensional model for the leaves Σ_s. Such a spacetime is also called a *globally hyperbolic* spacetime. As a matter of fact being globally hyperbolic is necessary in order to be able to write field equations in evolutionary form, thus we shall restrict to these cases.

The fibers $\Sigma_s = t^{-1}(s) \subset M$ are called *synchronous hypersurfaces* or *space hypersurfaces* at time s. The map $t : M \to \mathbb{R}$ is usually thought as a map attaching the time to each event that happens and events on the same fiber happen at the same time and are hence synchronous.

Of course, different observers may define different notions of synchronisation and then different *ADM* splittings.

First of all, we have to remark that, on the trivial bundle $\mathcal{B}$, one can define many different structures of product $\mathbb{R} \times \Sigma$, essentially one for any global trivialisation.

> To keep it simple, let us consider a fibered plane $p : \mathbb{R}^2 \to \mathbb{R} : (s, x) \mapsto x$. One has a global trivialisation $t : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R} : (s, x) \mapsto (s, x)$ which is compatible with the choice of fibered coordinates. This corresponds to embed back in $\mathbb{R}^2$ the spatial point x by the maps $h_x : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (s, x)$. If we are in a mechanical setting and s is time, then the curves $h_x(s)$ represent the rest motions at the point x (which, of course, depend on the observer).
>
> However, one can also define a different product structure on $\mathbb{R}^2$ which corresponds to a different global trivialisation, i.e. $t' : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R} : (s, x) \mapsto (s, x + vs)$. This corresponds to rest motions $h'_x : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (s, x + vs)$, which are rest motions for an observer in motion at velocity v with respect to the first one.
>
> As a matter of fact, different product structures compatible with an *ADM* slicing correspond to different observers. A single *ADM* slicing selects a class of preferred observers, called *ADM observers* with respect to that *ADM* slicing.

One has a correspondence between product structures and connections of the bundle $\mathcal{B}$. Let us consider a connection H on the *ADM* slicing, thought as a distribution of horizontal subspaces $H_x \in T_x M$. Since the base manifold $\mathbb{R}$ is of dimension 1, then the subspaces H_x are of dimension 1 as well. Then the horizontal distribution is of rank 1, hence it is always integrable. The integral curves of the horizontal distributions are in the form $h_x : s \mapsto (s, x(s))$ and are everywhere tangent to the horizontal distribution. One can choose initial conditions $x(0) = x \in \Sigma_0$. They are called *rest motions* and they foliate the spacetime M. Since they are horizontal the rest motions are everywhere transverse to the surfaces Σ_s.

Accordingly, we have two different foliations on M: the foliation Σ_s of synchronous hypersurfaces and the foliation h_x of rest motions (with $x \in \Sigma \simeq \Sigma_0$). Any event $e \in M$ belongs to precisely one leaf Σ_s and one leaf h_x and one can define a global diffeomorphism

$$\iota_H : M \to \mathbb{R} \times \Sigma : e \mapsto (s, x) \tag{2.3.2}$$

Of course, different connections H produce different diffeomorphisms $\iota_H : M \to \mathbb{R} \times \Sigma$.

In view of these correspondence, let us define a pair $(\mathcal{B}, H)$ of an *ADM* slicing and a connection H over it to be an *ADM space-time*. Choosing an *ADM* space-time corresponds to endow the spacetime M with a product structure $M \simeq \mathbb{R} \times \Sigma$ which in turn corresponds to an observer which splits time and space according to its own conventions.

> An *ADM* space-time is also called a $(3 + 1)$-*slicing* when the spacetime is of dimension $m = 4$. We shall use the dash *space-time* to denote a $(3 + 1)$-slicing of a spacetime, or more generally an *ADM* space-time, since space and time have been conventionally split.

It is important to understand that the *ADM* slicing is still a more general structure than the *ADM* space-time. The spacetime M is general and absolute. On M one can use any observer. Fixing an *ADM* slicing selects a class of *ADM* observers which agree on what it means to be contemporary but disagree on which space points correspond at different times, i.e. they disagree on rest motions. Finally, on an *ADM* space-time the observers agree on both contemporary and on how space points at different times are in correspondence.

The spacetime M can be covered with fibered coordinates (t, k^A) with respect to the *ADM* splitting. Such coordinate systems are called *ADM coordinates* or *ADM observers*. The transition functions among *ADM* observers are in the form

$$\begin{cases} t' = t'(t) \\ k'^A = k'^A(t, k) \end{cases} \tag{2.3.3}$$

and a globally hyperbolic spacetime can always be covered with a family of ADM observers which share the same definition of time. ADM coordinates are not the most general coordinate systems on M, since in general one can consider observers which do not preserve the time structure (i.e. they refer to different ADM splittings).

The ADM space-time is a rigid structure. It is in fact stricter than a trivial bundle, it corresponds to a trivial bundle with a global trivialisation fixed on it, i.e. a product structure. Being it a product, one has two projections, one on $\mathbb{R}$ which coincides with the original ADM projection $t : M \to \mathbb{R}$ and one on Σ which corresponds to the map $\chi := p_2 \circ \iota_H : M \to \Sigma$ where p_2 is the canonical projection on the second factor in a product.

Now suppose we are able to split the configuration bundle C on M as a bundle D on Σ such that $\chi^* D \simeq C$ is a bundle isomorphism, i.e.

$$
\begin{array}{ccccccc}
C & \dashrightarrow^{\sim} & \chi^* D & \xhookrightarrow{\iota_H^*} & p_2^* D & \xhookrightarrow{p_2^*} & D \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
M & = & M & \xrightarrow{\iota_H} & R \times \Sigma & \xrightarrow{p_2} & \Sigma
\end{array}
\tag{2.3.4}
$$

If we denote by (x^μ, y^i) as fibered coordinates on C and (k^A, z^i) as fibered coordinates on $\chi^* D$, then fibered coordinates on $\iota_H^* D$ are (s, k^A, z^i).

A section of the bundle $\iota_H^* D$ is locally described by functions $z^i(s, k)$ which can be interpreted as a 1-parameter family of fields $z_s^i(k)$ on D. The detail of the isomorphism $\Phi : C \to \chi^* D$ depends on which fields we are considering. We shall give plenty of examples below.

Ok, the construction is everything but simple and immediate. But I warned you that ADM foliations break general covariance and select special observers. The isomorphisms $\Phi : C \to \chi^* D$ and $\iota_H : M \to \mathbb{R} \times \Sigma$ is where the details about these special observers are hidden.

A bundle D over space Σ such that one can define the isomorphism $\Phi : C \to \chi^* D$ is called *adapted configuration bundle*.

To summarise, in a specific situation the point will be to fix a generic ADM foliation and to learn how to decompose fields on spacetime as fields on space, i.e. how to define the bundle D such that one has the diagram above. At that point, it will be equivalent to describe the system as a set of covariant fields (i.e. a section of C) or by a 1-parameter family of fields on space, i.e. a section of $\iota_H^* D$. Once this is somehow achieved, then, of course, by composition with any vertical isomorphism of $\iota_H^* D$, one obtains infinitely many ways of doing it. The hope is that, despite these infinitely many ways of doing it, in at least one of these splittings field equations will be arranged in a way in which equations are evolutionary.

Let us describe some simple examples of these decompositions.

Let us start from the trivial case of scalar field φ. The configuration bundle is $C = M \times \mathbb{R}$, as we already discussed. Let us suppose M is globally hyperbolic and let us fix an ADM space-time for it, i.e. a specific isomorphism $\iota : M \to \mathbb{R} \times \Sigma$ for some $(m-1)$-manifold Σ.

In general, we can set coordinates x^μ on M and coordinates (s, k^A) on $\mathbb{R} \times \Sigma$, so that the (inverse) isomorphism is locally expressed as

$$
\bar{\iota} : \mathbb{R} \times \Sigma \to M : (s, k) \mapsto x^\mu(s, k)
\tag{2.3.5}
$$

This isomorphism can be regarded in two ways: first, one can consider it as a change of coordinates in M to adapted coordinates to the foliation, i.e. one can use (s, k) as coordinates on M instead of x^μ. In these adapted coordinates the local expression of the isomorphism simplifies to $\bar{\iota} : \mathbb{R} \times \Sigma \to M : (s, k) \mapsto (s, k)$.

Second, one can associate to the isomorphism i_H above a family of embeddings

$$
\iota_s : \Sigma \to M : k \mapsto x^\mu(s, k)
\tag{2.3.6}
$$

which describes the foliation by selecting the leaves as $\Sigma_s = \mathrm{Im}(\iota_s)$.

Now, the scalar field φ pulls back nicely along embeddings. The scalar field $\varphi(x)$ defines a 1-parameter family of scalar fields $\iota_s^* \varphi(k)$ on Σ. Thus one can fix $D = \Sigma \times \mathbb{R}$ for the bundle of scalar fields on Σ. Let us fix, by an abuse of notation, (k, φ) as fibered coordinates on D so that we can describe the maps which appear in diagram (2.3.4) as

$$\Phi : C \to \chi^* D : (x, \varphi) \mapsto (x, \varphi) \qquad \iota_H^* : \chi^* D \to p_2 D : (x, \varphi) \mapsto (s, k, \varphi) \tag{2.3.7}$$

Thus a covariant scalar field $\varphi(x)$ corresponds to a 1-parameter family $\varphi_s = \iota_s^* \varphi : \Sigma \to R : x = (s.k) \mapsto \varphi_s(k)$, i.e., when $x = \iota_s(k)$, we obtain

$$\varphi_s(k) = \varphi(x) \tag{2.3.8}$$

Notice we have one function of x in the covariant setting, one function of (s, k) in the *ADM* setting. The correspondence between the covariant field φ and the space field φ is one-to-one.

As a second relatively less trivial example, let us consider a metric field $g_{\mu\nu}$ on a spacetime M of dimension $m = 4$. Since the components are symmetric, a metric field is described by 10 functions $g_{\mu\nu}(x)$. Being the metric a covariant tensor, it also nicely pulls back on space Σ to define a tensor $\gamma = i_s^* g$ on Σ, namely

$$\gamma_{AB}(s, k) = J_A^\alpha g_{\alpha\beta}(x) J_A^\beta \qquad J_A^\alpha := \partial_A x^\alpha(s, k) \tag{2.3.9}$$

If the metric g is strictly Riemannian, then the tensor $\gamma(s, \cdot)$ is also a 1-parameter family of metrics on Σ. However, γ is described by 6 functions. Thus, unlike what happened with scalar fields, by pulling back we lost 4 functions. There are other objects on Σ one can define with the metric g to account for the 4 extra functions. While γ accounts for inner products of vectors tangent to Σ, the inner products involving the normal vector have been lost. The normal covector is canonically defined as

$$\boldsymbol{u} = \tfrac{1}{3!} \epsilon^{ABC} J_A^\alpha J_B^\beta J_C^\gamma \epsilon_{\alpha\beta\gamma\mu} \, dx^\mu \tag{2.3.10}$$

while for the normal unit vector one needs to use the metric g:

$$u := \tfrac{1}{3!} \epsilon^{ABC} J_A^\alpha J_B^\beta J_C^\gamma \epsilon_{\alpha\beta\gamma}{}^\mu{}_. \partial_\mu \qquad n := \pm \frac{1}{\sqrt{g(u, u)}} u \tag{2.3.11}$$

One can also define the tangent vectors

$$e_A = J_A^\alpha \partial_\alpha \tag{2.3.12}$$

and prove that $g(e_A, n) = 0$ and $g(n, n) = 1$.

Let us now define the *lapse N* and the *shift* β_A so that one has

$$\partial_0 = Nn + \beta^A_. e_A \tag{2.3.13}$$

Then one has

$$g_{00} = g(Nn + \beta^A_. e_A, Nn + \beta^B_. e_B) = N^2 + \beta^A_. \beta_A = N^2 + |\vec{\beta}|^2 \qquad g_{0B} = g(Nn + \beta^A_. e_A, e_B) = \beta_B$$
$$g_{AB} = \gamma_{AB} \tag{2.3.14}$$

In other words we have now 10 functions $(N, \beta_A, \gamma_{AB})$ which are defined out of $g_{\mu\nu}$ and such that one can reconstruct the covariant metric g once the lapse N, shift β and space metric γ are given.

Thus we can define

$$D = (\Sigma \times \mathbb{R}) \times_\Sigma T^* \Sigma \times_\Sigma \mathrm{Euc}(\Sigma) \tag{2.3.15}$$

with fibered coordinates $(k^A, N, \beta_A, \gamma_{AB})$.

Evolutionary equations

By *evolutionary scheme* for covariant equations, we mean to fix and ADM space-time, to define the adapted configuration bundle D and write field equations in terms of the adapted fields and the adapted coordinates (s, k). Then one gives initial conditions on a space sub-manifold Σ_0 (also called the *Cauchy surface*) and, hopefully, field equations uniquely determine a solution which agrees with initial conditions.

Let us consider Klein–Gordon equations on Minkowski spacetime $M = \mathbb{R}^4$. Let us fix Cartesian coordinates x^μ on M so that the metric expression is

$$\eta = \eta_{\mu\nu}\, dx^\mu \otimes d\, dx^\nu = -\, dx^0 \otimes dx^0 +\, dx^1 \otimes dx^1 +\, dx^2 \otimes dx^2 +\, dx^3 \otimes dx^3 \tag{2.3.16}$$

and the ADM slicing to be $t : M \to \mathbb{R} : x^\mu \mapsto x^0$. The leaves of the ADM splicing are the hypersurfaces $\Sigma_s := \{(s, k^A)\}$ modelled by $\Sigma = \mathbb{R}^3$ with coordinates k^A. The Cartesian coordinates x^μ turn out to be adapted to the ADM slicing, since the embeddings

$$\iota_H : \Sigma \to M : k^A \mapsto (s, k^A) \qquad \begin{cases} x^0 = s \\ x^i = \delta^i_A k^A \end{cases} \tag{2.3.17}$$

are in standard form and

$$J^\alpha_A = \begin{pmatrix} \delta^0_A = 0 \\ \delta^i_A \end{pmatrix} =: \delta^\alpha_A \tag{2.3.18}$$

We can fix a connection $H_x = \mathrm{Span}\,(\partial_0) \subset T_x M$ so that the rest motions are the horizontal curves $h_x(s) = (s, x^i)$. This establishes the ADM space-time as

$$\iota_H : \mathbb{R} \times \Sigma \to M : (s, k) \mapsto (x^0 = s, k^A) \tag{2.3.19}$$

We already discussed the adapted configuration bundle: a covariant scalar field induces a space scalar field which is denoted by the same name φ. The covariant field equations are

$$\left(\Box - \mu^2\right) \varphi = 0 \tag{2.3.20}$$

which can be split as

$$\partial_{00}\varphi - \delta^{AB}\partial_{AB}\varphi + \mu^2\varphi = 0 \tag{2.3.21}$$

For this equation, one gives initial conditions $\varphi(0, k) = \phi(k)$ and $\partial_0\varphi(0, k) = \psi(k)$ and the equation selects one and only one solution $\varphi(s, k)$ which obeys initial conditions. This is a section of the bundle $\chi^* D \sim C$ which induces a section of the covariant configuration bundle C, namely $\varphi(x) = \varphi(x^0, x^i)$. The covariant scalar field $\varphi(x)$ is a solution of the original covariant equations.

Of course, the choice of the connection (2.3.19) made everything particularly simple. If one chooses a different connection, e.g. $H' = \mathrm{Span}\left(\partial_0 + \frac{v^i}{c}\partial_i\right)$, then a different product structure is induced

$$\bar{\iota}_{H'} : \mathbb{R} \times \Sigma \to M : (s, k^A) \mapsto \left(s, \delta^i_A k^A + \frac{v^i}{c}s\right) \tag{2.3.22}$$

By fixing k^A, one obtains rest motions, by fixing s one obtains Σ_s space surfaces.
Thus we have new adapted coordinates

$$\begin{cases} x'^0 = s = x^0 \\ x'^i = x^i - \frac{v^i}{c}x^0 \end{cases} \tag{2.3.23}$$

The metric in the new coordinates reads as

$$\eta = -(c^2 - |v|^2)\, dt' \otimes dt' + v_i \left(dt' \otimes dx'^i + dx'^i \otimes dt' \right) + \delta_{ij}\, dx'^i \otimes dx'^j \tag{2.3.24}$$

which corresponds to taking

$$N' = 1 \qquad \beta_i' = \frac{v_i}{c} \qquad \gamma_{ij}' = \delta_{ij} \tag{2.3.25}$$

The non-vanishing shift β^i accounts for how spaces are mapped one into the others as time passes by. Notice that the metric coefficients are still constant, thus Christoffel symbols $\{\eta\}'^{\alpha}_{\mu\nu}$ are zero.

The Klein–Gordon equation can be written in the new coordinates as

$$\Box'\varphi' - \mu^2\varphi' = g'^{\mu\nu}\left(\partial'_{\mu\nu}\varphi' + \{\eta\}'^{\alpha}_{\mu\nu}\partial'_{\alpha}\varphi'\right) - \mu^2\varphi' = -\left(\partial'_{00}\varphi' - 2\frac{v^i}{c}\partial'_{0i}\varphi' - \left(\delta^{ij} - \frac{v^iv^j}{c^2}\right)\partial'_{ij}\varphi' + \mu^2\varphi'\right) = 0 \tag{2.3.26}$$

Also for this equation, one gives initial conditions $\varphi'(0,k) = \phi(k)$ and $\partial_0\varphi'(0,k) = \psi(k) + \frac{v^i}{c}\partial_i\phi$ and the equation selects one and only one solution $\varphi'(x')$ which obeys initial conditions. This is a section of the bundle $\chi^*D \sim C$ which induces a section of the covariant configuration bundle C, namely $\varphi'(x') = \varphi'(x'^0, x'^i)$.

Again, the covariant scalar field $\varphi'(x')$ is a solution of the original covariant equation.

One can verify that since $\varphi(s,k)$ satisfies the adapted equation with respect to the *ADM* space-time above, then one has that

$$\varphi'(x'^0, x'^i) = \varphi(x'^0, x'^i + v^i x'^0) \tag{2.3.27}$$

satisfies the new equation (and initial conditions).

It is evident that we need a more general structure to deal with solutions in general when one considers fields more complicated than scalar fields.

In order to solve adapted field equations, we need a form of Cauchy theorem which guarantees existence and uniqueness of solutions for a given initial condition. Hereafter we shall sketch a framework which is able to encapsulate all analytical details in Cauchy theorem and leave us with algebraic conditions to be checked on equations.

Let us first notice that we are not interested in general equations; we are interested in field equations coming from a variational principle as Euler– *Review first this example* Lagrange equations. We can also restrict for simplicity to equations which are quasi-linear. This restricts allowed Lagrangians, though all Lagrangians which are relevant to fundamental physics lead to quasi-linear equations. By considering a Lagrangian $\boldsymbol{L}$, one can consider its Euler–Lagrange part $\mathbb{E}(L) = \mathbb{E}_i(L)\,\omega^i \wedge d\sigma$ as an element $\hat{\mathbb{E}}(L) = \mathbb{E}_i(L)\,\bar{d}y^i \otimes d\sigma$ in $V^*(C) \otimes A_m(M)$. In fact, for a vertical vector $X = \delta y^i \partial_i$ one has

$$i_X \mathbb{E}(L) = E_i(L)\delta y^i\, d\sigma = \hat{\mathbb{E}}(L)(X) \tag{2.3.28}$$

Thus we can restrict to equations $E_i = 0$ with $\hat{\mathbb{E}}(L) = E_i(L)\,\bar{d}y^i \otimes d\sigma \in V^*(C) \otimes A_m(M)$, which is in fact compatible with the transformation laws $E_i = J J_i^k E_k'$ that we proved above; see (1.3.32).

Let us start by considering a first order quasi-linear differential operator $\mathbb{E}$ on an *ADM* space-time; this can be written in *ADM* coordinates as

$$e_{ij}y_0^j + e_{ij}^a y_a^j + b_i(t, x^a, y^k) = 0 \tag{2.3.29}$$

A first order quasi-linear operator (2.3.29), with $e_{ij}^a = e_{ji}^a$ and a e_{ij} which is a non-degenerate definite positive bilinear form, is called a *symmetric hyperbolic* operator.

It would be better to define a *symmetric hyperbolic* operator as one for which a system of *ADM* coordinates exists so that $e_{ij}^a = e_{ji}^a$ and e_{ij} is a non-degenerate definite positive bilinear form.

If such *ADM* coordinates exists then one can check that the form of the operator is preserved in any *ADM* coordinates for a given splitting. However, if one changes the *ADM* splitting, such a form is not necessarily preserved. Of course, in order to solve equations, one just needs that *there exists* an *ADM* splitting in which the operator turns out to be symmetric hyperbolic.

We can state the Cauchy theorem as follows:

Theorem: *given a symmetric hyperbolic system with smooth initial conditions, then there exists a unique smooth solution satisfying the equations and the initial conditions.*

If, for any smooth initial conditions, there exists a unique smooth solution satisfying the equations and the initial conditions, we say that the Cauchy problem is *well-posed*.

Let us stress that being symmetric hyperbolic is *sufficient* for being well-posed. It is not necessary.

As a matter of fact, one just have to write down equations and check that the coefficients obey some algebraic conditions in order to guarantee that the Cauchy problem is well-posed.

Let us consider now a second order, quasi-linear differential operator $\mathbb{E}$ on an *ADM* space-time so that the operator can be written in *ADM* coordinates as

$$e_{ij}y^j_{00} + e^a_{ij}y^j_{0a} + e^{ab}_{ij}y^j_{ab} + b_i(t, x^a, y^k, y^k_0, y^k_a) = 0 \qquad (2.3.30)$$

As often done with ODE, one can write a second order equations as a system of first order equations by setting $v^i := y^i_0$ and $v^i_a := y^i_{0a}$. The last definition implies that $d_0 v^i_a - d_a v^i = 0$.

$$\begin{cases} d_0 y^i - v^i = 0 \\ e_{ij}d_0v^j + e^a_{ij}d_av^j + e^{ab}_{ij}d_av^j_b + b_i(t, x^a, y^k, v^k, v^k_a) = 0 \qquad v^i_a := d_a v^i \\ d_0 v^i_a - d_a v^i = 0 \end{cases} \qquad (2.3.31)$$

The equation $v^i_a := d_a v^i$ does not involve time derivatives and it will be called a *constraint*.

In fact, it can be seen as a constraint to allowed initial conditions $y^i(0, k) = f(k)$, $v^i(0, k) = g^i(k)$, $v^i_a(0, k) = h^i_a(k)$. The constraint implies that initial conditions must be chosen so that $h^i_a(k) = d_a f(k)$.

We have here to remark that the system (2.3.31) is a first order system, though not in the form of the first order system considered above. The fields are (y^i, v^i, v^i_a) though the equations do not live in the dual space. This can be corrected easily introducing two non-degenerate bilinear forms β_{ij} and β^{ab}_{ij} and writing the system (2.3.31) in the equivalent form

$$\begin{cases} \beta_{ij}d_0y^j - \beta_{ij}v^j = 0 \\ e_{ij}d_0v^j + e^a_{ij}d_av^j + e^{ab}_{ij}d_av^j_b + b_i(t, x^a, y^k, v^k, v^k_a) = 0 \qquad v^i_a := d_a v^i \\ \beta^{ab}_{ij}d_0v^j_b - \beta^{ab}_{ij}d_bv^j = 0 \end{cases} \qquad (2.3.32)$$

There is no loss of generality in choosing $\beta_{ij} = e_{ij}$.

Let us then consider the second order Cauchy problem (*CP2*)

$$e_{ij}y_{00}^j + e_{ij}^a y_{0a}^j + e_{ij}^{ab} y_{ab}^j + b_i(t, x^a, y^k, y_0^k, y_a^k) = 0$$
$$y^i(0, k) = f^i(k), \quad y_0^i(0, k) = g^i(k) \tag{2.3.33}$$

and the first order Cauchy problem (with constraint) (*CP1*)

$$\begin{cases} e_{ij}d_0 y^j - e_{ij}v^j = 0 \\ e_{ij}d_0 v^j + e_{ij}^a d_a v^j + e_{ij}^{ab} d_a v_b^j + b_i(t, x^a, y^k, v^k, v_a^k) = 0 \\ \beta_{ij}^{ab}d_0 v_b^j - \beta_{ij}^{ab}d_b v^j = 0 \\ y^i(0, k) = f(k), \quad v^i(0, k) = g^i(k), \quad v_a^i(0, k) = \partial_a f^i(k) \end{cases} \qquad v_a^i = d_a v^i \tag{2.3.34}$$

and show that there is a one-to-one correspondence between solutions.

Let $y^i(t, k)$ be a solution of *CP2* and let us define $v^i(t, k) := \partial_0 y^i(t, k)$ and $v_a^i(t, k) := \partial_a y^i(t, k)$. One has immediately that $e_{ij}\left(d_0 y^j - v^j\right) = 0$, $\beta_{ij}^{ab}\left(d_0 v_b^j - d_b v^j\right) = 0$, $v_a^i = d_a v^i$, and $v_a^i(0, k) = \partial_a f^i(k)$. Accordingly, one has a solution of *CP1* in the form $(y^i(t, k), \partial_0 y^i(t, k), \partial_a y^i(t, k))$

On the other hand, let us consider a solution $(y^i(t, k), v^i(t, k), v_a^i(t, k))$ of *CP1*. Because of equation $d_0 y^j = v^j$, the field v^i must be equal to $v^i(t, k) = \partial_0 y^i(t, k)$. Because of the equation $d_0 v_b^j = d_b v^j$, one has $v_a^i = \partial_a v^i + c(k)$ and because of the constraint one has $c(k) = 0$. Thus $y^i(t, k)$ is a solution of *CP2*.

The second order problem *CP2* is *well-posed* if the first order problem *CP1* is symmetric hyperbolic. That is symmetric when $\beta_{ij}^{ab} = e_{ji}^{ab}$ and it is symmetric hyperbolic when e_{ij}^{ab} is non-degenerate and definite positive. Accordingly, a second order quasi-linear operator (2.3.33) is *symmetric hyperbolic* iff e_{ij} is non-degenerate definite positive and $e_{ij}^{ab} = e_{ji}^{ab}$ is symmetric and non-degenerate definite positive.

The second order Cauchy problem *CP2* is well-posed if the operator is symmetric hyperbolic in some *ADM* space-time. Unlike for first order operators, an operator which is symmetric hyperbolic in a system of *ADM* coordinates may fail to be in another *ADM* coordinate system (e.g. if e_{ij}^a happens not to be symmetric).

Let us stress that these conditions are met for Klein–Gordon equation considered above. For the Klein–Gordon equation the evolutionary scheme applies directly and thoroughly.

Let us consider now vacuum electromagnetism on Minkowski. With the standard *ADM* space-time we have *ADM* coordinates (t, x^i), and the potential A_μ can be pulled back to space as

$$A_i = J_i^\alpha A_\alpha = A_i \tag{2.3.35}$$

and a space scalar field $A = A_\alpha n^\alpha = A_0$.

The adapted configuration bundle can be fixed to

$$D = (\Sigma \times \mathbb{R}) \times_M T^*\Sigma \tag{2.3.36}$$

Maxwell equations are written as

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.3.37}$$

which splits as

$$\partial_{j0}A^j_. = \eta^{kj}\partial_{jk}A \qquad\qquad \eta^{ik}\partial_{00}A_k - \eta^{ik}\partial_{0k}A + \left(\eta^{ik}\eta^{nl} - \eta^{il}\eta^{nk}\right)\partial_{nk}A_l = 0 \qquad\qquad (2.3.38)$$

This system is not symmetric hyperbolic. In particular, the first equation has no second time derivatives implying a degenerate coefficient e_{ij}. The first equation does not determine fields, it just constrains initial conditions. In other words, initial conditions must obey these equations.

The second equation are 3 equations which are symmetric hyperbolic for the field A_i. If initial conditions are provided, the field $A_i(s,k)$ is uniquely determined. In other words, Maxwell equations do not determine uniquely the potential A_μ. It determines A_i while A_0 is left undetermined. Thus field equations are *underdetermined*.

Moreover, one of the equations actually is a constraint for initial conditions, thus field equations are *overdetermined*.

In the next Section we shall show that the fact that equation is underdetermined is directly connected with covariance. Of course, in a variational setting there are as many equations as fields, and if some fields are left undetermined, the corresponding equations cannot determine anything and must be constraints. accordingly also overdetermination is (indirectly, in a variational framework) related to covariance.

This is a general feature of gauge theories and relativistic theories; field equations split in two groups. One group of constraints and one group of evolutionary equations which partially determine fields. The other fields are left undetermined and are the remnant of observer freedom. In this way one can count how many fields are determined by the evolutionary system. Their number is also called the number of *physical degrees of freedom*.

When there are no constraints and field equations are of order k, the sections of $J^{k-1}D$ uniquely determine solutions. Then $\mathrm{Sec}(J^{k-1}D)$ is the phase space of the system. When initial conditions are constrained, constraints identifies a subset $\mathbb{P} \subset \mathrm{Sec}(J^{k-1}D)$ which is the phase space of the system.

In general, field equations which contain time derivatives could fail to be symmetric hyperbolic, e.g. because e_{ij}^{AB} are not symmetric. This could be settled by introducing other constraints or partially fixing spatial coordinates. We shall discuss the general situation below.

For now, we shall only prove that this evolutionary scheme (i.e. the one without constraints) does not work in most theories of physical interest, i.e. every time one has a non-trivial symmetry group as a consequence of covariance.

4. Hole argument

Let us review now a modification of the *hole argument* presented by Einstein in 1913, though in this form it was discussed by Hilbert; see[15]. It is a fundamental piece of interpretation of relativistic theories. It will also give us a precise and general definition of *gauge theory*. Let us remark that the original hole argument is traditionally used to discuss whether the points in spacetime retain some direct physical interpretation, while here we use it only to discuss observability of the metric field, since we believe it makes it a clearer argument.

Let us first notice that, as a matter of fact, compact supported diffeomorphisms exist on any smooth manifold.

Let $D \subset M$ be a compact subset of spacetime.

Let us consider any vector field $\xi \in \mathfrak{X}(M)$ and $\varphi : M \to \mathbb{R}$ a function compactly supported in D. Such a function exists in view of paracompactness of manifolds (essentially because one can easily define them on $\mathbb{R}^m$).

Then define the vector field $\xi' = \varphi \cdot \xi$ which is zero outside D and different from zero in D. The flow Φ_s of ξ' is then the identity for any s and any x outside D, while it is different from identity within D. A diffeomorphsm Φ is called *compactly supported* if there exists a compact set $D \subset M$ such that Φ is the identity outside D while it is different from the identity within D. Thus the flow Φ_s is made of compactly supported diffeomorphisms of M.

In any natural theory, diffeomorphisms are symmetries, so they send solutions into solutions. Accordingly, if $\psi \in \mathrm{Sec}(C)$ is a solution, then $\psi' := \Phi_*\psi$ is a solution as well. Now the important issue is that, if Φ is supported in $D \subset M$, then ψ and ψ' are identical outside D and different within D.

To be true one does not need the diffeomorphism to be compactly supported. What is needed is that the support does not touch a Cauchy surface Σ_0. Thus, at a fixed *ADM* slicing, one can fix a Cauchy surface and define a *hole* (or a *Cauchy domain*) $D \subset M$ as a close set which does not intersect the Cauchy surface. A *Cauchy diffeomorphism* is a diffeomorphism of M which is supported into a Cauchy domain D. More generally, a *Cauchy transformation* is a symmetry (not necessarily a spacetime diffeomorphism depending on the theory, e.g. a vertical gauge transformation in a gauge-natural theory) which is supported into a Cauchy domain and the set of Cauchy transformations is denoted by $\mathcal{D}^1$. We shall also denote by $\mathcal{D}^n$ the transformations which can be expressed as the composition of n Cauchy transformations and by $\mathcal{D}$ the group generated by Cauchy transformations. Of course, one has

$$\mathcal{D}^1 \subset \mathcal{D}^2 \subset \ldots \subset \mathcal{D}^n \subset \ldots \mathcal{D} \subset \mathcal{S} \tag{2.4.1}$$

For any $\Phi \in \mathcal{D}^1$, the two solutions ψ and $\psi' = \Phi_*\psi$ are different (within the Cauchy domain D), they are both solutions, and they have the same initial conditions since they are identical (everywhere outside D and in particular) on the Cauchy surface Σ_0.

The Cauchy theorem does not hold true simply because, assuming existence of solutions, we constructed a counterexample to uniqueness. Thus field equations cannot uniquely determine solutions and are hence *underdetermined*.

The only two ways out are getting rid of determinism without a fight or assuming that the two mathematical objects ψ and ψ' in fact describe the same *physical state*.

In the second case we are saying that many different mathematical objects do describe the same physical state. One can try to make this idea precise: there must be some equivalence relation to be defined so that it implements the idea that $\psi \sim \psi'$ iff they represent the same *physical state*.

Here we have three groups which are relevant.

The first group is the *symmetry group* $\mathcal{S} = \mathrm{Diff}(M)$ of all transformations which are symmetries of the dynamics.

The second group is the one generated by Cauchy diffeomorphisms, denoted by $\mathcal{D}$. Transformations in $\mathcal{D}$ are compositions of a finite number of Cauchy diffeomorphisms. Notice that a composition of Cauchy diffeomoprhisms need not to be a Cauchy diffeomorphisms; in fact one can compose two Cauchy diffeomorphisms which reduce to the identity in a neighbourhood of two different Cauchy surfaces.

The third group is the group which defines physical states, denoted by $\mathcal{G}$. Two configurations define the same physical state iff there exists a transformation $\Phi \in \mathcal{G}$ mapping one configuration into the other. In other words, physical states are *defined* as equivalence classes, i.e. as the orbits of the group action of $\mathcal{G}$.

The field equation which is written for ψ is compatible with this equivalence relation iff whenever ψ is a solution and $\psi' \sim \psi$ then ψ' is a solution as well. Thus the transformations between different representations of the same physical state must be symmetries, i.e. $\mathcal{G} \subset \mathcal{S}$. Then the field equations pass to the quotient and become equations for the true physical states which are in fact equivalence classes $[\psi]$.

This idea solves the problem highlighted by the hole argument provided one has that if ψ and $\psi' = \Phi_*\psi$ differ by a Cauchy diffeomorphism, then they must represent the same physical situation, i.e. $\psi \sim \psi'$.

In other words, one must also have $\mathcal{D} \subset \mathcal{G}$. To summarise the results, then the situation must be

$$\mathcal{D} \subset \mathcal{G} \subset \mathcal{S} \tag{2.4.2}$$

In some cases, one might have $\mathcal{D} = \mathcal{S}$ so that there is only one possibility for $\mathcal{G} = \mathcal{S}$. However, if the inclusion $\mathcal{D} \subset \mathcal{S}$ is strict, one may have different possibilities for choosing $\mathcal{G}$. In this cases, different specification of the group $\mathcal{G}$ do define different systems. One can see it studying a simple example. Let us consider Kepler problem in the plane. The symmetry group is SO(2) and there is no symmetry which is compactly supported, thus $\mathcal{D} = \{e\}$. Then one has $\mathcal{D} \subsetneq \mathcal{S}$ and different options for choosing $\mathcal{G}$, e.g $\mathcal{G} = \{e\}$ or $\mathcal{G} = \text{SO}(2)$.

In the first case, by choosing $\mathcal{G} = \{e\}$ we are declaring that two planets with orbits which are rigidly rotated around the Sun are two different physical states. Since physical states are defined as what we are able to observe, that means, for example, that we can observe the direction of the orbital main axis, for example by referring it to fixed stars.

But suppose that for some reason there are no fixed stars. Then we are not able to observe a global rotation and only relative angles are physically defined. This corresponds to declare that rotations do not change the physical state and to choose $\mathcal{G} = \text{SO}(2)$.

None of the two choices is *a priori* right and it must be addressed on the physical stance. On the other hand, let us stress that different choices come with a consequence on what is to be considered *observable* in the theory.

More generally, whether $\mathcal{D} = \mathcal{S}$ or not, it depends on details and it will discussed below.

When $\mathcal{D}$ is not trivial, there is no choice: if one wants to keep some sort of determinism, then at least two sections which differ by (a finite composition of) Cauchy diffeomorphisms cannot describe different physical states.

One can get easily convinced that the hole argument can be expressed in a much more general way. Whenever one has symmetries which reduce to the identity in a neighbourhood of a Cauchy surface, then sections related by these transformations must represent the same physical state.

What is relevant is the equivalence class of sections (modulo transformations in $\mathcal{G}$). This applies to diffeomorphisms in natural theories as well as to Cauchy gauge transformations in gauge-natural theories.

Cauchy transformations are a kind of symmetries which are quite peculiar of field theory and quite rare in mechanics.

Consider the motion of a particle in a central force. For this system, rotations are symmetries. However, there is nothing like a compactly supported rotation. Either the rotation is non-trivial (then it moves almost any point in the space, namely all except the ones on its axis) or it is the identity everywhere. The same consideration applies to translations.

As a matter of fact, something similar to gauge transformations in mechanics is considered only when considering the motion of a relativistic point. In that case, one wants arbitrary reparameterisations of the worldlines to be symmetries. And one can easily see that one can define Cauchy reparameterisations (i.e. reparameterisations which reduce to the identity out of a Cauchy domain $D \subset \mathbb{R}$). We shall show below that any reparameterisation can be written as a composition of two Cauchy reparameterisations, so that in this case one has $\mathcal{D} = \mathcal{S}$ and one is forced to choose $\mathcal{G} = \mathcal{S}$.

In that situation, i.e. when reparameterisations are symmetries for the dynamics, one has no option. Different parameterisations of the same trajectories do define the same physical state. There is no model for parameterised curves in spacetime, which is covariant with respect to reparameterisations, in which different parameterisations describe different motions. And this has little to do with the physical meaning of the sentence, it is rather a pure mathematical claim.

We define a *gauge theory* to be a field theory in which one has Cauchy symmetries. This includes the relativistic point, natural theories (e.g. gravitational theories), and gauge-natural theories (e.g. Maxwell and Yang–Mills theories).

This is not optional. If there are Cauchy symmetries one is forced to purge them from physical states if determinism has to be saved. One cannot ignore if a field theory is a gauge theory and act as if it were not.

Accordingly, in gauge theories the physical state is always a class of configurations and field equations are always underdetermined since they cannot determine a representative within the gauge class.

As we discussed above, field equations are also overdetermined (there are constraints on allowed initial conditions).

Thus in general one needs to discuss whether the inclusions $\mathcal{D} \subset \mathcal{G} \subset \mathcal{S}$ are strict and how many possibilities one has in choosing $\mathcal{G}$ accordingly. Hereafter we shall discuss the case of vertical gauge transformations, of reparameterisation of trajectories on spacetime, and of Cauchy spacetime diffeomorphisms in natural theories with a compact and non-compact space Σ.

Step-like functions

Before proceeding with Cauchy transformations, let us here introduce a pair of smooth functions, called *step-like functions* and denoted by $\varphi_\pm : \mathbb{R} \to \mathbb{R}$, which will prove useful below. Let us first define $\varphi_+ : \mathbb{R} \to \mathbb{R}$ as

$$\varphi_+(s; a, \alpha) = \frac{1}{N} \int_{-\infty}^{s} \hat{\varphi}_+(x; a, \alpha) dx \tag{2.4.3}$$

where we set

$$\hat{\varphi}_+(x; a, \alpha) = \begin{cases} \exp\left(-\frac{a^2}{\alpha^2 - x^2}\right) & x \in (-\alpha, \alpha) \\ 0 & \text{otherwise} \end{cases} \tag{2.4.4}$$

The function $\varphi_+(s; a, \alpha)$ is smooth everywhere, it is identically $\varphi_+(s; a, \alpha) = 0$ for $s \leq -\alpha$, the parameter $a > 0$ can be fixed at will and the normalization factor is fixed as

$$N := \int_{-\infty}^{+\infty} \hat{\varphi}_+(x; a, \alpha) dx = \int_{-\alpha}^{+\alpha} \hat{\varphi}_+(x; a, \alpha) dx \tag{2.4.5}$$

so that the function $\varphi_+(s; a, \alpha) = 1$ for $s \geq \alpha$. Moreover, the derivative $\varphi'_+(s; a, \alpha)$ is positive and bounded by the value $\varphi'_+(0; a, \alpha) = 1/N$.

Then one can define

$$\varphi_-(s; a, \alpha) = 1 - \varphi_+(s; a, \alpha) \tag{2.4.6}$$

which is also everywhere positive and decreasing, it is $\varphi_-(s; a, \alpha) = 1$ for $s \leq -\alpha$, $\varphi_-(s; a, \alpha) = 0$ for $s \geq \alpha$ and its derivative is bounded from below by the value $\varphi'_-(0; a, \alpha) = -1/N$. For short we shall set

$$\varphi_\pm(x) = \varphi_\pm(x; 1, 1) \tag{2.4.7}$$

Fig. 2.1: Smooth step-like functions $\varphi_\pm$

The step-like functions $\varphi_\pm(s; a, \alpha)$ are a kind of partition of unity, though not compactly supported. We shall see in next Subsection how to use these functions to dump a symmetry to make it a Cauchy transformation in a number of different contexts. However, these step-like functions can be used to solve a number of different problems, for example, they can be used to smoothly connect two functions.

Let $f(x)$, $g(x) : \mathbb{R} \to \mathbb{R}$ be two functions and suppose we want to define a smooth connection between these two functions in the interval $[x_0, x_1]$. Let us set

$$s(x; x_0, x_1) := 2\frac{x - x_0}{x_1 - x_0} - 1 \tag{2.4.8}$$

which runs in the range $s \in (-1, 1)$ as $x \in (x_0, x_1)$, and consider the function $h(x) = f(x)\varphi_- (s(x; x_0, x_1)) + g(x)\varphi_+ (s(x; x_0, x_1))$.

One has that

$$h(x_0) = f(x_0) \qquad h(x_1) = g(x_1) \tag{2.4.9}$$

Thus the function defined as

$$H(x) = \begin{cases} f(x) & x \le x_0 \\ h(x) & x_0 < x < x_1 \\ g(x) & x \ge x_1 \end{cases} \tag{2.4.10}$$

turns out to be continuous. Moreover, one has that the derivative of $h(x)$ is

$$h'(x) = f'(x)\varphi_-(s) + \frac{2}{x_1 - x_0}f(x)\varphi'_-(s) + g'(x)\varphi_+(s) + \frac{2}{x_1 - x_0}g(x)\varphi'_+(s) \tag{2.4.11}$$

so that $h'(x_0) = f'(x_0)$ and $h'(x_1) = g'(x_1)$ and the function $H(x)$ is also $\mathcal{C}^1$. Then one can prove that all higher derivatives match at any order and the function $H(x)$ is hence, in fact, smooth.

Now let us suppose that $f(x_0) < g(x_0)$. One can expand $f(x)$ and $g(x)$ around the point $x = x_0$ to obtain

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)h + O(h^2) & \qquad g(x) = g(x_0) + g'(x_0)h + O(h^2) \\ f'(x) = f'(x_0) + f''(x_0)h + O(h^2) & \qquad g'(x) = g'(x_0) + g''(x_0)h + O(h^2) \end{aligned} \tag{2.4.12}$$

where we set $h := x - x_0 > 0$.

Then one has in $x \in (x_0, x_1)$

$$\begin{aligned} h'(x) =& f'(x_0)\varphi_-(s) + f''(x_0)h\varphi_-(s) + \frac{2}{x_1 - x_0}f(x_0)\varphi'_-(s) + \frac{2h}{x_1 - x_0}f'(x_0)\varphi'_-(s) + g'(x_0)\varphi_+(s) + g''(x_0)h\varphi_+(s) + \\ & + \frac{2}{x_1 - x_0}g(x_0)\varphi'_+(s) + \frac{2h}{x_1 - x_0}g'(x_0)\varphi'_+(s) + O(h^2) = \\ =& f'(x_0) - f'(x_0)\varphi_+(s) + f''(x_0)h - f''(x_0)h\varphi_+(s) - \frac{2}{x_1 - x_0}f(x_0)\varphi'_+(s) - \frac{2h}{x_1 - x_0}f'(x_0)\varphi'_+(s) + g'(x_0)\varphi_+(s) + g''(x_0)h\varphi_+(s) + \\ & + \frac{2}{x_1 - x_0}g(x_0)\varphi'_+(s) + \frac{2h}{x_1 - x_0}g'(x_0)\varphi'_+(s) + O(h^2) = \\ =& \frac{2\left(g(x_0) - f(x_0)\right)}{x_1 - x_0}\varphi'_+(s) + f'(x_0) + \left(g'(x_0) - f'(x_0)\right)\varphi_+(s) + \frac{2h\left(g'(x_0) - f'(x_0)\right)}{x_1 - x_0}\varphi'_+(s) + O(h) \end{aligned} \tag{2.4.13}$$

Thus we have a smooth link between the two functions and if $f(x_0) < g(x_0)$ one has that as x_1 is chosen near x_0 the first term grows unbounded and alone. If $f(x)$ and $g(x)$ are positive diffeomorphisms then one can choose $x_1 - x_0$ small enough so that $h(x)$ has positive derivative as well. Accordingly, $H(x)$ is a smooth diffeomorphism.

> Let us stress that we showed that we can define a smooth diffeomorphism $H(x)$ starting form any two positive diffeomorphisms $f(x)$ and $g(x)$, without any request about being homotopic to the identity. We shall use this result later on to discuss reparameterisations.

Physical states and Vertical gauge transformations

The issue about whether generic gauge transformations do preserve the physical state is particularly tricky. Of course, transitivity of equivalence relations implies that the composition of a finite number of Cauchy transformations does still preserve the physical state. However, it is unclear if *any* gauge transformation can be written as a *finite* composition of Cauchy ones.

In a gauge-natural theory, vertical gauge transformations are generated by vertical automorphisms of the structure bundle $\mathcal{P}$. The group of such transformations is denoted by $\mathrm{Aut}_V(\mathcal{P})$ and we restrict to transformations which can be connected by a flow to the identity, thus one has a flow of vertical gauge transformations Φ_s such that $\Phi = \Phi_{(s=1)}$. The corresponding group is denoted by $\mathcal{S} = \mathrm{Aut}_V^e(\mathcal{P})$.

Let us prove that any symmetry transformation $\Phi \in \mathcal{S}$ can be written as the composition of two Cauchy transformations $\Phi_\pm \in \mathcal{D}^1$.

Now, given a vertical gauge transformation flow $\Phi_s : P \to P$ one can decide to follow the flow first for a time $s = \varphi_+(t \circ \pi(p))$, then for a time $s = \varphi_-(t \circ \pi(p))$. This can be obtained by defining

$$\Phi_+(p) = \Phi_{\varphi_+(t\circ\pi(p))}(p) \qquad \Phi_-(p) = \Phi_{\varphi_-(t\circ\pi(p))}(p) \tag{2.4.14}$$

The composition of these two maps is

$$\Phi_- \circ \Phi_+(p) = \Phi_{\varphi_-(t\circ\pi(\Phi_{\varphi_+(t\circ\pi(p))}(p)))}\left(\Phi_{\varphi_+(t\circ\pi(p))}(p)\right) \tag{2.4.15}$$

Since the map Φ_+ is vertical one has $t \circ \pi(\Phi_{\varphi_+(t\circ\pi(p))}(p)) = t \circ \pi(p)$ so that

$$\Phi_- \circ \Phi_+(p) = \Phi_{\varphi_-(t\circ\pi(p))}\left(\Phi_{\varphi_+(t\circ\pi(p))}(p)\right) = \Phi_{\varphi_-(t\circ\pi(p))+\varphi_+(t\circ\pi(p))}(p) = \Phi_1(p) = \Phi(p) \tag{2.4.16}$$

> Analogously, one has

$$\Phi_+ \circ \Phi_-(p) = \Phi_{\varphi_+(t\circ\pi(\Phi_-(p)))}\left(\Phi_{\varphi_-(t\circ\pi(p))}(p)\right) = \Phi_{\varphi_+(t\circ\pi(p))}\left(\Phi_{\varphi_-(t\circ\pi(p))}(p)\right) = \Phi_{\varphi_+(t\circ\pi(p))+\varphi_-(t\circ\pi(p))}(p) = \Phi_1(p) = \Phi(p) \tag{2.4.17}$$

Both $\Phi_\pm$ are Cauchy transformations (they are identical for $t \circ \pi(p) < -\alpha$ and $t \circ \pi(p) > \alpha$, respectively) and any $\Phi \in \mathrm{Aut}_V^e(P)$ can always be written as a composition of two Cauchy transformations, namely $\Phi = \Phi_+ \circ \Phi_-$.

Thus one has $\mathcal{D} = \mathcal{S}$. Accordingly, one is forced to set $\mathcal{G} = \mathcal{S}$ and any two configurations differing by a vertical gauge transformation are always equivalent and they represent the same physical state.

Physical states and reparameterisation of trajectories

Let us here consider reparameterisations in the model for relativistic point. Let us consider a curve $\gamma : \mathbb{R} \to M$ in spacetime and $\phi : \mathbb{R} \to \mathbb{R} : \hat{s} \mapsto s(\hat{s})$ a (positively oriented) diffeomorphism. If $\gamma(s)$ is a local expression for the curve, one can define a reparameterisation by composition $\hat{\gamma}(\hat{s}) = \gamma(s(\hat{s}))$.

Also in this case we can prove that any (positive) reparameterisation ϕ can be obtained as a composition of two Cauchy reparameterisations $\phi = \phi_+ \circ \phi_-$. Actually, in this case, we can be more general than what we did for vertical gauge transformations, since we shall prove it for any positive reparameterisation, without resorting to flows at all, i.e. not only for reparameterisations in a flow.

If we show that $\phi = \phi_+ \circ \phi_-$ for two Cauchy reparameterisations $\phi_\pm \in \mathcal{D}^1$, then, given the curve γ, one can define $\gamma_+ = \gamma \circ \phi_+$ and $\hat{\gamma} = \gamma_+ \circ \phi_- = \gamma \circ \phi$ having $\gamma \sim \gamma_+$ and $\gamma_+ \sim \hat{\gamma}$, thus by transitivity property, one has $\gamma \sim \hat{\gamma}$. Accordingly, two curves differing by such a (positive) reparameterisation do in fact represent the same physical state.

Before starting proving this result, let us remark that it is necessary and sufficient to show that one can find a reparameterisation $\phi_- : \mathbb{R} \to \mathbb{R}$ which agrees with ϕ in an interval $I \subset \mathbb{R}$ and it reduces to the identity in another interval $J \subset \mathbb{R}$. In fact, once one has such a reparameterisation, which will be called a *splitter* for ϕ, one can define

$$\phi_+ = \phi \circ \phi_-^{-1} \quad \Rightarrow \phi = \phi_+ \circ \phi_- \tag{2.4.18}$$

and the map $\phi_+ : \mathbb{R} \to \mathbb{R}$ in fact reduces to the identity on I. Then both $\phi_\pm \in \mathcal{D}^1$ and hence $\phi \in \mathcal{D}^2$.

> Thus let us prove that one can build splitters ϕ_- for any (positive) diffeomorphism ϕ by using the step-like functions $\varphi_\pm$ defined above.
> Let $\phi : \mathbb{R} \to \mathbb{R} : \hat{s} \mapsto \phi(\hat{s})$ be a (positive) diffeomorphism, i.e. with $\dot{\phi}(\hat{s}) > 0$ everywhere in $\mathbb{R}$.
> If $\phi = \mathrm{id}_\mathbb{R}$, there is nothing to prove. If $\phi \neq \mathrm{id}_\mathbb{R}$ one can find a point $x_0 \in \mathbb{R}$ such that $\phi(x_0) \neq x_0$. Let us suppose that $\phi(x_0) > x_0$; then we can go back to when we proved smooth connection between diffeomorphisms and set $f = \phi$ and $g(x) = 0$. Then one can find $x_1 - x_0$ small enough to have a smooth ϕ_- which agrees with ϕ for $\hat{s} \geq x_1$, while it agrees with $\mathrm{id}_\mathbb{R}$ for $\hat{s} \leq x_0$.
> On the other hand, if $\phi(x_0) < x_0$ one can choose $f(x) = x$ and $g(x) = \phi(x)$. Accordingly, one can find a smooth splitter ϕ_+ for ϕ, which agrees with ϕ for $\hat{s} \leq x_0$, while it agrees with $\mathrm{id}_\mathbb{R}$ for $\hat{s} \geq x_1$.

Since any reparameterization $\phi \in \mathcal{S} = \mathrm{Diff}(\mathbb{R})$ is compositions of two Cauchy reparameterisations $\phi_\pm$, then $\phi \in \mathcal{D}^2 = \mathcal{D} = \mathcal{S}$ and one is forced again to set $\mathcal{G} = \mathcal{S}$.

Two curves differing by a reparameterisation do describe the same physical state and there is nothing like a theory for relativistic particles where state is described by a motion is spacetime. Only trajectories matter.

Physical states in spacetimes with compact space

We can now adapt the argument of the reparameterisations above to any natural theory defined on a spacetime M equipped with an *ADM* splitting $t : M \to \mathbb{R} \times \Sigma$ with a compact space Σ. Of course, this includes the case of reparameterisations where the space Σ was made of one point only.

Let us fix two disjoint closed intervals $I, J \subset \mathbb{R}$ and denote by $U = t^{-1}(I)$ and $V = t^{-1}(J)$ the corresponding sets in M. Let us consider a diffeomorphism $\phi : M \to M$ which can be deformed to the identity by a flow ϕ_s so that $\phi = \phi_{(s=1)}$ and show we can build a splitter for it, i.e. there exists a diffeomorphism $\phi_- : M \to M$ which reduces to the identity in V and it coincides with ϕ on U.

Once a splitter is produced, then one can define

$$\phi_+ = \phi \circ \phi_-^{-1} \quad \Rightarrow \phi = \phi_+ \circ \phi_- \tag{2.4.19}$$

and the map $\phi_+ : M \to M$ in fact reduces to the identity on U. Then both $\phi_\pm \in \mathcal{D}^1$ and hence $\phi \in \mathcal{D}^2$ so that one has $\mathcal{D}^2 = \mathcal{D} = \mathcal{S}$.

> In order to build a splitter ϕ_- for ϕ, let us consider the infinitesimal generator ξ of the flow ϕ_s and define the dumped vector field $\xi_- = \varphi_-(t(x))\xi$. Let us denote by ϕ_s^- the flow of ξ_- and set $\phi_- = \phi_{(s=1)}^-$, which is a diffeomorphism by construction.
>
> Let us then show that ϕ_- is a splitter for ϕ. The vector field ξ_- coincides with ξ where $t(x) < -1$. However, if one considers a point x with $t(x) < -1$ and one leaves it be dragged along the flow ϕ_s^- until $s = 1$, then the endpoint can be different from $\phi(x)$. In fact, it may happen that during the dragging the point enters the region $t(x) > -1$, it feels a dumped field, and shows down.
>
> In order to avoid such a problem, we should restrict to points that will not enter in the region $t(x) > -1$ when dragged until $s = 1$. For, let us consider the space Σ_{-1} at $t = -1$ and consider the map $\tau : \Sigma \to \mathbb{R} : x \mapsto t(\phi^{-1}(x))$. Since Σ is compact, the function τ attains its minimum at $t = t_1$. Now it is easy to check that by starting from a point in the region $t(x) < t_1$ one can follow the flow ϕ_s^- until $s = 1$ and stay safely in the region $t(x) < -1$, i.e. for $t(x) < t_1$ one has $\phi_{(s=1)}^-(x) = \phi(x)$.
>
> Then the map ϕ_- is a splitter for ϕ, since it coincides with ϕ in the region $t(x) < t_1$ and it reduces to the identity in the region $t(x) > 1$.

Since we showed that splitters exist, then one has $\mathcal{D} = \mathcal{G} = \mathcal{S}$. Accordingly, two configurations differing by any diffeomorphism (in the connected component to the identity) represent the same physical state.

> As we said this argument can be applied straight away to reparameterisations.
>
> However, in Subsection 2.4.3 we obtained the same result for *any* (positive) reparameterisation, not only for reparameterisation in the connected component to the identity.

Physical states in spacetimes with non-compact space

The proof in the compact-space case relies on the assumption that space is compact to guarantee that the map τ attains its minimum and one can find a region in which the transformation ϕ_- coincides with ϕ, which is essential to prove that $\phi_+ \in \mathcal{D}^1$. One could argue that one may have $\mathcal{D}^2 = \mathcal{S}$ in any case, though possibly the structure of the proof might need to be adapted to different situations.

This is not the case. We can show that when the space Σ is non-compact there are diffeomorphisms which are not in $\mathcal{D}^2$.

Let us consider $M = \mathbb{R}^2$ with the canonical *ADM* splitting on $\mathbb{R}$. Since $M = \mathbb{R} \times \mathbb{R}$ the space $\Sigma = \mathbb{R}$ is non-compact. Then define the diffeomorphism

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2 : (t, x) \mapsto (t', x') \qquad \begin{cases} t' = \cos(t^2 + x^2)t - \sin(t^2 + x^2)x \\ x' = \sin(t^2 + x^2)t + \cos(t^2 + x^2)x \end{cases} \tag{2.4.20}$$

which is in fact generated by the flow of the vector field $\xi = (t^2 + x^2)(x\partial_t - t\partial_x)$, i.e. one has $\phi = \phi_{(s=1)}$ where the flow of ξ is given by

$$\phi_s : \mathbb{R}^2 \to \mathbb{R}^2 : (t, x) \mapsto (t', x') \qquad \begin{cases} t' = \cos\left((t^2 + x^2)s\right)t - \sin\left((t^2 + x^2)s\right)x \\ x' = \sin\left((t^2 + x^2)s\right)t + \cos\left((t^2 + x^2)s\right)x \end{cases} \tag{2.4.21}$$

A proof along the lines for compact-space case does not work in this case. In fact, one has that a line $\Sigma_{t_0} = \{(t_0, x)\}$ is sent around in a spiral $\phi(\Sigma_{t_0})$ which has no maximum or minimum.

However, this does not mean that there could be other ways to build a splitter for ϕ. Thus we have to prove directly that $\phi \notin \mathcal{D}^2$.

Fig 2.2: The image (blue) of a vertical stripe (red)

under the transformation ϕ

> One can easily check that $\phi \notin \mathcal{D}^1$ since its fixed points are the origin and a discrete sets of circles around it (the ones of radius $r^2 = t^2 + x^2 = 2k\pi$ with $k \in \mathbb{Z}$) so that ϕ is not the identity on any vertical line.
>
> Let us first introduce two notations. For two subsets $I_1, I_2 \subset \mathbb{R}$ we write $I_1 < I_2$ iff $\forall t_1 \in I_1$ and $\forall t_2 \in I_2$ one has $t_1 < t_2$.
>
> Analogously, for two subsets $U_1, U_2 \subset M$ we write $U_1 < U_2$ iff $\forall x_1 \in U_1$ and $\forall x_2 \in U_2$ one has $t(x_1) < t(x_2)$.
>
> In both cases we shall say that I_1 (or U_1) is *on the left* of I_2 (or U_2) or equivalently that I_2 (or U_2) is *on the right* of I_1 (or U_1).
>
> Let us suppose, for the sake of argument, that $\phi \in \mathcal{D}^2$ and it can be written as $\phi = \phi_+ \circ \phi_-$ for two Cauchy transformations $\phi_\pm \in \mathcal{D}^1$.
>
> Let us denote by $U_i = \pi^{-1}(I_i)$ the opensets with I_i intervals in $\mathbb{R}$ so that ϕ_i is the identity on the vertical stripe U_i. The two intervals I_1 and I_2 are disjoint (if not ϕ would be in $\mathcal{D}^1$). With no loss of generality we can choose $I_1 < I_2$ (meaning that I_1 is on the left of I_2); if this were not the case one can repeat the argument for $\phi^{-1} = \phi_2^{-1} \circ \phi_1^{-1}$ (being that $\phi \in \mathcal{D}^2$ if and only if $\phi^{-1} \in \mathcal{D}^2$).
>
> Let us remark that a map $\phi_1 \in \mathcal{D}^1$ (which is the identity on a stripe U_1) maps the left (right) region into itself, thus preserving the ordering. In other words, one has that for any $\{p\} < U_1$ then one has $\{\phi_1(p)\} < U_1$ (as well as for any $\{p\} > U_1$ then one has $\{\phi_1(p)\} > U_1$).
>
> Now consider a point $p \in U_2$; one has $\phi_2(p) = p$ and $\phi_1(p) = \phi(p)$. Since ϕ_1 is the identity on U_1 and $U_1 < \{p\}$ then $U_1 < \{\phi(p)\}$. Thus the diffeomorphism ϕ maps the stripe U_2 in a stripe which lies on the right of U_1, i.e. we write $U_1 < \phi(U_2)$. However, since the stripes go up (and down) to infinity the spiral drives their image $\phi(U_2)$ around and sooner or later it will be forced to have some points on the left of U_1. By the contradiction one concludes that ϕ cannot the written as $\phi = \phi_1 \circ \phi_2$, i.e. $\phi \notin \mathcal{D}^2$.

One can easily check that $\phi \notin \mathcal{D}^1$ since its fixed points are the origin and a discrete sets of circles around it (the ones of radius $r^2 = t^2 + x^2 = 2k\pi$ with $k \in \mathbb{Z}$) so that ϕ is not the identity on any vertical line.

Let us first introduce a notation. For two subsets $I_1, I_2 \subset \mathbb{R}$ we write $I_1 < I_2$ iff $\forall t_1 \in I_1$ and $\forall t_2 \in I_2$ one has $t_1 < t_2$.

Analogously, for two subsets $U_1, U_2 \subset M$ we write $U_1 < U_2$ iff $\forall x_1 \in U_1$ and $\forall x_2 \in U_2$ one has $t(x_1) < t(x_2)$.

In both cases we shall say that I_1 (or U_1) is *on the left* of I_2 (or U_2) or equivalently that I_2 (or U_2) is *on the right* of I_1 (or U_1).

Let us suppose, for the sake of argument, that $\phi \in \mathcal{D}^2$ and it can be written as $\phi = \phi_+ \circ \phi_-$ for two Cauchy transformations $\phi_\pm \in \mathcal{D}^1$.

Let us denote by $U_i = \pi^{-1}(I_i)$ the open sets with I_i intervals in $\mathbb{R}$ so that ϕ_i is the identity on the vertical stripe U_i. The two intervals I_1 and I_2 are disjoint (if not, ϕ would be in $\mathcal{D}^1$). With no loss of generality, we can choose $I_1 < I_2$ (meaning that I_1 is on the left of I_2); if this were not the case one can repeat the argument for $\phi^{-1} = \phi_2^{-1} \circ \phi_1^{-1}$ (being that $\phi \in \mathcal{D}^2$ if and only if $\phi^{-1} \in \mathcal{D}^2$).

Let us remark that a map $\phi_1 \in \mathcal{D}^1$ (which is the identity on a stripe U_1) maps the left (right) region into itself, thus preserving the ordering. In other words, one has that for any $\{p\} < U_1$ then one has $\{\phi_1(p)\} < U_1$ (as well as for any $\{p\} > U_1$ then one has $\{\phi_1(p)\} > U_1$).

Now consider a point $p \in U_2$; one has $\phi_2(p) = p$ and $\phi_1(p) = \phi(p)$. Since ϕ_1 is the identity on U_1 and $U_1 < \{p\}$ then $U_1 < \{\phi(p)\}$. Thus the diffeomorphism ϕ maps the stripe U_2 in a stripe which lies on the right of U_1, i.e. we write $U_1 < \phi(U_2)$. However, since the stripes go up (and down) to infinity the spiral drives their image $\phi(U_2)$ around and sooner or later it will be forced to have some points on the left of U_1. By the contradiction, one concludes that ϕ cannot the written as $\phi = \phi_1 \circ \phi_2$, i.e. $\phi \notin \mathcal{D}^2$.

Thus, in the cases for which the space manifold is non-compact, one can have diffeomorphisms of M which are not in $\mathcal{D}^2$ and, accordingly, one has $\mathcal{D}^2 \subsetneq \mathrm{Diff}_e(M)$.

With some more effort, one can show that the diffeomorphism ϕ here considered is not in $\mathcal{D}^3$ either. Thus one has that $\mathcal{D}^3 \subsetneq \mathrm{Diff}_e(M)$. However, this direct approach to $\mathcal{D}^n$ grows soon too difficult since the direct proof needs to consider in $\mathcal{D}^n$ all the possible orderings of the stripes U_i (and find a contradiction for each of such orderings) on which the diffeomorphisms ϕ_i reduce to the identity once one assumes that $\phi = \phi_1 \circ \phi_2 \circ \ldots \circ \phi_n$. In other words, the length of the direct proof grows as $n!/2$ and it becomes soon too long.

We still miss to prove that this is a real counterexample, that $\phi \notin \mathcal{D}$, so that $\mathcal{D} \subsetneq \mathrm{Diff}_e(M)$ and one really has some freedom in choosing the gauge transformations $\mathcal{D} \subset \mathcal{G} \subset \mathcal{S}$ in these cases.

On the other hand, we have to remark that one alternatively would need a proof that $\mathcal{D} = \mathcal{S}$ in order to assume, as usually done, that two configurations differing by a diffeomorphism are necessarily representing the same physical state (i.e. general covariance principle). Such a proof is missing and one should study each case on a separate basis, possibly resorting to physical arguments, to justify the assumption.

We shall assume hereafter that the standard choice $\mathcal{G} = \mathrm{Diff}(M)$, which is always a possible choice, (and the consequent general covariance principle) holds true.

Local Lagrangians

If field equations are not global, then a section is a solution or not depending on the coordinate used. This is unacceptable for a fundamental field theory. For that reason, we required the Lagrangian to be global (so that field equations are automatically global as well).

However, one can obtain global field equations also from non-global Lagrangians.

Let us consider two observers (x^λ, y^k) and (x'^λ, y'^k) defined over U and U' (or, for simplicity, both defined on $U \cap U'$) which define two local Lagrangians

$$L(x^\lambda, y^k, y^k_\lambda, \ldots) \, d\sigma \qquad L'(x'^\lambda, y'^k, y'^k_\lambda, \ldots) \, d\sigma' \tag{2.4.22}$$

If the difference of the Lagrangian density is a pure divergence on $U \cap U'$, i.e. if

$$JL'(x'^\lambda, y'^k, y'^k_\lambda, \ldots) = L(x^\lambda, y^k, y^k_\lambda, \ldots) + d_\mu \alpha^\mu(x^\lambda, y^k, y^k_\lambda, \ldots) \tag{2.4.23}$$

then they induce the same field equations, i.e.

$$E_i(L') = JE'_k(L)J^k_i \tag{2.4.24}$$

Preserving the Lagrangian up to pure divergences is sufficient for having global field equations, not necessary. However, we restrict to this case.

Now, if one has a global covariant Lagrangian $\boldsymbol{L}$ (actively or passively, it does not matter, as we shall discuss below), locally one can write it as

$$\boldsymbol{L} = \left(L + d_\mu\lambda^\mu\right)\, d\sigma = \left(L' + d_\mu\lambda'^\mu\right)\, d\sigma' \qquad JL' = L + d_\mu\alpha^\mu$$

where we set $\alpha^\mu := \left(\lambda^\mu - J\bar{J}^\mu_\nu\,\lambda'^\nu\right)$, so that we are in the situation presented above and the two observers define the same field equations.

If $\lambda = \lambda^\mu\, d\sigma_\mu$ and $\lambda' = \lambda'^\mu\, d\sigma'_\mu$ are global $(m-1)$-forms then also $L\, d\sigma$ and $L'\, d\sigma'$ are global Lagrangians as well and they differ at most by a pure divergence $d\alpha$.

The interesting case is when each observer subtract to the global Lagrangian $\boldsymbol{L}$ a *local* divergence, i.e. a local $(m-1)$-forms defined on its domain. Since $\boldsymbol{L}$ is a global form and λ (λ', respectively) is local, then $L\, d\sigma$ ($L'\, d\sigma'$, respectively) is local. In this case, the observers define local Lagrangians which define the same field equations.

There are essentially two examples of this situation. One is Chern–Simons theories presented above in Section 1.10.10.
The other is the local variational principle which was used by Einstein in place of the Hilbert Lagrangian. If one considers the Hilbert Lagrangian $L_H = \frac{\sqrt{g}}{2\kappa}R$ (for simplicity with no cosmological constant) and tries to isolate second derivatives of the metric, one can recast it as

$$
\begin{aligned}
2\kappa L_H =& \sqrt{g}R = \sqrt{g}g^{\beta\mu}\delta^\epsilon_\alpha R^\alpha{}_{\beta\epsilon\mu} = \sqrt{g}g^{\beta\mu}\delta^\epsilon_\alpha\left(d_\epsilon\{g\}^\alpha_{\beta\mu} - d_\mu\{g\}^\alpha_{\beta\epsilon} + \{g\}^\alpha_{\lambda\epsilon}\{g\}^\lambda_{\beta\mu} - \{g\}^\alpha_{\lambda\mu}\{g\}^\lambda_{\beta\epsilon}\right) = \\
=& \sqrt{g}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)d_\rho\{g\}^\alpha_{\beta\sigma} + \sqrt{g}\left(g^{\beta\sigma}\delta^\rho_\alpha - g^{\beta\rho}\delta^\sigma_\alpha\right)\{g\}^\alpha_{\lambda\rho}\{g\}^\lambda_{\beta\sigma} = \\
=& d_\rho\left(\sqrt{g}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)\{g\}^\alpha_{\beta\sigma}\right) - d_\rho\left(\sqrt{g}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)\right)\{g\}^\alpha_{\beta\sigma} + \sqrt{g}\left(\delta^\mu_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\mu}\right)\{g\}^\epsilon_{\lambda\mu}\{g\}^\alpha_{\beta\sigma} = \\
=& d_\rho\left(\sqrt{g}g^{\beta\sigma}\left(\{g\}^\rho_{\beta\sigma} - \delta^\rho_{(\sigma}\{g\}^\epsilon_{\beta)\epsilon}\right)\right) - d_\rho\left(\sqrt{g}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)\right)\{g\}^\alpha_{\beta\sigma} + \sqrt{g}\left(\delta^\mu_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\mu}\right)\{g\}^\epsilon_{\lambda\mu}\{g\}^\alpha_{\beta\sigma} = \\
=& d_\rho\left(\sqrt{g}g^{\beta\sigma}u^\rho_{\beta\sigma}\right) - d_\rho\left(\sqrt{g}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)\right)\{g\}^\alpha_{\beta\sigma} + \sqrt{g}\left(\delta^\mu_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\mu}\right)\{g\}^\epsilon_{\lambda\mu}\{g\}^\alpha_{\beta\sigma}
\end{aligned}
\tag{2.4.25}
$$

Then, if one discards the local divergence $d_\rho\left(\sqrt{g}g^{\beta\sigma}u^\rho_{\beta\sigma}\right)$, one obtains a family of local Lagrangians of first order in the metric, which, in this case, are called the *Einstein Lagrangian(s)*. They can be recast as

$$
\begin{aligned}
L_E =& -d_\rho\left(\frac{\sqrt{g}}{2\kappa}\left(\delta^\rho_\alpha g^{\beta\sigma} - g^{\beta\rho}\delta^\sigma_\alpha\right)\right)\{g\}^\alpha_{\beta\sigma} + \frac{\sqrt{g}}{2\kappa}\left(\delta^\mu_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\mu}\right)\{g\}^\epsilon_{\lambda\mu}\{g\}^\alpha_{\beta\sigma} = \\
=& \frac{\sqrt{g}}{2\kappa}\left[-\tfrac{1}{2}\delta^\rho_\alpha g^{\mu\nu}g^{\beta\sigma} + \tfrac{1}{2}\delta^\sigma_\alpha g^{\mu\nu}g^{\beta\rho} + \delta^\rho_\alpha g^{\beta\mu}g^{\sigma\nu} - \delta^\sigma_\alpha g^{\beta\mu}g^{\rho\nu}\right]d_\rho g_{\mu\nu}\{g\}^\alpha_{\beta\sigma} + \frac{\sqrt{g}}{2\kappa}\left(\delta^\mu_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\mu}\right)\{g\}^\epsilon_{\lambda\mu}\{g\}^\alpha_{\beta\sigma} = \\
=& \frac{\sqrt{g}}{2\kappa}\left[-\tfrac{1}{2}\delta^\rho_\alpha g^{\mu\nu}g^{\beta\sigma} + \tfrac{1}{2}\delta^\sigma_\alpha g^{\mu\nu}g^{\beta\rho} + \delta^\rho_\alpha g^{\beta\mu}g^{\sigma\nu} - \delta^\sigma_\alpha g^{\beta\mu}g^{\rho\nu}\right]\left(g_{\nu\epsilon}\delta^\lambda_\mu + g_{\mu\epsilon}\delta^\lambda_\nu\right)\{g\}^\epsilon_{\lambda\rho}\{g\}^\alpha_{\beta\sigma} + \frac{\sqrt{g}}{2\kappa}\left(\delta^\rho_\epsilon\delta^\lambda_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\lambda_\alpha g^{\sigma\rho}\right)\{g\}^\epsilon_{\lambda\rho}\{g\}^\alpha_{\beta\sigma} = \\
=& \frac{\sqrt{g}}{2\kappa}\left[-\tfrac{1}{2}\delta^\lambda_\epsilon\delta^\rho_\alpha g^{\beta\sigma} + \tfrac{1}{2}\delta^\lambda_\epsilon\delta^\sigma_\alpha g^{\beta\rho} + \delta^\lambda_\epsilon\delta^\sigma_\alpha g^{\sigma\lambda} - \delta^\lambda_\epsilon\delta^\sigma_\alpha g^{\beta\rho} - \tfrac{1}{2}\delta^\lambda_\epsilon\delta^\rho_\alpha g^{\beta\sigma} + \tfrac{1}{2}\delta^\lambda_\epsilon\delta^\sigma_\alpha g^{\beta\rho} + \delta^\beta_\epsilon\delta^\rho_\alpha g^{\sigma\lambda} - \delta^\beta_\epsilon\delta^\sigma_\alpha g^{\rho\lambda} + \delta^\lambda_\epsilon\delta^\rho_\alpha g^{\beta\sigma} - \delta^\beta_\epsilon\delta^\rho_\alpha g^{\sigma\lambda}\right]\{g\}^\epsilon_{\lambda\rho}\{g\}^\alpha_{\beta\sigma} = \\
=& \frac{\sqrt{g}}{2\kappa}\left[\delta^\rho_\epsilon\delta^\rho_\alpha g^{\sigma\lambda} - \delta^\beta_\epsilon\delta^\sigma_\alpha g^{\rho\lambda}\right]\{g\}^\epsilon_{\lambda\rho}\{g\}^\alpha_{\beta\sigma} = \frac{\sqrt{g}}{2\kappa}g^{\rho\lambda}\left(\{g\}^\beta_{\sigma\lambda}\{g\}^\sigma_{\beta\rho} - \{g\}^\sigma_{\beta\sigma}\{g\}^\beta_{\lambda\rho}\right)
\end{aligned}
\tag{2.4.26}
$$

Notice that the Lagrangian(s) L_E is not a scalar density (just because the divergence we discarded was local).

In these cases, we usually try to go the other direction. When observers define local Lagrangians, we shall usually try to restore the local divergences to describe the same system by the global Lagrangian $\boldsymbol{L}$. Restoring divergences does not change field equations, and it restores globality of conservation laws.

5. Covariance principles

Until now, we have been quite sloppy, mixing the principles of general relativity (any observer has equal right to describe physics) with the general covariance (diffeomorphisms, or gauge transformations, are symmetries of the system) and globality. It is time here to stop and clarify the issue in detail.

Below we shall consider different conditions, investigating their geometric meaning and their mutual relations.

Globality

Let us start by considering a global Lagrangian on the configuration bundle $\mathcal{C} = (C, M, \pi, F)$.

Let us consider an open cover U_α of spacetime M and each open set U_α supports an observer which is defined on $\pi^{-1}(U_\alpha) \subset \mathcal{C}$ of the configuration bundle $\mathcal{C}$. Each observer is identified with a fibered chart (x^μ, y^i) and one can map one observer into another (x'^μ, y'^i) by transition functions

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \tag{2.5.1}$$

The observer can be induced by a chart on spacetime, as it happens in natural theories, or can be induced by a local trivialisation on the structure bundle $\mathcal{P}$, as it happens in gauge-natural theories, or, even not being induced by anything external to the configuration bundle, as in a generic field theory with no particular covariance required. In any event, the observer is a fibered chart (x^μ, y^i) supported on U_α.

> In natural theories, the transition functions (2.5.1) on $\mathcal{C}$ are induced by transition functions on M, namely $x'^\mu = x'^\mu(x)$. As it happens in the tangent bundle, there is a covariant functor which takes the transition functions on M and defines the transition functions (2.5.1) on $\mathcal{C}$. In other words, the maps $Y^i(x, y)$ are defined out of the functions $x'^\mu = x'^\mu(x)$. For example, on the tangent bundle, one has $v'^\mu = J^\mu_\nu(x)v^\nu$ which is determined by transition functions on M since $J^\mu_\nu(x)$ are the Jacobians of $x'^\mu = x'^\mu(x)$.
>
> In gauge-natural theories the transition functions (2.5.1) on $\mathcal{C}$ are induced by transition functions on the structure bundle $\mathcal{P}$
>
> $$\begin{cases} x'^\mu = x'^\mu(x) \\ g' = \phi(x) \cdot g \end{cases} \tag{2.5.2}$$
>
> This means that the maps $Y^i(x, y)$ are defined out of the transition functions on the structure bundle. In this case, the configuration bundle $\mathcal{C}$ is the image of the structure bundle through a covariant functor. The transition functions of $\mathcal{C}$ are the image of the transition functions of the structure bundle $\mathcal{P}$ through the same functor.

Then, each observer gives a local description of the system under consideration. In particular, each observer needs to select a dynamics which is done by choosing a local Lagrangian $\boldsymbol{L}_\alpha = L_\alpha(x^\lambda, y^k, y^k_\lambda, \ldots) \, d\sigma$.

Then one needs the Lagrangian of the system to be global. For, in the intersection of the observer domains one must have $\boldsymbol{L}_\alpha = \boldsymbol{L}_\beta$. For that one has to have

$$J L_\beta(x'^\lambda, y'^k, y'^k_\lambda, \ldots) = L_\alpha(x^\lambda, y^k, y^k_\lambda, \ldots) \tag{2.5.3}$$

where we denoted by (x^λ, y^i) the coordinates associated to the observer U_α and (x'^λ, y'^i) the coordinates associated to the observer U_β.

This has not much to do with covariance. Global Lagrangians are common in mechanics, as well. Of course, $t' = t + t_0$ is a symmetry only when the Lagrangian does not depend explicitly on time. Any time-dependent force gives us an example of a global Lagrangian which is not covariant with respect to time reparameterisations. [Not to mention that virtually any Newtonian Lagrangian is not covariant with respect to *arbitrary* time-reparameterisations. This is the consequence of the fact that Newtonian time is *absolute*. One can set the origin of times at will but non-linear reparameterisations introduce accelerations which are not a matter of conventions. They have a physical meaning of new forces acting on the system. For that reason, arbitrary time-reparameterisations are not symmetries.]

Then, the two observers compute their local field equations obtaining

$$E_i(L_\alpha) = 0 \qquad E_i'(L_\beta) = 0 \tag{2.5.4}$$

We already computed the relation between the two sets of field equations (see (1.3.32)) and we found that

$$E_i(L_\alpha) = J E_j'(L_\beta) J_i^j \tag{2.5.5}$$

This is just what it is needed to show that the two observers are defining as local field equations two patches of the same sub-manifold $S \subset J^{2k}\mathcal{C}$.

Let us consider a point $j_x^{2k}\sigma \in J^{2k}\mathcal{C}$ which is projecting over a point x in the intersection $U_{\alpha\beta}$ of the domains of the two observers which is assumed to be non-empty. One has $j_x^{2k}\sigma \in S \subset J^{2k}\mathcal{C}$ if for any $X \in \mathfrak{X}_V(\mathcal{C})$

$$(j^{2k}X \lrcorner \mathbb{E}(L_\alpha)) \circ j_x^{2k}\sigma = 0 \qquad \Longleftrightarrow \quad E_i(L_\alpha) \circ j_x^{2k}\sigma = 0 \tag{2.5.6}$$

where we defined the form $\mathbb{E}(L_\alpha) = E_i(L_\alpha)\, \omega^i \wedge\, d\sigma$.
If $j_x^{2k}\sigma \in S \subset J^{2k}\mathcal{C}$ for the first observer, then for the second observer one has

$$E_i(L_\alpha) \circ j_x^{2k}\sigma = J E_j'(L_\beta) J_i^j \circ j_x^{2k}\sigma = 0 \qquad \Longleftrightarrow \quad E_j'(L_\beta) \circ j_x^{2k}\sigma = 0 \tag{2.5.7}$$

The fact that $j_x^{2k}\sigma \in S \subset J^{2k}\mathcal{C}$ is independent of the observer since the two observers agree on which points are in S and on which are not. They define the same field equation S, just giving a different local expression for it.

Now, let the two observers look for local solutions of their field equations. The first observer finds a local solution $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$, i.e. a local section such that $E_j(L_\alpha) \circ j^{2k}\sigma_\alpha = 0$. Let the expression of σ_α be

$$\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) : x^\lambda \mapsto (x^\lambda, y^i(x)) \tag{2.5.8}$$

This local section is seen by the second observer as a local section $\sigma_\beta : U_\beta \to \pi^{-1}(U_\beta)$ locally expressed as

$$\sigma_\beta : U_\beta \to \pi^{-1}(U_\beta) : x'^\lambda \mapsto (x'^\lambda, y'^i(x')) \qquad \begin{cases} x'^\mu = x'^\mu(x) \\ y'^i(x') = Y^i(x, y(x)) \end{cases} \tag{2.5.9}$$

This is equivalent to say that the two local sections σ_α and σ_β glue together and are, in fact, two local expressions of a single global section σ defined on $U_\alpha \cup U_\beta$. Now, let us consider a point $j^{2k}\sigma(x)$ in the image of (prolongation of the) section σ. For what we said above, that point is a solution of the field equation for one observer iff it is for the other observer.

Then the map (2.5.1) between the observers sends solutions of equations $\mathbb{E}_i(L_\alpha)$ into solutions of equations $\mathbb{E}'_i(L_\beta)$. Then globality of the Lagrangian just means that a section either is a solution or it is not, and that does not depend on which observer is asked to address the issue.

> Of course, it may happen that the Lagrangian is not global while field equations are, as it happens in Chern–Simons theories. Just in these cases, one needs extra arguments to show that field equations are global, while when the Lagrangian is global, globality of field equations follows automatically.

Without globality of field equations, being a solution is something that depends on the section *and* on the observer. In mechanics, that would mean that one observer solves equations on a interval $t \in [t_0, t_1]$ until the system enters the overlap between two observers. Then the system evolves for time $t \geq t_1$ on a curve for the first observer and on a different curve for the other observers, which is clearly hard to be given a physical meaning. For this reason, globality is for us is a very minimal requirement for a theory which makes sense and which is physically sound.

Principle of general relativity (passive covariance)

First of all, unless notification to the contrary, *transformation* refers to an invertible map. We shall consider two types of transformations.

The first type are spacetime diffeomorphisms $\phi : M \to M$ or bundle automorphisms $\Phi : C \to C$. These are maps which move points on spacetime or bundle and are called *active maps*. They are *represented* in coordinates by local expressions in coordinates.

> Let us fix a spacetime transformation $\phi : M \to M$ and a coordinate chart, which is called the *origin chart* and which is defined on a domain $U \subset M$. A point x in the origin chart is represented by its coordinates x^μ. If the point $\phi(x)$ is still in the same chart it is represented by coordinates x'^μ and the local expression of the map is given by
>
> $$x'^\mu = \phi^\mu(x) \tag{2.5.10}$$
>
> which assigns the coordinates of the target point as a function of the origin coordinates.
> In general, not all points in the coordinate domain U are mapped in U. In general, one defines a subdomain
>
> $$U' = \{x \in U : \phi(x) \in U\} \subset U \subset M \tag{2.5.11}$$
>
> and, if U' is not the empty set, one restricts the map ϕ to U' and its local expression is given by (2.5.10).
> Or, more generally, if the point $x' = \phi(x)$ does not belong to the same chart U then one needs to choose a new chart with coordinates x'^μ, which is called the *target chart*, and again the local expression of the active map ϕ in coordinates is given by (2.5.10), though with a different meaning: the point corresponding to coordinates x^μ in the origin chart, is mapped on the point corresponding to coordinates x'^μ in the target chart.
> In any event, the local expression is a function $\phi^\mu : \mathbb{R}^m \to \mathbb{R}^m$, while the active transformation is a map $\phi : M \to M$.
>
> The same argument can be carried over on the configuration bundle C. Let us fix a bundle transformation $\Phi : C \to C$ and the origin fibered coordinate chart. A point $p \in C$ in the origin chart is represented by its coordinates (x^μ, y^i). If the point $\Phi(x)$ is still in the same chart, it is represented by coordinates (x'^μ, y'^i) and the local expression of the map is given by
>
> $$\begin{cases} x'^\mu = \phi^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \tag{2.5.12}$$
>
> which assigns the coordinates of the target point as a function of the coordinates of the origin point both in the origin chart. In (2.5.12) one needs to restrict the domain to be sure that the image point still belongs to the origin chart domain, or choosing a new *target fibered chart* around the target point. Either way, the local expression is a function $(\phi^\mu, Y^i) : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$, not on the bundle C.

The second type of transformations are *transition functions* on the spacetime M (or on a bundle C). These maps are called *passive transformations*. These are local maps $\phi^\mu : \mathbb{R}^m \to \mathbb{R}^m$ (or $(\phi^\mu, Y^i) : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$) which, in fact ,do not move anything. They simply map the *name* of a point $x \in M$ (i.e. its coordinates in the origin chart) to the *name* of the same point in the target chart.

> If one considers the identity (active) map $\mathrm{id}_M : M \to M : x \mapsto x$ and selects two different charts, x^μ for the origin chart and x'^μ for the target chart, then the local expression of the identity map is in the form (2.5.10) which coincides with a passive transformation.
> Transition functions are the local expressions of the identity map in two charts.

> Quite a similar phenomenon happens on vector spaces when one uses different basis to write the matrix representing an endomorphism. It happens both that the identity map can be represented by a matrix different from $\mathbb{I}$ and that any endomorphism $f : V \to V$ can be represented by the matrix $\mathbb{I}$ (by choosing a basis e_i for the origin and a basis $f_i = f(e_i)$ for the target vectors). The situation arises from the confusion between a map (or an endomorphism) and its local expression, which depends on the choice of coordinates (or a basis).

Active and passive transformations have two completely different natures, though in fact they share the same local expressions.

> This is not a coincidence. It happens on manifold, bundles and whenever one builds global objects by glueing together trivial models along a class of maps. Then a good definition for morphisms in that category is obtained by considering maps which have local expressions in the same class where glueing functions are.
> In such a category, morphisms (the active transformations) and transition functions (the passive transformations) share the same type of local expressions. In fact manifolds are obtained by glueing together patches of $\mathbb{R}^m$ along local diffeomorphisms, and bundles are obtained by glueing together tubelike products $U \times F$ along fibered morphisms. In both cases, active and passive transformations do share the same local expressions.

Let us now discuss the *principle of passive covariance* or *general relativity*.

In natural theories, (natural) observers are identified with spacetime charts. They are the conventions needed to give a name to events. A local observer x^μ relates to a nearby observer x'^μ when we are able to map the name of an event for the first observer into the name of the same event for the second event, i.e. when we define the transition functions

$$x'^\mu = x'^\mu(x) \tag{2.5.13}$$

If the first observer is defined on an open set U and the second is defined on U', then the transition functions (2.5.13) are defined on $U \cap U'$.

> In gauge-natural theories, gauge-natural observers are identified with local fibered charts on the structure bundle $\mathcal{P}$. Then transition functions are in the form (2.5.2).
> Again, in transition functions (2.5.2), (x, g) and (x', g') refer to the *same* point and its coordinate labels according to two different gauge-natural observers. There is nothing moving, just two names for the same point in $\mathcal{P}$.

Let us first give a heuristic formulation of the *general relativity principle* (also called *passive general covariance principle*). Let us require that physics can and ought to be described in an observer-independent fashion and there is no privileged class of (natural or gauge-natural) observers.

That in particular means that all observers set the same Lagrangian functional dependence on fields, i.e. passive covariance is obtained if each observer defines the same Lagrangian density:

$$\boldsymbol{L}_\alpha = L_\alpha(x^\lambda, y^k, y^k_\lambda, \ldots)\, d\sigma = L(x^\lambda, y^k, y^k_\lambda, \ldots)\, d\sigma \qquad \boldsymbol{L}_\beta = L_\beta(x'^\lambda, y'^k, y'^k_\lambda, \ldots)\, d\sigma' = L(x'^\lambda, y'^k, y'^k_\lambda, \ldots)\, d\sigma' \tag{2.5.14}$$

where L is the same Lagrangian density for all observers, just expressed using the coordinates of each observer.

If the Lagrangian of one observer has extra terms, these terms can usually be interpreted as interactions which is not there for the other observer and the two observers describe the same situation in different ways, one with different interactions than the other. This is against the general relativity principle.

General relativity alone does not say much. The function L can be *any* (local) function. It just prescribes total democracy among observers. It is only when general relativity is put together with globality and one has $\boldsymbol{L}_\alpha = \boldsymbol{L}_\beta$, that general relativity implies that

$$JL(x'^\lambda, y'^k, y'^k_\lambda, \ldots) = L(x^\lambda, y^k, y^k_\lambda, \ldots) \tag{2.5.15}$$

i.e. the function L behaves as a scalar density (and the changes of observer act as symmetries).

Probably, we should stress again that a genuinely non-global Lagrangian (one for which the local Lagrangians differ by more than a local divergences) would determine local field equations and solutions in one patch that would not be connected to solutions in the other nearby patches.

In other words, being a solution would not be a geometric condition on sections alone but a property of local sections *and* observers. Being a solution of field equation would depend on the coordinates chosen. That would mean to give up an absolute description of physical reality.

Physically speaking, the general relativity principle asserts that an absolute description of physics is in fact possible. And in principle, it must be preferred over relative descriptions.

Thus we can state the *general relativity principle* as:

General Relativity Principle: *the local representation of the (global) Lagrangian $\boldsymbol{L}$ given by each observer has to be in the form (2.5.14). Equivalently, the Lagrangian density chosen by any observer must be a scalar density as in (2.5.15) with respect to arbitrary changes of observer.*

Then *general relativity principle* (or, synonymously, *passive covariance principle*) is the *globality* of dynamics *plus* the *covariance* with respect to general changes of observer. Since different observers select the same Lagrangian density, they find the same field equations (just each written with the fields used by that observer).

Since equations are global the transformations on local sections defined as

$$y'^i(x') = Y^i(x, y(x)) \tag{2.5.16}$$

map local solutions in local solutions and since the transformations preserve the equations in form, they map local solutions of field equations into local solutions of the *same* field equations. In other words, the changes of observer are (local) *symmetries of the field equations*.

Each natural (gauge-natural, respectively) observer locally defines the same Lagrangian density (the same functional form just as a function of its fields). Hence physically speaking, each observer accounts for the same interactions and essentially provides the same description of physical reality.

General covariance (active covariance)

Active general covariance is a completely different property.

Let us start with natural theories. Here we are concerned with diffeomorphisms on spacetime M. These are real (and global) maps which actively act on spacetime points. Locally, the diffeomorphism is expressed as

$$\phi : M \to M : x^\mu \mapsto x'^\mu = \phi^\mu(x) \tag{2.5.17}$$

Notice that the local expression of diffeomorphism is similar to transition functions, just with a different meaning. That happens because manifolds are obtained by glueing patches of $\mathbb{R}^m$ exactly along (local) diffeomorphisms.

Let us stress that, unlike for principle of general relativity, here x^μ and x'^μ refer to coordinates of two different points. The map ϕ is actually regarded as a map which moves the points in M.

Since the configuration bundle $\mathcal{C}$ is natural, the spacetime diffeomorphism ϕ can be lifted to a natural transformation $\hat\phi : C \to C$ of the configuration bundle.

Being the configuration bundle $\mathcal{C}$ a natural bundle, there is a covariant functor F such that $\mathcal{C} = F(M)$. Then one has $\hat\phi = F(\phi)$. Locally, natural transformations are in the same form of (2.5.1), and the maps $Y^i(x,y)$ are constructed out of $\phi^\mu(x)$ in a canonical way dictated by F.

Let us remark once again that the local expression of a natural transformation $\hat\phi$ is formally similar to the expression of a change of observer. Again, that is true since bundles are obtained by glueing together trivial patches (namely, tubes $U \times F$) along natural transformations.

Then, the Lagrangian is chosen so that (it is global and) any such natural transformation is a symmetry, i.e. one has

$$(j^k \hat\phi)^* \boldsymbol{L} = \boldsymbol{L} \qquad \Longleftrightarrow \quad JL(x'^\lambda, y'^k, y'^k_\lambda, \ldots) = L(x^\lambda, y^k, y^k_\lambda, \ldots) \tag{2.5.18}$$

as in (1.4.1).

However, the active covariance condition (2.5.18) holds true *for all* diffeomorphism, i.e. for all functions $x'^\mu = \phi^\mu(x)$, including the same local maps which represent general relativity in a passive viewpoint.

Thus, with our definitions, a global Lagrangian is actively covariant iff it is passively covariant. For that reason, often one does not care much about the distinction between active and passive viewpoint, though, of course, they are different conditions concerning different kind of transformations and endowed with different physical meaning. Still, they select the same global dynamics.

A Lagrangian is *natural* if, for any spacetime vector field $\xi \in \mathfrak{X}(M)$, one has (2.1.13), i.e., in the case of a second order Lagrangian,

$$p_i \pounds_\xi y^i + p_i^\mu d_\mu \pounds_\xi y^i + p_i^{\mu\nu} d_{\mu\nu} \pounds_\xi y^i = d_\mu (\xi^\mu L) \tag{2.5.19}$$

A similar situation happens in gauge-natural theories. There one has a structure bundle $\mathcal{P}$ and the configuration bundle $\mathcal{C}$ is a gauge-natural bundle associated to $\mathcal{P}$. In other words, there exists a covariant functor F such that $\mathcal{C} = F(\mathcal{P})$. Any gauge transformation $\Phi : \mathcal{P} \to \mathcal{P}$ on the structure bundle induces a gauge transformation $F(\Phi) =: \hat\Phi : \mathcal{C} \to \mathcal{C}$ on the configuration bundle.

Then the Lagrangian is chosen so that (it is global and) any such gauge transformation is a symmetry, i.e. one has

$$(j^k \hat\Phi)^* \boldsymbol{L} = \boldsymbol{L} \qquad \Longleftrightarrow \quad JL(x'^\lambda, y'^k, y'^k_\lambda, \ldots) = L(x^\lambda, y^k, y^k_\lambda, \ldots) \tag{2.5.20}$$

Locally, a Lagrangian is *gauge-natural* if for any generator $\Xi \in \mathfrak{X}(\mathcal{P})$ of gauge transformations one has (2.1.23), i.e., in the case of a second order Lagrangian,

$$p_i \pounds_\Xi y^i + p_i^\mu d_\mu \pounds_\Xi y^i + p_i^{\mu\nu} d_{\mu\nu} \pounds_\Xi y^i = d_\mu (\xi^\mu L) \tag{2.5.21}$$

The situation, in the two cases of natural and gauge-natural theories, is essentially the same; the difference between them is encoded in the different definition of Lie derivatives that one uses in writing down the covariance principle.

To summarise, in both cases, natural (gauge, respectively) transformations are Lagrangian symmetries, thus they leave field equations invariant and they map solutions of field equations into solutions of the *same* field equations.

If one fixes the configuration bundle (which describes the fundamental fields and the whole structure of variations) and the order of the theory (i.e. how many derivatives enter the Lagrangian) then Utiyama arguments do in fact select covariant dynamics.

Those global dynamics are, at the same time, actively and passively (gauge-)natural Lagrangian dynamics.

This would be satisfactory, if were not that already back in 1917 Kretschmann observed that *any* set of equations can be put in generally covariant form at the price of introducing new fields; see[16][17]. Einstein almost immediately accepted Kretschmann's argument and agreed that general relativity was not a physical principle since it does not really select dynamics. He admitted that general relativity can be used only as a heuristic principle to select covariant dynamics which are *simple* in covariant form. That is probably the historical reason why *equivalence principle* took over as a physical root for GR.

It is here where we get into troubles! We showed that the principle of general relativity is equivalent (unless local Lagrangians are considered, as, by the way, Einstein was doing at that time) to the principle of general covariance. If the principle of general relativity has no physical content then the principle of general covariance has neither. On the other hand, we have examples of how general and gauge-natural covariance does select dynamics. That is what Utiyama-like theorems do.

> Utiyama arguments show that covariance principle (plus the choice of fundamental fields) does select dynamics. This, strictly speaking, does not contradict Kretschmann argument since he claimed that covariantisation can be achieved, *though* at the price of adding new fields or equations.
> Moreover, we shall also see below that the weak equivalence principle (in the *naive* form that all test particles in free fall are described by geodesics of a connection, or projective structure) can be proven on the basis of hypotheses which are essentially only covariance hypotheses.
> Thus equivalence principle itself is, as a matter of fact, a consequence of covariance principle. If covariance principle had no physical content, I see no reason why (weak) equivalence principle should have.

We have though to stress that the context of Utiyama-like arguments and the original Kretschmann's argument are quite different. While Kretschmann discussed field equations, Utiyama discusses action principles. While Kretschmann added fields to obtain general covariance, Utiyama works with a given set of fields and with a fixed order. Moreover, in 1917 the globality notion was quite obscure and it is unclear what survives of Kretschmann in a global and variational setting. Thus a direct comparison is difficult and currently unclear.

> We already discussed an example which partially enlightens some issues and the differences in context. We discussed the dynamics for a test particle on spacetime. We gave the homogeneous formulation of a Newtonian test particle in Subsection 1.11.1 and the homogeneous formulation of a relativistic test particle in Subsection 1.11.9.
> They are both given by an action principle, thus this does not seem to be a relevant difference. The relativistic Lagrangian is somehow *simpler* than the Newtonian one, though *simpler* does not mean anything precise. In this specific case, simplicity means that, as we already noticed, in the Newtonian formulation time does not completely disappear and survive in a factor u^0 which can be written intrinsically by introducing a vector field $\tau = -\frac{1}{2U}\partial_0$ (which makes perfect absolute sense in a Newtonian world, provided that one has forces acting) and writing $u^0 = \hat{g}(u, \tau)$.
> Unfortunately, this is done at the price of introducing a new structure τ on spacetime which is fixed instead being determined dynamically. Thus the difference between Newtonian and relativistic test particles can be traced back to the existence of a *background structure* τ which is not dynamically determined and it is forbidden in natural theories.

This does not close the issue though, since there is no precise, general definition of what is a *background structure*.

In particular, it is not clear if one can, for example, add a new Lagrangian depending on τ which selects $\tau = -(2U)^{-1}\partial_0$ as solution of the equations of motion.

We believe it is worth having both Utiyama and Kretschmann arguments in mind, even though, at the moment, we cannot trace precisely their mutual relation.

6. Physical content of general covariance

Already in 1917, E. Kretschmann argued that general covariance has no physical content; see[16][17]. Precisely, he argued that any theory can be formulated in a covariant form.

Kretschmann presented some examples which are worth keeping in mind.

Let us consider $M = \mathbb{R} \times \mathbb{R}^n$ with coordinates $(x^0 = t, x^i)$ where x^i are orthogonal Cartesian coordinates on $\mathbb{R}^n$. Let us also consider a scalar field $\Phi : M \to \mathbb{R}$. The field is a section of the (trivial) bundle $\mathcal{F} = (M \times \mathbb{R}, M, \pi, \mathbb{R})$ which then has fibered coordinates (t, x^i, ϕ).

Then consider the second order equation

$$\partial_t \phi = \kappa \Delta \phi \tag{2.6.1}$$

where κ is a real constant and Δ denotes the Laplacian operator in $\mathbb{R}^n$. The equation (2.6.1), which is more or less inspired to the Schrödinger equation for the free particle, is global, just because the coordinates are global on M. It is not covariant. It does not preserve its form with respect to arbitrary changes of coordinates. For example, if one considers a transformation changing x^0 and x^i the equation does not preserve its form.

Let us then define a metric on M, of signature $(n, 1)$ and form

$$\eta = -dt^2 + \delta_{ij} dx^i \otimes dx^j = \eta_{\mu\nu} dx^\mu \otimes dx^\nu \tag{2.6.2}$$

and define a time-like vector $n := \partial_0$. Then equation (2.6.1) can be written in the form

$$n^\mu \nabla_\mu \phi - \kappa(g^{\mu\nu} - n^\mu n^\nu)\nabla_{\mu\nu}\phi = 0 \tag{2.6.3}$$

which *is* covariant with respect to general changes of coordinates (or general diffeomorphisms from the active viewpoint). Of course, equation (2.6.3) *is* equation (2.6.1), since in orthonormal coordinates it reduces to that; it is covariant though it depends also on a metric η and the time-like unit vector n. These are to be considered as fixed structures on M. In fact, equation (2.6.3) is covariant provided that diffeomorphisms transform the components $\eta_{\mu\nu}$ and n^μ so that the corresponding tensors $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ and $n = n^\mu \partial_\mu$ are invariant.

Having fixed structures on spacetime is not allowed in a relativistic theory. All fields are dynamical, which in particular means that all fields should be determined by their field equations. Of course, when we selected g and n we could add field equations to justify our selection. For example, we can consider

$$\begin{cases} n^\mu \nabla_\mu \phi - \kappa(g^{\mu\nu} - n^\mu n^\nu)\nabla_{\mu\nu}\phi = 0 \\ g(n,n) = -1 \\ \nabla_\mu n^\nu = 0 \\ R^\alpha_{\beta\mu\nu} = 0 \end{cases} \tag{2.6.4}$$

where the Riemann tensor $R^\alpha_{\beta\mu\nu}$ and the covariant derivative ∇_μ are induced by the metric.

The fields $(g_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1), n^\mu = (1, 0, \ldots, 0), \phi(x))$ are a solution of the system (2.6.4) which is covariant. With these choices the original equation (2.6.1) is in fact recovered.

Of course, we would like that the system (2.6.4) came from a Lagrangian on a configuration bundle which accounts for the fields $(g_{\mu\nu}, n^\mu, \phi)$. Unfortunately, this is not the case since one should have $1 + (1 + n) + \frac{1}{2}(n+1)(n+2)$ equations, which is not the case since in fact (2.6.4) are $2 + (n+1)^2 + \frac{1}{12}(n+1)^2((n+1)^2 - 1)$ equations.

By setting $n = 3$ we have 15, instead of 38, equations.

However, we clearly see that an equation which does not seem to be covariant can be make up as covariant by adding extra fields and extra equations. This would be just a first step to show that any field theory can be formulated as a relativistic theory with extra fields, still the example is annoying and worth considering.

A possible way out is to make precise the notion of *absolute structure* and to require that the system has not absolute structures. Then one could show that g and n are absolute structures in the Kretschmann's example and in similar situations, even when they are determined by field equations. The problem with this program is that there is not universally accepted definition of *absolute structure* and on how this definition should be formulated. There are examples in which anyone agrees that some structure is absolute and some examples in which anyone agrees that there is no absolute structures.

We shall hereafter discuss some examples of background and non-background structures in order to make an explicit and precise list of what we want and what we do not want to happen in a relativistic theory. Unfortunately, at the present stage, we cannot say wether these conditions single out definitely what we should mean by a background structure. One can consider the following examples as tests that we developed the geometric framework enough to be able to precisely discuss situations which are allowed or have to be avoided.

Back to the homogeneous formulation of mechanics given in Subsection 1.11.1, one can introduce a time-like unit vector n as in the example above and make it covariant by replacing occurrences of u^0 with $u^0 = u^\mu n_\mu$.

Electromagnetism on Minkowski

We already discussed Minkowski electromagnetism above in Subsection 1.10.6. If we consider the Maxwell Lagrangian and fix the Minkowski metric, then Maxwell equations follows. Unfortunately, to fix the metric, one must consider the corresponding section of $\mathrm{Lor}(M)$ fixed. We can split the configuration bundle into the product of two bundles, one accounting for dynamical fields A_μ, the other for background fields $g_{\mu\nu}$.

> Let us consider a field theory with a configuration bundle which is the fibered product of two bundles $\mathcal{C} = \mathcal{F} \times_M \mathcal{B}$ with coordinates (x^μ, y^i, z^a). Sections of $\mathcal{F}$ are called *dynamical fields* while sections of $\mathcal{B}$ are called *background fields*. A section of $\mathcal{C}$ is a pair (y, z) of a collection of dynamical and background fields.
>
> In this setting, the Maxwell Lagrangian is written on a configuration bundle $\mathrm{Con}(P) \times_M \mathrm{Lor}(M)$, A is a dynamical field and g is a background field.

Let us assume all the bundles to be natural so that we have a group action of spacetime diffeomorphisms $\mathrm{Diff}(M)$ on fields, on both dynamical and background fields. Such a group action is the product of the natural action on $\mathcal{F}$ and the natural action on $\mathcal{B}$ and will be denoted by $\Phi_*(y, z) = (\Phi_* y, \Phi_* z)$. Such actions can be prolonged to any jet prolongation $J^s\mathcal{C}$. We shall still denote by Φ_* the prolonged group actions so that we can write, e.g., $\Phi_*(j^s y) = j^s(\Phi_* y)$.

Dynamics is specified by a Lagrangian which is k-order in the dynamical fields y and h-order in the background fields z.

> The Maxwell Lagrangian is first order with respect to the electromagnetic potential A and zero order in the metric field g.
>
> This theory is not a natural theory. If it were, then all fields would have to be dynamical, both (A, g). But as we discussed, one cannot take too seriously the field equations associated to g (unless one couples to a dynamics for the metric only, obtaining Einstein–Maxwell theory which is natural, but a different theory from the one on Minkowski, where the metric is not determined by some field equation but is instead fixed *by hand* to be the Minkowski metric $g = \eta$ and not even varied).
>
> If one fixes the metric g, then the Lagrangian depends on the dynamical field A only. Unfortunately, a general diffeomorphism does not fix the Minkowski metric and only transformations on isometries are symmetries of the theory. Hence the theory where one has substituted $g_{\mu\nu} = \eta_{\mu\nu}$ is not generally covariant.

Canonical objects

As a matter of fact, some natural bundles have canonical sections. The Kronecker delta δ^μ_ν and the Levi Civita antisymmetric tensor densities $\epsilon^{\mu_1\mu_2\cdots\mu_m}$ are examples.

Let us consider the bundle $T^1_1(M)$ with coordinates (x^μ, t^μ_ν). If in each patch we define a local section

$$\delta : M \to T^1_1(M) : x \mapsto (x^\mu, \delta^\mu_\nu) \tag{2.6.5}$$

these local sections do glue together to define a global section δ. This is true in view of invariance of δ^μ_ν with respect to general diffeomorphisms over M.

$$\delta'^\mu_\nu = J^\mu_\alpha \delta^\alpha_\beta \bar{J}^\beta_\nu = J^\mu_\alpha \bar{J}^\alpha_\nu = \delta^\mu_\nu \tag{2.6.6}$$

Let us consider the bundle $D^1_m(M)$ of antisymmetric tensor densities of weight 1 of rank $(0, m)$ on a manifold M of dimension m. This has *coordinates* $(x^\mu, t^{\mu_1\mu_2\cdots\mu_m})$ and transformation rules

$$\begin{cases} x'^\mu = x'^\mu(x) \\ t'^{\mu_1\mu_2\cdots\mu_m} = \bar{J} J^{\mu_1}_{\nu_1} J^{\mu_2}_{\nu_2} \ldots J^{\mu_m}_{\nu_m} \, t^{\nu_1\nu_2\cdots\nu_m} \end{cases} \tag{2.6.7}$$

As for the Kronecker delta, one can define local sections

$$t^{\mu_1\mu_2\cdots\mu_m} = \epsilon^{\mu_1\mu_2\cdots\mu_m} \tag{2.6.8}$$

and they glue together to define a global section ϵ which is called the *Levi Civita density* of weight 1.

Also in this case, the existence of ϵ is related to its invariance under diffeomorphisms (and the definition and properties of determinants)

$$\epsilon'^{\mu_1\mu_2\ldots\mu_m} = \bar{J}J^{\mu_1}_{\nu_1}J^{\mu_2}_{\nu_2}\ldots J^{\mu_m}_{\nu_m}\,\epsilon^{\nu_1\nu_2\ldots\nu_m} = \bar{J}J\epsilon^{\mu_1\mu_2\ldots\mu_m} = \epsilon^{\mu_1\mu_2\ldots\mu_m} \tag{2.6.9}$$

Thus we have these natural bundles equipped with special sections $\delta : M \to T^1_1(M)$ and $\epsilon : M \to D^1_m(M)$. And these sections are invariant with respect to diffeomorphisms $\varphi \in \mathrm{Diff}(M)$

$$\varphi^*\delta = \delta \qquad\qquad \varphi^*\epsilon = \epsilon \tag{2.6.10}$$

Accordingly, one can restrict the bundles to the image of these sections, obtaining the bundles $\delta(M) = (\mathrm{Im}(\delta), M, i, \{p\})$ and $\epsilon(M) = (\mathrm{Im}(\epsilon), M, i, \{p\})$, which are natural sub-bundles of $T^1_1(M)$ and $D^1_m(M)$. Although these are formally bundles, their standard fiber is made of one point only, the total space is diffeomorphic to the base. Accordingly, they have only one section. In view of the invariance of the sections, the bundles $\delta(M)$ and $\epsilon(M)$ are also natural bundles.

If one has a natural bundle C, one considers the bundle $C' = C \times_M \delta(M)$ which is again natural. A natural theory is associated to a natural Lagrangian which is in the form $L(x^\mu, y^i, \delta^\mu_\nu)$. In this natural theory, the δ^μ_ν is a *field* and it must be varied. However, since the standard fiber of the bundle $\delta(M)$ is discrete there is no non-zero variation and no field equations for δ. Accordingly, one should allow *any* section as solution, but $\delta(M)$ has only a unique section δ.

Thus we have two equivalent frameworks: on one hand, one can regard the field δ^μ_ν as a background: it is not varied, it is fixed to the value δ^μ_ν, it has not field equations. On the other hand, one can consider it as a field in a trivial natural theory on the bundle $\delta(M)$.

This can be repeated any time one has an *invariant* section of a natural bundle. In other words, one can use invariant objects in natural (and gauge-natural) theories without considering them as fields, still obtaining (gauge-)natural theories.

On the other hand, given a natural theory with a dynamics depending on an invariant object (e.g. δ^μ_ν), one *can* promote the object to a dynamical field (e.g. replacing δ^μ_ν with a field t^μ_ν in the dynamics) and then varying it as a dynamical field, endowing field equations and letting these field equations to determine solutions. If field equations select many possible solutions, one has defined a good (gauge-)natural theory, though *another* (gauge-)natural theory.

This is quite different from what happens, for example, to the metric in Minkowski electromagnetism. In that case, one is *forced* to promote the Minkowski metric to a dynamical field, since the Minkowski metric is not invariant with respect to general diffeomorphisms. If one built the sub-bundle $\eta(M) = (\mathrm{Im}(\eta), \mathbb{R}^m, i, \{p\}) \subset \mathrm{Lor}(\mathbb{R}^4)$ that sub-bundle would not be natural. Moreover, of course, one cannot do it on a general manifold M. In any event, one cannot use the trick used for the Kronecker delta.

Evolution of physical degrees of freedom

Another feature of dynamical field is that they must have a dynamics. In particular, field equations must endow a non-trivial dynamic with a non-trivial phase space. There must be some freedom in choosing the initial conditions and, correspondently, to allow initial conditions, and field equations must uniquely determine a solution.

However, as we already discussed, hoping that dynamics fully determines dynamical fields without constraints on initial conditions is too much in many relevant cases. If there are no constraints, field equations are not overdetermined, and, since in a variational setting one has as many equations as fundamental fields, they are not underdetermined either. However, this happens essentially only for the Klein–Gordon scalar field (which though needs to be coupled to a metric which is associated to degenerate dynamics, in any event).

Most of the relevant dynamics in fundamental physics are degenerate and constraints are there to account for observer freedom. The determinism needs to be traded with an absolute description. Covariant and gauge covariant dynamics are generically degenerate and there is nothing one can do about it.

As a consequence, one has to learn to live with fields which are only partially determined by field equations. However, a *physical field* must have a phase space which is not discrete.

> Of course, according to this definition, Kronecker delta δ, as well as Levi Civita tensor densities ϵ are *not* physical field.

If we add a field to a theory, we would like that the phase space of the theory becomes bigger than without the field. Technically, we still have to determine when a field comes with a non-trivial contribution to the phase space. However, we already discussed initial problems and phase spaces and how they are described in field theories. It may be difficult to explicit this information since one needs to define *ADM* space-time and adapted fields. Still, it is a property of field equations and hence a property of the Lagrangian which induces field equations.

> One needs to characterise Lagrangians which induce field equations which are quasi-linear with a principal symbol such that there exists a field transformation in which field equations decouple into an elliptic part ,which defines constraints, and in a symmetric hyperbolic part, which defines the Cauchy problem for some of the adapted fields. The other fields account for observer freedom.

We shall do it, but not now. It is equivalent to Hamiltonian analysis of the theory and it will be discussed below. We shall also consider a number of example of fields in which the phase space is well-defined and big enough. We already discussed Maxwell electromagnetism.

In view of the hole argument, what one needs is that solutions do not form a single orbit with respect to (finite compositions of) Cauchy (gauge) transformations. If they do, then there is only one point in the phase space of the theory. This is clearly an intrinsic property, though one would like to have a criterium to ensure it *a priori*.

7. Examples

Let us hereafter consider some examples. The first three examples (the relativistic point, Hilbert–Einstein GR and Maxwell–Einstein–Klein–Gordon theory) have already been computed in the previous Chapter. We shall here simply summarise and collect the results discussed above and focus on the natural or gauge-natural structure of the theory..

Further examples will be added from scratch. These examples are chosen to illustrate what we discussed in this Chapter and for future reference, as examples of (gauge-)natural theories. In particular, we shall consider Yang–Mils theories, which are an extension of Maxwell electromagnetism, *Vielbein* formalism for GR and Dirac spinor fields, which are example of gauge-natural theories, conformal gravity, in which we show how with some dynamics a natural theory can get extra symmetries and become gauge-natural.

In the next Chapter we shall compute conservation laws for these theories as an application of the general framework which we shall discuss for conservation laws and the techniques to compute superpotentials in (gauge-)natural theories.

Relativistic point

The dynamics for the relativistic point particle has been discussed in Subsection 1.11.9.

Let (M, g) be a spacetime equipped with a Lorentzian metric g. This is not a natural theory for two reasons: first, there is background metric g which is fixed and not dynamically determined and, second, it is a mechanical theory in which the fields x^μ are coordinates in spacetime and they need to be determined as a function of a parameter s.

However, it can be recast in a form which resembles a natural theory in which the covariance with respect to the reparameterisations is analogous to the general covariance theory.

The histories of particles are trajectories in the spacetime M and one can define a configuration bundle $\mathcal{C} = (C, \mathbb{R}, \pi, M)$. Since the basis $\mathbb{R}$ is contractible, then any bundle over it is trivial. This means that there exists a global trivialisation and $C \simeq \mathbb{R} \times M$.

Of course, global trivialisations of $\mathcal{C}$ are not unique and different choices of the product structures $C \simeq \mathbb{R} \times M$ correspond to different reference frames. Let us denote by (s, x^μ) fibered coordinates on $\mathcal{C}$.

> Let us remark that there is a one-to-one correspondence between curves $\gamma : \mathbb{R} \to M$ in M and sections of $\mathcal{C}$.
> In fact, to any curve $\gamma : \mathbb{R} \to M : s \mapsto x^\mu(s)$ one can associate a section $\sigma : \mathbb{R} \to \mathbb{R} \times M : s \mapsto (s, x^\mu(s))$ and vice versa.

The first jet prolongation is $J^1\mathcal{C} \simeq \mathbb{R} \times TM$ and one can choose a Lagrangian

$$\boldsymbol{L} = L(s, x, u)\, ds = mc\sqrt{|g_{\mu\nu}(x)u^\mu u^\nu|}\, ds \tag{2.7.1}$$

This Lagrangian is covariant with respect to arbitrary reparameterisations.

> Let $\sigma : \mathbb{R} \to C : s \mapsto (s, x^\mu(s))$ be a section of the configuration bundle and $\phi : \mathbb{R} \to \mathbb{R} : \hat{s} \mapsto s = \phi(\hat{s})$ be a positive diffeomorphism on the basis $\mathbb{R}$. This acts as $\hat{\phi} : \mathcal{C} \to \mathcal{C} : (s, x^\mu) \mapsto (\hat{s}(s), x^\mu)$ which acts on the section σ as
>
> $$\phi_*\sigma : \mathbb{R} \to C : \hat{s} \mapsto (\hat{s}, x^\mu(s(\hat{s}))) \tag{2.7.2}$$

The transformation can be prolonged to the jet bundle as

$$\begin{cases} \hat{s} = \hat{s}(s) \\ \hat{x}^\mu = x^\mu \\ \hat{u}^\mu = \dot{\phi}(\hat{s})u^\mu \end{cases} \tag{2.7.3}$$

With respect to this transformation the Lagrangian transforms as

$$\hat{\boldsymbol{L}} = mc\sqrt{|g_{\mu\nu}(\hat{x})\hat{u}^\mu\hat{u}^\nu|}\, d\hat{s} = mc\dot{\phi}\sqrt{|g_{\mu\nu}(x)u^\mu u^\nu|}\, d\hat{s} = mc\sqrt{|g_{\mu\nu}(x)u^\mu u^\nu|}\, ds = \boldsymbol{L} \tag{2.7.4}$$

Thus, positive reparameterisations are Lagrangian symmetries. By the discussion above, since one has reparameterisations supported in Cauchy domains, one is forced to define the physical state as curves modulo reparameterisations, i.e. *trajectories*.

Since the theory is covariant with respect to arbitrary (positive) diffemorphisms of the base, it implements a kind of general covariance principle.

The equation of motions are

$$\begin{cases} \dot{u}^\lambda + \{g\}^\lambda_{\alpha\beta}u^\alpha u^\beta = \frac{d}{ds}\left(\ln(|u|)\right)u^\lambda \\ \dot{x}^\mu = u^\mu \end{cases} \iff \ddot{x}^\lambda + \{g\}^\lambda_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{d}{ds}\left(\ln(|\dot{x}|)\right)\dot{x}^\lambda \tag{2.7.5}$$

Let us specialise M to Minkowski space $(\mathbb{R}^4, \eta)$ in Cartesian coordinates x^μ. In these coordinates, the metric is constant, the Christoffel symbols $\{g\}^\lambda_{\alpha\beta} = 0$ vanish, and equations can be recast as

$$\begin{cases} \ddot{x}^0 = \frac{d}{ds}\left(\ln(|\dot{x}|)\right)\dot{x}^0 = \dfrac{\eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu}{\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}\dot{x}^0 = \dfrac{\dot{x}^0\ddot{x}^0 - \delta_{ij}\dot{x}^i\ddot{x}^j}{\dot{x}^0\dot{x}^0 - \delta_{kh}\dot{x}^k\dot{x}^h}\dot{x}^0 \\[3ex] \ddot{x}^i = \frac{d}{ds}\left(\ln(|\dot{x}|)\right)\dot{x}^i = \dfrac{\eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu}{\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}\dot{x}^i = \dfrac{\dot{x}^0\ddot{x}^0 - \delta_{ij}\dot{x}^i\ddot{x}^j}{\dot{x}^0\dot{x}^0 - \delta_{kh}\dot{x}^k\dot{x}^h}\dot{x}^i \end{cases} \tag{2.7.6}$$

Let us now define the physical velocities and physical accelerations to be

$$\beta^i = \frac{\dot{x}^i}{\dot{x}^0} \qquad \alpha^i = \frac{\ddot{x}^i\dot{x}^0 - \dot{x}^i\ddot{x}^0}{(\dot{x}^0)^2}\frac{1}{\dot{x}^0} = \frac{\ddot{x}^i - \ddot{x}^0\beta^i}{(\dot{x}^0)^2} \tag{2.7.7}$$

so that one has equations in the form

$$\begin{cases} \ddot{x}^0 = \dfrac{\ddot{x}^0 - \delta_{ij}\beta^i\ddot{x}^j}{1 - |\beta|^2} \\[3ex] \ddot{x}^i = \dfrac{\ddot{x}^0 - \delta_{ij}\beta^i\ddot{x}^j}{1 - |\beta|^2}\beta^i \end{cases} \Rightarrow \ddot{x}^0\beta^i = \ddot{x}^i \quad \Rightarrow \alpha^i = 0 \tag{2.7.8}$$

Thus, among solutions, one has uniform straight motions, which are determined by these $m - 1$ equations.

However, in homogenous formalism one has m equations for four unknowns, while the solution $\alpha^i = 0$ are $m - 1$ equations. That means that one equation has been left unused. Potentially, the further equation could select initial conditions and reduce the space of solutions to *some* straight motions.

This is *not* what actually happens. In fact, by using $\ddot{x}^0 \beta^i = \ddot{x}^i$ in the first equation of motion one obtains the identity

$$\ddot{x}^0 = \frac{1 - \delta_{ij}\beta^i\beta^j}{1 - |\beta|^2}\ddot{x}^0 = \ddot{x}^0 \tag{2.7.9}$$

showing that the first equation is identically satisfied, once the other $m - 1$ are.

This a direct consequence of covariance with respect to reparameterisations: covariance implies that some of the variables cannot be determined by equations and the equations are not independent. The extra variable has no direct physical meaning and instead it represents the conventional freedom of the observer to describe the same situation.

In fact, one can describe the system with physical quantities (β^i, α^i) (which, unfortunately, is a description depending on the observer) or with unphysical quantities $(\dot{x}^\mu, \ddot{x}^\mu)$ which do not depend on the observer, though are partially left undetermined by the equations (which, in fact, do not allow to determine $\ddot{x}^0$).

Standard Hilbert–Einstein general relativity

We already discussed Hilbert–Einstein theory in Subsection 1.10.11 and found in Subsection 2.2.1 the most general second order Lagrangian. In Subsection 2.1.3, we also discussed how the bundle $\mathrm{Lor}(M)$ emerges as a natural bundle in view of transformation laws of the metric.

A spacetime vector field ξ induces a bundle vector field on $\mathrm{Lor}(M)$

$$\hat{\xi} = \xi^\mu \partial_\mu + (g^{\nu\alpha}\partial_\alpha\xi^\mu + g^{\mu\alpha}\partial_\alpha\xi^\nu)\frac{\partial}{\partial g^{\mu\nu}} \tag{2.7.10}$$

which leaves the Lagrangian invariant.

Dynamics is described by Hilbert Lagrangian

$$\boldsymbol{L}_H = \tfrac{\sqrt{g}}{2\bar{\kappa}}(R - 2\Lambda)\,d\sigma \tag{2.7.11}$$

or by local Einstein Lagrangians described in Subsection 2.4.6, which are equivalent to Hilbert global Lagrangian.

We still have to discuss why a metric on spacetime should have anything to do with gravity (which will be discussed in Part II below) and how *AddReference to ADM* Einstein equations can be put in evolutionary form (which is quite complicated, deserves a Chapter on its own and will be done in Chapter xx). Some physical consequences of this theory will be analysed in detail in Part III.

Standard Einstein–Maxwell–Klein–Gordon theory

If we want to couple Hilbert–Einstein GR to some matter field, all we have to do is adding terms to the Lagrangian. The matter Lagrangian to be added will depend on matter field *and* the metric. The variation of the matter Lagrangian with respect to the matter fields will produce field equations for matter fields (which, in general, depend on the metric as well). These field equations predict how matter evolve in a given gravitational field, which is described by the metric field.

Then, the variation of the matter Lagrangian with respect to the metric will define the energy-momentum stress tensor $T_{\mu\nu}$ which, together with the variation of the Hilbert Lagrangian with respect to the metric, will produce Einstein equations. The energy-momentum stress tensor describes the back reaction of matter fields on the gravitational field. It describes how matter fields act as a source of gravitational field.

The system of Einstein equations and matter field equations described the whole system of gravity *and* matter in their mutual interaction.

In Subsections 1.10.6 we computed what happens by coupling with electromagnetism.

The electromagnetic field is described by the local potential A_μ. Two nearby observers can describe the same electromagnetic field by two local potentials provided that in the overlap the local potentials are related by the transformation rules

$$A_\mu^{(\beta)} = \bar{J}_\mu^\nu \left(A_\nu^{(\alpha)} + \partial_\nu \alpha_{(\alpha\beta)} \right) \tag{2.7.12}$$

for some function $\alpha_{(\alpha\beta)}$ defined on the overlap.

The functions $\alpha_{(\alpha\beta)}$ defined in the overlaps of all pairs of observers contain physical information encoding the global properties of the electromagnetic field. They define a cocycle

$$\varphi_{(\alpha\beta)} : U_{(\alpha\beta)} \to U(1) : x \mapsto \exp\left(i\alpha_{(\alpha\beta)}(x) \right) \tag{2.7.13}$$

which, in turn, define a principal bundle $\mathcal{P} = (P, M, \pi, U(1))$ which is called the *structure bundle*.

One can define principal connections on the structure bundle and find that they are in the form

$$\omega = dx^\mu \otimes (\partial_\mu - A_\mu \rho) \tag{2.7.14}$$

where ρ is a right-invariant pointwise basis for vertical vectors on $\mathcal{P}$. The coefficients A_μ of the connection transform as local potential (2.7.12) and, as a consequence, there is a one-to-one correspondence between connections on structure bundles and electromagnetic potentials.

The electromagnetic field is described as the curvature of the connection ω

$$\mathbf{F} = \tfrac{1}{2} F_{\mu\nu} \, dx^\mu \wedge \, dx^\nu = \tfrac{1}{2} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \, dx^\mu \wedge \, dx^\nu \tag{2.7.15}$$

which, in this case of the commutative group $U(1)$, is in fact a 2-form on spacetime.

In view of transformation laws (2.7.12), one can guess a group action of the group $W^{(1,1)}U(1) = J^1 U(1) \rtimes \mathrm{GL}(m)$ with coordinates $(\alpha, \alpha_a, J_a^b)$ on the manifold $\mathbb{R}^m$ with coordinates A_a given by

$$\lambda : W^{(1,1)}U(1) \times \mathbb{R}^m \to \mathbb{R}^m : (\alpha, \alpha_a, J_a^b, A_a) \mapsto A_a' = (A_b + \alpha_b) \bar{J}_a^b \tag{2.7.16}$$

By this group action and the principal bundle $W^{(1,1)}\mathcal{P} = J^1 P \rtimes L(M)$, one can define the gauge-natural bundle

$$\mathrm{Con}(\mathcal{P}) = W^{(1,1)}\mathcal{P} \times_\lambda \mathbb{R}^m \tag{2.7.17}$$

which is defined so that there is a one-to-one correspondence between principal connections on $\mathcal{P}$ and (global) sections of $\mathrm{Con}(\mathcal{P})$.

The construction is somehow overshot for the group $U(1)$. The essential is that we end up with a bundle which parameterises principal connections to be used as configuration bundle. However, the construction is canonical with respect to the group and it will be repeated with no changes for Yang–Mills fields.

The configuration bundle for Maxwell–Einstein theory is chosen to be

$$\mathcal{C} = \mathrm{Lor}(M) \times_M \mathrm{Con}(\mathcal{P}) \tag{2.7.18}$$

which has fibered coordinates $(x^\mu, g_{\mu\nu}, A_\mu)$. Sections of $\mathcal{C}$ are in one-to-one correspondence with pairs (g, A) of a Lorentzian metric g and a connection ω on $\mathcal{P}$ which acts as potential of electromagnetic field $\mathbf{F}$.

The dynamics is described by the Lagrangian

$$\boldsymbol{L} = \left(\frac{\sqrt{g}}{2\kappa} \left(R - 2\Lambda \right) - \frac{\sqrt{g}}{4g} F_{\mu\nu} F^{\mu\nu} \right) \, d\sigma \tag{2.7.19}$$

In Subsections [1.10.6](), we already computed field equations to be

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T_{\mu\nu} \\ \nabla_\mu F^{\mu\nu} = 0 \end{cases} \tag{2.7.20}$$

The first field equation describes the evolution of the metric field (which will be recognised to be the gravitational field) when the electromagnetic field $F_{\mu\nu}$ acts as a gravitational source. The second field equation describes the evolution of electromagnetic field under the action of the gravitational field.

Mathematically speaking, the equations are coupled and can only be solved as a system. This means, in physical jargon, that field equations describe the *interaction* of the electromagnetic and gravitational fields.

In the case described above, there is no source for the electromagnetic field in the region of spacetime under consideration. If one wants to describe also the interaction between gravity, electromagnetism, and a charged field, a new Lagrangian for the charged matter field has to be added. In the simplest model, the charged matter field is a complex scalar field, denoted by φ; we refer to Subsection [1.10.7]().

The transformation laws for this matter field is assumed to be

$$\varphi'(x') = e^{iq\alpha(x)} \varphi(x) \tag{2.7.21}$$

Then one can guess a group action $\rho : U(1) \times \mathbb{C} \to \mathbb{C} : (e^{i\alpha}, \varphi) \mapsto e^{iq\alpha(x)} \varphi(x)$ and define a bundle $P \times_\rho \mathbb{C}$. The relevant things about such a bundle are that there is a one-to-one correspondence between global sections of $P \times_\rho \mathbb{C}$ and matter fields and that matter fields by construction transform as prescribed by [(2.7.21)](). The construction of associated bundles is not too esoteric; it is just a mean to obtain the prescribed transformation laws.

Then the configuration bundle is now chosen to be

$$\mathcal{C} = \mathrm{Lor}(M) \times_M \mathrm{Con}(\mathcal{P}) \times_M \left(P \times_\rho \mathbb{C} \right) \tag{2.7.22}$$

which has (real) fibered coordinates $(x^\mu, g_{\mu\nu}, A_\mu, \varphi, \varphi^\dagger)$.

The dynamics will be described by the Lagrangian

$$\boldsymbol{L} = \left(\frac{\sqrt{g}}{2\kappa} \left(R - 2\Lambda \right) - \frac{\sqrt{g}}{4g} F_{\mu\nu} F^{\mu\nu} - \frac{\sqrt{g}}{2} \left(\nabla_\mu \varphi^\dagger \nabla^\mu \varphi + \mu^2 \varphi^\dagger \varphi \right) \right) \, d\sigma \tag{2.7.23}$$

where we set $\nabla_\mu \varphi = d_\mu \varphi - iq A_\mu \varphi$ for the gauge-natural covariant derivative.

One can apply a Utiyama-like argument to show that if a Lagrangian on this bundle is gauge covariant, then it depends on the combination $\nabla_\mu \varphi = d_\mu \varphi - iq A_\mu \varphi$ only, i.e. it has no further dependences on A_μ other than those encoded into the gauge-natural covariant derivatives.

This Lagrangian is then covariant with respect to arbitrary automorphisms of the structure bundle and we have a gauge-natural theory.

For field equations ,one has to vary with respect to all fields, namely $(g_{\mu\nu}, A_\mu, \varphi, \varphi^\dagger)$, to obtain

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T^{em}_{\mu\nu} + \kappa T^{KG}_{\mu\nu} \\ \nabla_\mu F^{\mu\nu} = \mathrm{g}J^\nu \\ \Box\varphi - \mu^2\varphi = 0 \end{cases} \tag{2.7.24}$$

where the Klein–Gordon matter Lagrangian is responsible for the term $T^{KG}_{\mu\nu}$ (by variation with respect to $g^{\mu\nu}$) for the term J^ν (by variation with respect to A_μ which is contained in the covariant derivatives) and for the matter field equation. In the matter field equation, the operator $\Box$ is defined as

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = g^{\mu\nu}\left(d_{\mu\nu}\varphi - \left(\{g\}^\lambda_{\nu\mu} + 2iqA_\nu\delta^\lambda_\mu\right)d_\lambda\varphi + \left(iq\{g\}^\lambda_{\nu\mu}A_\lambda - iqd_\mu A_\nu - q^2A_\nu A_\mu\right)\varphi\right) \tag{2.7.25}$$

and it depends on the metric $g_{\mu\nu}$ (also though its Christoffel symbols in the second covariant derivative) and the electromagnetic potential A_μ.

Accordingly, both the electromagnetic field and the matter field act as sources of the gravitational field in the first equation (through their energy-momentum tensors $T^{em}_{\mu\nu}$ and $T^{KG}_{\mu\nu}$). The matter field acts as a source of the electromagnetic field (through the current tensor J^ν and the electromagnetic field also is affected by the gravitational field in its evolution. Finally, the matter field has no sources, though it is affected by the gravitational and the electromagnetic field in its evolution.

The theory then describes the interactions among gravitational,, electromagnetic and matter fields. The reader should refer to Subsections 1.10.6 for detailed computations, though we need to stress that most of the general characteristics of field equations (in particular which interactions are considered and by which terms in the Lagrangian they are described) can be inferred qualitatively by inspection of the Lagrangian.

Yang–Mills theories

In analogy with electromagnetism, we can fix a semisimple Lie group G and try to write a gauge-natural theory for connections on a structure bundle $\mathcal{P} = (P, M, \pi, G)$.

> One usually restricts to semisimple groups since, on these groups, one has the canonical Ad-invariant Cartan–Killing bilinear form which is non-degenerate positive-definite and can be used as a metric on the group (and as an inner product on the Lie algebra $\mathfrak{g}$).
> In this case, one chooses an orthonormal basis T_A in the Lie algebra $\mathfrak{g}$ and, in such a basis, the Cartan–Killing form is expressed as δ_{AB}. This is a canonical structure that, for example, can be used to write Lagrangians (e.g. contracting Lie algebra indices) to describe possible dynamics.

Thus let us fix a principal bundle $\mathcal{P} = (P, M, \pi, G)$ as a structure bundle and consider the group action

$$\lambda : W^{(1,1)}G \times (\mathfrak{g} \otimes \mathbb{R}^m) \to (\mathfrak{g} \otimes \mathbb{R}^m) : (g^i, g^i_a, J^b_a, \omega^B_a) \mapsto \omega'^A_a = \mathrm{Ad}^A_B(g)(\omega^B_b + \bar{R}^A_i(g)g^i_b)\bar{J}^b_a \tag{2.7.26}$$

One should check that this is, in fact, a left group action and define the bundle

$$\mathrm{Con}(\mathcal{P}) = (\mathcal{P} \times L(M)) \times_\lambda (\mathfrak{g} \otimes \mathbb{R}^m) \tag{2.7.27}$$

which has fibered coordinates (x^μ, ω_μ^A). The fields ω_μ^A transforms by construction as the coefficients of principal connections on $\mathcal{P}$ so that there is a one-to-one correspondence between global sections of $\mathrm{Con}(\mathcal{P})$ and principal connections on the structure bundle, i.e.

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^A(x)\rho_A \right) \tag{2.7.28}$$

where ρ_A is a right-invariant pointwise basis for vertical vectors on the structure bundle $\mathcal{P}$.

The curvature of the connection ω is a 2-form $\boldsymbol{F}$ with values in the Lie algebra

$$\boldsymbol{F} = \tfrac{1}{2} F_{\mu\nu}^A T_A \otimes dx^\mu \wedge dx^\nu \qquad\qquad F_{\mu\nu}^A = \partial_\mu \omega_\nu^A - \partial_\nu \omega_\mu^A + c^A{}_{BC}\omega_\mu^B \omega_\nu^C \tag{2.7.29}$$

The coefficients $F_{\mu\nu}^A$ of the curvature transform under generalised gauge transformations as

$$F'^A_{\mu\nu} = \mathrm{Ad}_B^A(\alpha(x)) F_{\rho\sigma}^B \bar{J}_\mu^\rho \bar{J}_\nu^\sigma \tag{2.7.30}$$

> Unlike in electromagnetism, where Ad reduces to the identity, the coefficients of the curvature, also called the *field strength*, are not gauge invariant. Accordingly, the hole argument applies to them as well and two field strengths which differ by a gauge transformation do define the same physical state.
>
> Thus, unlike in electromagnetism, the components of field strength are not observables. One cannot measure $F_{\mu\nu}^A$ and there is nothing like the electric or magnetic field, or at least they are not measurable quantities.
>
> However, if one selects a Ad-invariant polynomial $\ell(\boldsymbol{F})$, for example $\ell(\boldsymbol{F}) = \delta_{AB} F_{\mu\nu}^A F_{\rho\sigma}^B g^{\mu\rho} g^{\nu\sigma}$, this quantity is gauge invariant and observable. Thus in Yang–Mills theories the trace of powers of field strength are observables.

If we managed to write a Lagrangian which depends on the field ω_μ^A alone (as we did in Section 1.10.10) then we could describe a system with only the field ω on spacetime and no interaction with anything else.

> In the physical jargon there is one (and the only) field which can live without interacting with anything else, and this is the gravitational field. Thus any such theory would be interpreted as an alternative description of the gravitational field. This is essentially because it is believed that any other physical field has to interact at least with the gravitational field.
>
> That is however, another story, which contains a lot of physical motivations and it needs that we improve our physical intuition before we deal with that.

In general, we can easily define Lagrangians which depend on the field ω_μ^A and the metric field $g_{\mu\nu}$. Such a system needs a configuration bundle which accounts for both fields $(\omega_\mu^A, g_{\mu\nu})$, so let us define

$$\mathcal{C} = \mathrm{Lor}(M) \times_M \mathrm{Con}(\mathcal{P}) \tag{2.7.31}$$

with fibered coordinates $(x^\mu, g_{\mu\nu}, \omega_\mu^A)$.

As an example of Lagrangian, we can get inspiration from electromagnetism and we can consider

$$\boldsymbol{L}_{YM} = -\tfrac{\sqrt{g}}{4\mathrm{g}} F_{\mu\nu}^A F_A^{\mu\nu} \, d\sigma \tag{2.7.32}$$

where we set $F_A^{\mu\nu} := \delta_{AB} F_{\alpha\beta}^B g^{\alpha\mu} g^{\beta\nu}$. This is called the *Yang–Mills Lagrangian*.

The variation of the Yang–Mills Lagrangian is

$$\delta \boldsymbol{L}_{YM} = \frac{\sqrt{g}}{\mathrm{g}} \left(-\frac{1}{2} \left(F^A_{\mu\alpha} F_{A\nu}{}^{.\alpha} - \frac{1}{4} F^A_{\beta\alpha} F^{\beta\alpha}_A g_{\mu\nu} \right) \delta g^{\mu\nu} - F^{\mu\nu}_A \nabla_\mu \delta\omega^A_\nu \right) d\sigma =$$
$$= \left(-\frac{\sqrt{g}}{2} T_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{\mathrm{g}} \nabla_\mu \left(\sqrt{g} F^{\mu\nu}_A \delta\omega^A_\nu \right) + \frac{1}{\mathrm{g}} \nabla_\mu \left(\sqrt{g} F^{\mu\nu}_A \right) \delta\omega^A_\nu \right) d\sigma \qquad (2.7.33)$$

where we set $T^{(YM)}_{\mu\nu} := \frac{1}{\mathrm{g}} \left(F^A_{\mu\alpha} F_{A\nu}{}^{.\alpha} - \frac{1}{4} F^A_{\beta\alpha} F^{\beta\alpha}_A g_{\mu\nu} \right)$ for the Yang–Mills energy momentum stress tensor.

To take it seriously, one has to couple it to a gravitational Lagrangian (e.g. the Hilbert Lagrangian) to describe the interaction of the Yang–Mills field with the gravitational field. The Lagrangian to be considered is then

$$\boldsymbol{L} = \left(\frac{\sqrt{g}}{2\kappa}(R - 2\Lambda) - \frac{\sqrt{g}}{4\mathrm{g}} F^A_{\mu\nu} F^{\mu\nu}_A \right) d\sigma \qquad (2.7.34)$$

This Lagrangian is covariant with respect to generalised gauge transformations and the theory we are considering is a gauge-natural theory.

Let us consider a generalised gauge transformation on the structure bundle

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g' = \alpha(x) \cdot g \end{cases} \qquad (2.7.35)$$

This induces on the configuration bundle the transformation

$$\begin{cases} x'^\mu = x'^\mu(x) \\ \omega'^A_\mu = \mathrm{Ad}^A_B(\alpha)(\omega^B_\rho + \bar{R}^A_i(\alpha) d_\rho \alpha^i) \bar{J}^\rho_\mu \\ g'_{\mu\nu} = \bar{J}^\rho_\mu g_{\rho\sigma} \bar{J}^\sigma_\nu \end{cases} \qquad (2.7.36)$$

Under such a transformation, R is a scalar, as well as $F^A_{\mu\nu} F^{\mu\nu}_A$; accordingly, the Lagrangian density is a scalar density and the Lagrangian $\boldsymbol{L}$ is covariant with respect to generalised gauge transformations.

As usual, we can also check that the covariance identity holds true. For, we need to compute Lie derivatives

$$\mathcal{L}_\Xi g^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu} = -(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) \qquad \mathcal{L}_\Xi \omega^A_\mu = \xi^\lambda F^A_{\lambda\mu} - \nabla_\mu \xi^A_{(V)} \qquad (2.7.37)$$

where covariant derivatives are induced by the connection ω^A_μ on $\mathcal{P}$ and $\{g\}^\alpha_{\beta\mu}$ on M (and, of course, the covariant derivative $\nabla_\mu \xi_\nu$ does not depend on the gauge connection ω but on the Levi Civita connection $\{g\}$, only) and we set $\xi^A_{(V)} := \xi^A + \omega^A_\lambda \xi^\lambda$ for the vertical part with respect to the gauge connection ω.

Using Bianchi identities for the curvature $F^A_{\mu\nu}$, one has

$$0 = F^{\mu\nu}_A \nabla_{[\mu} F^A_{\lambda\nu]} = F^{\mu\nu}_A \left(\nabla_\mu F^A_{\lambda\nu} + \nabla_\lambda F^A_{\nu\mu} + \nabla_\nu F^A_{\mu\lambda} \right) = F^{\mu\nu}_A \left(2\nabla_\mu F^A_{\lambda\nu} + \nabla_\lambda F^A_{\nu\mu} \right) \qquad \Rightarrow 2F^{\mu\nu}_A \nabla_\mu F^A_{\lambda\nu} = F^{\mu\nu}_A \nabla_\lambda F^A_{\mu\nu} \qquad (2.7.38)$$

In view of this identity, we can prove the covariance identity for the Yang–Mills Lagrangian

$$-\frac{\sqrt{g}}{2\mathrm{g}} \left(F^A_{\mu\alpha} F_{A\nu}{}^{.\alpha} - \frac{1}{4} F^A_{\beta\alpha} F^{\beta\alpha}_A g_{\mu\nu} \right) \mathcal{L}_\xi g^{\mu\nu} - \frac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_A \nabla_\mu \mathcal{L}_\Xi \omega^A_\nu =$$
$$= \frac{\sqrt{g}}{\mathrm{g}} \left(F^A_{\mu\alpha} F_{A\nu}{}^{.\alpha} - \frac{1}{4} F^A_{\beta\alpha} F^{\beta\alpha}_A g_{\mu\nu} \right) \nabla^\mu \xi^\nu - \frac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_A \nabla_\mu \left(\xi^\lambda F^A_{\lambda\nu} - \nabla_\nu \xi^A_{(V)} \right) =$$
$$= \frac{\sqrt{g}}{\mathrm{g}} F^A_{\mu\alpha} F_{A\nu}{}^{.\alpha} \nabla^\mu \xi^\nu - \frac{\sqrt{g}}{4\mathrm{g}} F^A_{\beta\alpha} F^{\beta\alpha}_A g_{\mu\nu} \nabla^\mu \xi^\nu - \frac{\sqrt{g}}{\mathrm{g}} F^{\mu\nu}_A \left(\nabla_\mu \xi^\lambda F^A_{\lambda\nu} + \xi^\lambda \nabla_\mu F^A_{\lambda\nu} \right) + \frac{\sqrt{g}}{2\mathrm{g}} F^{\mu\nu}_A [\nabla_\mu, \nabla_\nu] \xi^A_{(V)} = \qquad (2.7.39)$$
$$= -\frac{\sqrt{g}}{4\mathrm{g}} F^A_{\beta\alpha} F^{\beta\alpha}_A \nabla_\mu \xi^\mu - \frac{\sqrt{g}}{2\mathrm{g}} 2 F^{\mu\nu}_A \nabla_\mu F^A_{\lambda\nu} \xi^\lambda + \boxed{\frac{\sqrt{g}}{4\mathrm{g}} F^{\mu\nu}_A c^A{}_{BC} F^B_{\mu\nu} \xi^C_{(V)}} = -\frac{\sqrt{g}}{4\mathrm{g}} F^A_{\beta\alpha} F^{\beta\alpha}_A \nabla_\mu \xi^\mu - \frac{\sqrt{g}}{2\mathrm{g}} F^{\alpha\beta}_A \nabla_\mu F^A_{\alpha\beta} \xi^\mu =$$
$$= d_\mu \left(\xi^\mu L_{YM} \right)$$

Then the Lagrangian is covariant with respect to all generalised gauge transformations.

Since the Lagrangian is a gauge-natural Lagrangian, Yang–Mills theory is a gauge-natural theory provided that all fields are dynamical. The variation of this Lagrangian is

$$\delta \boldsymbol{L} = \left[\frac{\sqrt{g}}{2\kappa} \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} + \nabla_\lambda \left(\frac{\sqrt{g}}{2\kappa} g^{\mu\nu} \delta u^\lambda_{\mu\nu} \right) + \left(-\frac{\sqrt{g}}{2} T^{(YM)}_{\mu\nu} \delta g^{\mu\nu} - \nabla_\mu \left(\frac{\sqrt{g}}{2g} F^{\mu\nu}_A \delta \omega^A_\nu \right) + \nabla_\mu \left(\frac{\sqrt{g}}{2g} F^{\mu\nu}_A \right) \delta \omega^A_\nu \right) \right] d\sigma \tag{2.7.40}$$

where we set $u^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \Gamma^\epsilon_{\nu)\epsilon}$.

Field equations are

$$\begin{cases} R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T^{(YM)}_{\mu\nu} \\ \nabla_\mu F^{\mu\nu}_A = 0 \end{cases} \tag{2.7.41}$$

The first variation formula then defines the Euler–Lagrange and the Liouville parts of the Lagrangian to be

$$\begin{aligned} \mathbb{E}(\boldsymbol{L}) &= \sqrt{g} \left[\frac{1}{2\kappa} \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T^{(YM)}_{\mu\nu} \right) \omega^{\mu\nu} + \frac{1}{2g} \nabla_\mu F^{\mu\nu}_A \omega^A_\nu \right] \wedge d\sigma \\ \mathbb{F}(\boldsymbol{L}) &= \sqrt{g} \left[\frac{1}{2\kappa} g^{\mu\nu} \omega^\lambda_{\mu\nu} - \frac{1}{2g} F^{\lambda\nu}_A \omega^A_\nu \right] \wedge d\sigma_\lambda \end{aligned} \tag{2.7.42}$$

where we defined $\omega^A_\mu := d\omega^A_\mu - d_\lambda \omega^A_\mu \, dx^\lambda$, $\omega^{\mu\nu} := dg^{\mu\nu} - d_\lambda g^{\mu\nu} \, dx^\lambda$, and $\omega^\lambda_{\mu\nu} := du^\lambda_{\mu\nu} - d_\epsilon u^\lambda_{\mu\nu} \, dx^\epsilon$ for the relevant contact forms.

◁

Vielbein formalism

Given a metric g of signature $\eta = (r, s)$ one can define orthonormal bases e_a at a point $x \in M$, i.e.

$$g(e_a, e_b) = \eta_{ab} \tag{2.7.43}$$

where η_{ab} is the standard bilinear form of signature $\eta = (r, s)$, namely $\eta_{ab} = \text{diag}(-1, \ldots, -1, 1, \ldots, 1)$ with s entries -1 and r entries $+1$.

Whether the Lorentzian signature corresponds to $(1, 3)$ or $(3, 1)$ depends on the book, as well as whether the pluses or the minuses come first in η_{ab} .
Our notation is that Lorentzian signature is $(3, 1)$ and minus comes first.

The set of all positive orthonormal frames for a metric g is denoted by $\text{SO}(M, g) \subset L(M)$ and it is a principal sub-bundle with the group $\text{SO}(\eta)$.

$$\begin{array}{ccc} \text{SO}(M, g) & \stackrel{\iota}{\hookrightarrow} & L(M) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ M & =\!=\!=\!= & M \end{array} \tag{2.7.44}$$

In particular, if one considers a local trivialisation of $\text{SO}(M, g)$, that is expressed by a local section $\sigma : U \to \text{SO}(M, g)$. Such a section can be composed with the canonical injection $\iota : \text{SO}(M, g) \hookrightarrow L(M)$ to obtain a local section $\iota \circ \sigma : U \to L(M) : x \mapsto V_a(x)$ of the frame bundle, which induces a local trivialisation of $L(M)$.

By construction, V_a is a positive orthonormal basis with respect to g and having two of such local trivialisations the transition functions are valued into the orthogonal group SO(η).

If there exists a metric g of signature η on (an orientable) M one can always define SO(M, g) and a trivialisation of this bundle always produce a trivialisation of $L(M)$ made by orthonormal bases, hence with transition functions in the smaller group SO(η) $\subset$ GL(m).

One metric g has many orthonormal bases. In fact, if e_a is an orthonormal basis and $\Lambda_b^a \in$ SO(η) then

$$e'_a = e_b \Lambda_a^b \tag{2.7.45}$$

is an orthonormal basis as well. However, given a basis e_a (*any* basis!) there is one and only one metric g (for each signature η) for which the basis $e_a = e_a^\mu \partial_\mu$ is an orthonormal basis, namely

$$g = g_{\mu\nu} \, dx^\mu \otimes \, dx^\nu \qquad g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b \tag{2.7.46}$$

where e_μ^a denotes the inverse matrix to e_a^μ

When using bases, moving frames, *Vielbein*, or spin frames, we have invertible matrices e_a^μ which carry a coordinate index μ and a frame index a. Both indices run in the same range $0..(m-1)$ since the elements in a basis are as many as the coordinates.

However, when one changes coordinates, the coordinate indices change by the Jacobian (or anti-Jacobian depending on the position). The frame indices do not transform with coordinates (unless they are a natural frame). Instead, they change with the matrix associated to the change of basis in GL(m) if one decides to change the basis (even if coordinates are unaffected). In other words, the family of an index is not determined by its range only, but also by its transformation rules.

We decide to raise and lower frame indices by the metric η and coordinate indices by the induced metric g. Since the matrix e_a^μ is invertible one should denote the inverse by $\bar{e}_\mu^a$ so that

$$e_a^\mu \bar{e}_\nu^a = \delta_\nu^\mu \qquad\qquad \bar{e}_\mu^b e_a^\mu = \delta_a^b \tag{2.7.47}$$

However, one can also start from e_a^μ, raise the index a and lower the index μ to obtain

$$e^a_{\cdot\mu} = g_{\mu\nu} e_b^\nu \eta^{ba} \tag{2.7.48}$$

The bar notation for the inverse would we justified to distinguish $\bar{e}_\mu^a$ from $e^a_{\cdot\mu}$. However, one has

$$e^a_{\cdot\mu} = g_{\mu\nu} e_b^\nu \eta^{ba} = \bar{e}_\mu^c \eta_{cd} \bar{e}_\nu^d e_b^\nu \eta^{ba} = \bar{e}_\mu^c \eta_{cd} \delta_b^d \eta^{ba} = \bar{e}_\mu^c \eta_{cb} \eta^{ba} = \bar{e}_\mu^c \delta_c^a = \bar{e}_\mu^a \tag{2.7.49}$$

Thus, in fact, one has $e^a_{\cdot\mu} = \bar{e}_\mu^a$ and one does not really need a notation to distinguish one object from the other. Accordingly, both objects are denoted simply by e_μ^a .

Analogously, by $e_{a\mu}$ one can denote equivalently both

$$e_{a\mu} := e_{a\cdot\mu} = e_a^\nu g_{\mu\nu} = e_a^\nu e_\nu^c \eta_{cd} e_\mu^d = \eta_{ad} e_\mu^d = e_{\cdot a\mu} \tag{2.7.50}$$

In other words, the position and family of the indices are sufficient to distinguish what object one refers to.

Let us also stress that η_{ab} is not a field. Here η_{ab} does not represents a metric field on a spacetime which, being written in Cartesian coordinante, happens to be constant. The quantities η_{ab} are instead *(integer) numbers* since they represent the signature which is fixed over the whole spacetime, not depending on the point. Equivalently, the indices of η_{ab} are frame indices, not coordinate indices and, as such, the quantities η_{ab} are invariant with respect to coordinate transformations. They transform according to the orthogonal group SO(η) and are invariant with respect to these gauge transformations. Accordingly, the object η_{ab} is a canonical object as δ_μ^ν and $\epsilon_{\alpha\beta\gamma\delta}$ are.

A basis e_a is a point in $L(M)$; in general, the bundle $L(M)$ has no global sections. However, one can consider a local section $e_a : M \to L(M) : x \mapsto e_a(x) = e_a^\mu(x)\partial_\mu$, which is called a *moving frame*.

If a moving frame $e_a(x)$ is given, then one can define a metric field

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes \, dx^\nu = e_\mu^a(x)\eta_{ab}e_\nu^b(x)\, dx^\mu \otimes \, dx^\nu \tag{2.7.51}$$

out of it. Then one could use moving frames to describe metrics.

However, there are many moving frames defining locally the same metric and, moreover, there are manifolds in which $L(M)$ has no global sections.

> If $L(M)$ has a global section then it is trivial, since it is principal. If $L(M)$ is trivial then any bundle associated to it (hence including the tangent and cotangent bundles) is trivial as well.
>
> Of course, there exist manifolds with a non-trivial tangent bundle (e.g, the sphere S_2). Accordingly, these manifolds do not allow global moving frames.
>
> A manifold M for which $L(M)$ is trivial is called *parallelisable*. Being parallelisable is a topological characteristics and if M is parallelisable then any manifold diffeomorphic to M is parallelisable as well.
>
> One could restrict spacetimes to be parallelisable manifolds. However, one can live without this assumption which is *hence* considered too strict. The price to be paid for it is to use a better, more elaborated, notion that moving frames, i.e. *Vielbein*.
>
> The word *Vielbein* is from German and it means *many legs*. Sometimes in dimension two they are called *Zweibein*, *Dreibein* in dimension three (*three legs*), and in dimension four they are also called *Vierbein* (*four legs*). *Vielbein* is useful because it does not focus on the specific dimension, as other English synonyms such as, e.g., *tetrad* do.

Beside existence of global moving frames on non-parallelisable manifolds, moving frames have also another reason to be abandoned. As long as what matters is the metric, then if two nearby observers set their moving frames to e_a and e_a', respectively, they define the same global metric g iff the two fames in the overlap are related as

$$e_a'^\mu(x') = J_\nu^\mu e_b^\nu(x)\bar\Lambda_a^b(x) \tag{2.7.52}$$

for some local map $\Lambda_a^b : U \to \mathrm{SO}(\eta)$ defined in the overlap U of the two observers. By taking this transformation law seriously, one recognises that the fields e_a^μ cannot be a natural object, but they are a gauge-natural object which is affected by gauge transformations Λ in the group $\mathrm{SO}(\eta)$.

Thus let us follow the standard procedure for building gauge-natural bundles out of transformation laws of their sections to be. Let us fix the gauge group $\mathrm{SO}(\eta)$ and a structure bundle $\mathcal{P} = (P, M, \pi, \mathrm{SO}(\eta))$.

A *Vielbein* is a vertical principal morphism $e : P \to L(M)$ which is hence given in the form

$$\pi \circ e(V) = p(V)$$
$$(\textit{verticality})$$

$$e(V \cdot g) = e(V) \cdot i(g)$$
$$(\textit{equivariance})$$

$$\tag{2.7.53}$$

where $i : \mathrm{SO}(\eta) \to \mathrm{GL}(m)$ denotes the canonical group embedding.

Let us consider a trivialisation of $\mathcal{P}$ associated to a family of local sections $\sigma_{(\alpha)} : U_{(\alpha)} \to P$. The section domains $U_{(\alpha)}$ form an open covering of the manifold M and, on the overlaps, one has transition functions

$$\sigma_{(\beta)} = \sigma_{(\alpha)} \cdot \varphi_{(\alpha\beta)} \tag{2.7.54}$$

which map one section into another by the canonical right action on P.

Let us set $e(\sigma_{(\alpha)}(x)) = (x, V_a^{(\alpha)})$. Once one knows the map e on the section $\sigma_{(\alpha)}$, then one knows it on the whole tube $p^{-1}(U_{(\alpha)})$. In fact, for any $V \in p^{-1}(U_{(\alpha)})$ one has $V = \sigma_{(\alpha)} \cdot g$ for some $g \in \mathrm{SO}(\eta)$ and consequently

$$e(V) = e(\sigma_{(\alpha)} \cdot g) = e(\sigma_{(\alpha)}) \cdot i(g) = (x, V_a^{(\alpha)} \cdot i(g)) = (x, V_b^{(\alpha)} g_a^b) \tag{2.7.55}$$

where we set $i(g) = g_a^b \in \mathrm{GL}(m)$.

Accordingly, all information about the map e is encoded in the family of moving fames $V_a^{(\alpha)} = e(\sigma_{(\alpha)}(x))$. On the overlaps, one has precisely

$$V_a^{(\beta)} = e(\sigma_{(\beta)}(x)) = e(\sigma_{(\alpha)}(x) \cdot \varphi_{(\alpha\beta)}(x)) = e(\sigma_{(\alpha)}(x)) \cdot i(\varphi_{(\alpha\beta)}(x)) = V_b \cdot (\varphi_{(\alpha\beta)})_a^b \tag{2.7.56}$$

where we set $i(\varphi_{(\alpha\beta)}) = (\varphi_{(\alpha\beta)})_a^b \in \mathrm{SO}(\eta) \subset \mathrm{GL}(m)$.

Thus the map e is precisely described by a family of frames which differ by orthonormal transformations on the overlaps. Vice versa, for any such a family one can define a *Vielbein* map e.

Thus *Vielbein* are in one-to-one correspondence with families of frames which differ by orthonormal transformations on the overlaps. And that is exactly what one needs to define a global metric.

Then the transformation laws above define for us a left group action on the manifold $\mathrm{GL}(m)$ (which is also a Lie group though that is irrelevant here)

$$\sigma : \mathrm{SO}(\eta) \times \mathrm{GL}(m) \times \mathrm{GL}(m) \to \mathrm{GL}(m) : (\Lambda_a^b, J_\alpha^\beta, e_a^\alpha) \mapsto e_a'^\beta = J_\alpha^\beta e_b^\alpha \bar{\Lambda}_a^b \tag{2.7.57}$$

This action allows us to define the gauge-natural bundle

$$F(\mathcal{P}) = (P \times_M L(M)) \times_\sigma \mathrm{GL}(m) \tag{2.7.58}$$

which is called the *bundle of Vielbein*, which has coordinates (x^μ, e_a^μ). By construction there is a one-to-one correspondence between sections of the bundle $F(\mathcal{P})$ and global *Vielbein* maps $e : P \to L(M)$.

Let us stress that the bundle $F(\mathcal{P})$ always allows global sections, even when M is not parallelisable. In fact, one can always find a family of local frames differing by orthogonal transformations, even when the manifold does not allow global moving frames (as a consequence of existence of orthonormal basis for inner products in arbitrary signature).

Now everything comes as expected, by construction. A generalised gauge transformation on $\mathcal{P}$ acts on $F(\mathcal{P})$ as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ S_b'^a = \Lambda_c^a(x) S_b^c \end{cases} \quad \rightsquigarrow \quad \begin{cases} x'^\mu = x'^\mu(x) \\ e_a'^\mu = J_\nu^\mu e_b^\nu \bar{\Lambda}_a^b(x) \end{cases} \tag{2.7.59}$$

One has a global bundle morphism $g : F(\mathcal{P}) \to \mathrm{Lor}(M) : (x^\mu, e_a^\mu) \mapsto (x^\mu, g_{\mu\nu})$ given by $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\mu^b$. Given a section e of $F(\mathcal{P})$, i.e. a *Vielbein*, one can take the composition $g \circ e : M \to \mathrm{Lor}(M)$ which is a global metric on M, which is called the *induced metric*.

Finally, any basis in the image of e is orthonormal with respect to the induced metric. Accordingly, one has $e(\mathcal{P}) = \mathrm{SO}(M, g) \subset L(M)$.

We have to stop and stress that here the order matters.

One starts from $\mathcal{P}$ and define the bundle of *Vielbein* $F(\mathcal{P})$. This is done without fixing a metric. The map $g : F(M) \rightarrow \mathrm{Lor}(M)$ is also obtained before fixing a metric. Then any section e of $F(\mathcal{P})$, i.e. any *Vielbein*, defines an induced metric by composition with the bundle map g. Only at this point, the frames in the image of the *Vielbein* $e : F(\mathcal{P}) \rightarrow L(M)$ becomes *orthonormal* bases for the induced metric.

In this framework, there is no metric at a fundamental level. The metric is defined as a by-product of *Vielbein* by (2.7.51). Of course, one can pull back any Lagrangian defined in terms of a metric to obtain a Lagrangian written in term of the *Vielbein*.

One has that

$$\sqrt{g} = \sqrt{e^2} = |e| \tag{2.7.60}$$

where we denoted by e the determinant of the inverse *Vielbein* e^a_μ.

For any *Vielbein* $e : M \rightarrow F(\mathcal{P})$, one can define a compatible connection defined by

$$\Gamma^a_{b\mu} = e^a_\alpha \left(\{g\}^\alpha_{\beta\mu} e^\beta_b + d_\mu e^\alpha_b \right) \tag{2.7.61}$$

which is called the *Vielbein connection* and has the property

$$\nabla_\mu e^a_\nu := d_\mu e^a_\nu + \Gamma^a_{b\lambda} e^b_\nu - \{g\}^\lambda_{\nu\mu} e^a_\lambda = 0 \tag{2.7.62}$$

The curvature of the *Vielbein* connection is defined as

$$R^a{}_{b\mu\nu} = d_\mu \Gamma^a_{b\nu} - d_\nu \Gamma^a_{b\mu} + \Gamma^a_{c\mu} \Gamma^c_{b\nu} - \Gamma^a_{c\nu} \Gamma^c_{b\mu} \tag{2.7.63}$$

and one has

$$R^a{}_{b\mu\nu} = e^a_\alpha R^\alpha{}_{\beta\mu\nu} e^\beta_b \qquad R^a{}_\mu = R^a{}_{b\mu\nu} e^{b\nu} = e^{a\alpha} R_{\alpha\mu} \qquad R = R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b \tag{2.7.64}$$

where $R^\alpha{}_{\beta\mu\nu}$ is the Riemann tensor and $R_{\alpha\mu}$ is the Ricci tensor of the Levi Civita connection $\{g\}$ of the induced metric g.

One can write the Hilbert Lagrangian as

$$\boldsymbol{L} = \frac{|e|}{2\kappa} (R - 2\Lambda) \; d\sigma = \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda) \; d\sigma \tag{2.7.65}$$

The variation of such Lagrangian is

$$\delta\boldsymbol{L} = \frac{\sqrt{g}}{2\kappa} \left[\left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} + g^{\mu\nu} \nabla_\lambda \delta u^\lambda_{\mu\nu} \right] \; d\sigma \tag{2.7.66}$$

However, our fundamental fields here are e^μ_a (or equivalently e^a_μ, but not $g_{\mu\nu}$), thus variation of the metric needs to be further expanded as

$$
\begin{aligned}
\delta g^{\mu\nu} = 2e^{a\mu} \delta e^\nu_a \qquad g^{\mu\nu} \delta u^\lambda_{\mu\nu} &= \left(\delta^\lambda_\epsilon g^{\sigma\rho} - g^{\lambda\rho} \delta^\sigma_\epsilon \right) \delta \Gamma^\epsilon_{\rho\sigma} = \tfrac{1}{2} \left(g^{\lambda\alpha} g^{\sigma\rho} - g^{\lambda\rho} g^{\sigma\alpha} \right) \left(-\underline{\nabla_\alpha \delta g_{\rho\sigma}} + \underline{\nabla_\rho \delta g_{\sigma\alpha}} + \boxed{\nabla_\sigma \delta g_{\alpha\rho}} \right) = \\
&= \left(g^{\lambda\alpha} g^{\sigma\rho} - g^{\lambda\rho} g^{\sigma\alpha} \right) \left(\nabla_\rho \delta e^a_\sigma e_{a\alpha} + e_{a\sigma} \nabla_\rho \delta e^a_\alpha \right) = \left(g^{\lambda\rho} g^{\sigma\alpha} - g^{\lambda\alpha} g^{\sigma\rho} \right) \left(e^a_\sigma g_{\beta\alpha} + g_{\sigma\beta} e^a_\alpha \right) \nabla_\rho \delta e^\beta_a = \\
&= \left(e^a_\sigma g_{\beta\alpha} g^{\lambda\rho} g^{\sigma\alpha} + g_{\sigma\beta} e^a_\alpha g^{\lambda\rho} g^{\sigma\alpha} - e^a_\sigma g_{\beta\alpha} g^{\lambda\alpha} g^{\sigma\rho} - g_{\sigma\beta} e^a_\alpha g^{\lambda\alpha} g^{\sigma\rho} \right) \nabla_\rho \delta e^\beta_a = \\
&= \left(2 e^a_\beta g^{\lambda\rho} - \delta^\lambda_\beta e^{a\rho} - \delta^\rho_\beta e^{a\lambda} \right) \nabla_\rho \delta e^\beta_a
\end{aligned}
\tag{2.7.67}
$$

We used the identity

$$e_a^\lambda e_\nu^a = \delta_\nu^\lambda \qquad \Rightarrow \delta e_a^\lambda = -e_b^\lambda \delta e_\nu^b e_a^\nu \qquad \Rightarrow \nabla_\mu \delta e_a^\lambda = -e_b^\lambda \nabla_\mu \delta e_\nu^b e_a^\nu \tag{2.7.68}$$

which is a consequence of compatibility condition $\nabla_\mu e_a^\nu = 0$ used to define the *Vielbein* connection.

Then the variation of the Lagrangian can be recast as

$$\delta \boldsymbol{L} = \tfrac{1}{\kappa} \left[\sqrt{g} \left(R_\nu^a - \tfrac{1}{2} R e_\nu^a + \Lambda e_\nu^a \right) \delta e_a^\nu + \nabla_\lambda \left(\tfrac{\sqrt{g}}{2} \left(2 e_\beta^a g^{\lambda\rho} - \delta_\beta^\lambda e^{a\rho} - \delta_\beta^\rho e^{a\lambda} \right) \nabla_\rho \delta e_a^\beta \right) \right] \, d\sigma \tag{2.7.69}$$

The Euler–Lagrange and Liouville parts are

$$\mathbb{E} = \tfrac{\sqrt{g}}{\kappa} \left(R_\nu^a - \tfrac{1}{2} R e_\nu^a + \Lambda e_\nu^a \right) \omega_a^\nu \wedge \, d\sigma \qquad \mathbb{F} = \tfrac{\sqrt{g}}{\kappa} \left(e_\beta^a g^{\lambda\rho} - e^{a(\rho} \delta_\beta^{\lambda)} \right) \nabla_\rho \omega_a^\beta \wedge \, d\sigma_\lambda \tag{2.7.70}$$

where we defined the relevant contact forms as $\omega_a^\mu := de_a^\mu - d_\lambda e_a^\mu \, dx^\lambda$.

The field equations are

$$R_\mu^a - \tfrac{1}{2} R e_\mu^a = -\Lambda e_\mu^a \tag{2.7.71}$$

Given a *Vielbein* $e : M \to F(\mathcal{P})$ one can define the action $\lambda : \mathrm{SO}(\eta) \times \mathbb{R}^m \to \mathbb{R}^m : (S, v^a) \mapsto v'^a = i_b^a(S) v^b$ and define the associated bundle $T = \mathcal{P} \times_\lambda \mathbb{R}^m$ and the *Vielbein* induces a map $\lambda(e) : T \to TM : (x^\mu, v^a) \mapsto v^a e_a$, which is called *soldering form*.

The *Vielbein* framework and the soldering forms are equivalent formulations, just one is done at the level of principal bundles, the other in terms of the corresponding vector bundles.

The Lagrangian (2.7.65) is clearly gauge-natural. The Lagrangian depends on the *Vielbein* through the induced metric only. A generalised gauge transformation Ξ acts on the *Vielbein* so that the metric only transforms with respect to the diffeomorphism ξ, i.e.

$$\pounds_\xi g^{\mu\nu} = 2 e^{a(\mu} \pounds_\Xi e_a^{\nu)} \tag{2.7.72}$$

Accordingly, the Lagrangian is a gauge-natural Lagrangian.

One can also show that any covariant Lagrangian depending on the *Vielbein* and derivatives up to order 2, then it necessarily depends on the frame through the induced metric and it is actually the pull-back of a metric Lagrangian.

Without any loss of generality the Lagrangian may be assumed locally to have the following form:

$$\boldsymbol{L} = \hat{L}(x^\mu, e_a^\mu, e_{a\sigma}^\mu, e_{a\rho\sigma}^\mu) \, d\sigma = L(x^\mu, e_a^\mu, \Gamma_{b\mu}^a, R^{ab}{}_{\mu\nu}, S^a{}_{b\mu\nu}) \, d\sigma \tag{2.7.73}$$

where we set $S^a{}_{b\mu\nu} := d_{(\mu} \Gamma_{b\nu)}^a$. Let us also define the *naive momenta* by the standard prescription:

$$p_\mu = \frac{\partial L}{\partial x^\mu} \qquad p_\mu^a = \frac{\partial L}{\partial e_a^\mu} \qquad p_a^{b\mu} = \frac{\partial L}{\partial \Gamma_{b\mu}^a} \qquad p_{ab}{}^{\mu\nu} = \frac{\partial L}{\partial R^{ab}{}_{\mu\nu}} \qquad P_a{}^{b\mu\nu} = \frac{\partial L}{\partial S^a{}_{b\mu\nu}} \tag{2.7.74}$$

Requiring this Lagrangian to be $\mathrm{Aut}(\mathcal{P})$-covariant amounts to the following covariant identity:

$$d_\sigma(L\xi^\sigma) = p_\mu^a \pounds_\Xi e_a^\mu + p_a^{b\mu} \pounds_\Xi \Gamma_{b\mu}^a + p_{ab}{}^{\mu\nu} \pounds_\Xi R^{ab}{}_{\mu\nu} + P_a{}^{b\mu\nu} \pounds_\Xi S^a{}_{b\mu\nu} \tag{2.7.75}$$

where $\pounds_\Xi$ denotes the Lie derivative of sections with respect to the infinitesimal generator $\Xi = \xi^\mu \delta_\mu + \xi^{ab}\sigma_{ab}$ of a one-parameter subgroup of automorphisms on $\mathcal{P}$. Let us denote by $\xi^{ab}_{(V)} = \xi^{ab} + \Gamma^{ab}_\mu \xi^\mu$ the vertical part of the infinitesimal generator Ξ with respect to the connections Γ^{ab}_μ,. The field $S^a{}_{b\mu\nu}$ transforms with second derivatives $\nabla_{\mu\nu}\xi^{ab}_{(V)}$. It is the only field transforming in such a way. Hence the corresponding momenta vanish and the Lagrangian cannot depend on it if it has to be gauge covariant.

The field $\Gamma^a_{b\mu}$ transforms with $\nabla_\mu \xi^{ab}_{(V)}$ and it is the only field transforming in such a way. Hence the Lagrangian cannot depend on it.

Thus the Lagrangian is in the form

$$\boldsymbol{L} = \ L(x^\mu, e^\mu_a, R^{ab}{}_{\mu\nu}) \ d\sigma \tag{2.7.76}$$

and the covariance identity simplifies to

$$\nabla_\sigma(L\xi^\sigma) = p^a_\mu \pounds_\Xi e^\mu_a + p_{ab}{}^{\mu\nu} \pounds_\Xi R^{ab}{}_{\mu\nu} \tag{2.7.77}$$

The Lie derivatives appearing in equation (2.7.77) may be explicitly expressed in the following form:

$$\begin{aligned}
\pounds_\Xi e^\mu_a &= e^\mu_b \left(\xi_{(V)}\right)^b{}_a - \nabla_\nu \xi^\mu e^\nu_a \\
\pounds_\Xi R^{ab}{}_{\mu\nu} &= \xi^\lambda \nabla_\lambda R^{ab}{}_{\mu\nu} + R^{ab}{}_{\lambda\nu}\nabla_\mu\xi^\lambda + R^{ab}{}_{\mu\lambda}\nabla_\nu\xi^\lambda - \left(\xi_{(V)}\right)^a{}_c R^{cb}{}_{\mu\nu} - \left(\xi_{(V)}\right)^b{}_c R^{ac}{}_{\mu\nu}
\end{aligned} \tag{2.7.78}$$

Hence for the Lagrangian to be gauge covariant one needs to have

$$\begin{cases}
\nabla_\lambda L = p_{ab}{}^{\mu\nu}\nabla_\lambda R^{ab}{}_{\mu\nu} \\
L\delta^\alpha_\lambda = -p^a_\lambda e^\alpha_a + p_{ab}{}^{\alpha\nu}R^{ab}{}_{\lambda\nu} + p_{ab}{}^{\mu\alpha}R^{ab}{}_{\mu\lambda} \\
p^{}_{[a\mu}e^\mu_{b]} = -2p_{c[a}{}^{\mu\nu}R^c{}_{b]\mu\nu}
\end{cases} \tag{2.7.79}$$

We want to show here how the Lagrangian (2.7.73) simplifies because of covariance. The first identity can be recast as

$$\begin{aligned}
d_\lambda L - \Gamma^\sigma_{\lambda\alpha}\delta^\alpha_\sigma L - p_{ab}{}^{\mu\nu}\nabla_\lambda R^{ab}{}_{\mu\nu} &= d_\lambda L - \Gamma^\sigma_{\lambda\alpha}\left(-p^a_\sigma e^\alpha_a + p_{ab}{}^{\alpha\nu}R^{ab}{}_{\sigma\nu} + p_{ab}{}^{\mu\alpha}R^{ab}{}_{\mu\sigma}\right) - p_{ab}{}^{\mu\nu}\nabla_\lambda R^{ab}{}_{\mu\nu} = \\
&= d_\lambda L - \Gamma^\sigma_{\lambda\alpha}\left(-p^a_\sigma e^\alpha_a + {\color{red}2p_{ab}{}^{\mu\alpha}R^{ab}{}_{\mu\sigma}}\right) - p_{ab}{}^{\mu\nu}d_\lambda R^{ab}{}_{\mu\nu} + {\color{red}2p_{ab}{}^{\mu\alpha}R^{ab}{}_{\mu\sigma}\Gamma^\sigma_{\alpha\lambda}} + 2p_{c[b}{}^{\mu\nu}R^c{}_{a]\mu\nu}\Gamma^{ab}_\lambda = \\
&= \partial_\lambda L + p^a_\mu d_\lambda e^\mu_a + {\color{red}p_{ab}{}^{\mu\nu}d_\lambda R^{ab}{}_{\mu\nu}} + p^a_\mu e^\alpha_a \Gamma^\mu_{\alpha\lambda} - {\color{red}p_{ab}{}^{\mu\nu}d_\lambda R^{ab}{}_{\mu\nu}} - 2p_{c[a}{}^{\mu\nu}R^c{}_{b]\mu\nu}\Gamma^{ab}_\lambda = \\
&= \partial_\lambda L - \left(\boxed{p^{}_{[a\mu}e^\mu_{b]} + 2p_{c[a}{}^{\mu\nu}R^c{}_{b]\mu\nu}}\right)\Gamma^{ab}_\lambda = \partial_\lambda L = 0
\end{aligned} \tag{2.7.80}$$

Accordingly, the Lagrangian cannot depend on x^μ either and we are left with a Lagrangian $\boldsymbol{L} = L(e^\mu_a, R^{ab}{}_{\mu\nu}) \ d\sigma$. With no loss of generality, we can introduce a new Lagrangian density

$$\tilde{L}(e^\mu_a, R^\alpha{}_{\beta\mu\nu}) = L(e^\mu_a, e^a_\alpha R^\alpha{}_{\beta\mu\nu}e^{b\beta}) \tag{2.7.81}$$

and the new momenta

$$\tilde{p}^a_\mu = p^a_\mu - \tilde{p}_\mu{}^{\beta\rho\sigma}e^a_\alpha R^\alpha{}_{\beta\rho\sigma} + \tilde{p}_\alpha{}^{\epsilon\rho\sigma}e^a_\epsilon R^\alpha{}_{\mu\rho\sigma} \qquad \tilde{p}_\alpha{}^{\beta\mu\nu} = p_{ab}{}^{\mu\nu}e^a_\alpha e^{b\beta} \tag{2.7.82}$$

Finally, we want to write the constraints (2.7.79) in terms of the new momenta

$$\tilde{p}^{}_{[a\mu}e^\mu_{b]} = p^{}_{[a\mu}e^\mu_{b]} + 2p_{c[a}{}^{\rho\sigma}R^c{}_{b]\rho\sigma} = 0 \qquad \tilde{p}^a_\mu e^\alpha_a = p^a_\mu e^\alpha_a + \tilde{p}_\mu{}^{\beta\rho\sigma}R^\alpha{}_{\beta\rho\sigma} + \tilde{p}^{\alpha\beta\rho\sigma}R_{\mu\beta\rho\sigma} \tag{2.7.83}$$

so that the constraints become

$$\begin{cases}
\tilde{L}\delta^\alpha_\lambda = -\tilde{p}^a_\lambda e^\alpha_a + \tilde{p}_\beta{}^{\beta\rho\sigma}R^\alpha{}_{\beta\rho\sigma} + 3\tilde{p}_\beta{}^{\alpha\rho\sigma}R^\beta{}_{\lambda\rho\sigma} \\
\tilde{p}^{}_{[a\mu}e^\mu_{b]} = 0
\end{cases} \tag{2.7.84}$$

The second constraint $\tilde{p}_{[a\mu}^{\cdot}e_{b]}^{\mu} = 0$ simply says that the Lagrangian density $\tilde{L}$ does depend on the *Vielbein* only through the combination $g^{\mu\nu} = e_a^\mu \eta^{ab} e_b^\nu$, i.e. it is satisfied if and only if the Lagrangian density $\tilde{L}$ is constant on the fibres of the projection $F(\mathcal{P}) \to \mathrm{Lor}(M)$, i.e. it is compatible with the quotient, so that a metric Lagrangian is induced. In other words, the Lagrangian depends only on the associated metric and not on others combinations of the *Vielbein*.

To prove our claim, let us recall that the projection is locally given by $g : F(\mathcal{P}) \to \mathrm{Lor}(M) : e_a^\mu \mapsto g_{\mu\nu} = e_\mu^a \, \eta_{ab} \, e_\nu^b$ and that the fibres of this projection are orbits of the following representation

$$\rho : O(\eta) \times F(\mathcal{P}) \to F(\mathcal{P}) : (a_b^a, e_a^\mu) \mapsto e_b^\mu a_a^b$$

Then $\tilde{L}$ is constant on fibres if and only if $\tilde{p}_\mu^a e_b^\mu \dot{a}_a^b = 0$ for all $\dot{a}_a^b(t) \in \mathfrak{o}(\eta)$ and each point $e_a^\mu \in F(\mathcal{P})$. This is, in turn, equivalent to the following identity

$$(\tilde{p}_\mu^a e^{b\mu})\eta_{bc}\dot{a}_a^c = 0$$

Since $a \in O(\eta)$, then $\eta_{bc}\dot{a}_e^c$ is skew-symmetric with respect to the indices $[be]$, thus this identity holds if and only if $\tilde{p}_\mu^{[a} e^{b]\mu} = 0$.

To summarise, any gauge-natural Lagrangian depending on the *Vielbein* (and their derivatives up to order 2) is, in fact, the pull back of a generally covariant metric Lagrangian.

Thus *Vielbein* formalism does not allow more dynamics than purely metric formalism. We shall see, however, that it allows more couplings to matter field. In particular, it allows couplings with spinor fields which do not factorise through metric formulations.

In other words, if one wishes to deal with spinors *Vielbein* formalism is needed (or more precisely, *spin frames* formalism). If one describes gravity in *Vielbein* formalism and then couples to tensor matter fields, then everything factorises through a metric theory and *Vielbein* formalism does not add new physics.

Spin frames and spinor fields

The group $\mathrm{Spin}(\eta)$ is a double covering of the orthogonal group $\mathrm{SO}(\eta)$, i.e. there exists a 2-to-1 group homomorphism $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta)$. When the spin group is defined within a Clifford algebra generated by a basis γ_a which obey the relations

$$\{\gamma_a, \gamma_b\} = -2\eta_{ab}\mathbb{I} \qquad \gamma_0\gamma_a^\dagger\gamma_0 = \gamma_a \tag{2.7.85}$$

then the covering map is defined as $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta) : S \mapsto \ell(S)$ such that

$$S\gamma_a S^{-1} = \gamma_b\ell_a^b(S) \qquad\qquad \left[\iff S\gamma^a S^{-1} = \ell_b^a(S^{-1})\gamma^b \right] \tag{2.7.86}$$

The map $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta)$ so defined is a group homomorphism. In fact:

$$TS\gamma_a S^{-1}T^{-1} = T\gamma_b T^{-1}\ell_a^b(S) = \gamma_c\ell_b^c(T)\ell_a^b(S) = \gamma_c\ell_a^c(TS) \tag{2.7.87}$$

Depending on the convention for the signature one assumes Dirac matrices' anticommutators to be as in (2.7.85) for the signature $(3,1)$ (the one that we are using) or

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{I} \tag{2.7.88}$$

for the signature $(1,3)$. The reason is to have the square of the Dirac operator equal to the Klein–Gordon operator in Minkowski which was a historical motivation for spinor fields.

One can repeat the argument for *Vielbein* starting from a structure bundle $\hat{\mathcal{P}} = (\hat{P}, M, \hat{p}, \mathrm{Spin}(\eta))$ for the spin group. Any principal spin bundle $\hat{\mathcal{P}}$ is a double covering of a principal orthogonal bundle $\mathcal{P} = (P, M, p, \mathrm{SO}(\eta))$ such that one has

$$
\begin{array}{ccc}
\hat{P} & \xrightarrow{\ \hat{\ell}\ } & P \\
\downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} \\
M & =\!=\!= & M
\end{array}
\qquad
\hat{\ell} : \hat{P} \to P : [x, S] \mapsto [x, \ell(S)]
\tag{2.7.89}
$$

This map $\hat{\ell}$ is global and independent of the trivialisation.

Let us define a *spin frame* to be a vertical principal morphism $\hat{e} : \hat{P} \to L(M)$ which is hence given in the form

$$
\begin{array}{c}
\pi \circ \hat{e}(\nu) = \hat{p}(\nu) \\
\textit{(verticality)}
\end{array}
\qquad
\begin{array}{ccc}
\hat{P} & \xrightarrow{\ \hat{e}\ } & L(M) \\
\downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \pi} \\
M & =\!=\!= & M
\end{array}
\qquad
\begin{array}{ccc}
\hat{P} & \xrightarrow{\ \hat{e}\ } & L(M) \\
\downarrow{\scriptstyle R_S} & & \downarrow{\scriptstyle R_{i\circ\ell(S)}} \\
\hat{P} & \xrightarrow{\ \hat{e}\ } & L(M)
\end{array}
\qquad
\begin{array}{c}
\hat{e}(\nu \cdot S) = \hat{e}(\nu) \cdot i \circ \ell(S) \\
\textit{(equivariance)}
\end{array}
\tag{2.7.90}
$$

where $i \circ \ell : \mathrm{Spin}(\eta) \to \mathrm{GL}(m)$ denotes the canonical group embedding.

As for *Vielbein*, a spin frame $\hat{e}$ is completely determined (by equivariance) once it is known on a section (e.g. the one associated to a local trivialisation). For example, let us fix a family of local sections $\hat{\sigma}_{(\alpha)} : U_{(\alpha)} \to \hat{P}$ which induces a trivialisation of $\hat{\mathcal{P}}$. Then, for any spin frame $\hat{e}$, let us define the local moving frames $\hat{e}(\hat{\sigma}_{(\alpha)}) = (x, e_a^{(\alpha)})$ with $e_a^{(\alpha)} = (e^{(\alpha)})_a^\mu \partial_\mu$ which completely determine the spin frame.

On the overlap of two local trivialisations, one has

$$
\hat{\sigma}_{(\beta)} = \hat{\sigma}_{(\alpha)} \cdot S^{(\alpha\beta)}(x)
\qquad
\Rightarrow e_a^{(\beta)} = e_b^{(\alpha)} \ell_a^b \left(S^{(\alpha\beta)}(x) \right)
\tag{2.7.91}
$$

so that two nearby spin frames differ by an orthogonal transformation. Then there exists one and only one metric

$$
g = e_\mu^a \eta_{ab} e_\nu^b \, dx^\mu \otimes \, dx^\nu
$$

which is called the *induced metric*.

The image $\hat{e}(\hat{\mathcal{P}}) \subset L(M)$ coincides with the sub-bundle $\hat{e}(\hat{\mathcal{P}}) = \mathrm{SO}(M, g)$ of orthonormal frames defined by the induced metric.

Any spin frame $\hat{e} : \hat{P} \to L(M)$ restricts to a map $\tilde{e} : \hat{P} \to \mathrm{SO}(M, g)$. The pair $(\hat{\mathcal{P}}, \tilde{e})$ is called a *spin structure* on the Riemannian manifold (M, g).

Spin structures are considered necessary to define global Dirac equations on the Riemannian manifold (M, g). This approach is good in geometry, not as good in GR, where the metric g is not a fixed structure and it is unknown until one solves Einstein equations. Spin frames are a rephrasing of the same topic, without starting with a fixed metric on the spacetime.

Of course, one fixes a spin frame instead of a metric. However, the construction is independent of which spin frame one fixes and the spin frame can be varied at will. On the contrary, if one fixes the metric, then the structure is too rigid to vary it (essentially because the variation of the metric does not single out uniquely a deformation of the spin frame).

The trivialisation on $\hat{\mathcal{P}}$ induces a trivialisation on $\mathcal{P}$ given by $\sigma_{(\alpha)} = \hat{\ell} \circ \hat{\sigma}_{(\alpha)} : U_{(\alpha)} \to P$.

If transition functions in the trivialisation induced by the sections $\hat{\sigma}_{(\alpha)}$ are denoted by $S^{(\alpha\beta)} : U_{(\alpha\beta)} \to \mathrm{Spin}(\eta)$, then the trivialisation induced on $\mathcal{P}$ has transition functions $\ell(S^{(\alpha\beta)}) : U_{(\alpha\beta)} \to \mathrm{SO}(\eta)$.
Since $S^{(\alpha\beta)}$ form a cocycle, then $\ell(S^{(\alpha\beta)}) : U_{(\alpha\beta)} \to \mathrm{SO}(\eta)$ is a cocycle as well, since $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta)$ is a group homomorphism. This cocycle defines the principal bundle for the orthogonal group, namely $\mathcal{P}$.

One can hence define a *Vielbein* $e : \mathcal{P} \to L(M)$ defined by

$$e(\sigma_{(\alpha)}) = \hat{e}(\hat{\sigma}_{(\alpha)}) \qquad \Rightarrow e \circ \hat{\ell} = \hat{e} \tag{2.7.92}$$

i.e. the spin frames factorises through the covering map $\hat{\ell}$:

$$\tag{2.7.93}$$

Since the covering map $\hat{\ell} : \hat{P} \to P$ is a local diffeomorphism, one can pull back connections (and then dynamics) from P to $\hat{P}$.

Given a connection $\Gamma^a_{b\mu}$ on $\mathcal{P}$, one can define the *spin connection* $\Gamma_\mu = -\frac{1}{4}\Gamma^{ab}_\mu \gamma_a \gamma_b$ (matrix indices understood) on $\hat{\mathcal{P}}$; see Appendix A below.

The definition of the bundle of spin frames $F(\hat{\mathcal{P}})$ goes along the same line as for *Vielbein*. One defines the left group action

$$\hat{\sigma} : \mathrm{Spin}(\eta) \times \mathrm{GL}(m) \times \mathrm{GL}(m) \to \mathrm{GL}(m) : (S, J^\beta_\alpha, e^\alpha_a) \mapsto e'^\beta_a = J^\beta_\alpha e^\alpha_b \ell^b_a(\bar{S}) \tag{2.7.94}$$

This action allows us to define the gauge-natural bundle

$$F(\hat{\mathcal{P}}) = \left(\hat{P} \times_M L(M)\right) \times_{\hat{\sigma}} \mathrm{GL}(m) \tag{2.7.95}$$

which is called the *bundle of spin frames*, which has coordinates (x^μ, e^μ_a). By construction there is a one-to-one correspondence between sections of the bundle $F(\hat{\mathcal{P}})$ and global spin frames maps $\hat{e} : \hat{P} \to L(M)$.

Since the spin frame $\hat{e}$ induces a metric, which defines a Levi Civita connection and can be then pulled back on $\hat{\mathcal{P}}$, then one has a spin connection Γ_μ on $\hat{P}$ and a connection $\{g\}^\alpha_{\beta\mu}$ on $L(M)$.

Consequently, one can define the covariant derivatives of the spin frame

$$\nabla_\mu e_a^\alpha = d_\mu e_a^\alpha - \Gamma_{a\mu}^b e_b^\alpha + \{g\}_{\beta\mu}^\alpha e_a^\beta = 0 \tag{2.7.96}$$

which vanishes in view of the definition (2.7.61) of the connection $\Gamma_{b\mu}^a$. In fact, the connection defined by (2.7.61) is the only connection for which $\nabla_\mu e_a^\alpha = 0$.

One can write a dynamics in terms of the spin frames only, e.g.

$$\boldsymbol{L} = \tfrac{|e|}{2\kappa} (R - 2\Lambda) \; d\sigma \tag{2.7.97}$$

The dynamics written in terms of *Vielbein* or spin frames are identical since the spin frame dynamics factorises through the *Vielbein* dynamics. This is true also with matter coupling until the representation $\hat{\lambda}$ of the spin group is a *tensor representation*, i.e. until there exists a representation λ of the orthogonal group $\text{SO}(\eta)$ such that

$$\hat{\lambda}(S) = \lambda(\ell(S)) \tag{2.7.98}$$

When this happens the matter dynamics written in terms of *Vielbein* or spin frames are identical.

However, the spin group also has representations which are not tensorial. For example, for any choice of matrices γ_a such that the anticommutation relations (2.7.85) are implemented, one has a representation of the whole Clifford algebra which restricts to a representation $\lambda : \text{Spin}(\eta) \times V \to V$ of the spin group. In such representations $\pm\mathbb{I}$ are represented by the matrix $\pm\mathbb{I}$, respectively, thus they are not compatible with the quotient induced by the covering map and the representation is not tensorial.

> For some obscure reason, the representations of the $\text{Spin}(\eta)$ group which are not tensorial are called *spin representations of the orthogonal group* in the literature. This is, of course, a notational abuse since a representation of the spin group is by no means a representation of the orthogonal group.
> Of course, the abuse of language relies on the fact that the two groups have the same algebra $\mathfrak{spin}(\eta) \simeq \mathfrak{so}(\eta)$ and there is the habit to confuse representations of the algebras with the representations of their group, which, in fact, is equally deplorable.

When considering matter couplings with spin frames, one has more freedom than with *Vielbein*. Every time a matter coupling is built using a non-tensorial representation of the spin group, that dynamics cannot be written in terms of an ordinary orthogonal representation (which hence factorises through the induced metric). Matter fields which couple in this way to spin frames are called *spinors*.

Let us consider a (non-tensorial) representation $\lambda : \text{Spin}(\eta) \times V \to V$ one can define the *spinor bundle*

$$S(\hat{\mathcal{P}}) = \hat{\mathcal{P}} \times_\lambda V \tag{2.7.99}$$

with coordinates (x^μ, ψ) (matrix indices understood in what should be ψ^i).

At this point, one should define connections of $\hat{\mathcal{P}}$ and covariant derivatives of spinor fields ψ. There is a long and correct way and a short and *naive* way to do that. The *naive* way is to define the covariant derivative of spinors to be

$$\nabla_\mu \psi = \partial_\mu \psi - \tfrac{1}{4} \Gamma_\mu^{ab} \gamma_a \gamma_b \psi \tag{2.7.100}$$

and check that it transforms well under gauge transformations $\psi' = S(x)\psi$.

In fact,

$$
\begin{aligned}
(\nabla_\mu \psi)' =& S\partial_\mu\psi + \partial_\mu S\psi - \tfrac{1}{4}\Gamma'^{ab}_\mu SS^{-1}\gamma_a SS^{-1}\gamma_b S\psi = S\left(\partial_\mu\psi + S^{-1}\partial_\mu S\psi - \tfrac{1}{4}\Gamma'^{ab}_\mu \bar{\ell}^c_a \bar{\ell}^d_b \gamma_c\gamma_d\psi\right) = S\left(\partial_\mu\psi - \tfrac{1}{4}\bar{\ell}^c_a\left(\Gamma'^{ab}_\mu\bar{\ell}^d_b + \partial_\mu\ell^a_b\eta^{bd}\right)\gamma_c\gamma_d\psi\right) = \\
=& S\left(\partial_\mu\psi - \tfrac{1}{4}\Gamma^{cd}_\mu\gamma_c\gamma_d\psi\right) = S\nabla_\mu\psi
\end{aligned}
\tag{2.7.101}
$$

Ok, we used the identities

$$
S^{-1}\partial_\mu S = -\bar{\ell}^c_a\partial_\mu\ell^a_b\eta^{bd}\gamma_c\gamma_d \tag{2.7.102}
$$

Except for that, it is just a computation.

One cannot really say that there is something wrong with this procedure. However, it is like having your grandmother living in Paris who gives you an airplane as a gift for allowing to visit her more often. And since you don't know how to pilot you enrol on a course in mechanics to tear the wings apart to be able to drive the airplane on the highway. :o)

Covariant derivatives are important and they transform well. They are important *because* they transform well and sometimes all you need is the fact they transform well. However, that is not why connections have been invented for or how they are meant to be used.

Since once in a lifetime everyone should be able to take the long and correct way, we do that for reference in Appendix A below. Of course, if one knows how to do it, the short and *naive* way will be fine as well.

Let us also compute the commutator of spinor covariant derivatives

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]\psi =& \nabla_\mu(\partial_\nu\psi - \tfrac{1}{4}\Gamma^{ab}_\nu\gamma_a\gamma_b\psi) - [\mu\nu] = \boxed{\partial_{\mu\nu}\psi} - \tfrac{1}{4}\partial_\mu\Gamma^{ab}_\nu\gamma_a\gamma_b\psi - \tfrac{1}{4}\Gamma^{ab}_\nu\gamma_a\gamma_b\partial_\mu\psi - \tfrac{1}{4}\Gamma^{ab}_\mu\gamma_a\gamma_b\nabla_\nu\psi - \boxed{\{g\}^\alpha_{\nu\mu}\nabla_\alpha\psi} - [\mu\nu] = \\
=& -\tfrac{1}{4}\partial_\mu\Gamma^{ab}_\nu\gamma_a\gamma_b\psi - \tfrac{1}{4}\Gamma^{ab}_\nu\gamma_a\gamma_b\left(\nabla_\mu\psi + \tfrac{1}{4}\Gamma^{cd}_\mu\gamma_c\gamma_d\psi\right) - \tfrac{1}{4}\Gamma^{ab}_\mu\gamma_a\gamma_b\nabla_\nu\psi - [\mu\nu] = \\
=& -\tfrac{1}{4}\partial_\mu\Gamma^{ab}_\nu\gamma_a\gamma_b\psi + \tfrac{1}{4}\Gamma^{ab}_\mu\gamma_a\gamma_b\nabla_\nu\psi + \tfrac{1}{16}\Gamma^{ab}_\mu\Gamma^{cd}_\nu\gamma_a\gamma_b\gamma_c\gamma_d\psi - \tfrac{1}{4}\Gamma^{ab}_\mu\gamma_a\gamma_b\nabla_\nu\psi - [\mu\nu] = \\
=& -\tfrac{1}{4}\left(\partial_\mu\Gamma^{ab}_\nu - \partial_\nu\Gamma^{ab}_\mu + \Gamma^a_{c\mu}\Gamma^{cb}_\nu - \Gamma^a_{c\nu}\Gamma^{cb}_\mu\right)\gamma_a\gamma_b\psi = -\tfrac{1}{4}R^{ab}{}_{\mu\nu}\gamma_a\gamma_b\psi
\end{aligned}
\tag{2.7.103}
$$

We used the identity on the Clifford algebra

$$
\begin{aligned}
\Gamma^{ab}_\mu\Gamma^{cd}_\nu\gamma_a\gamma_b\gamma_c\gamma_d - \Gamma^{ab}_\mu\Gamma^{cd}_\nu\gamma_a\gamma_b\gamma_c\gamma_d =& \Gamma^{ab}_\mu\Gamma^{cd}_\nu\left(abcd - cdab\right) = \\
=& \Gamma^{ab}_\mu\Gamma^{cd}_\nu\left(abcd - abcd - 2(bc)ad + 2(bd)ac - 2(ac)db + 2(ad)cb\right) = 4\Gamma^{ab}_\mu\Gamma^{cd}_\nu\left(-(bc)ad + (bc)da\right) = \\
=& -4\Gamma^a_{c\mu}\Gamma^{cb}_\nu\gamma_a\gamma_b + 4\Gamma^a_{c\mu}\Gamma^{cb}_\nu\gamma_b\gamma_a = -4\left(\Gamma^a_{c\mu}\Gamma^{cb}_\nu - \Gamma^a_{c\nu}\Gamma^{cb}_\mu\right)\gamma_a\gamma_b
\end{aligned}
\tag{2.7.104}
$$

In the proof, we did not used a particular representation of the Clifford algebra but only the anticommutation relations (2.7.85).

We introduced a shortcut notation for products of gamma matrices.

Exercise: guess how the shortcut notation works by knowing that the anticommutation relations read as

$$
ab + ba = -2(ab) \tag{2.7.105}
$$

If you cannot guess just repeat the computation above in full notation.

Let us define $\bar\psi := \psi^\dagger\gamma_0$, $\nabla_a\psi := e^\mu_a\nabla_\mu\psi$ and consider the Lagrangian

$$
\boldsymbol{L}_D = -\hbar|e|\left(\tfrac{i}{2}\bar\psi\gamma^a\nabla_a\psi - \tfrac{i}{2}\nabla_a\bar\psi\gamma^a\psi + \mu\bar\psi\psi\right)\,d\sigma \tag{2.7.106}
$$

which is called the *Dirac Lagrangian* and it depends on the spinor field ψ (and its first derivatives) and on the spin frame e_a^μ (and its first derivatives).

As usual let us spend some time for dimensional analysis. As in the metric case $[|e|\,d\sigma] = L^m$; the matrices γ^a and the spin frame e_a^μ are adimensional. Then $[\frac{i}{2}\bar\psi\gamma^a\nabla_a\psi] = L^{-1}[\psi]^2$; thus, if we want the Lagrangian to have the dimension of an action, we have to set $ML^{m+1}T^{-1}[\psi]^2 = ML^2T^{-1}$ and obtain $[\psi] = L^{\frac{1-m}{2}}$. For the mass term, now we have $[\mu] = L^{-1}$, which is thus not a mass, however, one can define $\mu = \frac{c}{\hbar}\hat\mu$ so that now $\hat\mu$ is. That is analogous to what we did for Klein–Gordon field.

The corresponding total Lagrangian will be

$$\boldsymbol{L} = |e|\left(\tfrac{1}{2\kappa}\left(R - 2\Lambda\right) - \tfrac{i\hbar}{2}\bar\psi\gamma^a\nabla_a\psi + \tfrac{i\hbar}{2}\nabla_a\bar\psi\gamma^a\psi - \mathrm{c}\hat\mu\bar\psi\psi\right)\,d\sigma \tag{2.7.107}$$

which is called the *Dirac–Einstein Lagrangian* and it describes interactions between a spinor field ψ and the gravitational field described by the spin frame e_a^μ (which induces a metric g on the spacetime M). Let us stress that here the metric field is not a fundamental field but it is a composite field which is induced by the spin frame which is to be considered the fundamental field.

The variation of the Dirac Lagrangian is

$$
\begin{aligned}
\delta\boldsymbol{L}_D = &-\hbar\tfrac{|e|}{2}\Big[-\delta e_a^\mu e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) + \delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi - i\delta\left(\nabla_a\bar\psi\right)\gamma^a\psi + i\bar\psi\gamma^a\delta\left(\nabla_a\psi\right) + \mu\left(\delta\bar\psi\psi + \bar\psi\delta\psi\right)\Big]\,d\sigma = \\
= &-\hbar\tfrac{|e|}{2}\Big[\delta e_a^\mu\left(-e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) + i\left(\bar\psi\gamma^a\nabla_b\psi - \nabla_b\bar\psi\gamma^a\psi\right)e_\mu^b\right) + \delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi + \mu\left(\delta\bar\psi\psi + \bar\psi\delta\psi\right) + \\
&+ i\left(-\nabla_a\bar\psi\gamma^a\delta\psi + \delta\bar\psi\gamma^a\nabla_a\psi\right) - \tfrac{i}{4}\bar\psi\left(\gamma_a\gamma_c\gamma_d - \gamma_d\gamma_c\gamma_a\right)\psi e^{a\mu}\delta\Gamma_\mu^{cd}\Big]\,d\sigma - \hbar\nabla_a\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^a\delta\psi - \delta\bar\psi\gamma^a\psi\right)\Big]\,d\sigma = \\
= &-\hbar\tfrac{|e|}{2}\Big[\delta e_a^\mu\left(-e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) + i\left(\bar\psi\gamma^a\nabla_b\psi - \nabla_b\bar\psi\gamma^a\psi\right)e_\mu^b\right) + 2\left(\delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi\right) - \tfrac{i}{2}\bar\psi\gamma_{acd}\psi\delta\Gamma_\mu^{cd}e^{a\mu}\Big]\,d\sigma + \\
&- \hbar\nabla_a\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^a\delta\psi - \delta\bar\psi\gamma^a\psi\right)\Big]\,d\sigma = \\
= &-\hbar\tfrac{|e|}{2}\Big[\delta e_a^\mu\left(-e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) + i\left(\bar\psi\gamma^a\nabla_b\psi - \nabla_b\bar\psi\gamma^a\psi\right)e_\mu^b\right) + 2\left(\delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi\right) + \\
&- \tfrac{i}{2}\bar\psi\gamma_{acd}\psi\left(-\delta_b^a e^{c[\alpha}e^{d\beta]} + \delta_b^c e^{d[\alpha}e^{a\beta]} + \delta_b^d e^{c[\beta}e^{a\alpha]}\right)\nabla_\alpha\delta e_\beta^b\Big]\,d\sigma - \hbar\nabla_a\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^a\delta\psi - \delta\bar\psi\gamma^a\psi\right)\Big]\,d\sigma = \\
= &-\hbar\tfrac{|e|}{2}\Big[\delta e_a^\mu\left(-e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) + i\left(\bar\psi\gamma^a\nabla_b\psi - \nabla_b\bar\psi\gamma^a\psi\right)e_\mu^b\right) + 2\left(\delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi\right) + \\
&- \tfrac{i}{2}\bar\psi\gamma_{acd}\psi\left(-\delta_b^a e^{c[\alpha}e^{d\beta]} + \delta_b^a e^{c[\alpha}e^{d\beta]} - \delta_b^a e^{c[\beta}e^{d\alpha]}\right)\nabla_\alpha\delta e_\beta^b\Big]\,d\sigma - \hbar\nabla_a\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^a\delta\psi - \delta\bar\psi\gamma^a\psi\right)\Big]\,d\sigma = \\
= &-\hbar\tfrac{|e|}{2}\Big[-\delta e_a^\mu\left(e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) - i\left(\bar\psi\gamma^a\nabla_b\psi - \nabla_b\bar\psi\gamma^a\psi\right)e_\mu^b - \tfrac{i}{2}\nabla_\alpha\left(\bar\psi\gamma_{bcd}\psi\right)\eta^{ca}e^{d\alpha}e_\mu^b\right) + 2\left(\delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi\right)\Big]\,d\sigma + \\
&- \hbar\nabla_\alpha\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^\alpha\delta\psi - \delta\bar\psi\gamma^\alpha\psi\right) - i\tfrac{|e|}{4}\left(\bar\psi\gamma_{bcd}\psi\right)\eta^{ca}e^{d\alpha}e_\mu^b\delta e_a^\mu\Big]\,d\sigma = \\
= &-|e|T_\mu^a\delta e_a^\mu\,d\sigma - \hbar|e|\left(\delta\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\delta\psi\right)\,d\sigma - \hbar\nabla_\alpha\Big[i\tfrac{|e|}{2}\left(\bar\psi\gamma^\alpha\delta\psi - \delta\bar\psi\gamma^\alpha\psi\right) - i\tfrac{|e|}{4}\left(\bar\psi\gamma_{bcd}\psi\right)\eta^{ca}e^{d\alpha}e_\mu^b\delta e_a^\mu\Big]\,d\sigma
\end{aligned}
\tag{2.7.108}
$$

where we set $\mathbb{D}\psi := i\gamma^c\nabla_c\psi + \mu\psi$, $\mathbb{D}\bar\psi := -i\nabla_c\bar\psi\gamma^c + \mu\bar\psi$ for the *Dirac operator* and

$$T_\mu^a := -\tfrac{\hbar}{2}\left(e_\mu^a\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) - i\left(\bar\psi\gamma^a\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^a\psi\right) - \tfrac{i}{2}\nabla_\alpha\left(\bar\psi\gamma_{bcd}\psi\right)\eta^{ca}e^{d\alpha}e_\mu^b\right) \tag{2.7.109}$$

for the energy-momentum stress tensor of spinors and we used the relations

$$
\begin{cases}
\delta\left(\nabla_a \psi\right) = \delta e_a^\epsilon e_\epsilon^d \nabla_d \psi + \nabla_a \delta\psi - \tfrac{1}{4} e_a^\mu \delta\Gamma_\mu^{cd} \gamma_c \gamma_d \psi \\
\delta\left(\nabla_a \bar\psi\right) = \delta\left(\nabla_a \psi^\dagger\right)\gamma_0 = \left(\delta e_a^\epsilon e_\epsilon^d \nabla_d \psi^\dagger + \nabla_a \delta\psi^\dagger - \tfrac{1}{4}\psi^\dagger \gamma_d^\dagger \gamma_c^\dagger e_a^\mu \delta\Gamma_\mu^{cd}\right)\gamma_0 = \delta e_a^\epsilon e_\epsilon^d \nabla_d \bar\psi + \nabla_a \delta\bar\psi - \tfrac{1}{4}\bar\psi \gamma_d \gamma_c e_a^\mu \delta\Gamma_\mu^{cd}
\end{cases}
\tag{2.7.110}
$$

as well as

$$
\begin{aligned}
\delta\Gamma_\mu^{cd} e^{a\mu} &= e_\alpha^c\left(\underline{-\delta e_b^\alpha \Gamma_\mu^{bd}} + \delta\Gamma_{\beta\mu}^\alpha e^{d\beta} + \underline{\Gamma_{\beta\mu}^\alpha \delta e^{d\beta}} + \underline{\partial_\mu \delta e^{d\alpha}}\right)e^{a\mu} = e_\alpha^c e^{d\beta}\left(\delta\Gamma_{\beta\mu}^\alpha + e_{b\beta}\nabla_\mu \delta e^{b\alpha}\right)e^{a\mu} = e_\alpha^c e^{d\beta}\left(\delta\Gamma_{\beta\mu}^\alpha - \nabla_\mu \delta e_\beta^b e_b^\alpha\right)e^{a\mu} = \\
&= \tfrac{1}{2}e^{c\alpha}e^{d\beta}\left(-\nabla_\alpha \delta e_\beta^b e_{b\mu} - e_{b\beta}\nabla_\alpha \delta e_\mu^b + \nabla_\mu \delta e_\alpha^b e_{b\beta} + \underline{e_{b\alpha}\nabla_\mu \delta e_\beta^b} + \nabla_\beta \delta e_\mu^b e_{b\alpha} + e_{b\mu}\nabla_\beta \delta e_\alpha^b - 2\underline{e_{b\alpha}\nabla_\mu \delta e_\beta^b}\right)e^{a\mu} = \\
&= \tfrac{1}{2}e^{c\alpha}e^{d\beta}\left(-\underline{\nabla_\alpha \delta e_\beta^b e_{b\mu}} - \underline{\underline{\nabla_\alpha \delta e_\mu^b e_{b\beta}}} + \nabla_\mu \delta e_\alpha^b e_{b\beta} + \underline{\nabla_\beta \delta e_\mu^b e_{b\alpha}} + \underline{\nabla_\beta \delta e_\alpha^b e_{b\mu}} - \nabla_\mu \delta e_\beta^b e_{b\alpha}\right)e^{a\mu} = \\
&= e^{[c\alpha}e^{d]\beta}\left(-\nabla_\alpha \delta e_\beta^b e_{b\mu} + \nabla_\mu \delta e_\alpha^b e_{b\beta} + \nabla_\beta \delta e_\mu^b e_{b\alpha}\right)e^{a\mu} = \left(-e^{c[\alpha}e^{d\beta]}\delta_b^a + \tfrac{1}{2}\delta_b^c e^{d\alpha}e^{a\beta} - \underline{\underline{\tfrac{1}{2}\delta_b^d e^{c\alpha}e^{a\beta}}} + \tfrac{1}{2}\delta_b^d e^{c\beta}e^{a\alpha} - \underline{\underline{\tfrac{1}{2}\delta_b^c e^{d\beta}e^{a\alpha}}}\right)\nabla_\alpha \delta e_\beta^b = \\
&= \left(-\delta_b^a e^{c[\alpha}e^{d\beta]} + \delta_b^c e^{d[\alpha}e^{a\beta]} + \delta_b^d e^{c[\beta}e^{a\alpha]}\right)\nabla_\alpha \delta e_\beta^b
\end{aligned}
\tag{2.7.111}
$$

We also used the identity in the Clifford algebra

$$
\delta\Gamma_\mu^{cd}\left(\gamma_a \gamma_c \gamma_d - \gamma_d \gamma_c \gamma_a\right) = \tfrac{1}{3}\delta\Gamma_\mu^{cd}\left(acd - dca - cad + cda - adc + dac - \textcolor{red}{2(ac)d} + \textcolor{red}{2(dc)a} - \textcolor{red}{2(dc)a} + \textcolor{red}{2(ac)d}\right) = 2\delta\Gamma_\mu^{cd}\gamma_{acd}
\tag{2.7.112}
$$

For future convenience, let us also compute the trace of the energy-momentum stress tensor

$$
T := T_\mu^a e_a^\mu = -\tfrac{\hbar}{2}\left(m\left(\bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \psi\right) - \bar\psi\left(\mathbb{D}\psi - \mu\psi\right) - \left(\mathbb{D}\bar\psi - \mu\bar\psi\right)\psi - \boxed{\tfrac{i}{2}\nabla_\alpha\left(\bar\psi \gamma_{bcd}\psi\right)\eta^{cb}e^{d\alpha}}\right) = -\tfrac{m-1}{2}\hbar\left(\bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \psi\right) - \hat\mu \mathbf{c}\bar\psi \psi
\tag{2.7.113}
$$

which on shell turns out to be proportional to the mass.

The first variation formula of the total Lagrangian (2.7.107) defines the Euler–Lagrange and Liouville parts as

$$
\begin{aligned}
\mathbb{E} &= |e|\left(\tfrac{1}{\kappa}\left(R_\mu^a - \tfrac{1}{2}Re_\mu^a + \Lambda e_\mu^a\right) - T_\mu^a\right)\omega_a^\mu \wedge d\sigma - \hbar|e|\left(\bar\omega\,\mathbb{D}\psi + \mathbb{D}\bar\psi\,\omega\right)\wedge d\sigma \\
\mathbb{F} &= \tfrac{|e|}{\kappa}\left(e_\beta^a g^{\alpha\rho} - e^{a(\rho}\delta_\beta^{\alpha)}\right)\nabla_\rho \omega_a^\beta \wedge d\sigma_\alpha - \hbar\tfrac{|e|}{2}\left[i\left(\bar\psi \gamma^\alpha \omega - \bar\omega \gamma^\alpha \psi\right) - \tfrac{i}{2}\left(\bar\psi \gamma_{bcd}\psi\right)\eta^{ca}e^{d\alpha}e_\mu^b \omega_a^\mu\right]\wedge d\sigma_\alpha
\end{aligned}
\tag{2.7.114}
$$

where we set $\omega_a^\mu := de_a^\mu - d_\lambda e_a^\mu\, dx^\lambda$, $\bar\omega := d\bar\psi - d_\lambda\bar\psi\, dx^\lambda$, $\omega := d\psi - d_\lambda\psi\, dx^\lambda$ for the relevant contact forms. Field equations are then

$$
\begin{cases}
R_\mu^a - \tfrac{1}{2}Re_\mu^a = -\Lambda e_\mu^a + \kappa T_\mu^a \\
i\gamma^c \nabla_c \psi + \mu\psi = 0
\end{cases}
\tag{2.7.115}
$$

that describe the interaction between the spinor field and gravity.

Field equations for the spinor field are $\mathbb{D}\psi = 0$ and $\mathbb{D}\bar\psi = 0$, which are two real-independent equations but two complex-dependent equations. Accordingly, one can regard them as one single complex field equation $\mathbb{D}\psi = 0$.

The automorphisms of the structure bundle $\hat{P}$ act on the configuration bundle by generalised gauge transformations

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
e'^\mu_a = J^\mu_\nu e^\nu_b \ell^b_a(\bar{S}) \\
\psi' = S\psi \quad \Rightarrow \bar{\psi}' = \bar{\psi}\gamma_0 S^\dagger \gamma_0 = \bar{\psi}S^{-1}
\end{cases}
\tag{2.7.116}
$$

We have to prove that $S^{-1} = \gamma_0 S^\dagger \gamma_0$. By definition, an element S of the spin group is a (even) product of unit vectors in the Clifford algebra.
Let $u = u^a \gamma_a \in V$ be a unit vector in the Clifford algebra $\mathcal{C}(V, \eta)$. Then one has

$$
\gamma_0 u^\dagger \gamma_0 = u^a \gamma_0 \gamma_a^\dagger \gamma_0 = u^a \gamma_a = u
\tag{2.7.117}
$$

Thus if $S = u_1 \cdot u_2 \cdot \ldots \cdot u_{2k}$ then

$$
\gamma_0 S^\dagger \gamma_0 = u_{2k} \cdot \ldots \cdot u_2 \cdot u_1
\tag{2.7.118}
$$

Hence one has

$$
S\gamma_0 S^\dagger \gamma_0 = u_1 \cdot u_2 \cdot \ldots \cdot u_{2k} u_{2k} \cdot \ldots \cdot u_2 \cdot u_1 = (-1)^{2k}\eta(u_1, u_1)\eta(u_2, \mathbf{u}_2)\ldots\eta(u_{2k}, u_{2k}) = \pm\mathbb{I} \quad \Rightarrow S^{-1} = \pm\gamma_0 S^\dagger \gamma_0
\tag{2.7.119}
$$

If S in the connected component to the identity, one can find a curve $S(t)$ in the spin group such that $S(0) = \mathbb{I}$ and $S(1) = S$. The inverse S^{-1} is a continuous function, the inverse of $\mathbb{I}$ is $\mathbb{I}$, a continuous function is locally constant, hence one has

$$
S^{-1}(t) = \pm\gamma_0 S^\dagger(t)\gamma_0 \quad \Rightarrow \mathbb{I} = \pm(\gamma_0)^2\mathbb{I} = \pm\mathbb{I} \quad \Rightarrow S^{-1} = \gamma_0 S^\dagger \gamma_0
\tag{2.7.120}
$$

The covariant derivative of spinors has been defined so that it transforms well with respect to generalised gauge transformations. In fact, one has

$$
\begin{aligned}
(\nabla_a\psi)' &= e'^\mu_a \left(\partial'_\mu\psi' - \tfrac{1}{4}\Gamma'^{cd}_\mu \gamma_c\gamma_d\psi'\right) = S\left(\partial_\nu\psi + S^{-1}\partial_\nu S\psi - \tfrac{1}{4}\ell^c_e\left(\Gamma^{ef}_\nu \ell^d_f + d_\nu\bar{\ell}^e_f\eta^{fd}\right)S^{-1}\gamma_c S S^{-1}\gamma_d S\psi\right)e^\nu_b\bar{\ell}^b_a = \\
&= S\left(\partial_\nu\psi + S^{-1}\partial_\nu S\psi - \tfrac{1}{4}\left(\Gamma^{eg}_\nu + d_\nu\bar{\ell}^e_f\eta^{fd}\bar{\ell}^g_d\right)\gamma_e\gamma_g\psi\right)e^\nu_b\bar{\ell}^b_a = S\nabla_b\psi\bar{\ell}^b_a + S\left(S^{-1}\partial_\nu S + \tfrac{1}{4}d_\nu\bar{\ell}^e_f\ell^f_d\eta^{gd}\gamma_e\gamma_g\right)\psi e^\nu_b\bar{\ell}^b_a = \\
&= S\nabla_b\psi\bar{\ell}^b_a
\end{aligned}
\tag{2.7.121}
$$

Here we used the identity $S^{-1}\partial_\nu S = -\tfrac{1}{4}\partial_\nu\bar{\ell}^a_c\ell^c_d\eta^{bd}\gamma_a\gamma_b$ the proof of which is analogous to the proof of (2.A.10).

One starts from a function $S : U \to \mathrm{Spin}(\eta)$, and hence has $\bar{S}\gamma_a S = \gamma_b\bar{\ell}^b_a$. By taking derivative of both sides one has

$$
\begin{aligned}
&d_\nu\bar{S}\gamma_a S + \bar{S}\gamma_a d_\nu S = \gamma_b d_\nu\bar{\ell}^b_a \quad \Rightarrow d_\nu\bar{S}S(\bar{S}\gamma_a S) + (\bar{S}\gamma_a S)\bar{S}d_\nu S = \gamma_b d_\nu\bar{\ell}^b_a \quad \Rightarrow -\bar{S}d_\nu S\gamma_c + \gamma_c\bar{S}d_\nu S = \gamma_b d_\nu\bar{\ell}^b_a\ell^a_c \quad \Rightarrow \\
&\Rightarrow \quad [\gamma_c, \bar{S}d_\nu S] = \gamma_b d_\nu\bar{\ell}^b_a\ell^a_c \quad \Rightarrow [\bar{S}d_\nu S, \gamma_c] = \gamma_b\bar{\ell}^b_a d_\nu\ell^a_c
\end{aligned}
\tag{2.7.122}
$$

Then by the same argument used in (2.A.10) we have

$$
\bar{S}d_\nu S = s^{ab}\gamma_a\gamma_b \qquad s^{ab} = -\tfrac{1}{4}d_\nu\bar{\ell}^a_c\ell^c_d\eta^{bd} \quad \Rightarrow S^{-1}d_\nu S = -\tfrac{1}{4}d_\nu\bar{\ell}^a_c\ell^c_d\eta^{bd}\gamma_a\gamma_b
\tag{2.7.123}
$$

Let us stress that the coefficient $-\tfrac{1}{4}$ does not depend on the dimension.

The bilinear $\bar{\psi}\psi$ is invariant with respect to generalised gauge transformations. We are now able to check that the bilinear $\bar{\psi}\gamma^a\nabla_a\psi$ is invariant as well.

We already checked that γ^a is invariant; see (2.A.33). One has

$$\bar{\psi}'\gamma^a(\nabla_a\psi)' = \bar{\psi}\bar{S}\gamma^a S\nabla_b\psi\bar{\ell}^b_a = \bar{\psi}\gamma^c\ell^a_c\nabla_b\psi\bar{\ell}^b_a = \bar{\psi}\gamma^b\,\nabla_b\psi \tag{2.7.124}$$

Accordingly, the Dirac–Einstein Lagrangian (2.7.107) is invariant under generalised gauge transformations. Since all fields are dynamical, it is in fact a gauge-natural Lagrangian. As usual, invariance with respect to generalised gauge transformations implies the covariance identity

$$p^a_\mu\pounds_\Xi e^\mu_a + p_{ab}{}^{\mu\nu}\pounds_\Xi R^{ab}{}_{\mu\nu} + p\pounds_\Xi\psi + \bar{p}\pounds_\Xi\bar{\psi} + p^a\pounds_\Xi(\nabla_a\psi) + \bar{p}_a\pounds_\Xi(\nabla_a\bar{\psi}) = d_\alpha\left(\xi^\lambda L\right) \tag{2.7.125}$$

which, however, can be proved directly.

We already know that covariance identity holds true for the purely gravitational Lagrangian. Thus we only have to check it for the Dirac Lagrangian (2.7.106). The relevant Lie derivatives are:

$$\begin{aligned}
&\pounds_\Xi e^\mu_a = e^\mu_b(\xi_{(V)})^b{}_a - \nabla_\alpha\xi^\mu e^\alpha_a \\
&\pounds_\Xi\psi = \xi^\alpha\nabla_\alpha\psi + \tfrac{1}{4}\xi^{cd}_{(V)}\gamma_c\gamma_d\psi && \pounds_\Xi\bar{\psi} = \xi^\alpha\nabla_\alpha\bar{\psi} - \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_c\gamma_d \\
&\pounds_\Xi(\nabla_a\psi) = \xi^\alpha\nabla_\alpha\nabla_a\psi + \tfrac{1}{4}\xi^{cd}_{(V)}\gamma_c\gamma_d\nabla_a\psi + (\xi_{(V)})^c{}_a\nabla_c\psi && \pounds_\Xi(\nabla_a\bar{\psi}) = \xi^\alpha\nabla_\alpha\nabla_a\bar{\psi} - \tfrac{1}{4}\xi^{cd}_{(V)}\nabla_a\bar{\psi}\gamma_c\gamma_d + (\xi_{(V)})^c{}_a\nabla_c\bar{\psi}
\end{aligned} \tag{2.7.126}$$

Then we have

$$\begin{aligned}
&p^a_\mu\pounds_\Xi e^\mu_a + p\pounds_\Xi\psi + \bar{p}\pounds_\Xi\bar{\psi} + p^a\pounds_\Xi(\nabla_a\psi) + \bar{p}_a\pounds_\Xi(\nabla_a\bar{\psi}) = \hbar|e|e^a_\mu\left(\tfrac{i}{2}\bar{\psi}\gamma^a\nabla_a\psi - \tfrac{i}{2}\nabla_a\bar{\psi}\gamma^a\psi + \mu\bar{\psi}\psi\right)\left(e^\mu_b\boxed{(\xi_{(V)})^b{}_a} - \nabla_\alpha\xi^\mu e^\alpha_a\right) + \\
&- \hbar|e|\left(-\tfrac{i}{2}\nabla^a\bar{\psi}\gamma_a + \mu\bar{\psi}\right)\left(\xi^\alpha\nabla_\alpha\psi + \tfrac{1}{4}\xi^{cd}_{(V)}\gamma_c\gamma_d\psi\right) - \hbar|e|\left(\xi^\alpha\nabla_\alpha\bar{\psi} - \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_c\gamma_d\right)\left(\tfrac{i}{2}\gamma_a\nabla^a\psi + \mu\psi\right) + \\
&- \tfrac{i\hbar}{2}|e|\left(\xi^\alpha\bar{\psi}\gamma_a\nabla_\alpha\nabla_a\psi + \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_a\gamma_c\gamma_d\nabla^a\psi + (\xi_{(V)})^{ca}\bar{\psi}\gamma_a\nabla_c\psi\right) - \tfrac{i\hbar}{2}|e|\left(-\xi^\alpha\nabla_\alpha\nabla_a\bar{\psi}\gamma_a\psi + \tfrac{1}{4}\xi^{cd}_{(V)}\nabla^a\bar{\psi}\gamma_c\gamma_d\gamma_a\psi - (\xi_{(V)})^{ca}\nabla_c\bar{\psi}\gamma_a\psi\right) = \\
&= L\nabla_\alpha\xi^\alpha - \hbar\mu|e|\left(\xi^\alpha\nabla_\alpha\left(\bar{\psi}\psi\right) + \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_c\gamma_d\psi - \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_c\gamma_d\psi\right) - \tfrac{i\hbar}{2}|e|\left(\underline{\xi^\alpha\nabla_\alpha\bar{\psi}\gamma_a\nabla^a\psi} - \underline{\xi^\alpha\nabla^a\bar{\psi}\gamma_a\nabla_\alpha\psi} - \tfrac{1}{4}\xi^{cd}_{(V)}\nabla^a\bar{\psi}\gamma_a\gamma_c\gamma_d\psi - \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_c\gamma_d\gamma_a\nabla^a\psi\right) + \\
&- \tfrac{i\hbar}{2}|e|\left(\underline{\xi^\alpha\bar{\psi}\gamma_a\nabla_\alpha\nabla_a\psi} + \tfrac{1}{4}\xi^{cd}_{(V)}\bar{\psi}\gamma_a\gamma_c\gamma_d\nabla^a\psi + (\xi_{(V)})^{cd}\bar{\psi}\gamma_d\nabla_c\psi\right) - \tfrac{i\hbar}{2}|e|\left(-\underline{\xi^\alpha\nabla_\alpha\nabla_a\bar{\psi}\gamma_a\psi} + \tfrac{1}{4}\xi^{cd}_{(V)}\nabla^a\bar{\psi}\gamma_c\gamma_d\gamma_a\psi - (\xi_{(V)})^{cd}\nabla_c\bar{\psi}\gamma_d\psi\right) = \\
&= L\nabla_\alpha\xi^\alpha - \hbar|e|\xi^\alpha\nabla_\alpha\left(\mu\bar{\psi}\psi + \tfrac{i}{2}\bar{\psi}\gamma_a\nabla_a\psi - \tfrac{i}{2}\nabla^a\bar{\psi}\gamma_a\psi\right) - \tfrac{i\hbar}{2}|e|\xi^{cd}_{(V)}\left(\tfrac{1}{4}\nabla^a\bar{\psi}[\gamma_c\gamma_d,\gamma_a]\psi - \tfrac{1}{4}\bar{\psi}[\gamma_c\gamma_d,\gamma_a]\nabla^a\psi + \bar{\psi}\gamma_d\nabla_c\psi - \nabla_c\bar{\psi}\gamma_d\psi\right) =^{(2.A.9)} \\
&= L\nabla_\alpha\xi^\alpha - \hbar|e|\xi^\alpha\nabla_\alpha\left(\mu\bar{\psi}\psi + \tfrac{i}{2}\bar{\psi}\gamma_a\nabla_a\psi - \tfrac{i}{2}\nabla^a\bar{\psi}\gamma_a\psi\right) - \tfrac{i\hbar}{2}|e|\xi^{cd}_{(V)}\left(\nabla_c\bar{\psi}\gamma_d\psi - \bar{\psi}\gamma_d\nabla_c\psi + \bar{\psi}\gamma_d\nabla_c\psi - \nabla_c\bar{\psi}\gamma_d\psi\right) = d_\alpha\left(\xi^\alpha L\right)
\end{aligned} \tag{2.7.127}$$

Let us remark that, in order to have covariance of the Dirac(–Einstein) Lagrangian, one is forced to transform spinors and spin frames at the same time. Let us also stress that spinors, as we introduced them here, are gauge-natural objects. They do not transform under spacetime diffeomorphisms (contrary to the usual Minkowski spinors which at least transform under Minkowski isometries). On a general spacetime, generically, there are no isometries at all and spinors became gauge. Once again, the group action of the isometry group is something which is lost on a general curved spacetime, or, to say it better, it is a free token one gets on flat spacetimes only.

As we declared in the beginning, for us, the flat case is a special case to be obtained by specialisation of the general curved space, not something more fundamental to be generalised from it. Accordingly, for us *true spinors* are gauge-natural objects and they have to be treated as such also when special structures can be defined in view of the specific solution under consideration.

As a side comment, since it has a special historical motivation, let us compute the square of the Dirac operator. When Dirac defined spinors on Minkowski space it was important that they somehow can be considered as a square root of a Klein–Gordon field. This feature is *anyway* lost on a generic curved background.

We can compute the square of the Dirac operator.

$$\mathbb{D}^2\psi = \mathbb{D}\left(i\gamma^c\nabla_c\psi + \mu\psi\right) = -\gamma^{[d}\gamma^{c]}\nabla_d\nabla_c\psi + \Box\psi + 2\mu i\gamma^d\nabla_d\psi + \mu^2\psi = -\tfrac{1}{2}\gamma^d\gamma^c[\nabla_d,\nabla_c]\psi + \Box\psi + 2\mu\left(\mathbb{D}\psi - \mu\psi\right) + \mu^2\psi =$$
$$= \tfrac{1}{8}R^{ab}{}_{cd}\gamma^c\gamma^d\gamma_a\gamma_b\psi + 2\mu\mathbb{D}\psi + \left(\Box\psi - \mu^2\psi\right) \tag{2.7.128}$$

On Minkowski space, one has no curvature and a solution ψ of Dirac equation is also a solution of the Klein–Gordon operator. Let us remark that the box operator for spinor fields is defined as

$$\Box\psi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = \eta^{ab}\nabla_a\nabla_b\psi = \eta^{ab}e_b^\beta e_a^\alpha\nabla_\alpha\left(d_\beta\psi - \tfrac{1}{4}\Gamma_\beta^{cd}\gamma_c\gamma_d\psi\right) =$$
$$= g^{\alpha\beta}\left(d_{\alpha\beta}\psi - \tfrac{1}{4}\Gamma_\beta^{cd}\gamma_c\gamma_d\left(\nabla_\alpha\psi + \tfrac{1}{4}\Gamma_\alpha^{ef}\gamma_e\gamma_f\psi\right) - \tfrac{1}{4}d_\alpha\Gamma_\beta^{cd}\gamma_c\gamma_d\psi - \tfrac{1}{4}\Gamma_\alpha^{cd}\gamma_c\gamma_d\nabla_\beta\psi - \{g\}_{\beta\alpha}^\epsilon\nabla_\epsilon\psi\right) = {}^{(2.7.104)}$$
$$= g^{\alpha\beta}\left(d_{\alpha\beta}\psi - \{g\}_{\beta\alpha}^\epsilon\nabla_\epsilon\psi\right) + g^{\alpha\beta}\left(-\tfrac{1}{2}\Gamma_\beta^{cd}\gamma_c\gamma_d\nabla_\alpha\psi + \tfrac{1}{4}\boxed{\left(\Gamma_{c\alpha}^a\Gamma_\beta^{cb} - \Gamma_{c\beta}^a\Gamma_\alpha^{cb}\right)}\gamma_a\gamma_b\psi - \tfrac{1}{4}d_\alpha\Gamma_\beta^{ab}\gamma_a\gamma_b\psi\right) \tag{2.7.129}$$

Again, this reduces to the box operator on each component ψ^i of the spinor field only in Minkowski space, in Cartesian coordinates, with an orthonormal frame. In general, the splitting is not even covariant since the operator $g^{\alpha\beta}\left(d_{\alpha\beta}\psi - \{g\}_{\beta\alpha}^\epsilon\nabla_\epsilon\psi\right)$ is not covariant with respect to generalised gauge transformations (and in general pure diffeomorphisms do not even act on spinor fields).

Even if this property is a historical motivation for Dirac equation, it does not survive on a general curved background.

Conformal gravity

A *conformal structure* on a spacetime M is an equivalence class $[g]$ of metrics (of signature η) with respect to conformal equivalence relation. Two metrics g, g' on M are said to be *conformally equivalent* iff there exists a function $\omega : M \to \mathbb{R}$ such that

$$g' = e^{\omega(x)} \cdot g \tag{2.7.130}$$

By a conformal structure, one can define angles but not distances. Distances depend on the representative of the conformal class $[g]$; different representatives define different distances and lengths for the same curve $\gamma : \mathbb{R} \to M$. On the contrary, (for strictly Riemannian signature) any representative of $g' \in [g]$ defines the same angle θ between two vectors $v, w \in T_xM$:

$$\cos(\theta) = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(w,w)}} = \frac{g'(v,w)}{\sqrt{g'(v,v)}\sqrt{g'(w,w)}} \tag{2.7.131}$$

Analogously, in the Lorentzian signature, the notion of time-like ($g(v,v) < 0$), light-like ($g(v,v) = 0$), and space-like ($g(v,v) > 0$) just depend on the conformal structure, not on the specific representative chosen. Also the set of light-like vectors, which is also called the *light cone* $C = \{v \in TM : g(v,v) = 0\} \subset TM$, just depends on the conformal structure, being the same for all its representatives.

We want to define a field theory for conformal stuctures. In such a theory, one can use fields $g_{\mu\nu}$ which transform as

$$g'_{\mu\nu}(x') = e^{\omega(x)} \, \bar{J}^\alpha_\mu g_{\alpha\beta}(x) \bar{J}^\beta_\nu \tag{2.7.132}$$

This is, of course, not a natural object. The transformation laws depend on gauge transformations parameterised by the local function $\omega : U \to \mathbb{R}$. Accordingly, here fields are gauge-natural objects.

In view of transformation laws, one defines the left group action

$$\lambda : \mathbb{R} \times \mathrm{GL}(m) \times L \to L : (\omega, J, g_{ab}) \mapsto g'_{ab} = e^\omega \, \bar{J}^c_a g_{cd} \bar{J}^d_b \tag{2.7.133}$$

where L is the set of non-degenerate bilinear forms of signature η, endowed with coordinates g_{ab}.

Thus one can start from a structure bundle $\mathcal{P} = (P, M, \pi, \mathbb{R})$ and define the configuration bundle as

$$W(\mathcal{P}) = (P \times_M L(M)) \times_\lambda L \tag{2.7.134}$$

with coordinates $(x^\mu, g_{\mu\nu})$.

The structure bundles $\mathcal{P}$ have fibered coordinates (x, l) and transition functions are in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ l' = \omega(x) + l \end{cases} \tag{2.7.135}$$

Because of the particular group $G = (\mathbb{R}, +)$ chosen, transition functions are principal and at the same time affine transformations. Then the structure bundle $\mathcal{P}$ is principal as usual, though in this case it is also an affine bundle. This combination is particularly interesting. The bundle $\mathcal{P}$ has global sections since it is affine, and it is trivial since it has global sections and it is principal.

Accordingly, structure bundles for the group $(\mathbb{R}, +)$ are always trivial. Then all associated bundles, including $W(\mathcal{P})$, are trivial as well. That, in turn, means that any global sections of $W(\mathcal{P})$ can be represented by a global metric on M.

A gauge transformation (which, in this case, is also called a *Weyl conformal transformation*) on the structure bundle is generated by

$$\Xi = \xi^\mu(x)\partial_\mu + \zeta(x)\rho \tag{2.7.136}$$

where $\rho = \partial_l$ is the usual right-invariant pointwise basis for vertical vectors on $\mathcal{P}$. The Lie derivatives of fields are in the form

$$\pounds_\Xi g_{\mu\nu} = \nabla_\mu \xi^\alpha g_{\alpha\nu} + g_{\nu\alpha}\nabla_\nu \xi^\alpha - \zeta g_{\mu\nu} \tag{2.7.137}$$

The Hilbert Lagrangian is not gauge covariant with respect to Weyl conformal transformations. However, we can consider a Lagrangian quadratic in the curvature, i.e. in the form

$$\boldsymbol{L} = \tfrac{\sqrt{g}}{2\kappa}\left(aR_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + bR_{\beta\nu}R^{\beta\nu} + cR^2\right) d\sigma = \tfrac{\sqrt{g}}{2\kappa}\left(ag^{\alpha\gamma}g^{\beta\delta}g^{\mu\rho}g^{\nu\sigma} + bg^{\alpha\mu}g^{\gamma\rho}g^{\delta\beta}g^{\sigma\nu} + cg^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}\right)R_{\alpha\beta\mu\nu}R_{\gamma\delta\rho\sigma}\, d\sigma \tag{2.7.138}$$

Let us define $K := R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ and $Q := R_{\beta\nu}R^{\beta\nu}$ so that the more general Lagrangian quadratic in the curvature can be split as

$$\boldsymbol{L} = \tfrac{\sqrt{g}}{2\kappa}\left(aK + bQ + cR^2\right)d\sigma = a\boldsymbol{L}_K + b\boldsymbol{L}_Q + c\boldsymbol{L}_R \tag{2.7.139}$$

By variation of the Lagrangian $\boldsymbol{L}_K$, one obtains

$$\delta L_K = \tfrac{\sqrt{g}}{2\kappa}\left(-\tfrac{1}{2}Kg_{\mu\nu} + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4\nabla_\lambda\nabla_\epsilon R^\lambda{}_{\mu\nu}{}^\epsilon\right)\delta g^{\mu\nu} + \nabla_\lambda\left[\tfrac{\sqrt{g}}{2\kappa}\left(-4\nabla_\epsilon R^{\lambda\mu\nu\epsilon}\delta g_{\mu\nu} + 4R^{\rho(\mu\nu)\lambda}\nabla_\rho\delta g_{\mu\nu}\right)\right] \tag{2.7.140}$$

For the Lagrangian $\boldsymbol{L}_Q$, one obtains

$$\begin{aligned}
\delta L_Q = {}&\tfrac{\sqrt{g}}{2\kappa}\left(2R_{\mu\alpha}R_\nu{}^\alpha - \tfrac{1}{2}Qg_{\mu\nu} - 2\nabla_\lambda\nabla_{(\mu}R_{\nu)}{}^\lambda + \Box R_{\mu\nu} + \tfrac{1}{2}\Box Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&+\nabla_\lambda\left[\tfrac{\sqrt{g}}{2\kappa}\left(\left(-2\nabla_{(\mu}R^\lambda{}_{\nu)} + \nabla^\lambda R_{\mu\nu} + \tfrac{1}{2}\nabla^\lambda Rg_{\mu\nu}\right)\delta g_{\mu\nu} + \left(2R^{\rho\nu}g^{\lambda\mu} - R^{\mu\nu}g^{\lambda\rho} - R^{\lambda\rho}g^{\mu\nu}\right)\nabla_\rho\delta g_{\mu\nu}\right)\right]
\end{aligned} \tag{2.7.141}$$

Finally, by varying the Lagrangian $\boldsymbol{L}_R$, one has

$$\delta L_R = \tfrac{\sqrt{g}}{2\kappa}\left(2RR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^2 + 2\Box Rg_{\mu\nu} - 2\nabla_{(\mu}\nabla_{\nu)}R\right)\delta g^{\mu\nu} + \nabla_\lambda\left[\tfrac{\sqrt{g}}{2\kappa}\left(2\nabla_\eta R\left(g^{\eta\lambda}g^{\mu\nu} - g^{\eta(\nu}g^{\mu)\lambda}\right)\delta g_{\mu\nu} + 2R\left(g^{\rho(\mu}g^{\nu)\lambda} - g^{\lambda\rho}g^{\mu\nu}\right)\nabla_\rho\delta g_{\mu\nu}\right)\right] \tag{2.7.142}$$

Field equations for the total quadratic Lagrangian are then

$$\begin{aligned}
&a\left(-\tfrac{1}{2}Kg_{\mu\nu} + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4\nabla_\lambda\nabla_\epsilon R^\lambda{}_{(\mu\nu)}{}^\epsilon\right) + b\left(2R_{\mu\alpha}R_\nu{}^\alpha - \tfrac{1}{2}Qg_{\mu\nu} - 2\nabla_\lambda\nabla_{(\mu}R_{\nu)}{}^\lambda + \Box R_{\mu\nu} + \tfrac{1}{2}\Box Rg_{\mu\nu}\right) + \\
&+ c\left(2RR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^2 + 2\Box Rg_{\mu\nu} - 2\nabla_{(\mu}\nabla_{\nu)}R\right) = 0
\end{aligned} \tag{2.7.143}$$

Before simplifying it, let us prove some identities. First of all, in view of second Bianchi identities

$$\nabla_\alpha R^\alpha{}_{\beta\mu\nu} = \nabla_\mu R_{\beta\nu} - \nabla_\nu R_{\beta\mu} \quad \Rightarrow \nabla_\nu\nabla_\alpha R^\alpha{}_{\beta\mu}{}^\nu = \nabla_\nu\nabla_\mu R_\beta{}^\nu_. - \Box R_{\beta\mu} \quad \Rightarrow \nabla_\nu\nabla_\alpha R^\alpha{}_{(\beta\mu)}{}^\nu_. = \nabla_\alpha\nabla_\nu R^\alpha{}_{(\beta\mu)}{}^\nu_. = \nabla_\nu\nabla_{(\mu}R_{\beta)}{}^\nu_. - \Box R_{\beta\mu} \tag{2.7.144}$$

Also, one has

$$[\nabla_\mu, \nabla_\nu]R_{\alpha\beta} = -R^\lambda{}_{\alpha\mu\nu}R_{\lambda\beta} - R^\lambda{}_{\beta\mu\nu}R_{\alpha\lambda} \tag{2.7.145}$$

from which one can easily expand to

$$\nabla_\beta\nabla_\mu R_\nu{}^\beta_. = \nabla_\mu\nabla_\beta R_\nu{}^\beta_. + R^\lambda{}_{\nu\mu}{}^\beta_. R_{\lambda\beta} + R^\lambda{}_\mu R_{\nu\lambda} = \tfrac{1}{2}\nabla_\mu\nabla_\nu R + R^\lambda{}_{\nu\mu}{}^\beta_. R_{\lambda\beta} + R^\lambda{}_\mu R_{\nu\lambda} \quad \Rightarrow \nabla_\beta\nabla_{(\mu}R_{\nu)}{}^\beta_. = \tfrac{1}{2}\nabla_{(\mu}\nabla_{\nu)}R + R^\lambda{}_{(\mu\nu)}{}^\beta_. R_{\lambda\beta} + R^\lambda{}_\mu R_{\nu\lambda} \tag{2.7.146}$$

Then, field equations (2.7.143) can be recast as

$$\begin{aligned}
&a\left(2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - \tfrac{1}{2}Kg_{\mu\nu} - 4R_{\alpha\mu}R_\nu{}^\alpha_.\right) - \tfrac{1}{2}bQg_{\mu\nu} + c\left(2RR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^2\right) - 2(2a+b)R^\alpha{}_{(\mu\nu)}{}^\beta_. R_{\alpha\beta} + \\
&+ (4a+b)\Box R_{\mu\nu} + \left(\tfrac{1}{2}b + 2c\right)\Box Rg_{\mu\nu} - (2c + 2a + b)\nabla_{(\mu}\nabla_{\nu)}R = 0
\end{aligned} \tag{2.7.147}$$

The quadratic Lagrangian $\boldsymbol{L}$ is generally covariant and we can focus on pure Weyl conformal transformations to check if and when such a Lagrangian is a gauge covariant Lagrangian. Since that result will depend on dimension, we shall eventually discuss the case $m = 4$.

Let us consider two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = e^{\omega} g_{\mu\nu}$ which are conformally equivalent. Their Christoffel symbols are

$$\{\tilde{g}\}^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2} \left(g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^{\alpha}_{(\mu}\delta^{\epsilon}_{\beta)} \right) \overset{*}{\nabla}_{\epsilon}\omega = \{g\}^{\alpha}_{\beta\mu} + K^{\alpha}_{\beta\mu} \tag{2.7.148}$$

where we set $K^{\alpha}_{\beta\mu} := -\tfrac{1}{2} \left(g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^{\alpha}_{(\mu}\delta^{\epsilon}_{\beta)} \right) \overset{*}{\nabla}_{\epsilon}\omega$, which is a tensor.
The corresponding Riemann tensors are

$$\tilde{R}^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu}K^{\alpha}_{\beta\nu} - \nabla_{\nu}K^{\alpha}_{\beta\mu} + K^{\alpha}_{\epsilon\mu}K^{\epsilon}_{\beta\nu} - K^{\alpha}_{\epsilon\nu}K^{\epsilon}_{\beta\mu} \tag{2.7.149}$$

$$\begin{aligned}
\tilde{R}_{\alpha\beta\mu\nu} =& e^{\omega} \Big(R_{\alpha\beta\mu\nu} - \tfrac{1}{2}\left(\delta^{\epsilon}_{\alpha}g_{\beta\nu} - 2g_{\alpha(\nu}\delta^{\epsilon}_{\beta)} \right)\nabla_{\mu\epsilon}\omega + \tfrac{1}{2}\left(\delta^{\epsilon}_{\alpha}g_{\beta\mu} - 2g_{\alpha(\mu}\delta^{\epsilon}_{\beta)} \right)\nabla_{\nu\epsilon}\omega + \\
&+ \tfrac{1}{4}\left(\delta^{\rho}_{\alpha}g_{\epsilon\mu} - 2g_{\alpha(\mu}\delta^{\rho}_{\epsilon)} \right)\left(g^{\epsilon\sigma}g_{\beta\nu} - 2\delta^{\epsilon}_{(\nu}\delta^{\sigma}_{\beta)} \right)\nabla_{\rho}\omega\nabla_{\sigma}\omega - \tfrac{1}{4}\left(\delta^{\rho}_{\alpha}g_{\epsilon\nu} - 2g_{\alpha(\nu}\delta^{\rho}_{\epsilon)} \right)\left(g^{\epsilon\sigma}g_{\beta\mu} - 2\delta^{\epsilon}_{(\mu}\delta^{\sigma}_{\beta)} \right)\nabla_{\rho}\omega\nabla_{\sigma}\omega \Big) = \\
=& e^{\omega} \Big(R_{\alpha\beta\mu\nu} - \tfrac{1}{2}\left(\delta^{\epsilon}_{\alpha}g_{\beta\nu} - g_{\alpha\nu}\delta^{\epsilon}_{\beta} - g_{\alpha\beta}\delta^{\epsilon}_{\nu} \right)\nabla_{\mu\epsilon}\omega + \tfrac{1}{2}\left(\delta^{\epsilon}_{\alpha}g_{\beta\mu} - g_{\alpha\mu}\delta^{\epsilon}_{\beta} - g_{\alpha\beta}\delta^{\epsilon}_{\mu} \right)\nabla_{\nu\epsilon}\omega + \\
&+ \tfrac{1}{4}\left(\delta^{\rho}_{\alpha}\delta^{\sigma}_{\mu}g_{\beta\nu} - \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}g_{\nu\mu} - \delta^{\rho}_{\alpha}\delta^{\sigma}_{\nu}g_{\beta\mu} - g_{\alpha\mu}g^{\rho\sigma}g_{\beta\nu} + \delta^{\rho}_{\nu}\delta^{\sigma}_{\beta}g_{\alpha\mu} + \delta^{\rho}_{\beta}\delta^{\sigma}_{\nu}g_{\alpha\mu} - \delta^{\rho}_{\mu}\delta^{\sigma}_{\alpha}g_{\beta\nu} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\beta}g_{\alpha\nu} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}g_{\alpha\beta} \right)\nabla_{\rho}\omega\nabla_{\sigma}\omega + \\
&- \tfrac{1}{4}\left(\delta^{\rho}_{\alpha}\delta^{\sigma}_{\nu}g_{\beta\mu} - \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}g_{\mu\nu} - \delta^{\rho}_{\alpha}\delta^{\sigma}_{\mu}g_{\beta\nu} - g_{\alpha\nu}g^{\rho\sigma}g_{\beta\mu} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\beta}g_{\alpha\nu} + \delta^{\rho}_{\beta}\delta^{\sigma}_{\mu}g_{\alpha\nu} - \delta^{\rho}_{\nu}\delta^{\sigma}_{\alpha}g_{\beta\mu} + \delta^{\rho}_{\nu}\delta^{\sigma}_{\beta}g_{\alpha\mu} + \delta^{\rho}_{\nu}\delta^{\sigma}_{\mu}g_{\alpha\beta} \right)\nabla_{\rho}\omega\nabla_{\sigma}\omega \Big) = \\
=& e^{\omega} \Big(R_{\alpha\beta\mu\nu} + 2\delta^{\epsilon}_{[\alpha}g_{\beta][\mu}\nabla_{\nu]\epsilon}\omega + \left(\delta^{\rho}_{[\alpha}g_{\beta][\nu}\delta^{\sigma}_{\mu]} - \tfrac{1}{2}g^{\rho\sigma}g_{\nu[\beta}g_{\alpha]\mu} \right)\nabla_{\rho}\omega\nabla_{\sigma}\omega \Big)
\end{aligned} \tag{2.7.150}$$

This is relatively ugly, though it improves a bit when one considers the Ricci tensor

$$\tilde{R}_{\beta\nu} = R_{\beta\nu} - \tfrac{1}{2}g_{\beta\nu}\square\omega - \tfrac{m-2}{2}\nabla_{\nu\beta}\omega + \tfrac{m-2}{4}\nabla_{\beta}\omega\nabla_{\nu}\omega - \tfrac{m-2}{4}g_{\nu\beta}\nabla_{\rho}\omega\nabla^{\rho}\omega \tag{2.7.151}$$

and the Ricci scalar

$$e^{\omega}\tilde{R} = R - (m-1)\square\omega - \tfrac{(m-1)(m-2)}{4}\nabla_{\rho}\omega\nabla^{\rho}\omega \tag{2.7.152}$$

For later convenience, we should also write these in terms of the conformal factor $\varphi = e^{\omega}$, i.e. $\omega = \ln\varphi$. For that, just remember that $\nabla_{\mu}\omega = \varphi^{-1}\nabla_{\mu}\varphi$ and

$$\nabla_{\mu\nu}\omega = \varphi^{-1}\nabla_{\mu\nu}\varphi - \varphi^{-2}\nabla_{\mu}\varphi\nabla_{\nu}\varphi \qquad\qquad \square\omega = \varphi^{-1}\square\varphi - \varphi^{-2}\nabla_{\mu}\varphi\nabla^{\mu}\varphi \tag{2.7.153}$$

Then, we have the Ricci tensor

$$\tilde{R}_{\beta\nu} = R_{\beta\nu} - \tfrac{1}{2\varphi}g_{\beta\nu}\square\varphi - \tfrac{m-2}{2\varphi}\nabla_{\beta\nu}\varphi + \left(\tfrac{m-2}{2\varphi^2} + \tfrac{m-2}{4\varphi^2} \right)\nabla_{\beta}\varphi\nabla_{\nu}\varphi + \left(\tfrac{1}{2\varphi^2} - \tfrac{m-2}{4\varphi^2} \right)\nabla_{\rho}\varphi\nabla^{\rho}\varphi g_{\nu\beta} = R_{\beta\nu} - \tfrac{1}{2\varphi}g_{\beta\nu}\square\varphi - \tfrac{m-2}{2\varphi}\nabla_{\beta\nu}\varphi + \tfrac{3(m-2)}{4\varphi^2}\nabla_{\beta}\varphi\nabla_{\nu}\varphi - \tfrac{m-4}{4\varphi^2}\nabla_{\rho}\varphi\nabla^{\rho}\varphi g_{\nu\beta} \tag{2.7.154}$$

and the Ricci scalar

$$\varphi\tilde{R} = R - \left(\tfrac{m}{2\varphi} + \tfrac{m-2}{2\varphi} \right)\square\varphi + \left(\tfrac{3(m-2)}{4\varphi^2} - \tfrac{m(m-4)}{4\varphi^2} \right)\nabla_{\rho}\varphi\nabla^{\rho}\varphi = R - \tfrac{m-1}{\varphi}\square\varphi - \tfrac{(m-1)(m-6)}{4\varphi^2}\nabla_{\rho}\varphi\nabla^{\rho}\varphi \tag{2.7.155}$$

However, even if the expression is not too simple, the action of Weyl conformal transformations on the Riemann tensor (2.7.150) are enough to compute the Lie derivative. A generator of pure Weyl conformal transformations is $\Xi = \zeta(x)\rho$ and the relevant Lie derivatives are

$$\pounds_{\Xi}g^{\mu\nu} = \zeta g^{\mu\nu} \qquad\qquad \pounds_{\Xi}R_{\alpha\beta\mu\nu} = -\zeta R_{\alpha\beta\mu\nu} + \delta^{\epsilon}_{[\alpha}g_{\beta]\nu}\nabla_{\mu\epsilon}\zeta - \delta^{\epsilon}_{[\alpha}g_{\beta]\mu}\nabla_{\nu\epsilon}\zeta \tag{2.7.156}$$

For future reference, we can also compute the Lie derivative of the Ricci tensor and the Ricci scalar

$$\pounds_\Xi R_{\beta\nu} = \tfrac{1}{2}g_{\beta\nu}\Box\zeta + \tfrac{m-2}{2}\nabla_{\nu\beta}\zeta \qquad \pounds_\Xi R = \zeta R + (m-1)\Box\zeta \tag{2.7.157}$$

The covariance identity associated to Weyl conformal transformations to be proven is

$$p_{\mu\nu}\pounds_\Xi g^{\mu\nu} + p^{\alpha\beta\mu\nu}\pounds_\Xi R_{\alpha\beta\mu\nu} = 0 \tag{2.7.158}$$

Using the *naive* momenta which have been obtained in the variation of the Lagrangian, we obtain

$$\begin{aligned}
\left(-\tfrac{m}{2}+4\right)L\zeta - 2L\zeta + 4a\sqrt{g}R^{\beta\nu}\nabla_{\beta\nu}\zeta + b\sqrt{g}\left(R\Box\zeta + (m-2)R^{\beta\nu}\nabla_{\nu\beta}\zeta\right) + c\sqrt{g}R\left(2m\Box\zeta - 2\Box\zeta\right) = \\
= -\tfrac{m-4}{2}L\zeta + \sqrt{g}\left(4a + (m-2)b\right)R^{\beta\nu}\nabla_{\nu\beta}\zeta + \sqrt{g}\left(b + 2c(m-1)\right)R\Box\zeta
\end{aligned} \tag{2.7.159}$$

This must vanish for any ζ. Hence one needs

$$\begin{cases} m = 4 \\ 4a + (m-2)b = 0 \\ b + 2c(m-1) = 0 \end{cases} \tag{2.7.160}$$

Accordingly, Lagrangians which are quadratic in the curvature can be Weyl covariant only in dimension $m = \dim(M) = 4$. In that case

$$\begin{cases} 2a + b = 0 \\ b + 6c = 0 \end{cases} \Rightarrow \begin{cases} b = -6c \\ a = 3c \end{cases} \tag{2.7.161}$$

Thus, in dimension 4, the only covariant quadratic Lagrangian is

$$\boldsymbol{L}_W = 3c\tfrac{\sqrt{g}}{2\kappa}\left(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\beta\nu}R^{\beta\nu} + \tfrac{1}{3}R^2\right)d\sigma = 3c\tfrac{\sqrt{g}}{2\kappa}\,W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu}\,d\sigma \tag{2.7.162}$$

where we set

$$W_{\alpha\beta\mu\nu} := R_{\alpha\beta\mu\nu} - \left(g_{\alpha[\mu}R_{\nu]\beta} - g_{\beta[\mu}R_{\nu]\alpha}\right) + \tfrac{1}{3}Rg_{\alpha[\mu}g_{\nu]\beta} \tag{2.7.163}$$

for the well-known *Weyl tensor*.

In fact, being in dimension 4, we have:

$$\begin{aligned}
W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu} &= \left(R_{\alpha\beta\mu\nu} - g_{\alpha[\mu}R_{\nu]\beta} + g_{\beta[\mu}R_{\nu]\alpha} + \tfrac{1}{3}Rg_{\alpha[\mu}g_{\nu]\beta}\right)\left(R^{\alpha\beta\mu\nu} - g^{\alpha\mu}R^{\nu\beta} + g^{\beta\mu}R^{\nu\alpha} + \tfrac{1}{3}Rg^{\alpha\mu}g^{\nu\beta}\right) = \\
&= R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\beta\nu}R^{\nu\beta} + \tfrac{1}{3}R^2 + \textcolor{red}{(R^{\beta\nu}R_{\nu\beta} - \tfrac{2}{3}R^2)} - \textcolor{red}{(R^{\beta\mu}R_{\mu\beta} + \tfrac{2}{3}R^2)} + \\
&\quad + \textcolor{green}{\tfrac{1}{2}(2R^{\nu\alpha}R_{\nu\alpha} + \tfrac{1}{3}R^2)} - \textcolor{green}{\tfrac{1}{2}R_{\mu\alpha}(2R^{\alpha\mu} + \tfrac{1}{3}Rg^{\alpha\mu})} + \textcolor{green}{\tfrac{1}{6}Rg_{\alpha\mu}(2R^{\alpha\mu} + \tfrac{1}{3}Rg^{\alpha\mu})} - \textcolor{green}{\tfrac{1}{6}Rg_{\alpha\nu}(2R^{\nu\alpha} + \tfrac{1}{3}Rg^{\nu\alpha})} = \\
&= R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\beta\nu}R^{\nu\beta} + \tfrac{1}{3}R^2
\end{aligned} \tag{2.7.164}$$

In dimension $m = 4$, there is one notable quantity in this family, namely the Gauss–Bonnet form

$$\mathbf{G} = \tfrac{\sqrt{g}}{2\kappa}\epsilon_{\alpha\beta\gamma\delta}\mathbf{R}^{\alpha\beta}\wedge\mathbf{R}^{\gamma\delta} = \tfrac{\sqrt{g}}{8\kappa}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma}\,d\sigma = \tfrac{\sqrt{g}}{2\kappa}\left(K - 4Q + R^2\right)d\sigma \tag{2.7.165}$$

where we set $\mathbf{R}^{\alpha\beta} := \frac{1}{2}R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu \wedge dx^\nu$ for the curvature 2-form.

In fact, one has

$$
\begin{aligned}
\tfrac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} &= \tfrac{4!}{4}\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\delta_{\sigma]}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \left(\delta^\mu_{[\alpha}\delta^\nu_{\beta]}\delta^\rho_{[\gamma}\delta^\sigma_{\delta]} + \delta^\mu_{[\alpha}\delta^\rho_{\beta]}\delta^\sigma_{[\gamma}\delta^\nu_{\delta]} + \delta^\mu_{[\alpha}\delta^\sigma_{\beta]}\delta^\nu_{[\gamma}\delta^\rho_{\delta]} + \delta^\nu_{[\alpha}\delta^\rho_{\beta]}\delta^\mu_{[\gamma}\delta^\sigma_{\delta]} + \delta^\nu_{[\alpha}\delta^\sigma_{\beta]}\delta^\rho_{[\gamma}\delta^\mu_{\delta]} + \delta^\rho_{[\alpha}\delta^\sigma_{\beta]}\delta^\mu_{[\gamma}\delta^\nu_{\delta]}\right)R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \\
&= R^{\mu\nu}{}_{\mu\nu}R^{\rho\sigma}{}_{\rho\sigma} + R^{\mu\rho}{}_{\mu\nu}R^{\sigma\nu}{}_{\rho\sigma} + R^{\mu\sigma}{}_{\mu\nu}R^{\nu\rho}{}_{\rho\sigma} + R^{\nu\rho}{}_{\mu\nu}R^{\mu\sigma}{}_{\rho\sigma} + R^{\nu\sigma}{}_{\mu\nu}R^{\rho\mu}{}_{\rho\sigma} + R^{\rho\sigma}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma} = \\
&= R^2 - R^\rho{}_\nu R^\nu{}_\rho - R^\sigma{}_\nu R^\nu{}_\sigma - R^\rho{}_\mu R^\mu{}_\rho - R^\sigma{}_\mu R^\mu{}_\sigma + R^{\rho\sigma}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma} = R^2 - 4R_{\rho\nu}R^{\nu\rho} + R^{\rho\sigma\mu\nu}R_{\mu\nu\rho\sigma} = R^2 - 4Q + K
\end{aligned}
\tag{2.7.166}
$$

Sometimes, in the literature, one sees that the Gauss–Bonnet form is used to cancel the K term in any quadratic Lagrangian. The Gauss–Bonnet form is known to endow identically satisfied field equations (as we shall see in a while), so adding it to a quadratic Lagrangian will not change field equations. And one can fix the coefficient of the linear combination to cancel the term K.

This simplifies the computation of field equations,, since the Lagrangian from K does not need to be varied any longer. However, Gauss–Bonnet form is not conformally invariant (since it is not the $\boldsymbol{L}_W$ which is the *only* quadratic Lagrangian which is conformally invariant). Then one is subtracting from the quadratic Lagrangian a divergence which is not conformally invariant, hence compromising the conformal covariance of the quadratic Lagrangian, if it originally was conformally invariant.

This trick may be ok, if we are interested to field equations only, though it makes then difficult to discuss conservation laws. It is not much different than subtracting non-covariant divergences from a covariant Lagrangian, producing local Lagrangian(s) which are however still producing global field equations.

Since we are interested in conservation laws, we shall not go with this trick. Of course, we shall prove that Gauss–Bonnet form does in fact preserves field equations to support what we here claimed.

Before considering what happens for the Gauss–Bonnet form, let us consider the following identity in dimension 4

$$
0 = \delta^\epsilon_{[\theta}\delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} = \delta^\epsilon_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\nu_\theta\delta^\mu_{[\alpha}\delta^\epsilon_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\rho_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\epsilon_\gamma\delta^\sigma_{\delta]} - \delta^\sigma_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\epsilon_{\delta]}
\tag{2.7.167}
$$

which we can then contract with

$$
\begin{aligned}
&-3!g_{\lambda\epsilon}\left(\delta^\epsilon_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\nu_\theta\delta^\mu_{[\alpha}\delta^\epsilon_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} - \delta^\rho_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\epsilon_\gamma\delta^\sigma_{\delta]} - \delta^\sigma_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\epsilon_{\delta]}\right)R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \\
&= 3!g_{\lambda\epsilon}\left(-\delta^\epsilon_\theta\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} + \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} + \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} + \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]} + \delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]}\right)R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \\
&= -3!g_{\lambda\theta}\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} + 3!\,4g_{\lambda\epsilon}\delta^\mu_\theta\delta^\epsilon_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \\
&= \left(-Kg_{\lambda\theta} + 4Qg_{\lambda\theta} - g_{\lambda\theta}R^2\right) + 4g_{\lambda\epsilon}\delta^\mu_\theta\left(\delta^\epsilon_{[\alpha}\delta^\nu_{\beta]}\delta^\rho_{[\gamma}\delta^\sigma_{\delta]} - \delta^\epsilon_{[\alpha}\delta^\rho_{\beta]}\delta^\nu_{[\gamma}\delta^\sigma_{\delta]} - \delta^\epsilon_{[\alpha}\delta^\sigma_{\beta]}\delta^\rho_{[\gamma}\delta^\nu_{\delta]} + \delta^\rho_{[\alpha}\delta^\sigma_{\beta]}\delta^\epsilon_{[\gamma}\delta^\nu_{\delta]} - \delta^\nu_{[\alpha}\delta^\sigma_{\beta]}\delta^\epsilon_{[\gamma}\delta^\rho_{\delta]} - \delta^\rho_{[\alpha}\delta^\nu_{\beta]}\delta^\epsilon_{[\gamma}\delta^\sigma_{\delta]}\right)R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma} = \\
&= \left(-Kg_{\lambda\theta} + 4Qg_{\lambda\theta} - g_{\lambda\theta}R^2\right) + 4g_{\lambda\epsilon}\delta^\mu_\theta\left(R^{\epsilon\nu}{}_{\mu\nu}R^{\rho\sigma}{}_{\rho\sigma} - R^{\epsilon\rho}{}_{\mu\nu}R^{\nu\sigma}{}_{\rho\sigma} - R^{\epsilon\sigma}{}_{\mu\nu}R^{\rho\nu}{}_{\rho\sigma} + R^{\rho\sigma}{}_{\mu\nu}R^{\epsilon\nu}{}_{\rho\sigma} - R^{\nu\sigma}{}_{\mu\nu}R^{\epsilon\rho}{}_{\rho\sigma} - R^{\rho\nu}{}_{\mu\nu}R^{\epsilon\sigma}{}_{\rho\sigma}\right) = \\
&= \left(-Kg_{\lambda\theta} + 4Qg_{\lambda\theta} - g_{\lambda\theta}R^2\right) + 4\left(RR_{\lambda\theta} + \underline{\underline{R^\rho{}_{\lambda\theta}{}^\nu R_{\nu\rho}}} + \underline{\underline{R^\sigma{}_{\lambda\theta}{}^\nu R_{\nu\sigma}}} + R_\theta{}^{\nu\rho\sigma}R_{\lambda\nu\rho\sigma} - \underline{R_{\lambda\sigma}R^\sigma{}_\theta} - \underline{R_{\lambda\rho}R^\rho{}_\theta}\right) = \\
&= -Kg_{\lambda\theta} + 4Qg_{\lambda\theta} - g_{\lambda\theta}R^2 + 4RR_{\lambda\theta} + 8R^\alpha{}_{(\lambda\theta)}{}^\beta{}_{.}R_{\alpha\beta} + 4R_{\lambda\alpha\beta\gamma}R_\theta{}^{\alpha\beta\gamma} - 8R_{\alpha\lambda}R_\theta{}^\alpha{}_{.} = 0
\end{aligned}
\tag{2.7.168}
$$

By using this identity, we can see directly what we claimed about Gauss-Bonnet form not affecting field equations. In fact, if we specialise the general field equations (2.7.147) to the Gauss–Bonnet case ($a = 1, b = -4, c = 1$) one obtains

$$4R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - Kg_{\mu\nu} - 8R_{\alpha\mu}R_\nu{}^{.\alpha}_{.} + 4Qg_{\mu\nu} + 4RR_{\mu\nu} - g_{\mu\nu}R^2 + 8R^\alpha{}_{(\mu\nu)}{}^{.\beta}_{.}R_{\alpha\beta} = 0 \tag{2.7.169}$$

which, in fact, vanishes as a consequence of identity (2.7.168). Accordingly, as we claimed, field equations of Gauss–Bonnet form in dimension 4 are identically satisfied.

The reason why Gauss-Bonnet form does not affect field equations is that it is locally (though not globally) a pure divergence.

This can be shown directly. It is not easy, nor particularly useful to what we shall do below. However, it is a good practice in tensor calculus, so we shall do it anyway.

Let $e_a = e_a^\mu \partial_\mu$ be a *Vielbein* inducing g and let us denote by e_μ^a the inverse matrix of e_a^μ. Thus we have $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$.

The *spin connection* is defined to be

$$\omega^a{}_{b\mu} = e_\alpha^a \left(\{g\}^\alpha_{\beta\mu} e_b^\beta + \partial_\mu e_b^\alpha \right) \tag{2.7.170}$$

and its curvature is defined to be

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu \tag{2.7.171}$$

One can show by direct computation that $R^{ab}{}_{\mu\nu} = e_\alpha^a e_\beta^b R^{\alpha\beta}{}_{\mu\nu}$ and define the local forms

$$\omega^{ab} := \omega^{ab}{}_\mu \, dx^\mu \qquad\qquad \mathbf{R}^{ab} := \frac{1}{2} R^{ab}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{2.7.172}$$

The Gauss–Bonnet form is defined to be

$$\mathbf{G} := \tfrac{1}{2\kappa}\epsilon_{abcd}\mathbf{R}^{ab} \wedge \mathbf{R}^{cd} = \frac{\sqrt{g}}{2\kappa} \left[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \right] d\sigma \tag{2.7.173}$$

Before proceeding, let us prove some lemmas.

Lemma: The identity $\epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} = -\epsilon_{abcd} \, \omega^a{}_e \wedge d\omega^{eb} \wedge \omega^{cd}$ holds true.

Proof: *one has:*

$$\epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} = \epsilon_{abcd} \, \omega^{eb} \wedge d\omega^a{}_e \wedge \omega^{cd} = \epsilon_{abcd} \, \omega^b{}_e \wedge d\omega^{ea} \wedge \omega^{cd} = -\epsilon_{abcd} \, \omega^a{}_e \wedge d\omega^{eb} \wedge \omega^{cd} \tag{2.7.174}$$

∎

Lemma: The identity $\epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} = \epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge d\omega^{cd}$ holds true.

Proof: *one has:*

$$\epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} = -\frac{1}{4}\epsilon_{abcd}\epsilon^{aehi}\epsilon_{fghi} \, d\omega^{fg} \wedge \omega^b{}_e \wedge \omega^{cd} = \epsilon_{abcd} \, d\omega^{cd} \wedge \omega^a{}_e \wedge \omega^{eb} = \epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge d\omega^{cd} \tag{2.7.175}$$

∎

Lemma: The identity $d\left(\epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} \right) = 3\epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge d\omega^{cd}$ holds true.

Proof: *one has*

$$d\left(\epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} \right) = \epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} - \epsilon_{abcd} \, \omega^a{}_e \wedge d\omega^{eb} \wedge \omega^{cd} + \epsilon_{abcd} \, \omega^a{}_e \wedge \omega^{eb} \wedge d\omega^{cd} = 3\epsilon_{abcd} \, d\omega^a{}_e \wedge \omega^{eb} \wedge \omega^{cd} \tag{2.7.176}$$

∎

One can also prove by direct computation that $\epsilon_{abcd}\, \omega^a{}_e \wedge \omega^{eb} \wedge \omega^c{}_f \wedge \omega^{fd} = 0$, and

Lemma: $\epsilon_{abcd}\, \omega^{ab} \wedge \omega^{cd} = 0$

Proof: *one has*

$$\epsilon_{abcd}\, \omega^{ab} \wedge \omega^{cd} = -\epsilon_{abcd}\, \omega^{cd} \wedge \omega^{ab} = -\epsilon_{abcd}\, \omega^{ab} \wedge \omega^{cd} \tag{2.7.177}$$

∎

By using these lemmas, one can easily show that

$$\mathbf{G} = \tfrac{1}{2\kappa} d\left(\epsilon_{abcd}\left(\mathbf{R}^{ab} - \frac{1}{3}\omega^a{}_e \wedge \omega^{eb}\right) \wedge \omega^{cd}\right) \tag{2.7.178}$$

Thus once a *Vielbein* e_a has been selected we are able to find a local potential $\mathbf{G} = d\,\boldsymbol{\Omega}$ defined by

$$\boldsymbol{\Omega} = \epsilon_{abcd}\left(\mathbf{R}^{ab} - \frac{1}{3}\omega^a{}_e \wedge \omega^{eb}\right) \wedge \omega^{cd} = \epsilon_{abcd}\left(d\omega^{ab} + \frac{2}{3}\omega^a{}_e \wedge \omega^{eb}\right) \wedge \omega^{cd} \tag{2.7.179}$$

This potential depends on the *Vielbein* and its derivatives up to order 2. Such a potential is not covariant since it depends directly on the spin connection, which is not a tensor.

In the special case of the Lagrangian $\boldsymbol{L}_W$ ($a = 1, b = -2, c = \frac{1}{3}$), field equations (2.7.147) specify to

$$6R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 12R_{\alpha\mu}R_\nu^{.\,\alpha} + 2RR_{\mu\nu} - \frac{3}{2}\left(K - 2Q + \frac{1}{3}R^2\right)g_{\mu\nu} + 6\Box R_{\mu\nu} - \Box Rg_{\mu\nu} - 2\nabla_{(\mu}\nabla_{\nu)}R = 0 \tag{2.7.180}$$

One can check directly that Weyl conformal transformations are symmetries for these equations by checking that if $g_{\mu\nu}$ is a solution then any conformal metric $\tilde{g}_{\mu\nu} = e^\omega g_{\mu\nu}$ is a solution as well. However, it is much easier to check covariance identities for $\boldsymbol{L}_W$ and compute its conservation laws with respect to Weyl conformal transformations.

Once you know that Weyl conformal transformations are symmetries, then the theory ensures that they send solutions into solutions.

Finally, let us remark that, since $\boldsymbol{L}_W$ is covariant with respect to Weyl conformal transformations, one can define a Noether current and the divergence of the Noether current equals the work current, which, in this case, is given by

$$\mathcal{W} = -\mathbb{E}_{\mu\nu}\pounds_\Xi g^{\mu\nu} = -\mathbb{E}_{\mu\nu}g^{\mu\nu}\zeta \tag{2.7.181}$$

This current generally vanishes on shell (i.e. along solutions). However, in this special case, the trace of field equations $\mathbb{E}_{\mu\nu}g^{\mu\nu}$ vanishes identically, so that the Noether current is conserved off shell (i.e. along any configurations, not only along solutions of field equations).

Among quadratic Lagrangians, there is a class of models for which the trace of field equations is identically satisfied.
This class corresponds to the theories for which

$$a\left(2K - 2K - \underline{4Q}\right) - \underline{2bQ} + c\left(2R^2 - 2R^2\right) + 2(2a + b)Q + \underline{\underline{(4a + b)\Box R}} + \underline{\underline{2(b + 4c)\Box R}} - \underline{\underline{(2c + 2a + b)\Box R}} = 0 \qquad \Rightarrow$$
$$\Rightarrow \quad -2(2a + b)Q + 2(2a + b)Q + 2(a + b + 3c)\Box R = 0 \qquad \Rightarrow a + b + 3c = 0 \tag{2.7.182}$$

This class includes the Weyl Lagrangian $\boldsymbol{L}_W$ ($a = 1, b = -2, c = \frac{1}{3}$), as well as the Gauss–Bonnet form ($a = 1, b = -4, c = 1$) (as well as other quadratic Lagrangians).
Of course, only the Weyl Lagrangian is covariant and produces a Noether current conserved, hence conserved off shell.

Let us summarise the situation we discovered. The Lagrangian $\boldsymbol{L}_W$ is covariant with respect to Weyl conformal transformations. Even if we did not define it as a gauge-natural theory, starting by *assuming* that we want a theory in which the difference among representatives of a conformal structure is not observable, hence with a structure bundle P, and the endowed vertical gauge transformations, but simply as a generalisation of standard GR, when we select that particular dynamics, it is a matter of fact that the theory earns Weyl conformal transformations as extra symmetries. Symmetries are there, they are a property of the Lagrangian and its field equations, and it is not something we select. One cannot select the Lagrangian $\boldsymbol{L}_W$ and forget to mention that it is covariant with respect to Weyl conformal transformations.

In view of what we said about the hole argument, we do not have much of a choice there: if one wants to maintain a sort of determinism, we are forced to purge conformal transformations from physical states. In this case, the transformations are vertical gauge transformations (i.e. they are vertical transformations) and we showed in general that any two configurations which are conformally equivalent must be considered as two different representatives of the same physical state.

Accordingly, just in view of our choice of the dynamics $\boldsymbol{L}_W$, we conclude that one cannot observe any different between two conformally equivalent metrics and that we are dealing with a theory for conformal structures rather than a theory for metrics.

Weyl electromagnetism

You may have noticed that one could interpret the electromagnetic potential as a connection on a $\mathbb{R}$-principal bundle, instead of a $U(1)$-principal *(to be done)* bundle. As a matter of fact this was the original proposal by Weyl which proved to be wrong but the underlying idea was the first gauge theory. It is interesting to try to develop such a theory from scratch in a modern language and trace the differences with electromagnetism.

Appendix A. Inducing connections around and defining covariant derivatives

Let us here introduce connections and covariant derivatives of spinors in the proper way. It will not be a short procedure, so take a comfortable seat.   :o)

All we have in the beginning is a *spin bundle* $\hat{\mathcal{P}} = (\hat{P}, M, \hat{p}, \mathrm{Spin}(\eta))$ and a *spin frame* $\hat{e} : \hat{P} \to L(M)$. Let us consider a trivialisation of $\hat{\mathcal{P}}$ given by a family of local sections $\hat{\sigma}_{(\alpha)}$ which are related by transition functions

$$\hat{\sigma}_{(\beta)} = \hat{\sigma}_{(\alpha)} \cdot S^{(\alpha\beta)} \qquad\qquad S^{(\alpha\beta)} : U_{(\alpha\beta)} \to \mathrm{Spin}(\eta) \tag{2.A.1}$$

on the patches overlap. These transition functions form a cocycle.

Since $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta)$ is a group homomorphism, then $O^{(\alpha\beta)} := \ell(S^{(\alpha\beta)}) : U_{(\alpha\beta)} \to \mathrm{SO}(\eta)$ is a cocycle as well and it defines a principal bundle $\mathcal{P} = (P, M, p, \mathrm{SO}(\eta))$. Since the two bundles $\hat{\mathcal{P}}$ and $\mathcal{P}$ come with two trivialisations with transition functions $S^{(\alpha\beta)}$ and $O^{(\alpha\beta)}$ which are the image through a group homomorphism ℓ. Then one can define a principal morphism

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\;\hat{\ell}\;} & P \\ {\scriptstyle\hat{p}}\downarrow & & \downarrow{\scriptstyle p} \\ M & =\!=\!= & M \end{array} \qquad\qquad \hat{\ell} : \hat{P} \to P : [x, S] \mapsto [x, \ell(S)]$$

$$\tag{2.A.2}$$

which is global and independent of the trivialisation.

The local sections $\hat{\sigma}_{(\alpha)}$ of $\hat{\mathcal{P}}$ induce local sections $\sigma_{(\alpha)} = \hat{\ell} \circ \hat{\sigma}_{(\alpha)}$. Accordingly, any trivialisation of $\hat{\mathcal{P}}$ induces a trivialisation of $\mathcal{P}$ and if $S^{(\alpha\beta)}$ are transition functions on $\hat{\mathcal{P}}$, the transition functions of the induced trivialisation on $\mathcal{P}$ are $O^{(\alpha\beta)} := \ell(S^{(\alpha\beta)})$. Unless we say otherwise, trivialisations on $\mathcal{P}$ will always be induced by trivialisations on $\hat{\mathcal{P}}$.

If we consider a generalised gauge transformation $\hat{\Phi} : \hat{P} \to \hat{P}$ that is locally given as

$$
\begin{cases}
x'^{\mu} = x'^{\mu}(x) \\
S' = \hat{s}(x) \cdot S
\end{cases}
\qquad \hat{s} : U \to \mathrm{Spin}(\eta) \tag{2.A.3}
$$

The transformation $\hat{\Phi}$ induces a transformation $\Phi : P \to P$ locally given by

$$
\begin{cases}
x'^{\mu} = x'^{\mu}(x) \\
O' = s(x) \cdot O
\end{cases}
\qquad s = \ell \circ \hat{s} : U \to \mathrm{SO}(\eta) \tag{2.A.4}
$$

For any $p \in P$, there are two preimages $\hat{p}_{\pm} \in \hat{P}$ such that $\hat{\ell}(\hat{p}_{\pm}) = p$. Their images $\hat{\Phi}(\hat{p}_{\pm})$ are different, however, $\hat{\ell} \circ \hat{\Phi}(\hat{p}_+) = \hat{\ell} \circ \hat{\Phi}(\hat{p}_-) =: \Phi(p)$.
Thus one can define the map $\Phi : P \to P : p \mapsto \Phi(p)$ intrinsically.

Let us now consider tangent vectors to the bundle. This is a particularly tricky issue since usually coordinates on groups are quite complicated to be defined. For that reason, one usually relies on the fact that matrix groups can be seen as sub-manifolds of $\mathrm{GL}(k)$ every time one has a representation. In $\mathrm{GL}(k)$, the theory for right-invariant fields is easy since the matrix entries are coordinates and $r^i_j = S^i_k \partial^k_j$ form a right-invariant basis for tangent vectors to $\mathrm{GL}(k)$.

If $G \subset \mathrm{GL}(k)$ is a subgroup, then one can represent vectors on G as combinations of r^i_j which are tangent to the sub-manifold G.

The group $\mathrm{SO}(\eta)$ is a subgroup of $\mathrm{GL}(m)$ identified by the constraints

$$
O^a_c \eta^{cd} O^b_d = \eta^{ab} \quad \Rightarrow \dot{O}^a_c \bar{O}^c_e O^e_f \eta^{fd} O^b_d + O^a_c \eta^{cf} O^e_f \bar{O}^d_e \dot{O}^b_d = \dot{O}^a_c \bar{O}^c_d \eta^{db} + \dot{O}^b_c \bar{O}^c_d \eta^{ad} = 0 \tag{2.A.5}
$$

Thus a vector $v = \dot{O}^a_d \partial^d_a = \dot{O}^a_c \bar{O}^c_d O^d_b \partial^b_a = \dot{O}^a_c \bar{O}^c_d \rho^d_a = \dot{O}^a_c \bar{O}^c_d \eta^{db} \rho_{ba}$ is tangent to $\mathrm{SO}(\eta) \subset \mathrm{GL}(m)$ iff its components $v^{ab} = \dot{O}^a_c \bar{O}^c_d \eta^{db}$ are skew-symmetric with respect to $[ab]$ (as it happens if it is represented by a curve in the subgroup).
Thus the combinations $s_{ab} = r_{[ba]}$ form a basis of vectors tangent to to $\mathrm{SO}(\eta)$ which are hence in the form

$$
v = v^{ab} s_{ab} \tag{2.A.6}
$$

The same procedure can be applied to the group $\mathrm{Spin}(\eta)$ by fixing some matrix representation in $\mathrm{GL}(k)$ where the group $\mathrm{Spin}(\eta)$ is defined by the constraints

$$
S \gamma_a S^{-1} = \gamma_b \ell^b_a(S) \tag{2.A.7}
$$

with $\ell^b_a(S) \in \mathrm{SO}(\eta)$. Then, by taking the derivative of the constraint, one has

$$
\dot{S} \gamma_a S^{-1} + S \gamma_a \dot{S}^{-1} = \gamma_b \dot{\ell}^b_a \quad \Rightarrow \dot{S} S^{-1} S \gamma_a S^{-1} + S \gamma_a S^{-1} S \dot{S}^{-1} = \dot{S} S^{-1} \gamma_c \ell^c_a + \ell^c_a \gamma_c S \dot{S}^{-1} = \gamma_b \dot{\ell}^b_a \quad \Rightarrow \dot{S} S^{-1} \gamma_c - \gamma_c \dot{S} S^{-1} = [\dot{S} S^{-1}, \gamma_c] = \gamma_b \dot{\ell}^b_a \bar{\ell}^a_c \tag{2.A.8}
$$

One can look for the element $\dot{S}S^{-1}$ in the Clifford algebra which fulfils this relation.

If $\dot{S}S^{-1}$ had a component along $\mathbb{I}$ that would not affect the commutator $[\dot{S}S^{-1}, \gamma^c]$. However, $\dot{S}S^{-1} \in \mathfrak{spin}(\eta)$ and cannot have components along $\mathbb{I}$. If it had a component along a odd product of gamma matrices, the commutator would be even and cannot be in the span of γ^b. If it had a component along a product of $2n$ gamma matrices (with $n \neq 1$) then the commutator would be a product of $2n+1$ or $2n-1$ and, since $n \neq 0, 1$, this cannot be 1.

Accordingly, $\dot{S}S^{-1}$ must be proportional to the product of 2 (different) gammas, i.e. $\dot{S}S^{-1} = v^{ab}\gamma_a\gamma_b$. Since one has

$$[\gamma_a\gamma_b, \gamma_c] = abc - cab = \textcolor{red}{abc} - \textcolor{red}{abc} - 2(bc)a + 2(ac)b = 2\left(\eta_{ac}\gamma_b - \eta_{bc}\gamma_a\right) \tag{2.A.9}$$

the commutator is

$$[\dot{S}S^{-1}, \gamma_c] = 4v^{ab}\eta_{ca}\gamma_b = \gamma_b\dot{\ell}_a^b\bar{\ell}_c^a \qquad \Rightarrow v^{ab} = \tfrac{1}{4}\dot{\ell}_d^b\bar{\ell}_c^d\eta^{ca} = -\tfrac{1}{4}\dot{\ell}_d^a\bar{\ell}_c^d\eta^{cb} \qquad \Rightarrow \dot{S}S^{-1} = -\tfrac{1}{4}\dot{\ell}_d^a\bar{\ell}_c^d\eta^{cb}\gamma_a\gamma_b \tag{2.A.10}$$

Then a vector $\hat{v} = v_j^i\partial_i^j = \hat{v}_k^i(S^{-1})_h^k S_j^h\partial_i^j = \dot{S}_k^i(S^{-1})_h^k r_i^h = \dot{S}S^{-1}r$ tangent to $\mathrm{GL}(k)$ happens to be tangent to $\mathrm{Spin}(\eta) \subset \mathrm{GL}(k)$ iff one has

$$\dot{S}S^{-1} = -\tfrac{1}{4}\dot{\ell}_c^a\bar{\ell}_d^c\eta^{db}\gamma_a\gamma_b = -\tfrac{1}{4}v^{ab}\gamma_a\gamma_b \tag{2.A.11}$$

for some vector $v = v^{ab}s_{ab}$ tangent to $\mathrm{SO}(\eta)$. Accordingly, the vectors

$$\hat{s}_{ab} = -\tfrac{1}{4}\gamma_{[a}\gamma_{b]}r \tag{2.A.12}$$

form a basis of tangent vectors to $\mathrm{Spin}(\eta)$.

The covering map $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta)$ is a local diffeomorphism and establishes an isomorphism between the corresponding tangent spaces

$$T\ell : T_S\mathrm{Spin}(\eta) \to T_{\ell(S)}\mathrm{SO}(\eta) : v^{ab}\hat{s}_{ab} \mapsto v^{ab}s_{ab} = v \cdot s \qquad \left(v^{ab} \mapsto v = -\tfrac{1}{4}v^{ab}\gamma_a\gamma_b\right) \tag{2.A.13}$$

One can define a right-invariant pointwise basis $\hat{\sigma}_{ab}$ for vertical vectors on $\hat{P}$. Since we have a map $\hat{\ell} : \hat{P} \to P$ one can define $\hat{\ell}_*(\hat{\sigma}_{ab}) = \sigma_{ab}$, which are right-invariant vector fields on P.

Let us remark that the map $\hat{\ell}$ is a local diffeomorphism, thus one can also pull back vectors from P to $\hat{P}$ (as well as vice versa). Hence, if we get a vertical vector $v = v^{ab}\sigma_{ab}$ at the point $(x^\mu, \ell_b^a) \in P$ that can be pulled back to the point $(x^\mu, S) \in \hat{P}$ and one has

$$\hat{v} := \hat{\ell}^*(v) = -\tfrac{1}{4}v^{ab}\gamma_a\gamma_b\rho = v^{ab}\hat{\sigma}_{ab} \tag{2.A.14}$$

We did some extra work which is, however, quite useful. For example, given a connection on $\mathcal{P}$ locally represented as

$$\omega = dx^\mu \otimes (\partial_\mu - \Gamma_\mu^{ab}\sigma_{ab}) \tag{2.A.15}$$

then that induces a connection on $\hat{P}$ given by

$$\omega = dx^\mu \otimes (\partial_\mu + \tfrac{1}{4}\Gamma_\mu^{ab}\gamma_a\gamma_b\rho) = dx^\mu \otimes (\partial_\mu - \Gamma_\mu^{ab}\hat{\sigma}_{ab}) \tag{2.A.16}$$

The spin frame $\hat{e} : \hat{P} \to L(M)$ we fixed above allows us to define a lot of further structures.

The spin frame $\hat{e}$ defines the induced metric $g_{\mu\nu}$ which in turn defines the Levi Civita connection $\{g\}^\alpha_{\beta\mu}$ on $L(M)$. It also defines the sub-bundle $\hat{e}(\hat{P}) = \mathrm{SO}(M, g) \subset L(M)$ of orthonormal frames.

Usually, a connection on $L(M)$ does not restrict to the sub-bundle $\mathrm{SO}(M, g)$, but the Levi Civita connection of g does.

Let us consider a trivialisation of $\mathrm{SO}(M, g)$ induced by orthonormal moving frames $V_A = V_A^\mu \partial_\mu$ which induces coordinates (x^μ, ϵ_a^A) on $L(M)$. These are *adapted to the sub-bundle* since a general frame $\epsilon_a = \epsilon_a^A V_A = \epsilon_a^\mu \partial_\mu \in L(M)$ belongs to the sub-bundle $\mathrm{SO}(M, g)$ iff $\epsilon_a^A \in \mathrm{SO}(\eta)$.

One can write the Levi Civita connection in this adapted trivialisation to obtain

$$\Gamma^A_{B\mu} = V_\alpha^A \left(\{g\}^\alpha_{\beta\mu} V_B^\beta + d_\mu V_B^\alpha \right) \tag{2.A.17}$$

By taking into account that the frames V_A have been chosen to be orthonormal with respect to the induced metric, i.e. $g_{\mu\nu} = V_\mu^A \eta_{AB} V_\nu^B$, one can expand the connection to

$$\begin{aligned}
\Gamma^{AB}_\mu =& \tfrac{1}{2} V^{A\alpha} V^{B\beta} \left(-d_\alpha V_\beta^C V_{C\mu} - \underline{V_{C\beta} d_\alpha V_\mu^C} + \underline{d_\beta V_\mu^C V_{C\alpha}} + \underline{V_{C\mu} d_\beta V_\alpha^C} + d_\mu V_\alpha^C V_{C\beta} + {\color{red}V_{C\alpha} d_\mu V_\beta^C} - {\color{red}V_{C\alpha} d_\mu V_\beta^C} - V_{C\alpha} d_\mu V_\beta^C \right) = \\
=& V^{A[\alpha} V^{B\beta]} \left(d_\beta V_\mu^C V_{C\alpha} + V_{C\mu} d_\beta V_\alpha^C + d_\mu V_\alpha^C V_{C\beta} \right) = V^{[A\alpha} V^{B]\beta} \left(d_\beta V_\mu^C V_{C\alpha} + V_{C\mu} d_\beta V_\alpha^C + d_\mu V_\alpha^C V_{C\beta} \right)
\end{aligned} \tag{2.A.18}$$

Thus Γ^{AB}_μ is antisymmetric with respect to the indices $[AB]$. Then the Levi Civita connection, in the adapted frame, reads as

$$\omega = dx^\mu \otimes (\partial_\mu - \Gamma^{AB}_\mu \rho_{\dot{B}A}) = dx^\mu \otimes (\partial_\mu - \Gamma^{AB}_\mu \sigma_{AB}) \tag{2.A.19}$$

which is hence a connection on the sub-bundle $\mathrm{SO}(M, g)$.

Then the Levi Civita connection $\{g\}$ on $L(M)$ does induce on $\mathrm{SO}(M, g)$ a connection

$$\omega = dx^\mu \otimes \left(\partial_\mu - \Gamma^A_{B\mu} \rho^B_A \right) \qquad \begin{cases} \Gamma^A_{B\mu} = V_\alpha^A \left(\{g\}^\alpha_{\beta\mu} V_B^\beta + d_\mu V_B^\alpha \right) \\ \rho^B_A = V_\alpha^A \rho^\alpha_\beta V_B^\beta \end{cases} \tag{2.A.20}$$

Then the *Vielbein* $e : P \to L(M) : (x, \ell_a^b) \mapsto (x, e_b^A \ell_a^b)$ allows to pull back the connection on P.

$$\omega = dx^\mu \otimes \left(\partial_\mu - \Gamma^{ab}_\mu \sigma_{ab} \right) \qquad \begin{cases} \Gamma^{ab}_\mu = e_A^a \left(\Gamma^A_{B\mu} e^{bB} + d_\mu e^{bA} \right) = e_\alpha^a \left(\{g\}^\alpha_{\beta\mu} e^{b\beta} + d_\mu e^{b\alpha} \right) \\ \sigma_{ab} = \eta_{c[a} e_A^c \rho^A_B e_{b]}^B = \eta_{c[a} e_\alpha^c \rho^\alpha_\beta e_{b]}^\beta \end{cases} \tag{2.A.21}$$

Once the spin frame has induced a connection on P that can be lifted to $\hat{P}$

$$\hat{\omega} = dx^\mu \otimes \left(\partial_\mu - \Gamma^{ab}_\mu \hat{\sigma}_{ab} \right) = dx^\mu \otimes \left(\partial_\mu - \Gamma^j_{i\mu} \rho^i_j \right) \qquad \begin{cases} \Gamma_\mu = -\tfrac{1}{4} \Gamma^{ab}_\mu \gamma_a \gamma_b \\ \rho^i_j = S^i_k \dfrac{\partial}{\partial S^j_k} \end{cases} \tag{2.A.22}$$

Now we have a connection on $\hat{\mathcal{P}}$ and one on $L(M)$ induced by the spin frame $\hat{e}$. Since the bundle $F(\hat{\mathcal{P}})$ of spin frames is associated to $\hat{\mathcal{P}} \times L(M)$, then the connection

$$dx^\mu \otimes \left(\partial_\mu - \Gamma^{ab}_\mu \hat{\sigma}_{ab} - \{g\}^\alpha_{\beta\mu} \rho^\beta_\alpha \right) \tag{2.A.23}$$

induces a connection on $F(\hat{\mathcal{P}})$.

Let us denote by $\epsilon_a = \epsilon_a^\mu \partial_\mu$ the points in $L(M)$, by $\sigma \cdot S$ the points in $\hat{\mathcal{P}}$, and by $e_a^b \in \mathrm{GL}(m)$. A point in $F(\hat{\mathcal{P}})$ is an equivalence class

$$[\sigma \cdot S, \epsilon_a, e_a^b]_{\hat{\sigma}} = [\sigma, \partial_\mu, e_a^\mu]_{\hat{\sigma}} \in F(\hat{\mathcal{P}}) \qquad e_a^\mu := \epsilon_d^\mu e_c^d \ell_a^c(\bar{S}) \tag{2.A.24}$$

For any matrix $e_a^b \in \mathrm{GL}(m)$ one can define a map $\Phi_e : \hat{\mathcal{P}} \times L(M) \to F(\hat{\mathcal{P}}) : [\hat{\sigma} \cdot S, \epsilon_a] \mapsto [\sigma \cdot S, \epsilon_a, e_a^b]_{\hat{\sigma}}$ which, in coordinates, reads as

$$\Phi_e : (x, S, \epsilon) \mapsto (x, e_a^\mu = \epsilon_c^\mu e_b^c \ell_a^b(\bar{S})) \tag{2.A.25}$$

The tangent map of this is given by

$$\begin{cases} \partial_\mu = \partial_\mu \\ \dfrac{\partial}{\partial \ell_a^b} = -\epsilon_f^\mu e_d^f \bar{\ell}_b^d \bar{\ell}_c^a \dfrac{\partial}{\partial e_c^\mu} = -e_b^\mu \bar{\ell}_c^a \dfrac{\partial}{\partial e_c^\mu} \\ \dfrac{\partial}{\partial \epsilon_a^\mu} = e_b^a \bar{\ell}_c^b \dfrac{\partial}{\partial e_c^\mu} = \epsilon_\nu^a e_c^\nu \dfrac{\partial}{\partial e_c^\mu} \end{cases} \Rightarrow \begin{cases} \partial_\mu = \partial_\mu \\ \sigma_{ba} = \eta_{c[a} \ell_{d]}^c \dfrac{\partial}{\partial \ell_d^{b]}} = -\eta_{c[a} e_{b]}^\mu \dfrac{\partial}{\partial e_c^\mu} \qquad \Rightarrow \sigma_{ab} = \eta_{c[a} e_{b]}^\mu \dfrac{\partial}{\partial e_c^\mu} \\ \rho_\mu^\nu = \epsilon_a^\nu \dfrac{\partial}{\partial \epsilon_a^\mu} = e_c^\nu \dfrac{\partial}{\partial e_c^\mu} \end{cases} \tag{2.A.26}$$

Hence we are able to induce the connection on $F(\hat{\mathcal{P}})$ as

$$\omega_{\hat{\sigma}} = dx^\mu \otimes \left(\partial_\mu - \left(\{g\}_{\beta\mu}^\alpha e_c^\beta + \Gamma_\mu^{ab} \eta_{c[a} e_{b]}^\alpha \right) \dfrac{\partial}{\partial e_c^\alpha} \right) \tag{2.A.27}$$

and such a connection induces the covariant derivative of sections (i.e. of spin frames) as

$$\nabla_\mu e_c^\alpha = d_\mu e_c^\alpha + \{g\}_{\beta\mu}^\alpha e_c^\beta + \Gamma_\mu^{ab} \eta_{ca} e_b^\alpha = d_\mu e_c^\alpha + \{g\}_{\beta\mu}^\alpha e_c^\beta - \Gamma_{c\mu}^a e_a^\mu \tag{2.A.28}$$

The connection on $\hat{\mathcal{P}}$ also induces a connection on the spinor bundle $S(\hat{\mathcal{P}})$.

Let us denote by v the coordinates on V and one can define the map

$$\Phi_v : \hat{P} \to S(\hat{\mathcal{P}}) : \sigma \cdot S \mapsto [\sigma \cdot S, v]_\lambda \qquad \Phi_v : S \mapsto \psi := S \cdot v \tag{2.A.29}$$

The tangent map of Φ_v is

$$\begin{cases} \partial_\mu = \partial_\mu \\ \partial_i^j = v^j \partial_i \end{cases} \Rightarrow \begin{cases} \partial_\mu = \partial_\mu \\ \rho_i^k = S_j^k \partial_i^j = \psi^k \partial_i \end{cases} \tag{2.A.30}$$

Then, the connection (2.A.22) induces the connection ω_λ on $S(\hat{\mathcal{P}})$ which is given by

$$\omega_\lambda = dx^\mu \otimes \left(\partial_\mu - \Gamma_{j\mu}^i \psi^j \dfrac{\partial}{\partial \psi^i} \right) = dx^\mu \otimes \left(\partial_\mu + \tfrac{1}{4} \Gamma_\mu^{ab} \gamma_a \gamma_b \psi \dfrac{\partial}{\partial \psi} \right) \tag{2.A.31}$$

The corresponding covariant derivative for spinors reads as

$$\nabla_\mu \psi = d_\mu \psi - \tfrac{1}{4} \Gamma_\mu^{ab} \gamma_a \gamma_b \psi \tag{2.A.32}$$

The same procedure should be used to check that the Dirac matrices γ^a are covariantly constant.

These matrices have 3 indices (a frame index a and two spinor indices which are understood) and they transform as

$$\gamma'^{a\,i}_{\ \ j} = \ell^a_b S^i_k \gamma^{b\,k}_{\ \ h} \bar{S}^h_j \quad (= \ell^a_b \bar{\ell}^b_c \gamma^{c\,i}_{\ \ j} = \gamma^{a\,i}_{\ \ j}) \tag{2.A.33}$$

Accordingly, the covariant derivative is defined as (mind the order or write the matrix indices explicitly):

$$\nabla_\mu \gamma^a = \boxed{d_\mu \gamma^a} + \Gamma^a_{c\mu} \gamma^c + \Gamma_\mu \gamma^a - \gamma^a \Gamma_\mu = \Gamma^a_{c\mu} \gamma^c - \tfrac{1}{4} \Gamma^{cd}_\mu [\gamma_c \gamma_d, \gamma^a] = \Gamma^a_{c\mu} \gamma^c - \Gamma^{ad}_\mu \gamma_d = 0 \tag{2.A.34}$$

Of course, this is long and detailed. On the other hand, there is nothing to be guessed or assumed. This is a canonical construction, which being canonical can be skipped once one knows how to do it and how to guess the result.

Moreover, the abundance of details to be kept under control is a good thing since it prevents inconsistencies. We can suggest to try to introduce some error.

Exercise: try to repeat the computations in this Appendix in all possible wrong ways and check that inconsistencies force you to spot the mistakes.

Let me tell you that the first time I did it I considered ℓ to be defined as $S\gamma^a S^{-1} = \ell^a_b(S)\gamma^b$, instead of $S\gamma^a S^{-1} = \ell^a_b(S^{-1})\gamma^b$. That is, of course, wrong as one is forced to eventually accept by skipping the check that ℓ is a group homomorphism and try to fight (in vain) to get a meaning of the rest of the computation.

References

Add references

For (2.1.30):

Kolár, I., Michor, P.W., Slovák, J., (1993), *Natural Operations in Differential Geometry*, Springer–Verlag, NY.

García, P.L., (1972), *Connections and 1–jet Fiber Bundles*, Rend. Sem. Mat. Univ. Padova, **47** 227–242.

8. Exercises

Exercise 1: Show that, for the metric Lagrangian $L = \frac{\sqrt{g}}{2\kappa} R$, the correct expression for momenta $p_\alpha^{\ \beta\mu\nu}$ is the one given in (2.2.20) and not simply

$$\tilde{p}_\alpha^{\ \beta\mu\nu} = \frac{\sqrt{g}}{2\kappa} g^{\beta\nu} \delta^\mu_\alpha \tag{2.8.1}$$

Exercise 2: Show that also Lagrangians (2.2.23) satisfy covariance conditions (2.2.17).

Exercise 3: Compute field equations for the Gauss-Bonnet Lagrangian

$$L = \tfrac{\sqrt{g}}{2\kappa}\left(K - 4Q + R^2\right) \tag{2.8.2}$$

Exercise 4: Show that, on a spacetime of dimension $m = 4$, the Lagrangian

$$\boldsymbol{L} = \tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\,d\sigma \tag{2.8.3}$$

is a gauge-natural Lagrangian which depends on A_μ and its first derivatives only. Compute its field equations and discuss conservation laws.

Exercise 5: Define a Lagrangian to describe a spinor in interaction with electromagnetism and gravity; show that it is a gauge-natural Lagrangian and compute its first variation formula.

Hint: The spinor field is meant to be charged if it transforms as

$$\psi' = e^{iq\alpha}S\psi \tag{2.8.4}$$

In order to account for $U(1)$-gauge transformations, the covariant derivative must depend on A_μ as well.

Exercise 6: Write down covariance identities for a second order Lagrangian depending on a metric and compare with (2.7.84).

Exercise 7: Define the bundle where $\bar\psi$ lives as an associated bundle to $\hat{\mathcal{P}}$, induce a connection on it and show that

$$\nabla_\mu\bar\psi = d_\mu\bar\psi + \tfrac{1}{4}\Gamma^{ab}_\mu\bar\psi\gamma_a\gamma_b \tag{2.8.5}$$

Chapter 3. Conservation laws

> *Pure mathematics is, in its way, the poetry of logical ideas. One seeks the most general ideas of operation which will bring together in simple, logical and unified form the largest possible circle of formal relationships. In this effort toward logical beauty spiritual formulas are discovered necessary for the deeper penetration into the laws of nature.*
>
> (Albert Einstein, *Obituary for Emmy Noether (1935)*)

1. Introduction

There is a beautiful, complete and quite satisfactory theory of conservation laws for relativistic theories. In particular one can prove that all relativistic theories allow superpotentials, i.e. any Noether current is not only closed (i.e. *conserved*) on shell (i.e. along solutions) but it is in fact *exact* on shell. As a consequence all conserved quantities can be computed by integrating a $(m-2)$-superpotential on a closed $(m-2)$-region. This is how one uses Gauss theorem to evaluate the electric charge in a spatial region by computing the flow of the electric field through the boundary of that region (which is in fact a $(m-2)$-surface in spacetime). Whenever, conserved quantities are computed in this way we shall say they are evaluated *à la Gauss*.

In this Chapter we are not interested in the physical meaning of conserved quantities which will be discussed later on. Here we are interested in extracting from a field theory some geometrically well-defined quantities which are characteristics of the theory. The physical quantities are expected to be geometrically well-defined quantities, thus whatever they are, they should be searched among the numbers we are here going to define. This is a consequence of the principle of relativity: physics should be described in a covariant way in order not to privilege any special class of observers. Thus they have to be independent of the coordinates used. Thus they have to be geometrically well-defined quantities. Once we defined conserved quantities the physical interpretation will follow from their properties and use.

In view of this consideration, the correct way of proceeding is analysing conserved quantities in general without selecting particular coordinates or specific symmetries. Then one can use the general results to specify to particular situations, for example considering the symmetries a specific solution may have.

Accordingly, one should not discuss symmetry in Minkowski space then trying to generalise the discussion to a general spacetime. On the contrary, one should obtain conserved quantities in Minkowski by specialising the results obtained for a general spacetime. In particular, Killing vectors should not play any fundamental role in the discussion, since the generic spacetime has no Killing vector.

In a natural theory *any* spacetime vector field is a generator of symmetries, thus one can associate conservation laws to any spacetime vector fields, without any need to restrict to Killing vectors. It is only in Minkowski, when discussing a field theory in which the Lagrangian depends on the metric which is treated as a background field, thus it is considered fixed, the variation with respect to the metric vanishes everywhere to cancel the corresponding term in the variation of the Lagrangian, thus the metric does not obey to field equations, then one needs to restrict to symmetries for

which $\pounds_\xi g_{\mu\nu} = 0$ (i.e. ξ is a Killing vector) in order to cancel the spurious term in Noether current. In natural theories there is no such spurious terms because all fields are dynamical and no (non-dynamical) background field is allowed.

For a matter Lagrangian $\boldsymbol{L} = L(g, j^1\phi)\, d\sigma$ the variation is

$$\delta L = -\frac{\sqrt{g}}{2} T_{\mu\nu} \delta g^{\mu\nu} + \mathbb{E}\delta\phi + d\mathbb{F} \tag{3.1.1}$$

The last term $d\mathbb{F}$ does not contribute to the variation of the action in view of boundary conditions. The second term $\mathbb{E}\delta\phi$ vanishes in view of matter field equations $\mathbb{E} = 0$. The first term vanishes since the metric is considered a background and one sets $\delta g^{\mu\nu} = 0$. As a consequence $T_{\mu\nu}$ is not constrained to be zero.

When one considers conservation laws Noether theorem for a transformation ξ one obtains

$$d\mathcal{E} = -\mathbb{E}\pounds_\xi\phi + \frac{\sqrt{g}}{2} T_{\mu\nu}\pounds_\xi g^{\mu\nu} \tag{3.1.2}$$

In order for $\mathcal{E}$ to be a conserved current, the right hand side must vanish on shell. The first term on the right hand side $\mathbb{E}\pounds_\xi\phi$ vanishes because of field equations $\mathbb{E} = 0$. Since the metric has been treated as a background and $T_{\mu\nu}$ does not vanish in general, the second term on the right hand side $T_{\mu\nu}\pounds_\xi g^{\mu\nu}$ vanishes only by restricting the symmetries to Killing vectors, for which $\pounds_\xi g^{\mu\nu} = 0$.

Notice that the restriction to Killing vectors is directly related to the background nature of the metric field. In a covariant theory no background field is allowed. In a physically sound extension of the theory above, one regards g as a dynamical field, usually adding a Lagrangian for the metric alone (e.g. the Hilbert Lagrangian $L_H(j^2 g)$) to get a total Lagrangian $\boldsymbol{L}_T = \boldsymbol{L}_H + \boldsymbol{L}$. Then the variation of the total Lagrangian reads as

$$\delta\boldsymbol{L}_T = \frac{\sqrt{g}}{2\kappa}\left(G_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} + \mathbb{E}\delta\phi + d\left(\mathbb{F} + \mathbb{F}_H\right) \tag{3.1.3}$$

and field equations are now

$$\begin{cases} G_{\mu\nu} = \kappa T_{\mu\nu} \\ \mathbb{E} = 0 \end{cases} \tag{3.1.4}$$

which describe the interaction between the gravitational field g and the matter field ϕ.

When one considers conservation laws Noether theorem for a transformation ξ one obtains

$$d\mathcal{E} = -\mathbb{E}\pounds_\xi\phi - \frac{\sqrt{g}}{2\kappa}\left(G_{\mu\nu} - \kappa T_{\mu\nu}\right)\pounds_\xi g^{\mu\nu} \tag{3.1.5}$$

and the right hand side vanishes on shell without restricting the symmetry generator in any way.

Killing vectors play no fundamental role in generally covariant conservation laws which then apply also to spacetimes with no Killing vectors. On the other hand this fantastic abundance of conservation laws poses an interpretation problem in selecting which of the infinitely many conservation laws play the role of specific (and finite in number) physical quantities.

In Minkowski one has 10 linearly independent Killing vectors. Then there are 10 conservation laws. The corresponding conserved quantities are interpreted as 1 energy, 3 momenta (which are together called the *covariant momentum vector*) 3 components of the angular momentum and 3 components of *boost momenta* (which together defines the 6-component *covariant momentum tensor*).

If this is a problem when generalising to spacetimes with less Killing vectors, it is also a problem when one has infinitely many conservation laws and wants to single out which ones generate these quantities, as well as the physical meaning of the others.

This interpretation problem will be also an opportunity to discuss whether a particular conservation quantity (for example the energy) plays a distinctive physical role over the others or rather all conservation laws have to be treated on equal footing and they play their role collectively while

the physical interpretation of which is the energy and which is the angular momentum is more related to our Newtonian intuition, and as such it is observer dependent *and* not fundamental.

2. Noether currents in natural theories

We already specialised the general covariance identity to natural (2.1.13) and gauge-natural theories (2.1.23). We also discussed the form of the Noether current (1.4.12) associated to an infinitesimal Lagrangian symmetry in Section 1.4. The Noether current is in the form $\mathcal{E}(L, \Xi) = (j^{k-1} \pounds_\Xi) \lrcorner \mathbb{F}(L) - \xi \lrcorner \boldsymbol{L}$, while the work current is $\mathcal{W}(L, \Xi) = -(\pounds_\Xi) \lrcorner \mathbb{E}(L)$.

On the other hand, in Section 18.9 we discuss the theory of Lie derivatives and computed the Lie derivatives of many different fields. We also proved that, given the relevant connections, the Lie derivative of sections of natural (and gauge-natural) bundles can be written as linear combinations of the symmetry generators and their symmetrised covariant derivatives up to some finite order which depends on the specific (gauge) natural object.

Let us restrict by now to the natural case. Gauge-natural theories will be considered below. We fix a connection $\hat{\Gamma}^\alpha_{\beta\mu}$ on spacetime. In general the connection does not need to be torsionless. However, we know that one can define the transposed connection ${}^t\hat{\Gamma}^\alpha_{\beta\mu} = \hat{\Gamma}^\alpha_{\mu\beta}$. Then one can decompose the original connection as

$$\hat{\Gamma}^\alpha_{\beta\mu} = \hat{\Gamma}^\alpha_{(\beta\mu)} + \hat{\Gamma}^\alpha_{[\beta\mu]} = \Gamma^\alpha_{\beta\mu} + \tfrac{1}{2} T^\alpha_{\beta\mu} \tag{3.2.1}$$

where $\Gamma^\alpha_{\beta\mu} = \hat{\Gamma}^\alpha_{(\beta\mu)}$ is a torsionless connection (which is called the *symmetrised connection*) and $T^\alpha_{\beta\mu} = \hat{\Gamma}^\alpha_{\beta\mu} - \hat{\Gamma}^\alpha_{\mu\beta}$ is the *torsion* tensor field.

Let us use the symmetrised connection Γ and denote by ∇_μ the corresponding covariant derivatives (acting on all kind of first order geometric objects). Both the Noether current and the work current are linear combinations of symmetrised covariant derivatives of the symmetry generators up to some finite order. In natural theories then Noether currents and work currents can be written as

$$\begin{cases} \mathcal{E} = \left(\mathcal{E}^\mu_\epsilon \xi^\epsilon + \mathcal{E}^{\mu\alpha}_\epsilon \nabla_\alpha \xi^\epsilon + \mathcal{E}^{\mu\alpha\beta}_\epsilon \nabla_{\alpha\beta} \xi^\epsilon + \ldots + \mathcal{E}^{\mu\alpha_1\ldots\alpha_s}_\epsilon \nabla_{\alpha_1\ldots\alpha_s} \xi^\epsilon \right) d\sigma_\mu \\ \mathcal{W} = \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}^\alpha_\epsilon \nabla_\alpha \xi^\epsilon + \mathcal{W}^{\alpha\beta}_\epsilon \nabla_{\alpha\beta} \xi^\epsilon + \ldots + \mathcal{W}^{\alpha_0\alpha_1\ldots\alpha_s}_\epsilon \nabla_{\alpha_0\ldots\alpha^s} \xi^\epsilon \right) d\sigma \end{cases} \tag{3.2.2}$$

Here the coefficients $\mathcal{E}^\mu_\epsilon$, $\mathcal{E}^{\mu\alpha}_\epsilon$, $\mathcal{E}^{\mu\alpha\beta}_\epsilon$, ..., $\mathcal{E}^{\mu\alpha_1\ldots\alpha_s}_\epsilon$ are tensor densities of weight 1. They are symmetric in all upper indices but μ. These are called *canonical stress tensor densities*.

Analogously, the coefficients $\mathcal{W}_\epsilon$, $\mathcal{W}^\alpha_\epsilon$, $\mathcal{W}^{\alpha\beta}_\epsilon$, ..., $\mathcal{W}^{\alpha_0\alpha_1\ldots\alpha_s}_\epsilon$ are tensor densities of weight 1 and they are symmetric in their upper indices. These are called *canonical work tensor densities*. Notice that the work current vanishes on shell by construction. Consequently, and in view of independence of symmetrised covariant derivatives of symmetry generators, all canonical work tensor densities vanish on shell as well.

More generally, we define *currents* to be horizontal $(m - d)$-forms which depend linearly on a symmetry generator ξ. Currents can be expanded as linear combinations of ξ and its (symmetrised covariant) derivatives up to some finite order s. The integer $m - d$ is called the *degree* of the current (and d the *codegree*), while the integer s is called the *order* of the current. The currents are horizontal forms on some jet prolongation J^kC of the configuration bundle, thus coefficients depend on fields and their derivatives up to order k. The integer k is called the *rank* of the current.

Essentially, a single current is a collection of infinitely many horizontal forms, one for any symmetry generator ξ. This infinite family is described by a single map from some jet prolongation of the bundle of symmetry generators (in this case $J^k TM$) to the bundle of horizontal forms of degree $m - d$, i.e. by a single map between finite dimensional spaces.

This is not very different from how the action functional (which is a function of the infinite dimensional space of sections of configuration bundle) is generated by the Lagrangian which is a (horizontal) form over a finite dimensional space $J^k C$.

In general, the coefficients of a current are not required to be all non-zero and can be padded so that the order of the Noether current is 1 less than the order of the work current. The form of these currents is a characteristic of the theory and in (gauge-)natural theories one has a pair of currents for any infinitesimal symmetry generator.

Let us stress that we are working with currents. These are finite linear conbinations of the symmetry generator ξ and its derivative up to some finite order. In this set, the symmetrised covariant derivatives form a basis.

In particular, the covariant derivative of a symmetrised covariant derivative can be uniquely expanded as a linear combination of symmetrised covariant derivatives by using the commutation relation of covariant derivatives and their relation with the curvature.

For example,

$$\nabla_\mu \nabla_\nu \xi^\epsilon = \nabla_{\mu\nu}\xi^\epsilon + \tfrac{1}{2}[\nabla_\mu, \nabla_\nu]\xi^\epsilon = \nabla_{\mu\nu}\xi^\epsilon + \tfrac{1}{2}R^\epsilon{}_{\rho\mu\nu}\xi^\rho \tag{3.2.3}$$

where $R^\epsilon{}_{\rho\mu\nu}$ denotes the Riemann tensor of the torsionless connection Γ. Analogously, one can compute

$$
\begin{aligned}
\nabla_\rho \nabla_{\mu\nu}\xi^\epsilon =& \tfrac{1}{6}\left(3\nabla_\rho\nabla_\mu\nabla_\nu + 3\nabla_\rho\nabla_\nu\nabla_\mu\right)\xi^\epsilon = \tfrac{1}{6}\left(\nabla_\rho\nabla_\mu\nabla_\nu + \nabla_\rho\nabla_\nu\nabla_\mu + 2\nabla_\mu\nabla_\rho\nabla_\nu + 2\nabla_\nu\nabla_\rho\nabla_\mu + 2[\nabla_\rho, \nabla_\mu]\nabla_\nu + 2[\nabla_\rho, \nabla_\nu]\nabla_\mu\right)\xi^\epsilon = \\
=& \tfrac{1}{6}\left(\nabla_\rho\nabla_\mu\nabla_\nu + \nabla_\rho\nabla_\nu\nabla_\mu + \nabla_\mu\nabla_\rho\nabla_\nu + \nabla_\nu\nabla_\rho\nabla_\mu + \nabla_\mu\nabla_\nu\nabla_\rho + \nabla_\nu\nabla_\mu\nabla_\rho\right)\xi^\epsilon + \tfrac{1}{6}\left(2[\nabla_\rho, \nabla_\mu]\nabla_\nu + 2[\nabla_\rho, \nabla_\nu]\nabla_\mu + \nabla_\mu[\nabla_\rho, \nabla_\nu] + \nabla_\nu[\nabla_\rho, \nabla_\mu]\right)\xi^\epsilon = \\
=& \nabla_{\rho\mu\nu}\xi^\epsilon + \tfrac{1}{3}R^\epsilon{}_{\sigma\rho\mu}\nabla_\nu\xi^\sigma - \tfrac{1}{3}R^\lambda{}_{\nu\rho\mu}\nabla_\lambda\xi^\epsilon + \tfrac{1}{3}R^\epsilon{}_{\sigma\rho\nu}\nabla_\mu\xi^\sigma - \tfrac{1}{3}R^\lambda{}_{\mu\rho\nu}\nabla_\lambda\xi^\epsilon + \tfrac{1}{6}\nabla_\mu\left(R^\epsilon{}_{\sigma\rho\nu}\xi^\sigma\right) + \tfrac{1}{6}\nabla_\nu\left(R^\epsilon{}_{\sigma\rho\mu}\xi^\sigma\right) = \\
=& \nabla_{\rho\mu\nu}\xi^\epsilon + \left(\tfrac{1}{3} + \tfrac{1}{6}\right)R^\epsilon{}_{\sigma\rho\mu}\nabla_\nu\xi^\sigma - \tfrac{1}{3}R^\lambda{}_{\nu\rho\mu}\nabla_\lambda\xi^\epsilon + \left(\tfrac{1}{3} + \tfrac{1}{6}\right)R^\epsilon{}_{\sigma\rho\nu}\nabla_\mu\xi^\sigma - \tfrac{1}{3}R^\lambda{}_{\mu\rho\nu}\nabla_\lambda\xi^\epsilon + \tfrac{1}{6}\nabla_\mu R^\epsilon{}_{\sigma\rho\nu}\xi^\sigma + \tfrac{1}{6}\nabla_\nu R^\epsilon{}_{\sigma\rho\mu}\xi^\sigma = \\
=& \nabla_{\rho\mu\nu}\xi^\epsilon + \left(R^\epsilon{}_{\sigma\rho(\mu}\delta^\lambda_{\nu)} + \tfrac{2}{3}R^\lambda{}_{(\nu\mu)\rho}\delta^\epsilon_\sigma\right)\nabla_\lambda\xi^\sigma + \tfrac{1}{6}\left(\nabla_\mu R^\epsilon{}_{\sigma\rho\nu} + \nabla_\nu R^\epsilon{}_{\sigma\rho\mu}\right)\xi^\sigma
\end{aligned}
\tag{3.2.4}
$$

and so on. In each natural field theory, one needs a finite number of these formulae which depends on the degree of Noether currents.

One can produce these formulae by iteration, though the procedure is somehow painfully involved; see[3].

Reduced currents

We say that a current of degree $(m-1)$ is *reduced* iff its components are symmetric in all their upper indices.

Reduced currents have a distinctive property: a closed reduced current of codegree 1 necessarily vanishes.

Let us consider a reduced current $\mathcal{C}$ of order 1 and degree $m-1$

$$\mathcal{C} = \left(\mathcal{C}^{\ \mu}_{\epsilon}\xi^\epsilon + \mathcal{C}^{\ \mu\alpha}_{\epsilon}\nabla_\alpha\xi^\epsilon\right)d\sigma_\mu = \mathcal{C}^\mu\, d\sigma_\mu \tag{3.2.5}$$

which is closed by hypothesis, i.e.

$$\nabla_\mu\left(\mathcal{C}^{\ \mu}_{\epsilon}\xi^\epsilon + \mathcal{C}^{\ \mu\alpha}_{\epsilon}\nabla_\alpha\xi^\epsilon\right) = 0 \tag{3.2.6}$$

Then we have

$$
\begin{aligned}
d_\mu \mathcal{C}^\mu =& \nabla_\mu \mathcal{C}_\epsilon{}^\mu \xi^\epsilon + \left(\mathcal{C}_\epsilon{}^\alpha + \nabla_\mu \mathcal{C}_\epsilon{}^{\mu\alpha}\right) \nabla_\alpha \xi^\epsilon + \mathcal{C}_\epsilon{}^{\mu\alpha} \left(\nabla_{\mu\alpha}\xi^\epsilon + \tfrac{1}{2} R^\epsilon{}_{\sigma\mu\alpha}\xi^\sigma\right) = \\
=& \left(\nabla_\mu \mathcal{C}_\epsilon{}^\mu + \tfrac{1}{2}\mathcal{C}_\sigma{}^{\mu\alpha} R^\sigma{}_{\epsilon\mu\alpha}\right)\xi^\epsilon + \left(\mathcal{C}_\epsilon{}^\alpha + \nabla_\mu \mathcal{C}_\epsilon{}^{\mu\alpha}\right)\nabla_\alpha \xi^\epsilon + \mathcal{C}_\epsilon{}^{\mu\alpha}\nabla_{\mu\alpha}\xi^\epsilon = 0
\end{aligned}
\tag{3.2.7}
$$

which, in view of the independence of the symmetrised covariant derivatives, implies

$$
\begin{cases}
\nabla_\mu \mathcal{C}_\epsilon{}^\mu + \tfrac{1}{2}\mathcal{C}_\sigma{}^{\mu\alpha} R^\sigma{}_{\epsilon\mu\alpha} = 0 \\
\mathcal{C}_\epsilon{}^\alpha + \nabla_\mu \mathcal{C}_\epsilon{}^{\mu\alpha} = 0 \\
\mathcal{C}_\epsilon{}^{(\mu\alpha)} = 0
\end{cases}
\tag{3.2.8}
$$

However, since the current is reduced, the coefficient $\mathcal{C}_\epsilon{}^{\mu\alpha}$ coincides with its symmetrisation and hence it vanishes by the third equation.

Once one knows that $\mathcal{C}_\epsilon{}^{\mu\alpha} = 0$, the second equation says also $\mathcal{C}_\epsilon{}^\alpha = 0$ vanishes and the first equation is identically satisfied. Then the reduced current $\mathcal{C}$ vanishes.

If we now consider a contribution to the reduced current for the order 2, namely $\mathcal{C}_\epsilon{}^{\mu\alpha\beta}\nabla_{\alpha\beta}\xi^\epsilon$, by closure one has

$$
\nabla_\mu \mathcal{C}_\epsilon{}^{\mu\alpha\beta}\nabla_{\alpha\beta}\xi^\epsilon + \mathcal{C}_\epsilon{}^{\mu\alpha\beta}\nabla_\mu \nabla_{\alpha\beta}\xi^\epsilon = 0
\tag{3.2.9}
$$

and one can expand $\nabla_\mu \nabla_{\alpha\beta}\xi^\epsilon$ in the basis of symmetrised covariant derivatives. One does not need here to know the details of this expansion, since one has a single contribution to third order, namely

$$
\mathcal{C}_\epsilon{}^{\mu\alpha\beta}\nabla_{\mu\alpha\beta}\xi^\epsilon = 0
\tag{3.2.10}
$$

which alone prove that $\mathcal{C}_\epsilon{}^{(\mu\alpha\beta)} = 0$ and, in view of reduction, $\mathcal{C}_\epsilon{}^{\mu\alpha\beta} = 0$.

Once we know that, all contributions from $\mathcal{C}_\epsilon{}^{\mu\alpha\beta}$ to lower order vanish and one repeats the proof for first order currents.

This technique can be iterated at will since at any order one can express $\nabla_\mu \nabla_{\alpha_1 \dots \alpha_s}\xi^\epsilon$ as a single contribution at order $s+1$ and a tail of contributions at lower orders.

However, the condition to order $s+1$ together with reduction proves that the coefficient $\mathcal{C}_\epsilon{}^{\mu\alpha_1 \dots \alpha_s} = 0$.

Thus, at any order, closed reduced currents vanish.

The condition of being reduced is trivial for currents of degree m (since they are symmetric in their upper indices by construction). On the other hand, currents of degrees $(m-2)$ are in the form

$$
\mathcal{C} = \left(\mathcal{C}_\epsilon{}^{\mu\nu}\xi^\epsilon + \mathcal{C}_\epsilon{}^{\mu\nu\alpha}\nabla_\alpha \xi^\epsilon + \mathcal{C}_\epsilon{}^{\mu\nu\alpha\beta}\nabla_{\alpha\beta}\xi^\epsilon + \dots\right)d\sigma_{\mu\nu}
\tag{3.2.11}
$$

and cannot be symmetric in their upper indices, since they are antisymmetric in $[\mu\nu]$. One can extend reductions to currents of degree $(m-2)$ by requiring maximal non-trivial symmetry, i.e.

$$
\mathcal{C}_\epsilon{}^{\mu\nu\alpha} = \mathcal{C}_\epsilon{}^{\mu(\nu\alpha)} \qquad \mathcal{C}_\epsilon{}^{\mu\nu\alpha\beta} = \mathcal{C}_\epsilon{}^{\mu(\nu\alpha\beta)} \qquad \dots
\tag{3.2.12}
$$

Anologously, at lower degrees.

We shall not need such general reduced currents, however, also in this case one can prove that closed reduced currents vanish.

Instead, let us show that being reduced depends on the connection one has decided to use.

Let us fix a connection $\Gamma^\alpha_{\beta\mu}$ and consider a reduced $(m-1)$-current of order 1, namely

$$
\mathcal{C} = \left(\mathcal{C}_\epsilon^\mu \xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha}\nabla_\alpha \xi^\epsilon\right)d\sigma_\mu
\tag{3.2.13}
$$

(thus with $\mathcal{C}_\epsilon^{\mu\alpha} = \mathcal{C}_\epsilon^{(\mu\alpha)}$).

Now let us define another (torsionless) connection $\hat\Gamma_{\beta\mu}^\alpha = \Gamma_{\beta\mu}^\alpha + K_{\beta\mu}^\alpha$ for which one has $\hat\nabla_\alpha \xi^\epsilon = \nabla_\alpha \xi^\epsilon + K_{\beta\alpha}^\epsilon \xi^\beta$. Thus the same current $\mathcal{C}$, expanded using the new connection, reads as

$$\mathcal{C} = \left(\mathcal{C}_\epsilon^\mu \xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha} \hat\nabla_\alpha \xi^\epsilon - \mathcal{C}_\epsilon^{\mu\alpha} K_{\beta\alpha}^\epsilon \xi^\beta\right) d\sigma_\mu = \left((\mathcal{C}_\epsilon^\mu - \mathcal{C}_\beta^{\mu\alpha} K_{\epsilon\alpha}^\beta)\xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha} \hat\nabla_\alpha \xi^\epsilon\right) d\sigma_\mu \qquad \begin{cases} \hat{\mathcal{C}}_\epsilon^\mu = \mathcal{C}_\epsilon^\mu - \mathcal{C}_\beta^{\mu\alpha} K_{\epsilon\alpha}^\beta \\ \hat{\mathcal{C}}_\epsilon^{\mu\alpha} = \mathcal{C}_\epsilon^{\mu\alpha} \end{cases} \tag{3.2.14}$$

Thus, when the current is reduced for the connection Γ, it is reduced also for the connection $\hat\Gamma$.

However, this is just because we chose a current of order 1. If we consider a current of order 2, i.e.

$$\mathcal{C} = \left(\mathcal{C}_\epsilon^\mu \xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon\right) d\sigma_\mu \tag{3.2.15}$$

(thus with $\mathcal{C}_\epsilon^{\mu\alpha} = \mathcal{C}_\epsilon^{(\mu\alpha)}$ and $\mathcal{C}_\epsilon^{\mu\alpha\beta} = \mathcal{C}_\epsilon^{(\mu\alpha\beta)}$) one has $\hat\nabla_\alpha \xi^\epsilon = \nabla_\alpha \xi^\epsilon + K_{\beta\alpha}^\epsilon \xi^\beta$

$$\begin{aligned} \hat\nabla_\alpha \hat\nabla_\beta \xi^\epsilon =& \nabla_\alpha \nabla_\beta \xi^\epsilon + \nabla_\alpha K_{\lambda\beta}^\epsilon \xi^\lambda + K_{\lambda\beta}^\epsilon \nabla_\alpha \xi^\lambda + K_{\beta\alpha}^\gamma \nabla_\gamma \xi^\epsilon + K_{\beta\alpha}^\gamma K_{\lambda\gamma}^\epsilon \xi^\lambda - K_{\gamma\alpha}^\epsilon \nabla_\beta \xi^\gamma - K_{\gamma\alpha}^\epsilon K_{\lambda\beta}^\gamma \xi^\lambda = \\ =& \nabla_\alpha \nabla_\beta \xi^\epsilon + \left(K_{\lambda\beta}^\epsilon \delta_\alpha^\gamma + K_{\beta\alpha}^\gamma \delta_\lambda^\epsilon - K_{\lambda\alpha}^\epsilon \delta_\beta^\gamma\right) \nabla_\gamma \xi^\lambda + \left(K_{\beta\alpha}^\gamma K_{\lambda\gamma}^\epsilon - K_{\gamma\alpha}^\epsilon K_{\lambda\beta}^\gamma + \nabla_\alpha K_{\lambda\beta}^\epsilon\right) \xi^\lambda \quad \Rightarrow \\ \Rightarrow \quad \hat\nabla_{\alpha\beta} \xi^\epsilon =& \nabla_{\alpha\beta} \xi^\epsilon + K_{\beta\alpha}^\gamma \nabla_\gamma \xi^\epsilon + \left(K_{\beta\alpha}^\gamma K_{\lambda\gamma}^\epsilon - K_{\gamma(\alpha}^\epsilon K_{\beta)\lambda}^\gamma + \nabla_{(\alpha} K_{\beta)\lambda}^\epsilon\right) \xi^\lambda \end{aligned} \tag{3.2.16}$$

When expanding the second order term of the current, one gets

$$\hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} \hat\nabla_{\alpha\beta} \xi^\epsilon = \hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} K_{\beta\alpha}^\gamma \nabla_\gamma \xi^\epsilon + \hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} \left(K_{\beta\alpha}^\gamma K_{\lambda\gamma}^\epsilon - K_{\gamma(\alpha}^\epsilon K_{\beta)\lambda}^\gamma + \nabla_{(\alpha} K_{\beta)\lambda}^\epsilon\right) \xi^\lambda \tag{3.2.17}$$

Thus the second order term contributes to second, first and zeroth order terms.

At second order, one has $\hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} = \mathcal{C}_\epsilon^{\mu\alpha\beta}$ and if the old term is reduced the new one is reduced as well.

However, at first order, one has

$$\mathcal{C}_\epsilon^{\mu\gamma} = \hat{\mathcal{C}}_\epsilon^{\mu\gamma} + \hat{\mathcal{C}}_\epsilon^{\mu\alpha\beta} K_{\beta\alpha}^\gamma \tag{3.2.18}$$

In fact, even if the old current is reduced, one has

$$\hat{\mathcal{C}}_\epsilon^{[\mu\gamma]} = K_{\alpha\beta}^{[\mu} \hat{\mathcal{C}}_\epsilon^{\gamma]\alpha\beta} \neq 0 \tag{3.2.19}$$

Thus $\hat{\mathcal{C}}_\epsilon^{\mu\gamma}$ is non-symmetric, in general. Thus, in general being, reduced depends on the connection, at least for currents of order higher than 1.

Conservation laws with currents

Let us now fix for simplicity the order of Noether current to 2 (and accordingly the order of the work current to 3). Thus we have

$$\begin{cases} \mathcal{E} = \left(\mathcal{E}_\epsilon^\mu \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon\right) d\sigma_\mu \\ \mathcal{W} = \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^\epsilon\right) d\sigma \end{cases} \tag{3.2.20}$$

The conservation laws $d\mathcal{E} = \mathcal{W}$ expands to

$$\nabla_\mu \left(\mathcal{E}_\epsilon^\mu \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon\right) = \mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^\epsilon \tag{3.2.21}$$

which is equivalent to the following conditions

$$
\begin{cases}
\nabla_\mu \mathcal{E}_\epsilon^{\ \mu} + \frac{1}{2}\mathcal{E}_\sigma^{\ \beta\alpha} R^\sigma_{\ \epsilon\beta\alpha} + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} = \mathcal{W}_\epsilon \\
\mathcal{E}_\epsilon^{\ \alpha} + \nabla_\mu \mathcal{E}_\epsilon^{\ \mu\alpha} + \mathcal{E}_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho} = \mathcal{W}_\epsilon^\alpha \\
\mathcal{E}_\epsilon^{\ (\beta\alpha)} + \nabla_\mu \mathcal{E}_\epsilon^{\ \mu\alpha\beta} = \mathcal{W}_\epsilon^{\alpha\beta} \\
\mathcal{E}_\epsilon^{\ (\gamma\alpha\beta)} = \mathcal{W}_\epsilon^{\alpha\beta\gamma}
\end{cases}
\tag{3.2.22}
$$

These conditions are just another equivalent form of conservation laws.

One can also try to integrate by parts the right hand side of conservation laws (3.2.21)

$$
\begin{aligned}
\mathcal{W} :={}& \mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta}\nabla_{\alpha\beta}\xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\alpha\beta\gamma}\xi^\epsilon = \\
={}& (\mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^\alpha)\xi^\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^{\alpha\beta}\nabla_\beta \xi^\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon + \nabla_\alpha\left(\mathcal{W}_\epsilon^\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta}\nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon\right) = \\
={}& \left(\mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^\alpha + \nabla_{\alpha\beta}\mathcal{W}_\epsilon^{\alpha\beta}\right)\xi^\epsilon + \nabla_{\alpha\beta}\mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_\gamma \xi^\epsilon + + \nabla_\alpha\left(\left(\mathcal{W}_\epsilon^\alpha - \nabla_\beta \mathcal{W}_\epsilon^{\alpha\beta}\right)\xi^\epsilon + \left(\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_\gamma \mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon\right) = \\
={}& \left(\mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^\alpha + \nabla_{\alpha\beta}\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\xi^\epsilon + + \nabla_\alpha\left(\left(\mathcal{W}_\epsilon^\alpha - \nabla_\beta \mathcal{W}_\epsilon^{\alpha\beta} + \nabla_{\beta\gamma}\mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\xi^\epsilon + \left(\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_\gamma \mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon\right)
\end{aligned}
\tag{3.2.23}
$$

This is particularly easy since the canonical work tensor densities are symmetric in upper indices. Let us set

$$
\begin{aligned}
\mathcal{B}_\epsilon :={}& \mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^\alpha + \nabla_{\alpha\beta}\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{W}_\epsilon^{\alpha\beta\gamma} \\
\mathcal{C}^\alpha :={}& \left(\mathcal{W}_\epsilon^\alpha - \nabla_\beta \mathcal{W}_\epsilon^{\alpha\beta} + \nabla_{\beta\gamma}\mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\xi^\epsilon + \left(\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_\gamma \mathcal{W}_\epsilon^{\alpha\beta\gamma}\right)\nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon
\end{aligned}
\tag{3.2.24}
$$

so to define the forms $\mathcal{B} := \mathcal{B}_\epsilon \xi^\epsilon \, d\sigma$, which is called the *Bianchi current*, and $\mathcal{C} := \mathcal{C}^\alpha \, d\sigma_\alpha$ which is called the *reduced current*. Let us stress that the reduced current is built with canonical work tensor densities which vanish on shell; it consequently vanishes on shell as well.

In other words, we just showed that one can recast the work current by covariant integration by parts as

$$
\mathcal{W} = \mathcal{B} + d\mathcal{C}
\tag{3.2.25}
$$

Now let us compute the Bianchi current

$$
\begin{aligned}
\mathcal{B}_\epsilon ={}& \mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^\alpha + \nabla_{\alpha\beta}\mathcal{W}_\epsilon^{\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{W}_\epsilon^{\alpha\beta\gamma} = \\
={}& \nabla_\mu \mathcal{E}_\epsilon^{\ \mu} + \frac{1}{2}\mathcal{E}_\sigma^{\ \beta\alpha} R^\sigma_{\ \epsilon\beta\alpha} + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} - \nabla_\alpha\left(\mathcal{E}_\epsilon^{\ \alpha} + \underline{\nabla_\mu \mathcal{E}_\epsilon^{\ \mu\alpha}} + \mathcal{E}_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho}\right) + + \nabla_{\alpha\beta}\left(\underline{\mathcal{E}_\epsilon^{\ (\beta\alpha)}} + \nabla_\mu \mathcal{E}_\epsilon^{\ \mu\alpha\beta}\right) - \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ (\gamma\alpha\beta)} = \\
={}& \frac{1}{2}\mathcal{E}_\sigma^{\ \alpha\beta} R^\sigma_{\ \epsilon\alpha\beta} + \frac{1}{2}[\nabla_\alpha,\nabla_\beta]\mathcal{E}_\epsilon^{\ \alpha\beta} + + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} - \nabla_\alpha\left(\mathcal{E}_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho}\right) + \nabla_{\alpha\beta}\nabla_\gamma \mathcal{E}_\epsilon^{\ \gamma\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} = \\
={}& \frac{1}{2}\left(\mathcal{E}_\sigma^{\ \alpha\beta} R^\sigma_{\ \epsilon\alpha\beta} + R^\alpha_{\ \sigma\alpha\beta}\mathcal{E}_\epsilon^{\ \sigma\beta} + R^\beta_{\ \sigma\alpha\beta}\mathcal{E}_\epsilon^{\ \alpha\sigma} - R^\sigma_{\ \epsilon\alpha\beta}\mathcal{E}_\sigma^{\ \alpha\beta} - 2R_{[\alpha\beta]}\mathcal{E}_\epsilon^{\ \alpha\beta}\right) + \\
& + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} - \nabla_\alpha\left(\mathcal{E}_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho}\right) + \nabla_{\alpha\beta}\nabla_\gamma \mathcal{E}_\epsilon^{\ \gamma\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} = \\
={}& \frac{1}{2}\left(R_{\alpha\beta} - R_{\beta\alpha} - 2R_{[\alpha\beta]}\right)\mathcal{E}_\epsilon^{\ \alpha\beta} + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} - \nabla_\alpha\left(\mathcal{E}_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho}\right) + \nabla_{\alpha\beta}\nabla_\gamma \mathcal{E}_\epsilon^{\ \gamma\alpha\beta} - \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} = \\
={}& -\frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\beta\gamma} - \nabla_\alpha\left(-\mathcal{E}_\sigma^{\ \gamma\alpha\beta} R^\sigma_{\ \epsilon\beta\gamma} + \frac{2}{3}\mathcal{E}_\epsilon^{\ \gamma\sigma\beta} R^\alpha_{\ \sigma\beta\gamma}\right) + \nabla_\alpha\left(\frac{2}{3}\mathcal{E}_\epsilon^{\ \gamma\sigma\beta} R^\alpha_{\ \sigma\beta\gamma} - \mathcal{E}_\sigma^{\ \gamma\alpha\beta} R^\sigma_{\ \epsilon\beta\gamma}\right) + \frac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\beta\gamma} \equiv 0
\end{aligned}
\tag{3.2.26}
$$

In the above proof, we used the identity

$$[\nabla_\alpha, \nabla_\beta]\mathcal{E}_\epsilon^{\ \alpha\beta} = R^\alpha_{\ \sigma\alpha\beta}\mathcal{E}_\epsilon^{\ \sigma\beta} + R^\beta_{\ \sigma\alpha\beta}\mathcal{E}_\epsilon^{\ \alpha\sigma} - R^\sigma_{\ \epsilon\alpha\beta}\mathcal{E}_\sigma^{\ \alpha\beta} - 2R_{[\alpha\beta]}\mathcal{E}_\epsilon^{\ \alpha\beta} = -R^\sigma_{\ \epsilon\alpha\beta}\mathcal{E}_\sigma^{\ \alpha\beta} \tag{3.2.27}$$

which holds true since $\mathcal{E}_\epsilon^{\ \alpha\beta}$ is a tensor density of weight 1 and the connection Γ is a generic torsionless connection (i.e. it does not need to be the Levi Civita connection of a metric and consequently its Ricci tensor does not need to be symmetric).

We also used the identity

$$\begin{aligned}
\nabla_{\alpha\beta}\nabla_\gamma\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} &= \tfrac{1}{3}\left(\nabla_\alpha\nabla_\beta\nabla_\gamma + \nabla_\beta\nabla_\gamma\nabla_\alpha + \nabla_\gamma\nabla_\alpha\nabla_\beta + \nabla_\beta[\nabla_\alpha,\nabla_\gamma] + \nabla_\alpha[\nabla_\beta,\nabla_\gamma] + [\nabla_\alpha,\nabla_\gamma]\nabla_\beta\right)\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} = \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} + \tfrac{1}{3}\left(2\nabla_\alpha[\nabla_\beta,\nabla_\gamma] + [\nabla_\alpha,\nabla_\gamma]\nabla_\beta\right)\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} = \\
&= \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} + \tfrac{2}{3}\nabla_\alpha\left(-\mathcal{E}_\epsilon^{\ \sigma\alpha\beta}R_{\sigma\beta} + \mathcal{E}_\epsilon^{\ \gamma\sigma\beta}R^\alpha_{\ \sigma\beta\gamma} + \mathcal{E}_\epsilon^{\ \gamma\alpha\sigma}R_{\sigma\gamma} - \mathcal{E}_\sigma^{\ \gamma\alpha\beta}R^\sigma_{\ \epsilon\beta\gamma} - 2R_{[\beta\gamma]}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta}\right) + \\
&\quad + \tfrac{1}{3}\left(-\nabla_\beta\mathcal{E}_\epsilon^{\ \sigma\alpha\beta}R_{\sigma\alpha} - \nabla_\beta\mathcal{E}_\sigma^{\ \gamma\alpha\beta}R^\sigma_{\ \epsilon\alpha\gamma} + \nabla_\beta\mathcal{E}_\epsilon^{\ \gamma\sigma\beta}R_{\sigma\gamma} + 2R_{[\gamma\alpha]}\nabla_\beta\mathcal{E}_\epsilon^{\ \gamma\alpha\beta}\right) = \\
&= \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} + \nabla_\alpha\left(\tfrac{2}{3}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta}R^\alpha_{\ \sigma\beta\gamma} - \tfrac{2}{3}\underline{\mathcal{E}_\sigma^{\ \gamma\alpha\beta}R^\sigma_{\ \epsilon\beta\gamma}}\right) - \tfrac{1}{3}\nabla_\alpha\left(\underline{\mathcal{E}_\sigma^{\ \gamma\alpha\beta}R^\sigma_{\ \epsilon\beta\gamma}}\right) + \tfrac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\beta\gamma} = \\
&= \nabla_{\alpha\beta\gamma}\mathcal{E}_\epsilon^{\ \gamma\alpha\beta} + \nabla_\alpha\left(\tfrac{2}{3}\mathcal{E}_\epsilon^{\ \gamma\sigma\beta}R^\alpha_{\ \sigma\beta\gamma} - \mathcal{E}_\sigma^{\ \gamma\alpha\beta}R^\sigma_{\ \epsilon\beta\gamma}\right) + \tfrac{1}{3}\mathcal{E}_\sigma^{\ \gamma\alpha\beta}\nabla_\alpha R^\sigma_{\ \epsilon\beta\gamma}
\end{aligned} \tag{3.2.28}$$

The vanishing of Bianchi current $\mathcal{B}_\epsilon = 0$ is called *(generalised) Bianchi identity*. Bianchi identities are characteristic of the natural field theory under consideration and any natural field theory produces its own Bianchi identities. We shall see that if one considers Hilbert–Einstein standard GR, generalised Bianchi identities are standard contracted Bianchi identities for the Riemann tensor $R^\alpha_{\ \beta\mu\nu}$.

Thus, while the reduced current vanishes on shell, the Bianchi current vanishes *identically*. We also say that the Bianchi identities vanish off shell (i.e. along any configuration not necessarily along solutions of field equations). Hence the work current is exact (*off-shell* exact!) and the reduced current is a potential for it; the work current can be recast as

$$\mathcal{W} = d\mathcal{C} \tag{3.2.29}$$

Superpotentials

By Noether theorem, we have that $d\mathcal{E} = \mathcal{W} \equiv d\mathcal{C}$. This is called a *weak conservation law* since, for the Noether current, $d\mathcal{E}$ vanishes on shell, though, in general, not off shell.

In view of Bianchi identities, one can recast Noether theorem in the form

$$d\mathcal{G} := d\left(\mathcal{E} - \mathcal{C}\right) = 0 \tag{3.2.30}$$

which instead holds off shell and it is hence called a *strong conservation law*.

The current $\mathcal{G} := \mathcal{E} - \mathcal{C}$ is strongly conserved and, since the reduced current $\mathcal{C}$ vanishes on shell, it coincides with the original Noether current on shell. The current $\mathcal{G}$ is called the *Gauss current*. The Gauss current coincides with the Noether current on shell, while it is modified off shell. On the other hand, the Gauss current $\mathcal{G}$ is enhanced in its off-shell behaviour since it is conserved off shell.

We shall show that the Gauss current is exact, i.e. there exists a $(m-2)$-current $\mathcal{U}$ called the *superpotential* such that

$$\mathcal{G} = d\mathcal{U} \qquad \Rightarrow \mathcal{E} = \mathcal{C} + d\mathcal{U} \tag{3.2.31}$$

Let us start by considering a Noether current of order 1, i.e.

$$
\begin{cases}
\mathcal{E} = \left(\mathcal{E}_\epsilon{}^\mu \xi^\epsilon + \mathcal{E}_\epsilon{}^{\mu\alpha} \nabla_\alpha \xi^\epsilon \right) d\sigma_\mu \\
\mathcal{W} = \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon{}^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon{}^{\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon \right) d\sigma
\end{cases}
\tag{3.2.32}
$$

One can recast Noether current as

$$
\begin{aligned}
\mathcal{E} &= \left(\mathcal{E}_\epsilon{}^\mu \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon{}^{[\mu\alpha]} \nabla_\alpha \xi^\epsilon \right) d\sigma_\mu = \left(\left(\mathcal{E}_\epsilon{}^\mu - \nabla_\alpha \mathcal{E}_\epsilon{}^{[\mu\alpha]} \right) \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon + \nabla_\alpha \left(\mathcal{E}_\epsilon{}^{[\mu\alpha]} \xi^\epsilon \right) \right) d\sigma_\mu = \\
&= \left(\left(\mathcal{E}_\epsilon{}^\mu - \nabla_\alpha \mathcal{E}_\epsilon{}^{[\mu\alpha]} \right) \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon \right) d\sigma_\mu + d \left(\tfrac{1}{2} \mathcal{E}_\epsilon{}^{[\mu\alpha]} \xi^\epsilon \, d\sigma_{\mu\alpha} \right)
\end{aligned}
\tag{3.2.33}
$$

and define

$$
\tilde{\mathcal{C}} = \left(\left(\mathcal{E}_\epsilon{}^\mu - \nabla_\alpha \mathcal{E}_\epsilon{}^{[\mu\alpha]} \right) \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon \right) d\sigma_\mu \qquad \mathcal{U} = \tfrac{1}{2} \mathcal{E}_\epsilon{}^{\mu\alpha} \xi^\epsilon \, d\sigma_{\mu\alpha}
\tag{3.2.34}
$$

Notice that the current $\tilde{\mathcal{C}}$ is reduced. Then one has for the Gauss current

$$
\mathcal{G} = \tilde{\mathcal{C}} - \mathcal{C} + d\mathcal{U}
\tag{3.2.35}
$$

and, from strong conservation laws, $d(\mathcal{C} - \tilde{\mathcal{C}}) = 0$. Since the current $\mathcal{C} - \tilde{\mathcal{C}}$ is reduced and closed, it vanishes and $\tilde{\mathcal{C}} = \mathcal{C}$.

In fact one can also check this directly

$$
\mathcal{C} = \left(\mathcal{W}_\epsilon{}^\alpha - \nabla_\beta \mathcal{W}_\epsilon{}^{\alpha\beta} \right) \xi^\epsilon + \mathcal{W}_\epsilon{}^{\alpha\beta} \nabla_\beta \xi^\epsilon = \left(\mathcal{E}_\epsilon{}^\alpha + \nabla_\beta \mathcal{E}_\epsilon{}^{\beta\alpha} - \nabla_\beta \mathcal{E}_\epsilon{}^{(\beta\alpha)} \right) \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\beta\alpha)} \nabla_\beta \xi^\epsilon = \left(\left(\mathcal{E}_\epsilon{}^\mu - \nabla_\alpha \mathcal{E}_\epsilon{}^{[\mu\alpha]} \right) \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon \right) d\sigma_\mu \equiv \mathcal{C}'
\tag{3.2.36}
$$

Thus one has

$$
\mathcal{E} = \mathcal{C} + d\mathcal{U} \qquad \mathcal{W} = d\mathcal{C} \qquad \mathcal{B} = 0
\tag{3.2.37}
$$

which tells us pretty much everything about conservation laws.

This scheme applies to higher orders as well.

For example, by considering a Noether current of order 2, namely

$$
\begin{aligned}
\mathcal{E} &= \mathcal{E}_\epsilon{}^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon = \left(\mathcal{E}_\epsilon{}^{(\mu\alpha\beta)} + \tfrac{2}{3} \left(\mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} + \mathcal{E}_\epsilon{}^{[\mu\beta]\alpha} \right) \right) \nabla_{\alpha\beta} \xi^\epsilon = \mathcal{E}_\epsilon{}^{(\mu\alpha\beta)} \nabla_{\alpha\beta} \xi^\epsilon + \tfrac{2}{3} \left(2\mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} \nabla_\alpha \nabla_\beta \xi^\epsilon + \mathcal{E}_\epsilon{}^{[\mu\beta]\alpha} R^\epsilon{}_{\sigma\alpha\beta} \xi^\sigma \right) = \\
&= \mathcal{E}_\epsilon{}^{(\mu\alpha\beta)} \nabla_{\alpha\beta} \xi^\epsilon + \tfrac{4}{3} \nabla_\alpha \left(\mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} \nabla_\beta \xi^\epsilon \right) + \tfrac{2}{3} \mathcal{E}_\epsilon{}^{[\mu\beta]\alpha} R^\epsilon{}_{\sigma\alpha\beta} \xi^\sigma - \tfrac{4}{3} \nabla_\alpha \mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} \nabla_\beta \xi^\epsilon = \\
&= \mathcal{E}_\epsilon{}^{(\mu\alpha\beta)} \nabla_{\alpha\beta} \xi^\epsilon + \tfrac{4}{3} \nabla_\alpha \left(\mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} \nabla_\beta \xi^\epsilon \right) + \mathcal{E}'_\epsilon{}^\mu \xi^\epsilon + \mathcal{E}'_\epsilon{}^{\mu\alpha} \nabla_\alpha \xi^\epsilon
\end{aligned}
\tag{3.2.38}
$$

where we set $\mathcal{E}'_\epsilon{}^\mu := \tfrac{2}{3} \mathcal{E}_\sigma{}^{[\mu\beta]\alpha} R^\sigma{}_{\epsilon\alpha\beta}$ and $\mathcal{E}'_\epsilon{}^{\mu\alpha} := -\tfrac{4}{3} \nabla_\beta \mathcal{E}_\epsilon{}^{[\mu\beta]\alpha} = -\tfrac{2}{3} \left(\nabla_\beta \mathcal{E}_\epsilon{}^{\mu\beta\alpha} - \nabla_\beta \mathcal{E}_\epsilon{}^{\beta\mu\alpha} \right)$ and we obtain $\mathcal{C}' = \mathcal{E}'_\epsilon{}^\mu \xi^\epsilon + \mathcal{E}'_\epsilon{}^{\mu\alpha} \nabla_\alpha \xi^\epsilon$ which is a current of order 1, which can hence be manipulated as above. Hence overall we obtain

$$
\mathcal{E} = \left(\left(\mathcal{E}'_\epsilon{}^\mu - \nabla_\alpha \mathcal{E}'_\epsilon{}^{[\mu\alpha]} \right) \xi^\epsilon + \mathcal{E}'_\epsilon{}^{(\mu\alpha)} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon{}^{(\mu\alpha\beta)} \nabla_{\alpha\beta} \xi^\epsilon \right) d\sigma_\mu + d \left(\tfrac{1}{2} \left(\tfrac{4}{3} \mathcal{E}_\epsilon{}^{[\mu\alpha]\beta} \nabla_\beta \xi^\epsilon + \mathcal{E}'_\epsilon{}^{[\mu\alpha]} \xi^\epsilon \right) d\sigma_{\mu\alpha} \right)
\tag{3.2.39}
$$

Thus a leading term of order 2 contributes to the superpotential by a current

$$\mathcal{U}' = \tfrac{1}{2}\left(\tfrac{4}{3}\mathcal{E}_\epsilon^{\,[\mu\alpha]\beta}\nabla_\beta\xi^\epsilon + \mathcal{E}'_{\;\epsilon}^{\,[\mu\alpha]}\xi^\epsilon\right)d\sigma_{\mu\alpha} = \tfrac{1}{2}\left(\tfrac{4}{3}\mathcal{E}_\epsilon^{\,[\mu\alpha]\beta}\nabla_\beta\xi^\epsilon - \tfrac{2}{3}\nabla_\beta\mathcal{E}_\epsilon^{\,[\mu\alpha]\beta}\xi^\epsilon\right)d\sigma_{\mu\alpha} \tag{3.2.40}$$

and to the reduced current by a current

$$\mathcal{C}' = \left(\left(\tfrac{2}{3}\mathcal{E}_\sigma^{\,[\mu\beta]\alpha}R^\sigma_{\;\epsilon\alpha\beta} + \tfrac{2}{3}\nabla_\alpha\nabla_\beta\mathcal{E}_\epsilon^{\,[\mu\alpha]\beta}\right)\xi^\epsilon - \tfrac{2}{3}\left(\nabla_\beta\mathcal{E}_\epsilon^{\,(\mu\alpha)\beta} - \nabla_\beta\mathcal{E}_\epsilon^{\,\beta\mu\alpha}\right)\nabla_\alpha\xi^\epsilon + \mathcal{E}_\epsilon^{\,(\mu\alpha\beta)}\nabla_{\alpha\beta}\xi^\epsilon\right)d\sigma_\mu \tag{3.2.41}$$

Thus, if one starts by a general Noether current of order 2, namely

$$\mathcal{E} = \left(\mathcal{E}_\epsilon^{\,\mu}\xi^\epsilon + \mathcal{E}_\epsilon^{\,\mu\alpha}\nabla_\alpha\xi^\epsilon + \mathcal{E}_\epsilon^{\,\mu\alpha\beta}\nabla_{\alpha\beta}\xi^\epsilon\right)d\sigma_\mu \tag{3.2.42}$$

this can be decomposed as $\mathcal{E} = \mathcal{C} + d\mathcal{U}$, where the reduced current is

$$\mathcal{C} = \left(\left(\mathcal{E}_\epsilon^{\,\mu} - \nabla_\alpha\mathcal{E}_\epsilon^{\,[\mu\alpha]} + \tfrac{1}{3}\mathcal{E}_\sigma^{\,\alpha\beta\mu}R^\sigma_{\;\epsilon\alpha\beta} + \tfrac{2}{3}\nabla_\alpha\nabla_\beta\mathcal{E}_\epsilon^{\,[\mu\alpha]\beta}\right)\xi^\epsilon + \left(\mathcal{E}_\epsilon^{\,(\mu\alpha)} + \tfrac{2}{3}\nabla_\beta\mathcal{E}_\epsilon^{\,\beta\mu\alpha} - \tfrac{2}{3}\nabla_\beta\mathcal{E}_\epsilon^{\,(\mu\alpha)\beta}\right)\nabla_\alpha\xi^\epsilon + \mathcal{E}_\epsilon^{\,(\mu\alpha\beta)}\nabla_{\alpha\beta}\xi^\epsilon\right)d\sigma_\mu \tag{3.2.43}$$

and the superpotential is

$$\mathcal{U} = \tfrac{1}{2}\left(\tfrac{4}{3}\mathcal{E}_\epsilon^{\,\mu\alpha\beta}\nabla_\beta\xi^\epsilon + \left(\mathcal{E}_\epsilon^{\,\mu\alpha} - \tfrac{2}{3}\nabla_\beta\mathcal{E}_\epsilon^{\,\mu\alpha\beta}\right)\xi^\epsilon\right)d\sigma_{\mu\alpha} \tag{3.2.44}$$

Usually, this will be enough for us. If we are considering a first order theory, the Liouville part $\mathbb{F}$ depends on variations at order zero and we can treat geometric objects of order 2 (thus including connections). If we are considering a second order theory the Liouville part $\mathbb{F}$ depends on first derivatives of variations and we can deal with fields which are first order (e.g. the metric). Since we shall consider first order theories depending on connections (which are second order objects) or second order theories depending on a metric (which is first order) the Noether current will be at most of order two and the decomposition above will be enough for our purposes.

However, this can be iterated up to the order needed. For example, at order 3 one considers the leading term

$$\mathcal{E}_\epsilon^{\,\mu\alpha\beta\gamma}\nabla_{\alpha\beta\gamma}\xi^\epsilon \tag{3.2.45}$$

All one has to do is find a relation between the coefficient and its symmetrised part, something like

$$\mathcal{E}_\epsilon^{\,(\mu\alpha\beta\gamma)} = \tfrac{1}{4}\left(\mathcal{E}_\epsilon^{\,\mu\alpha\beta\gamma} + \mathcal{E}_\epsilon^{\,\gamma\mu\alpha\beta} + \mathcal{E}_\epsilon^{\,\beta\gamma\mu\alpha} + \mathcal{E}_\epsilon^{\,\alpha\beta\gamma\mu}\right) = \mathcal{E}_\epsilon^{\,\mu\alpha\beta\gamma} + \tfrac{1}{2}\left(\mathcal{E}_\epsilon^{\,[\gamma\mu]\alpha\beta} + \mathcal{E}_\epsilon^{\,[\beta\mu]\gamma\alpha} + \mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\right) \quad \Rightarrow \mathcal{E}_\epsilon^{\,\mu\alpha\beta\gamma} = \mathcal{E}_\epsilon^{\,(\mu\alpha\beta\gamma)} - \tfrac{1}{2}\left(\mathcal{E}_\epsilon^{\,[\gamma\mu]\alpha\beta} + \mathcal{E}_\epsilon^{\,[\beta\mu]\gamma\alpha} + \mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\right) \tag{3.2.46}$$

We should not worry about the symmetrised term $\mathcal{E}_\epsilon^{\,(\mu\alpha\beta\gamma)}\nabla_{\alpha\beta\gamma}\xi^\epsilon$, since it contributes to the reduced current. Thus we are left with

$$-\tfrac{1}{2}\left(\mathcal{E}_\epsilon^{\,[\gamma\mu]\alpha\beta} + \mathcal{E}_\epsilon^{\,[\beta\mu]\gamma\alpha} + \mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\right)\nabla_{\alpha\beta\gamma}\xi^\epsilon = -\tfrac{1}{2}\mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\left(\nabla_\alpha\nabla_{\beta\gamma}\xi^\epsilon + \nabla_\beta\nabla_{\gamma\alpha}\xi^\epsilon + \nabla_\gamma\nabla_{\alpha\beta}\xi^\epsilon\right) =$$

$$= -\tfrac{1}{2}\mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\left(3\nabla_\alpha\nabla_\beta\nabla_\gamma\xi^\epsilon + \underline{R^\epsilon_{\;\sigma\beta\alpha}\nabla_\gamma\xi^\sigma} - \underline{R^\sigma_{\;\gamma\beta\alpha}\nabla_\sigma\xi^\epsilon} + \nabla_\beta R^\epsilon_{\;\sigma\gamma\alpha}\xi^\sigma + \underline{R^\epsilon_{\;\sigma\gamma\alpha}\nabla_\beta\xi^\sigma} + \underline{R^\epsilon_{\;\sigma\gamma\alpha}\nabla_\beta\xi^\sigma} - \underline{R^\sigma_{\;\beta\gamma\alpha}\nabla_\sigma\xi^\epsilon}\right) =$$

$$= \nabla_\alpha\left(-\tfrac{3}{2}\mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\nabla_\beta\nabla_\gamma\xi^\epsilon\right) + \tfrac{3}{2}\nabla_\alpha\mathcal{E}_\epsilon^{\,[\alpha\mu]\beta\gamma}\nabla_{\beta\gamma}\xi^\epsilon + \left(\tfrac{3}{2}\mathcal{E}_\sigma^{\,[\mu\nu]\beta\alpha}R^\sigma_{\;\epsilon\beta\nu} - \mathcal{E}_\epsilon^{\,[\mu\nu]\beta\gamma}R^\alpha_{\;\gamma\beta\nu}\right)\nabla_\alpha\xi^\epsilon - \tfrac{1}{2}\mathcal{E}_\sigma^{\,[\alpha\mu]\beta\gamma}\nabla_\beta R^\sigma_{\;\epsilon\gamma\alpha}\xi^\epsilon \tag{3.2.47}$$

The first term on the right hand side contributes to superpotential. What is then left is a current of order 2 which can be treated as above.

3. Noether currents gauge-natural theories

In a gauge-natural theory, one starts from a structure bundle $\mathcal{P} = (P, M, p, G)$. A generator of gauge symmetries is a right-invariant vector field on
the structure bundle, namely

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^A(x)\rho_A \tag{3.3.1}$$

where ρ_A is a right-invariant pointwise basis of vertical vectors on $\mathcal{P}$. By choosing a principal connection ω_μ^A on the structure bundle and a (torsionless) connection $\Gamma^\alpha_{\beta\mu}$ on spacetime, one can define covariant derivatives of tensor densities.

Let us set $\xi^A_{(V)} = \xi^a + \omega_\mu^A \xi^\mu$ for the vertical part of the infinitesimal symmetry generator. Since Lie derivatives are linear in $(\xi^\mu, \xi^A_{(V)})$, the current can be expanded as linear combinations of $(\xi^\mu, \xi^A_{(V)})$ and their symmetrised covariant derivatives.

For example, the work current in a gauge-natural theory of order 3 can be expressed as

$$\mathcal{W} = \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^\epsilon + \mathcal{W}_A \xi^A_{(V)} + \mathcal{W}_A^\alpha \nabla_\alpha \xi^A_{(V)} + \mathcal{W}_A^{\alpha\beta} \nabla_{\alpha\beta} \xi^A_{(V)} + \mathcal{W}_A^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^A_{(V)} \right) d\sigma \tag{3.3.2}$$

while the Noether current can be expressed as

$$\mathcal{E} = \left(\mathcal{E}_\epsilon^\mu \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{E}_A^\mu \xi^A_{(V)} + \mathcal{E}_A^{\mu\alpha} \nabla_\alpha \xi^A_{(V)} + \mathcal{E}_A^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^A_{(V)} \right) d\sigma_\mu \tag{3.3.3}$$

We can hence repeat the argument done for natural theories, just using the correct relation among higher order covariant derivatives. For example, we can compute

$$[\nabla_\alpha, \nabla_\beta] \xi^A_{(V)} = c^A{}_{BC} F^B_{\alpha\beta} \xi^C_{(V)} \tag{3.3.4}$$

where $F^B_{\alpha\beta}$ denotes the curvature of the gauge connection and $c^A{}_{BC}$ are the structure constants of the gauge group.

Reduced currents are defined as in the natural case. A current of order 1 and degree $m-1$

$$\mathcal{C} = \left(\mathcal{C}_\epsilon^\mu \xi^\epsilon + \mathcal{C}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{C}_A^\mu \xi^A_{(V)} + \mathcal{C}_A^{\mu\alpha} \nabla_\alpha \xi^A_{(V)} \right) d\sigma_\mu = \mathcal{C}^\mu d\sigma_\mu \tag{3.3.5}$$

is reduced iff $\mathcal{C}_\epsilon^{\mu\alpha} = \mathcal{C}_\epsilon^{(\mu\alpha)}$ and $\mathcal{C}_A^{\mu\alpha} = \mathcal{C}_A^{(\mu\alpha)}$.

Let us now fix, for simplicity, the order of Noether current to 2 (and accordingly the order of the work current to 3). Thus we have

$$\begin{cases} \mathcal{E} = \left(\mathcal{E}_\epsilon^\mu \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha} \nabla_\alpha \xi^\epsilon + \mathcal{E}_\epsilon^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{E}_A^\mu \xi^A_{(V)} + \mathcal{E}_A^{\mu\alpha} \nabla_\alpha \xi^A_{(V)} + \mathcal{E}_A^{\mu\alpha\beta} \nabla_{\alpha\beta} \xi^A_{(V)} \right) d\sigma_\mu \\ \mathcal{W} = \left(\mathcal{W}_\epsilon \xi^\epsilon + \mathcal{W}_\epsilon^\alpha \nabla_\alpha \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta} \nabla_{\alpha\beta} \xi^\epsilon + \mathcal{W}_\epsilon^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^\epsilon + \mathcal{W}_A \xi^A_{(V)} + \mathcal{W}_A^\alpha \nabla_\alpha \xi^A_{(V)} + \mathcal{W}_A^{\alpha\beta} \nabla_{\alpha\beta} \xi^A_{(V)} + \mathcal{W}_A^{\alpha\beta\gamma} \nabla_{\alpha\beta\gamma} \xi^A_{(V)} \right) d\sigma \end{cases} \tag{3.3.6}$$

The conservation laws $d\mathcal{E} = \mathcal{W}$ expand to

$$\begin{cases} \nabla_\mu \mathcal{E}_\epsilon^\mu + \frac{1}{2} \mathcal{E}_\sigma^{\beta\alpha} R^\sigma{}_{\epsilon\beta\alpha} + \frac{1}{3} \mathcal{E}_\sigma^{\gamma\alpha\beta} \nabla_\alpha R^\sigma{}_{\epsilon\gamma\beta} = \mathcal{W}_\epsilon \\ \mathcal{E}_\epsilon^\alpha + \nabla_\mu \mathcal{E}_\epsilon^{\mu\alpha} + \mathcal{E}_\sigma^{\rho\mu\alpha} R^\sigma{}_{\epsilon\rho\mu} + \frac{2}{3} \mathcal{E}_\epsilon^{\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} = \mathcal{W}_\epsilon^\alpha \\ \mathcal{E}_\epsilon^{(\beta\alpha)} + \nabla_\mu \mathcal{E}_\epsilon^{\mu\alpha\beta} = \mathcal{W}_\epsilon^{\alpha\beta} \\ \mathcal{E}_\epsilon^{(\gamma\alpha\beta)} = \mathcal{W}_\epsilon^{\alpha\beta\gamma} \end{cases} \qquad \begin{cases} \nabla_\mu \mathcal{E}_A^\mu + \frac{1}{2} c^B{}_{CA} \mathcal{E}_B^{\beta\alpha} F^C_{\beta\alpha} + \frac{1}{3} c^B{}_{CA} \mathcal{E}_B^{\gamma\alpha\beta} \nabla_\alpha F^C_{\gamma\beta} = \mathcal{W}_A \\ \mathcal{E}_A^\alpha + \nabla_\mu \mathcal{E}_A^{\mu\alpha} + c^B{}_{CA} \mathcal{E}_B^{\rho\mu\alpha} F^C_{\rho\mu} + \frac{2}{3} \mathcal{E}_A^{\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} = \mathcal{W}_A^\alpha \\ \mathcal{E}_A^{(\beta\alpha)} + \nabla_\mu \mathcal{E}_A^{\mu\alpha\beta} = \mathcal{W}_A^{\alpha\beta} \\ \mathcal{E}_A^{(\gamma\alpha\beta)} = \mathcal{W}_A^{\alpha\beta\gamma} \end{cases} \tag{3.3.7}$$

By integrating the work current by parts, one has

$$
\begin{aligned}
\mathcal{W} := & \left(\mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^{\,\alpha} + \nabla_{\alpha\beta} \mathcal{W}_\epsilon^{\,\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \right) \xi^\epsilon + \left(\mathcal{W}_A - \nabla_\alpha \mathcal{W}_A^{\,\alpha} + \nabla_{\alpha\beta} \mathcal{W}_A^{\,\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{W}_A^{\,\alpha\beta\gamma} \right) \xi_{(V)}^A + \\
& + \nabla_\alpha \left(\left(\mathcal{W}_\epsilon^{\,\alpha} - \nabla_\beta \mathcal{W}_\epsilon^{\,\alpha\beta} + \nabla_{\beta\gamma} \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \right) \xi^\epsilon + \left(\mathcal{W}_\epsilon^{\,\alpha\beta} - \nabla_\gamma \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \right) \nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \nabla_{\beta\gamma} \xi^\epsilon \right) + \\
& + \nabla_\alpha \left(\left(\mathcal{W}_A^{\,\alpha} - \nabla_\beta \mathcal{W}_A^{\,\alpha\beta} + \nabla_{\beta\gamma} \mathcal{W}_A^{\,\alpha\beta\gamma} \right) \xi_{(V)}^A + \left(\mathcal{W}_A^{\,\alpha\beta} - \nabla_\gamma \mathcal{W}_A^{\,\alpha\beta\gamma} \right) \nabla_\beta \xi_{(V)}^A + \mathcal{W}_A^{\,\alpha\beta\gamma} \nabla_{\beta\gamma} \xi_{(V)}^A \right)
\end{aligned}
\tag{3.3.8}
$$

Let us set

$$
\begin{aligned}
\mathcal{B}_\epsilon := & \, \mathcal{W}_\epsilon - \nabla_\alpha \mathcal{W}_\epsilon^{\,\alpha} + \nabla_{\alpha\beta} \mathcal{W}_\epsilon^{\,\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \\
\mathcal{B}_A := & \, \mathcal{W}_A - \nabla_\alpha \mathcal{W}_A^{\,\alpha} + \nabla_{\alpha\beta} \mathcal{W}_A^{\,\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{W}_A^{\,\alpha\beta\gamma} \\
\mathcal{C}^{\,\alpha} := & \, \left(\mathcal{W}_\epsilon^{\,\alpha} - \nabla_\beta \mathcal{W}_\epsilon^{\,\alpha\beta} + \nabla_{\beta\gamma} \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \right) \xi^\epsilon + \left(\mathcal{W}_\epsilon^{\,\alpha\beta} - \nabla_\gamma \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \right) \nabla_\beta \xi^\epsilon + \mathcal{W}_\epsilon^{\,\alpha\beta\gamma} \nabla_{\beta\gamma} \xi^\epsilon \\
\mathcal{D}^{\,\alpha} := & \, \left(\mathcal{W}_A^{\,\alpha} - \nabla_\beta \mathcal{W}_A^{\,\alpha\beta} + \nabla_{\beta\gamma} \mathcal{W}_A^{\,\alpha\beta\gamma} \right) \xi_{(V)}^A + \left(\mathcal{W}_A^{\,\alpha\beta} - \nabla_\gamma \mathcal{W}_A^{\,\alpha\beta\gamma} \right) \nabla_\beta \xi_{(V)}^A + \mathcal{W}_A^{\,\alpha\beta\gamma} \nabla_{\beta\gamma} \xi_{(V)}^A
\end{aligned}
\tag{3.3.9}
$$

so to define the forms $\mathcal{B} := \left(\mathcal{B}_\epsilon \xi^\epsilon + \mathcal{B}_A \xi_{(V)}^A \right) d\sigma$, which is called the *Bianchi current*, and $\mathcal{C} := (\mathcal{C}^\alpha + \mathcal{D}^\alpha) \, d\sigma_\alpha$ which is called the *reduced current*. Let us stress that the reduced current vanishes on shell.

Thus the work current, by covariant integration by parts, can be recast as

$$
\mathcal{W} = \mathcal{B} + d\mathcal{C}
\tag{3.3.10}
$$

While one has $\mathcal{B}_\epsilon = 0$ by essentially the same computation done for natural theories, let us compute the gauge part of the Bianchi current

$$
\begin{aligned}
\mathcal{B}_A = & \, \mathcal{W}_A - \nabla_\alpha \mathcal{W}_A^{\,\alpha} + \nabla_{\alpha\beta} \mathcal{W}_A^{\,\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{W}_A^{\,\alpha\beta\gamma} = \\
= & \, \nabla_\mu \mathcal{E}_A^{\,\mu} + \tfrac{1}{2} \mathcal{E}_B^{\,\beta\alpha} c^B{}_{CA} F_{\beta\alpha}^C + \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\gamma\beta}^C - \nabla_\alpha \left(\mathcal{E}_A^{\,\alpha} + \underline{\nabla_\mu \mathcal{E}_A^{\,\mu\alpha}} + \mathcal{E}_B^{\,\rho\mu\alpha} c^B{}_{CA} F_{\rho\mu}^C + \tfrac{2}{3} \mathcal{E}_A^{\,\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} \right) + \nabla_{\alpha\beta} \left(\underline{\mathcal{E}_A^{\,(\beta\alpha)}} + \nabla_\mu \mathcal{E}_A^{\,\mu\alpha\beta} \right) - \nabla_{\alpha\beta\gamma} \mathcal{E}_A^{\,(\gamma\alpha\beta)} = \\
= & \, \tfrac{1}{2} \mathcal{E}_B^{\,\alpha\beta} c^B{}_{CA} F_{\alpha\beta}^C + \tfrac{1}{2} [\nabla_\alpha, \nabla_\beta] \mathcal{E}_A^{\,\alpha\beta} + + \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\gamma\beta}^C - \nabla_\alpha \left(\mathcal{E}_B^{\,\rho\mu\alpha} c^B{}_{CA} F_{\rho\mu}^C + \tfrac{2}{3} \mathcal{E}_A^{\,\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} \right) + \nabla_{\alpha\beta} \nabla_\gamma \mathcal{E}_A^{\,\gamma\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{E}_A^{\,\gamma\alpha\beta} = \\
= & \, \tfrac{1}{2} \left(\mathcal{E}_B^{\,\alpha\beta} c^B{}_{CA} F_{\alpha\beta}^C + R^\alpha{}_{\sigma\alpha\beta} \mathcal{E}_A^{\,\sigma\beta} + R^\beta{}_{\sigma\alpha\beta} \mathcal{E}_A^{\,\alpha\sigma} - c^B{}_{CA} F_{\alpha\beta}^C \mathcal{E}_B^{\,\alpha\beta} - 2 R_{[\alpha\beta]} \mathcal{E}_A^{\,\alpha\beta} \right) + \\
& \, + \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\gamma\beta}^C - \nabla_\alpha \left(\mathcal{E}_B^{\,\rho\mu\alpha} c^B{}_{CA} F_{\rho\mu}^C + \tfrac{2}{3} \mathcal{E}_A^{\,\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} \right) + \nabla_{\alpha\beta} \nabla_\gamma \mathcal{E}_A^{\,\gamma\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{E}_A^{\,\gamma\alpha\beta} = \\
= & \, \tfrac{1}{2} \left(R_{\alpha\beta} - R_{\beta\alpha} - 2 R_{[\alpha\beta]} \right) \mathcal{E}_A^{\,\alpha\beta} + \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\gamma\beta}^C - \nabla_\alpha \left(\mathcal{E}_B^{\,\rho\mu\alpha} c^B{}_{CA} F_{\rho\mu}^C + \tfrac{2}{3} \mathcal{E}_A^{\,\rho\mu\nu} R^\alpha{}_{\mu\nu\rho} \right) + \nabla_{\alpha\beta} \nabla_\gamma \mathcal{E}_A^{\,\gamma\alpha\beta} - \nabla_{\alpha\beta\gamma} \mathcal{E}_A^{\,\gamma\alpha\beta} = \\
= & \, - \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\beta\gamma}^C - \nabla_\alpha \left(- \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} F_{\beta\gamma}^C + \tfrac{2}{3} \mathcal{E}_A^{\,\gamma\sigma\beta} R^\alpha{}_{\sigma\beta\gamma} \right) + \nabla_\alpha \left(\tfrac{2}{3} \mathcal{E}_A^{\,\gamma\sigma\beta} R^\alpha{}_{\sigma\beta\gamma} - \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} F_{\beta\gamma}^C \right) + \tfrac{1}{3} \mathcal{E}_B^{\,\gamma\alpha\beta} c^B{}_{CA} \nabla_\alpha F_{\beta\gamma}^C \equiv 0
\end{aligned}
\tag{3.3.11}
$$

In the above proof, we used the identity

$$
[\nabla_\alpha, \nabla_\beta] \mathcal{E}_A^{\,\alpha\beta} = R^\alpha{}_{\sigma\alpha\beta} \mathcal{E}_A^{\,\sigma\beta} + R^\beta{}_{\sigma\alpha\beta} \mathcal{E}_A^{\,\alpha\sigma} - c^B{}_{CA} F_{\alpha\beta}^C \mathcal{E}_B^{\,\alpha\beta} - 2 R_{[\alpha\beta]} \mathcal{E}_A^{\,\alpha\beta} = - c^B{}_{CA} F_{\alpha\beta}^C \mathcal{E}_B^{\,\alpha\beta}
\tag{3.3.12}
$$

which holds true since $\mathcal{E}_A^{\alpha\beta}$ is a tensor density of weight 1 and the connection Γ is a generic torsionless connection (i.e. it does not need to be the Levi Civita connection of a metric and consequently its Ricci tensor does not need to be symmetric). We also used the identity

$$
\begin{aligned}
\nabla_{\alpha\beta}\nabla_\gamma\mathcal{E}_A^{\gamma\alpha\beta} =& \tfrac{1}{3}\left(\nabla_\alpha\nabla_\beta\nabla_\gamma + \nabla_\beta\nabla_\gamma\nabla_\alpha + \nabla_\gamma\nabla_\alpha\nabla_\beta + \nabla_\beta[\nabla_\alpha,\nabla_\gamma] + \nabla_\alpha[\nabla_\beta,\nabla_\gamma] + [\nabla_\alpha,\nabla_\gamma]\nabla_\beta\right)\mathcal{E}_A^{\gamma\alpha\beta} = \nabla_{\alpha\beta\gamma}\mathcal{E}_A^{\gamma\alpha\beta} + \tfrac{1}{3}\left(2\nabla_\alpha[\nabla_\beta,\nabla_\gamma] + [\nabla_\alpha,\nabla_\gamma]\nabla_\beta\right)\mathcal{E}_A^{\gamma\alpha\beta} = \\
=& \nabla_{\alpha\beta\gamma}\mathcal{E}_A^{\gamma\alpha\beta} + \tfrac{2}{3}\nabla_\alpha\left(-\mathcal{E}_A^{\sigma\alpha\beta}R_{\sigma\beta} + \mathcal{E}_A^{\gamma\sigma\beta}R^\alpha{}_{\sigma\beta\gamma} + \mathcal{E}_A^{\gamma\alpha\sigma}R_{\sigma\gamma} - \mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}F^C_{\beta\gamma} - 2R_{[\beta\gamma]}\mathcal{E}_A^{\gamma\alpha\beta}\right) + \\
& + \tfrac{1}{3}\left(-\nabla_\beta\mathcal{E}_A^{\sigma\alpha\beta}R_{\sigma\alpha} - \nabla_\beta\mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}F^C_{\alpha\gamma} + \nabla_\beta\mathcal{E}_A^{\gamma\sigma\beta}R_{\sigma\gamma} + 2R_{[\gamma\alpha]}\nabla_\beta\mathcal{E}_A^{\gamma\alpha\beta}\right) = \\
=& \nabla_{\alpha\beta\gamma}\mathcal{E}_A^{\gamma\alpha\beta} + \nabla_\alpha\left(\tfrac{2}{3}\mathcal{E}_A^{\gamma\sigma\beta}R^\alpha{}_{\sigma\beta\gamma} - \tfrac{2}{3}\mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}F^C_{\beta\gamma}\right) - \tfrac{1}{3}\nabla_\alpha\left(\mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}F^C_{\beta\gamma}\right) + \tfrac{1}{3}\mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}\nabla_\alpha F^C_{\beta\gamma} = \\
=& \nabla_{\alpha\beta\gamma}\mathcal{E}_A^{\gamma\alpha\beta} + \nabla_\alpha\left(\tfrac{2}{3}\mathcal{E}_A^{\gamma\sigma\beta}R^\alpha{}_{\sigma\beta\gamma} - \mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}F^C_{\beta\gamma}\right) + \tfrac{1}{3}\mathcal{E}_B^{\gamma\alpha\beta}c^B{}_{CA}\nabla_\alpha F^C_{\beta\gamma}
\end{aligned}
\tag{3.3.13}
$$

The off-shell vanishing of Bianchi current $\mathcal{B}_\epsilon = 0$ and $\mathcal{B}_A = 0$ are called *(generalised) Bianchi identities*. Hence the work current can be recast as

$$
\mathcal{W} = d\mathcal{C}
\tag{3.3.14}
$$

Also, existence of superpotentials is analogous to the natural case. The Noether current of order 2 can always be recast as $\mathcal{E} = \mathcal{C} + d\mathcal{U}$ where the gauge part $\mathcal{D} = \mathcal{D}^\alpha\, d\sigma_\alpha$ of reduced current is

$$
\mathcal{D} = \left(\left(\mathcal{E}_A^\mu - \nabla_\alpha\mathcal{E}_A^{[\mu\alpha]} + \tfrac{1}{3}c^B{}_{CA}\mathcal{E}_B^{\alpha\beta\mu}F^C_{\alpha\beta} + \tfrac{2}{3}\nabla_\alpha\nabla_\beta\mathcal{E}_A^{[\mu\alpha]\beta}\right)\xi^A_{(V)} + \left(\mathcal{E}_A^{(\mu\alpha)} + \tfrac{2}{3}\nabla_\beta\mathcal{E}_A^{\beta\mu\alpha} - \tfrac{2}{3}\nabla_\beta\mathcal{E}_A^{(\mu\alpha)\beta}\right)\nabla_\alpha\xi^A_{(V)} + \mathcal{E}_A^{(\mu\alpha\beta)}\nabla_{\alpha\beta}\xi^A_{(V)}\right)\, d\sigma_\mu
\tag{3.3.15}
$$

and the superpotential is

$$
\mathcal{U} = \tfrac{1}{2}\left(\tfrac{4}{3}\mathcal{E}_\epsilon^{\mu\alpha\beta}\nabla_\beta\xi^\epsilon + \left(\mathcal{E}_\epsilon^{\mu\alpha} - \tfrac{2}{3}\nabla_\beta\mathcal{E}_\epsilon^{\mu\alpha\beta}\right)\xi^\epsilon + \tfrac{4}{3}\mathcal{E}_A^{\mu\alpha\beta}\nabla_\beta\xi^A_{(V)} + \left(\mathcal{E}_A^{\mu\alpha} - \tfrac{2}{3}\nabla_\beta\mathcal{E}_A^{\mu\alpha\beta}\right)\xi^A_{(V)}\right)\, d\sigma_{\mu\alpha}
\tag{3.3.16}
$$

If needed, superpotentials can be found by iteration at any order.

Unifying notation

The natural and gauge-natural theories can be treated on equal footing as far as conservation laws are concerned. For, it is convenient to have a standardised way of refer to symmetry generators. To this end ,we can define a new index family $\mathbf{a}, \mathbf{b}, \ldots$ such that $\xi^{\mathbf{a}} = (\xi^\mu)$ in natural theories, and $\xi^{\mathbf{a}} = (\xi^\mu, \xi^A_{(V)})$ in gauge-natural theories. Let us also set

$$
R^A{}_{B\mu\nu} := c^A{}_{CB}F^C_{\mu\nu}
\tag{3.3.17}
$$

Thus one has

$$
[\nabla_\mu,\nabla_\nu]\xi^{\mathbf{a}} = R^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}\xi^{\mathbf{b}} \qquad \Longleftrightarrow \qquad
\begin{cases}
[\nabla_\mu,\nabla_\nu]\xi^\epsilon = R^\epsilon{}_{\sigma\mu\nu}\xi^\sigma \\
[\nabla_\mu,\nabla_\nu]\xi^A_{(V)} = c^A{}_{CB}F^C_{\mu\nu}\xi^B_{(V)}
\end{cases}
\tag{3.3.18}
$$

By using this notation, a Noether current of order 2 is

$$
\mathcal{E} = \left(\mathcal{E}_{\mathbf{a}}^\mu\xi^{\mathbf{a}} + \mathcal{E}_{\mathbf{a}}^{\mu\alpha}\nabla_\alpha\xi^{\mathbf{a}} + \mathcal{E}_{\mathbf{a}}^{\mu\alpha\beta}\nabla_{\alpha\beta}\xi^{\mathbf{a}}\right)\, d\sigma_\mu
\tag{3.3.19}
$$

:Notation: :Symbols: :AIndex: :Index:

the work form is

$$\mathcal{W} = \left(\mathcal{W}_{\mathbf{a}}\xi^{\mathbf{a}} + \mathcal{W}_{\mathbf{a}}^{\alpha}\nabla_{\alpha}\xi^{\mathbf{a}} + \mathcal{W}_{\mathbf{a}}^{\alpha\beta}\nabla_{\alpha\beta}\xi^{\mathbf{a}} + \mathcal{W}_{\mathbf{a}}^{\alpha\beta\gamma}\nabla_{\alpha\beta\gamma}\xi^{\mathbf{a}} \right) d\sigma \tag{3.3.20}$$

and the corresponding superpotential is

$$\mathcal{U} = \tfrac{1}{2} \left(\tfrac{4}{3}\mathcal{E}_{\mathbf{a}}^{\mu\alpha\beta}\nabla_{\beta}\xi^{\mathbf{a}} + \left(\mathcal{E}_{\mathbf{a}}^{\mu\alpha} - \tfrac{2}{3}\nabla_{\beta}\mathcal{E}_{\mathbf{a}}^{\mu\alpha\beta} \right)\xi^{\mathbf{a}} \right) d\sigma_{\mu\alpha} \tag{3.3.21}$$

Also, if one has a metric g in the theory (be it fundamental or composite as in spin frame formulation), then one can systematically turn densities into tensors, e.g.

$$\mathcal{E}_{\mathbf{a}}^{\mu} := \sqrt{g}\, E_{\mathbf{a}}^{\mu} \qquad \mathcal{E}_{\mathbf{a}}^{\mu\alpha} := \sqrt{g}\, E_{\mathbf{a}}^{\mu\alpha} \qquad \mathcal{E}_{\mathbf{a}}^{\mu\alpha\beta} := \sqrt{g}\, E_{\mathbf{a}}^{\mu\alpha\beta} \tag{3.3.22}$$

or

$$\mathcal{W}_{\mathbf{a}} := \sqrt{g}\, W_{\mathbf{a}} \qquad \mathcal{W}_{\mathbf{a}}^{\alpha} := \sqrt{g}\, W_{\mathbf{a}}^{\alpha} \qquad \mathcal{W}_{\mathbf{a}}^{\alpha\beta} := \sqrt{g}\, W_{\mathbf{a}}^{\alpha\beta} \qquad \mathcal{W}_{\mathbf{a}}^{\alpha\beta\gamma} := \sqrt{g}\, W_{\mathbf{a}}^{\alpha\beta\gamma} \tag{3.3.23}$$

The tensors $(E_{\mathbf{a}}^{\mu}, E_{\mathbf{a}}^{\mu\alpha}, E_{\mathbf{a}}^{\mu\alpha\beta}, \dots)$ are also called *canonical stress tensors* while the tensors $(W_{\mathbf{a}}^{\mu}, W_{\mathbf{a}}^{\alpha}, W_{\mathbf{a}}^{\alpha\beta}, W_{\mathbf{a}}^{\alpha\beta\gamma}, \dots)$ are called the *canonical work tensors*.

4. Conserved quantities

Once in a field theory we computed Noether currents and their superpotentials, one can pull them back along a configuration to obtain forms on spacetime. If the work current is

$$\mathcal{W} = \sqrt{g} \left(W_{\mathbf{a}}\xi^{\mathbf{a}} + W_{\mathbf{a}}^{\alpha}\nabla_{\alpha}\xi^{\mathbf{a}} + W_{\mathbf{a}}^{\alpha\beta}\nabla_{\alpha\beta}\xi^{\mathbf{a}} + W_{\mathbf{a}}^{\alpha\beta\gamma}\nabla_{\alpha\beta\gamma}\xi^{\mathbf{a}} \right) d\sigma \tag{3.4.1}$$

the Noether current is in the form

$$\mathcal{E} = \sqrt{g} \left(E_{\mathbf{a}}^{\mu}\xi^{\mathbf{a}} + E_{\mathbf{a}}^{\mu\alpha}\nabla_{\alpha}\xi^{\mathbf{a}} + E_{\mathbf{a}}^{\mu\alpha\beta}\nabla_{\alpha\beta}\xi^{\mathbf{a}} \right) d\sigma_{\mu} \tag{3.4.2}$$

and the superpotential is

$$\mathcal{U} = \tfrac{\sqrt{g}}{2} \left(U_{\mathbf{a}}^{\mu\nu}\xi^{\mathbf{a}} + U_{\mathbf{a}}^{\mu\nu\alpha}\nabla_{\alpha}\xi^{\mathbf{a}} \right) d\sigma_{\mu\nu} \tag{3.4.3}$$

then their pull-back along a section σ in the configuration bundle will be denoted by

$$\mathcal{W}(\sigma) = (j^{2k}\sigma)^{*}\mathcal{W} \qquad \mathcal{E}(\sigma) = (j^{2k-1}\sigma)^{*}\mathcal{E} \qquad \mathcal{U}(\sigma) = (j^{2k-2}\sigma)^{*}\mathcal{U} \tag{3.4.4}$$

which are spacetime forms of degree m, $(m-1)$, and $(m-2)$, respectively. Of course, if σ is a solution of field equations, then $\mathcal{W}(\sigma) = 0$ and $\mathcal{E}(\sigma) = d\mathcal{U}(\sigma)$.

Now, if we consider an $(m-1)$-region Ω of spacetime the integrals

$$\tilde{Q} = \int_{\Omega} (j^{2k-1}\sigma)^{*}\mathcal{E} \tag{3.4.5}$$

are well-defined and independent of coordinates, as well as, for any $(m-2)$-region D the integrals

$$Q = \int_D (j^{2k-2}\sigma)^* \mathcal{U} \tag{3.4.6}$$

Thus, as we fix a solution σ and suitable regions, we have two numbers for any symmetry generator $\xi^{\mathbf{a}}$. In a (gauge-)natural theory, there are infinitely many symmetry generators, thus we have plenty of well-defined numbers to play with. Not that these numbers are all relevant or independent from each other, however, we are not short of them. We can instead easily predict that our problem will not be finding the numbers but selecting the relevant ones and giving them their correct physical meaning.

First of all, let us remark that, if we integrate the Noether current on an open region, the solution can be unbounded and the integral can diverge.

> If we have fields on spacetime and a pointlike source at a point $x \in \bar{M}$ usually fields diverge as approaching to x. For that reason, one usually purges the point x from spacetime (to have fields which are smooth on the base manifold $M = \bar{M} - \{x\}$).
> Thus one is left with a manifold $M \subset \bar{M}$ and smooth fields on it. However, fields still goes to infinity as one gets closed to what was the point x, even if x does not belong to M any longer.
> Accordingly, one can choose a hypersurface $\Omega \subset M$ going close to the point x, on which fields are unbounded. The integral for $\tilde{Q}$ is an improper integral and may diverge depending on how fast fields blow up.

Let us then start by inspecting some relation among these numbers and considering an $(m-1)$-region Ω with no boundary. Then

$$\tilde{Q} = \int_\Omega (j^{2k-1}\sigma)^* \mathcal{E} = \int_\Omega d(j^{2k-2}\sigma)^* \mathcal{U} = \int_{\partial\Omega \equiv \varnothing} (j^{2k-2}\sigma)^* \mathcal{U} = 0 \tag{3.4.7}$$

Consequently, if one has two regions Ω_1 and Ω_2 with the same boundary $\partial\Omega_1 = \partial\Omega_2 \equiv \partial\Omega$, then one has

$$\tilde{Q}_1 - \tilde{Q}_2 = \int_{\Omega_1 - \Omega_2} (j^{2k-1}\sigma)^* \mathcal{E} = \int_{\Omega_1 - \Omega_2} d(j^{2k-2}\sigma)^* \mathcal{U} = \int_{\partial\Omega - \partial\Omega} (j^{2k-2}\sigma)^* \mathcal{U} = 0 \qquad \Rightarrow \tilde{Q}_1 = \tilde{Q}_2 \tag{3.4.8}$$

> In general one has a similar, though weaker, result with a more general proof. If the region $\Omega_1 - \Omega_2 = \partial V$ is the boundary of an m-region V then one has
>
> $$\tilde{Q}_1 - \tilde{Q}_2 = \int_{\Omega_1 - \Omega_2} (j^{2k-1}\sigma)^* \mathcal{E} = \int_{\partial V} (j^{2k-1}\sigma)^* \mathcal{E} = \int_V (j^{2k-1}\sigma)^* d\mathcal{E} = 0 \qquad \Rightarrow \tilde{Q}_1 = \tilde{Q}_2 \tag{3.4.9}$$
>
> in view of conservation on the Noether current.
> This argument uses conservation of the Noether current instead of superpotentials, however, it proves $\tilde{Q}_1 = \tilde{Q}_2$ only when the two regions $\Omega_1 - \Omega_2 = \partial V$ form a boundary. Using superpotentials, one can prove it for any closed region $\Omega_1 - \Omega_2$.

Accordingly, the integral of the Noether current on a region Ω actually just depends on the boundary $D = \partial\Omega$. This can be seen directly, by considering an $(m-2)$-region $D = \partial\Omega$ which is the boundary of an $(m-1)$-region Ω, then, in view of Stokes theorem, one has

$$Q = \int_{D \equiv \partial\Omega} (j^{2k-2}\sigma)^* \mathcal{U} = \int_\Omega d(j^{2k-2}\sigma)^* \mathcal{U} = \int_\Omega (j^{2k-1}\sigma)^* \mathcal{E} = \tilde{Q} \tag{3.4.10}$$

Under these assumptions, the integrals (3.4.10) are the most general ones to be considered as integrals of the Noether currents.

However, we have a mismatch. While we have one integral of the Noether current on any $(m-1)$-region Ω, which can be recast as integral on a boundary $D = \partial\Omega$ of the superpotential, the superpotentials can be still integrated on any $(m-2)$-region D, not necessarily a boundary.

Let then D be an $(m-2)$-region (not necessarily a boundary) and consider

$$Q = \int_D (j^{2k-2}\sigma)^*\mathcal{U} \tag{3.4.11}$$

Let us consider two regions D_1 and D_2 such that $D_1 - D_2 = \partial\Omega$ is the boundary of some $(m-1)$-region Ω. Then one has

$$Q_1 - Q_2 = \int_{D_1 - D_2}(j^{2k-2}\sigma)^*\mathcal{U} = \int_{\partial\Omega}(j^{2k-2}\sigma)^*\mathcal{U} = \int_{\Omega}(j^{2k-1}\sigma)^*d\mathcal{U} = \int_{\Omega}(j^{2k-1}\sigma)^*\mathcal{E} \tag{3.4.12}$$

For example, let us consider two spheres D_1 and D_2 at different radii around the point $x \in \bar{M}$ as discussed above. Then imagine to shrink the sphere D_2 on the point x. At any stage $Q_1 - Q_2 = \int_{\Omega}(j^{2k-1}\sigma)^*\mathcal{E}$ and, as the sphere D_2 shrinks around the point x, that becomes what one would define as the integral of the Noether current on the region Ω with a hole at x. This is how one would try to give a meaning to improper integrals and the integral of the superpotential on the sphere D_2 would be called the *residual contribution* at the point x.

This is the loose motivation why the integral of the superpotential on the sphere D_1 (which is well-defined)

$$Q_1 = \int_{D_1}(j^{2k-2}\sigma)^*\mathcal{U} = \int_{\Omega}(j^{2k-1}\sigma)^*\mathcal{E} + Q_2 \tag{3.4.13}$$

is considered to be the *"value of the integral of the Noether current on the open region $\Omega \cup \{x\}$"*.

In view of this discussion, we can forget about integrals of the Noether currents and focus on integrals of the superpotentials which are more general.

In relativistic theories then, for any region D with no boundary, one can consider the integrals

$$Q = \int_D (j^{2k-2}\sigma)^*\mathcal{U} \tag{3.4.14}$$

which are (infinitely many, one for each symmetry generator) good numbers associated to any solution σ. These are called the *conserved quantities* associated to the symmetry generator considered within the surface D.

> For example, one considers a solution of Einstein–Maxwell equations corresponding to a pointlike particle falling around. These solutions are singular on the worldline of the charge particle so that that worldline is purged from spacetime to guarantee smooth fields.
> Then, consider the $(m-2)$-region defined at some time $t = t_0$ around the point particle by $r = r_0$. This is a surface D with the topology of the sphere, which surrounds the particle at time $t = t_0$, on which fields are well-defined. One can compute the integral of the superpotential associated to vertical gauge transformations (which, in that case, we already discussed to be the flow of the electric field; see (1.10.99)) and, at least on Minkowski space, we are ready to accept it defines the electric charge of the pointlike particle. Of course, one expects to find the same result by saying that the electric field is generated by a *suitable* distributional charge density supported on the worldline and integrating the Noether current (which is the divergence of the electric field, which in view of field equations and Stokes theorem can be written as the integral of the charge density distribution).
> Here, in fact, there is nothing really to prove since the *suitable* charge density distribution is exactly the one which makes this miracle to happen. In other words, one could say that superpotentials are a way to avoid the details in defining distributional sources by just focusing on well-defined integrals.

Our standard attitude will be to recover physical quantities as integrals of suitable superpotentials on suitable regions with no boundary. Searching for physical quantities as integrals of superpotentials may seem optimistic (as well hopeless since, typically, we have infinitely many superpotentials attached to each solution). On the other hand, one can regard the issue as an application of the general relativity principle. According to this principle, a physical quantity is an important part of the description of the real world if and only if it can be expressed as something which is independent of the observer conventions. Since the integrals of superpotentials are known to be independent of the observer conventions they are the right place to look for physical quantities.

There is another viewpoint which is popular in the literature which goes in another direction. It has been argued that physical quantities are *by their own nature* observer dependent since they are measured by observers. For example, the *energy* of a system is reasonably expected to depend on the observer conventions (and I agree with this remark). Consequently, it has been argued that one can define physical quantities as integrals of objects that are not forms but they are considered to be written in the coordinates defining the observers. These quantities are called *pseudo-potentials* and, of course, geometrically speaking, they make no sense.

However, when one has two observers each computing the same physical quantity and obtaining two different results then one should discuss whether the difference is justified by considering that the physical quantity under consideration has been measured by two different observers. In order to discuss that, one would need a complete control on the relation between the observers and their conventions, something which is often beyond the case, even in simple situations.

Moreover, one should pay attention that the procedure does not, in fact, select a special class of observers which would spoil general relativity (as well as they are known not be traditionally well-defined in physics; in some sense relativity—all Galileian, Special and General relativities—exactly originated by the original sin of privileging inertial frames without too much and fundamental support).

I have to confess, I have never managed to give a fundamental meaning to pseudo-potentials, nor a mathematical or a physical meaning. It seems to me that, if they are the solution, then that is simply declaring that the physical quantities one defines by using them, even though possibly traditionally important, are unessential to the description of the physical world. The good news is that one does not need to use pseudo-potentials. Some argued that the integrals of superpotential, being intrinsic, do not depend on the observer and as such cannot represent quantities as the energy which *is* expected to depend on the observer after all!

Well, this is flat wrong.

Superpotentials are intrinsic forms, though they do depend on the symmetry generator under consideration. If each observer selects a different generator to define its own energy, then the energy can depend on the observer even being well defined as a geometric quantity.

We shall see that, in many situations, this can be done and it gives the expected results whenever we are able to predict what should be expected.

We shall see some of such machinery at work below in the examples.

5. Augmented variational principles

In some sense, the Lagrangian of a system is defined modulo pure divergences, meaning that adding a pure divergence to the Lagrangian does not affect field equations though it changes Noether currents and superpotentials.

Let us consider a (gauge) natural Lagrangian $\boldsymbol{L}$, a field-dependent (global) horizontal $(m-1)$-form $\boldsymbol{\alpha} = \alpha^\mu \, d\sigma_\mu$ and a new Lagrangian

$$\boldsymbol{L}' = \boldsymbol{L} + d\boldsymbol{\alpha} \tag{3.5.1}$$

The pure divergence $d\boldsymbol{\alpha} = d_\mu \alpha^\mu \, d\sigma$ does not affect field equations, though it contributes to Noether currents and superpotentials.

Assuming for the sake of simplicity that the form $\boldsymbol{\alpha}$ depends on fields and their derivatives up to order 1, the variation of $d\boldsymbol{\alpha} = d_\mu \alpha^\mu\, d\sigma = \left(\partial_i \alpha^\mu y^i_\mu + \partial^\alpha_i \alpha^\mu y^i_{\alpha\mu} \right) d\sigma$ is

$$
\begin{aligned}
\delta d_\mu \alpha^\mu =& \underline{\partial_{ik} \alpha^\mu y^i_\mu \delta y^k} + \underline{\underline{\partial^\alpha_k \partial_i \alpha^\mu y^i_\mu d_\alpha \delta y^k}} + \partial_i \alpha^\mu d_\mu \delta y^i + \partial_k \partial^\alpha_i \alpha^\mu y^i_{\alpha\mu} \delta y^k + \underline{\underline{\partial^\alpha_k \partial^\epsilon_i \alpha^\mu y^i_{\epsilon\mu} d_\alpha \delta y^k}} + \partial^\alpha_i \alpha^\mu \delta y^i_{\alpha\mu} = \\
=& \underline{d_\mu \partial_k \alpha^\mu \delta y^k} + \underline{\underline{d_\mu \partial^\alpha_k \alpha^\mu d_\alpha \delta y^k}} + \underline{\partial_i \alpha^\mu d_\mu \delta y^i} + \underline{\underline{\partial^\alpha_i \alpha^\mu d_\mu \delta y^i_\alpha}} = d_\mu \left(\partial_k \alpha^\mu \delta y^k + \partial^\alpha_k \alpha^\mu \delta y^k_\alpha \right)
\end{aligned}
\tag{3.5.2}
$$

Thus, as expected, the pure divergence does not contribute to the Euler–Lagrange part, while it contributes to the Liouville one by

$$
\mathbb{F}' = \mathbb{F} + \left(\partial_k \alpha^\mu \omega^k \wedge\, d\sigma_\mu + \partial^\alpha_k \alpha^\mu \omega^k_\alpha \wedge\, d\sigma_\mu \right)
\tag{3.5.3}
$$

The pure divergence Lagrangian automatically obeys covariance identity

$$
\underline{d_\mu \partial_k \alpha^\mu \pounds_\Xi y^k} + \underline{\underline{d_\mu \partial^\alpha_k \alpha^\mu d_\alpha \pounds_\Xi y^k}} + \underline{\partial_i \alpha^\mu d_\mu \pounds_\Xi y^i} + \underline{\underline{\partial^\alpha_i \alpha^\mu d_\mu \pounds_\Xi y^i_\alpha}} = d_\mu \left(\partial_k \alpha^\mu \pounds_\Xi y^k + \partial^\alpha_k \alpha^\mu \pounds_\Xi y^k_\alpha \right) = \pounds_\Xi d_\mu \alpha^\mu = d_\epsilon \left(\xi^\epsilon d_\mu \alpha^\mu \right)
\tag{3.5.4}
$$

so that $\boldsymbol{L}'$ is a (gauge-)natural Lagrangian as well.
The Noether current for the Lagrangian $\boldsymbol{L}'$ is

$$
\begin{aligned}
\mathcal{E}' =& \mathcal{E} + \left(\partial_k \alpha^\mu \pounds_\Xi y^k + \partial^\alpha_k \alpha^\mu d_\alpha \pounds_\Xi y^k \right)\, d\sigma_\mu - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \mathcal{E} + \pounds_\xi \boldsymbol{\alpha} - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \mathcal{E} + \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu + d \left(\xi^\nu \alpha^\mu\, d\sigma_{\mu\nu} \right) - \xi^\mu d_\nu \alpha^\nu\, d\sigma_\mu = \\
=& \mathcal{E} + \left({\color{red}\xi^\mu d_\nu \alpha^\nu} + d_\nu \left(\xi^\nu \alpha^\mu \right) - d_\nu \left(\xi^\mu \alpha^\nu \right) - {\color{red}\xi^\mu d_\nu \alpha^\nu} \right) d\sigma_\mu = \mathcal{E} + 2 d_\nu \left(\xi^{[\nu} \alpha^{\mu]} \right) d\sigma_\mu = \mathcal{E} + d_\lambda \left(\xi^{[\nu} \alpha^{\mu]} \right) dx^\lambda \wedge\, d\sigma_{\mu\nu} = \mathcal{E} + d \left(\xi^{[\nu} \alpha^{\mu]}\, d\sigma_{\mu\nu} \right)
\end{aligned}
\tag{3.5.5}
$$

Thus the superpotential is

$$
\mathcal{U}' = \mathcal{U} - \xi^{[\mu} \alpha^{\nu]}\, d\sigma_{\mu\nu}
\tag{3.5.6}
$$

Accordingly, as a consequence of adding a pure divergence $d\boldsymbol{\alpha}$ to the Lagrangian $\boldsymbol{L}$, one gets an extra contribution to the superpotential

$$
\Delta = -\xi^{[\mu} \alpha^{\nu]}\, d\sigma_{\mu\nu} \qquad\qquad \mathcal{U}' = \mathcal{U} + \Delta
\tag{3.5.7}
$$

When computing the conserved quantities, this extra contribution is integrated on a $(m-2)$-region D with no boundary and has no reason to vanish. In fact, one can check in examples that it does not vanish in general.

> Let us remark that in mechanics this is not the case. By adding a total time-derivative to a mechanical Lagrangian does not modify the Noether current at all, as it is manifest by specialising equation (3.5.5) to dimension $\dim(M) = m = 1$.
>
> What we are discussing hereafter has no counterpart in mechanics and it is specific of field theories. Consequently, one should not trust too much physical intuition for it, when it comes from mechanics, since its generalisation to fields theory may not be harmless.

Thus whenever integrals of superpotential define quantities which are not what one expects, one can still hope that adding a suitable divergence to the Lagrangian the new superpotential accounts for the expected quantities. This is what one does defining the so-called *augmented variational principles* which are a canonical choice of the divergence term which produces better conserved quantities than the original theories.

Thus we have to find reasons to complain about integrals of superpotential to fix this ambiguity. In order to keep the discussion focused, let us consider *energy* and *angular momentum*, though, in the end, most of the arguments will hold in general.

In mechanics, *energy* is associated by Noether theorem to invariance with respect to time translations. In a relativistic field theory, one has fields which live on spacetime and observers represented by coordinate systems on spacetime. *ADM* observers correspond to coordinates (t, x^i) in which one coordinate comes with the meaning of a time. One can consider the corresponding vector field $\xi = \partial_t$ which is the nearest thing to a time translation.

The vector field ξ, as any other spacetime vector field, is a symmetry generator and one can compute the superpotential for it and take the integral on a $(m-2)$-region with no boundary. We shall call it the *energy* of the fields within the region surrounded by D.

If one considers spacetimes which support an action of the group of spatial rotations, there are vector fields ζ which are generators of spatial rotations. The integral of the superpotential of such generators of rotations are called *angular momentum.*

Analogously, space translations generate *momentum* and one can include boosts to define the *energy-momentum tensor* which assembles the conserved quantities associated to rotations (angular momentum) and boosts in a symmetric spacetime tensor $T_{\mu\nu}$.

Of course, different observers define a different transformation as *time translation* and this accounts for dependence of energy on the observer.

Relative conservation laws

First of all, one should not talk about *the* energy of a system. This is an unphysical notion in relativistic theories as it was in Newtonian physics.

> The physical interpretation of these multiple energies defined in field theory goes back to thermodynamics where one has different quantities (the *internal energy*, the *Helmholtz free energy*, the *Gibbs free energy*, the *enthalpy*, and so on) each associated to different boundary conditions which, in turn, express which variables are controlled at the boundary. Boundary conditions are associated to boundary counterterms, thus one expects *an energy* of the system for any boundary term added to the Lagrangian, which is, in fact, what we have.
>
> This is why one expects many energies in a field theory and what we are up to discuss hereafter is how to choose *an energy* in this set that can represent what one usually expects as *the energy*.

There is nothing like a scale to measure *the energy* of a system but all forms of energy are *relative* to something else, being it a reference frame or another state of the system. Only differences of energy matter. Potential energy is measured relative to a conventional zero level. Kinetic energy is relative to the rest frame which is conventionally chosen by the observer, as was already known by Galilean physics before Newton came.

The *reference frames* used in mechanics strongly rely on the affine structure on space(time) which does not survive in relativistic theories. However, one can try to assume the differences of energies as fundamental. If a system undergoes a transformation and goes from one state to another, its energy changes. The amount of energy needed to go from one state to another should be better defined on a physical stance than *the* energy of the system.

This kind of energy difference will be called the *relative conserved quantity* between two solutions. One should double the configuration bundle $_*\mathcal{C} = \mathcal{C} \times_M \mathcal{C}$ which will be called the *extended configuration bundle* and it has coordinates $(x^\mu, y^i, \bar{y}^i)$ to account for the two states. A section of the extended bundle $_*\sigma : M \to {}_*\mathcal{C}$ is a pair of sections $\sigma, \bar{\sigma} : M \to \mathcal{C}$ which, in fact, are two configurations of the original system. Conventionally, $\sigma : M \to \mathcal{C} : x \mapsto y(x)$ will be called the *(real) field* while $\bar{\sigma} : M \to \mathcal{C} : x \mapsto \bar{y}(x)$ will be called the *reference field.*

Of course, the evolution of the two fields should be independent, since we do not want the real field to behave differently depending on how we fix the reference field. Accordingly, if the original Lagrangian was $\boldsymbol{L}(j\sigma)$, the *augmented Lagrangian* will be something like

$$_*\boldsymbol{L}(j\sigma, j\bar{\sigma}) = \boldsymbol{L}(j\sigma) - \boldsymbol{L}(j\bar{\sigma}) + d\boldsymbol{\alpha}(j\sigma, j\bar{\sigma}) \tag{3.5.8}$$

Both the real and reference fields will be treated as dynamical, and they both obey the same field equations as in the original theory.

The conserved quantity obtained in the theory for the Lagrangian $_*\boldsymbol{L}$ (which will depend on both σ and $\bar{\sigma}$) is called the *relative conserved quantity* between σ and $\bar{\sigma}$ and it is

$$Q_D(_*\boldsymbol{L}, \Xi, \sigma, \bar{\sigma}) = \boxed{\int_\Omega (j^{2k-1}\sigma)^* \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi)} + \int_D (j^{2k-2}\sigma)^* \mathcal{U}(\boldsymbol{L}, \Xi) - \int_D \xi^{[\mu}\alpha^{\nu]}(\sigma, \bar{\sigma})\, d\sigma_{\mu\nu} - \boxed{\int_\Omega (j^{2k-1}\bar{\sigma})^* \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi)} - \int_D (j^{2k-2}\bar{\sigma})^* \mathcal{U}(\boldsymbol{L}, \Xi) \tag{3.5.9}$$

We are going to investigate if one can choose the boundary term $d\boldsymbol{\alpha}$ somehow canonically so that the behaviour of conserved quantities is improved.

Variation of conserved quantities

Let us consider a Lagrangian $\boldsymbol{L}(j\sigma)$, a 1-parameter family of configurations σ_s generated by the vertical field X on the configuration bundle $\mathcal{C}$, and a symmetry generator Ξ which projects on a spacetime field ξ. Let us set $\bar{\sigma} = \sigma_0$ for the reference field and $\sigma = \sigma_1$ for the real field. Conserved quantities within a regular $(m-1)$-region Ω can be (naively) defined as

$$Q_\Omega(\boldsymbol{L}, \Xi, \sigma) = \int_\Omega (j^{2k-1}\sigma)^* \mathcal{E}(\boldsymbol{L}, \Xi) = \int_\Omega (j^{2k-1}\sigma)^* \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi) + \int_{\partial\Omega} (j^{2k-2}\sigma)^* \mathcal{U}(\boldsymbol{L}, \Xi) \tag{3.5.10}$$

On the other hand, by considering variation along the family of solutions generated by some vertical vector field X, we obtain

$$\delta_X Q_\Omega(\boldsymbol{L}, \Xi, \sigma) = \int_\Omega \delta_X \mathcal{E}(\boldsymbol{L}, \Xi, \sigma) = \int_\Omega \delta_X \left(j^{k-1}\pounds_\Xi \lrcorner \mathbb{F}(\boldsymbol{L}) \right) - \xi \lrcorner \left(j^k X \lrcorner d\boldsymbol{L} \right) = \int_\Omega \delta_X \left(j^{k-1}\pounds_\Xi \lrcorner \mathbb{F}(\boldsymbol{L}) \right) - \xi \lrcorner \left(X \lrcorner \mathbb{E}(\boldsymbol{L}) + d(j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L})) \right) =$$

$$= \int_\Omega \delta_X \left(j^{k-1}\pounds_\Xi \lrcorner \mathbb{F}(\boldsymbol{L}) \right) - \xi \lrcorner X \lrcorner \mathbb{E}(\boldsymbol{L}) - \pounds_\xi(j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L})) + d(\xi \lrcorner j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L})) = \tag{3.5.11}$$

$$= \int_\Omega \omega(X, \pounds_\Xi) - \xi \lrcorner X \lrcorner \mathbb{E}(\boldsymbol{L}) + d(\xi \lrcorner j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L}))$$

where all currents are avaluated along σ. We have used the identity $\pounds_\xi \theta = \xi \lrcorner d\theta + d(\xi \lrcorner \theta)$ holding for operators on spacetime forms; we set $\omega(X, \pounds_\Xi) := \delta_X \left(j^{k-1}\pounds_\Xi \lrcorner \mathbb{F}(\boldsymbol{L}) \right) - \pounds_\xi(j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L}))$ for the so-called *symplectic form*.

Comparing the two expression (3.5.10) and (3.5.11) one obtains

$$d(\delta_X \mathcal{U}(\boldsymbol{L}, \Xi) - \xi \lrcorner j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L})) = \omega(X, \pounds_\Xi) - \xi \lrcorner X \lrcorner \mathbb{E}(\boldsymbol{L}) - \delta_X \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi) \tag{3.5.12}$$

Notice that $\xi \lrcorner X \lrcorner \mathbb{E}(\boldsymbol{L})$ vanishes on shell and $\delta_X \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi)$ vanishes because X is tangent to the space of solutions.

> The quantity $\delta_X \tilde{\mathcal{E}}(\boldsymbol{L}, \Xi)$ is the infinitesimal generator of the quantities $\tilde{\mathcal{E}}(\boldsymbol{L}, \Xi, \sigma_s)$. However, since each σ_s is a solution and the reduced current vanishes along solutions, one has to take the derivative of $\tilde{\mathcal{E}}(\boldsymbol{L}, \Xi, \sigma_s) = 0$ with respect to the parameter s.

Essentially, what we are doing by correcting the (variation of the) Noether current is adding a pure divergence to it. The Noether current is originally defined up to an exact current and adding it does not affect the integrals along $(m-1)$-regions with no boundaries. Moreover, such corrections can be obtained by adding to the Lagrangian a suitable pure divergence, as we shall show in a while.

This suggests to define the *corrected conserved quantity density* as

$$\delta_X \mathcal{Q}(\boldsymbol{L}, \Xi) = \delta_X \mathcal{U}(\boldsymbol{L}, \Xi) - \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F}(\boldsymbol{L}) \tag{3.5.13}$$

giving the *corrected conserved quantity* when integrated on an $(m-2)$-region D.

Let us stress that this is a *definition* and as such, strictly speaking, it does not need to be proven. This choice will (or will not) be justified by the examples of application to different situations and by the extent to which it will produce the *expected* results.

We remark that the quantity (3.5.13) is a cohomological invariant; in fact, it is independent of any pure divergence term added to the Lagrangian, which does not change Euler–Lagrange field equations but changes the Liouville part $\mathbb{F}(\boldsymbol{L})$ and, consequently, the definition of conserved quantities through (3.5.11).

By adding a pure divergence $d\boldsymbol{\alpha}$ to the Lagrangian, the superpotential will change by

$$\mathcal{U}' = \mathcal{U} - \xi^\mu \alpha^\nu \, d\sigma_{\mu\nu} \qquad \Rightarrow \delta_X \mathcal{U}' = \delta_X \mathcal{U} - \xi \lrcorner \delta_X \boldsymbol{\alpha} \tag{3.5.14}$$

while the Liouville part will be affected as

$$j^{k-1} X \lrcorner \mathbb{F}' = j^{k-1} X \lrcorner \mathbb{F} + \delta_X \alpha \qquad \Rightarrow \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F}' = \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F} + \xi \lrcorner \delta_X \alpha \tag{3.5.15}$$

Hence one has

$$\delta_X \mathcal{U}' - \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F}' = \delta_X \mathcal{U} - \textcolor{red}{\xi \lrcorner \delta_X \boldsymbol{\alpha}} - \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F} - \textcolor{red}{\xi \lrcorner \delta_X \alpha} = \delta_X \mathcal{U} - \xi \lrcorner j^{k-1} X \lrcorner \mathbb{F} \tag{3.5.16}$$

Schwarzschild anomalous factor

Relative conserved quantities originated from the need to correct the so-called *anomalous factor* (which more properly should be called *anomalous term*). In order to keep it simple, let us consider standard GR (with cosmological constant) and a solution which is called the *Schwarzschild (A)dS metric*

$$g = -A(r)\mathrm{c}^2 \, dt \otimes \, dt + \frac{dr \otimes \, dr}{A(r)} + r^2 d\Omega^2 \qquad A(r) := 1 - \frac{2Gm}{\mathrm{c}^2 r} - \frac{\Lambda}{3} r^2 \tag{3.5.17}$$

where $d\Omega^2 = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$ is the area element on a sphere.

The function $A(r)$ needs to be non-zero in order to have g non-degenerate. Of course, one has

$$\lim_{r \to 0} A(r) = -\infty \tag{3.5.18}$$

Then, if $\Lambda \leq 0$ then the function $A(r)$ is always increasing and it is positive in the interval $r \in [r_m, +\infty]$.
If the cosmological constant $\Lambda =: \lambda^2 > 0$ is positive then

$$\lim_{r \to \infty} A(r) = -\infty \tag{3.5.19}$$

it has a single maximum at $r_* = \frac{3Gm}{\lambda^2 c^2}$ at which the function A takes the value

$$A(r_*) = 1 - \frac{2Gm}{c^2}\left(\frac{\lambda^2 c^2}{3Gm}\right)^{\frac{1}{3}} - \frac{\lambda^2}{3}\left(\frac{3Gm}{\lambda^2 c^2}\right)^{\frac{2}{3}} \tag{3.5.20}$$

which is positive when

$$\lambda^2 = \Lambda < \left(\frac{c^2}{3Gm}\right)^2 \tag{3.5.21}$$

Since we need the function $A(r)$ to be positive in the metric, we assume that the cosmological constant, when positive, satisfies this constraint and it is not too big. The radial coordinate r extends to a maximal value r_M so that, in the interval $r \in [r_m, r_M]$, one has $A(r) > 0$.

In other words, when the cosmological constant is positive (but not too big), one has three numbers, two positive $0 < r_m < r_M$ and a negative one $r_- < 0$ so that the function $A(r)$ can be written as

$$A(r) = \frac{-\Lambda}{3c^2 r}(r - r_-)(r - r_m)(r - r_M) \tag{3.5.22}$$

and it is positive in the interval $r \in [r_m, r_M]$.

Of course, in this case it is meaningless to compute limits to $r \to +\infty$, since $r \to r_M$ is the maximal value allowed.

The metric (3.5.17) is a solution of Einstein equations with a cosmological constant

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{3.5.23}$$

and are considered to represent the gravitational field generated by a mass m at the origin.

Let us here consider solutions with no cosmological constant $\Lambda = 0$ and radial motions of test particles, starting at infinite distance and at rest.

Test particles motion is described by the Lagrangian

$$\boldsymbol{L} = mc\sqrt{-g_{\mu\nu}u^\mu u^\nu}\,ds = mc\sqrt{A(u^0)^2 - \frac{(u^1)^2}{A} - r^2\left((u^2)^2 + \sin^2(\theta)(u^3)^2\right)}\,ds \tag{3.5.24}$$

One can collect $(u^0)^2$ in the square root and remember that $x^0 = ct$ to rewrite the Lagrangian as

$$\boldsymbol{L} = mc\sqrt{A - \frac{(v^1)^2}{A} - r^2\left((v^2)^2 + \sin^2(\theta)(v^3)^2\right)}\,u^0\,ds = mc\sqrt{c^2 A - \frac{\dot{r}^2}{A} - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}\,dt \tag{3.5.25}$$

where we set

$$v^i := \frac{u^i}{u^0} \qquad \dot{r} = cv^1 \quad \dot{\theta} = cv^2 \quad \dot{\phi} = cv^3 \tag{3.5.26}$$

Since we are considering radial motions, we can fix $\theta(t) = \frac{\pi}{2}$ and $\phi(t) = \phi_0$ and the Lagrangian simplifies to

$$\boldsymbol{L} = mc\sqrt{c^2 A - \tfrac{1}{A}\dot{r}^2}\,dt \tag{3.5.27}$$

This is a Lagrangian depending on r only and independent of time, so that its total energy is conserved, i.e.:

$$\frac{\dot{r}^2}{A\sqrt{c^2 A - \tfrac{1}{A}\dot{r}^2}} + \sqrt{c^2 A - \tfrac{1}{A}\dot{r}^2} = Kc \qquad \Rightarrow \dot{r}^2 + c^2 A^2 - \dot{r}^2 = KcA\sqrt{c^2 A - \tfrac{1}{A}\dot{r}^2}$$

$$\Rightarrow c^2 A^4 = K^2 A^2\left(c^2 A - \tfrac{1}{A}\dot{r}^2\right) \qquad \Rightarrow c^2 A^3 = K^2\left(c^2 A^2 - \dot{r}^2\right) \qquad \Rightarrow K^2\dot{r}^2 = A^2 c^2\left(K^2 - A\right) \tag{3.5.28}$$

If we focus on test particles that at $t = 0$ are at infinity ($r(0) = +\infty$) and at rest $\dot{r}(0) = 0$, then one has $A(r(0)) = 1$, $K = 1$. For these motions, one has

$$\dot{r}^2 = A^2 \mathrm{c}^2 (1 - A) = \frac{2Gm}{r} \left(1 - \frac{4Gm}{\mathrm{c}^2 r} + \frac{4G^2 m^2}{\mathrm{c}^4 r^2} \right) \tag{3.5.29}$$

Far away from the origin, one has

$$2\dot{r}\ddot{r} \simeq -\frac{2Gm}{r^2}\dot{r} \quad \Rightarrow \ddot{r} \simeq -G\frac{m}{r^2} \tag{3.5.30}$$

Thus one obtains that, far away from the mass at the origin, the test particle feels an acceleration of

$$\vec{g} = -G\frac{m}{r^2}\vec{u}_r \tag{3.5.31}$$

which reproduces the Newton law of gravity.

That is why one sets $A = 1 - \frac{2Gm}{\mathrm{c}^2 r}$ and not $A = 1 - \frac{6Gm}{\mathrm{c}^2 r}$ or $A = 1 + \frac{Gm}{\mathrm{c}^2 r}$.

Let us also stress that (3.5.29) is an *exact*s account of GR effects on radial free fall on a spherically symmetric mass as a semiclassical modification of the gravitational potential. Of course, the effects become soon negligible as r grows.

We already computed in (1.10.167) the superpotential to be

$$\mathcal{U} = \frac{\sqrt{g}}{2\kappa} \nabla^{[\rho}\xi^{\lambda]} \, d\sigma_{\lambda\rho} \tag{3.5.32}$$

Now we can fix the symmetry generator to be $\xi = \partial_t = \mathrm{c}\partial_0$, the $(m-2)$-region D to be $t = t_0$, $r = r_0$ and compute the integral to be

$$Q = \int_D \mathcal{U} = \frac{m}{2}\mathrm{c}^2 + \frac{\Lambda\mathrm{c}}{6\kappa}r_0^3 \tag{3.5.33}$$

We can expand

$$\mathcal{U} = \frac{\sqrt{g}}{2\kappa} \nabla^{[\rho}\xi^{\lambda]} \, d\sigma_{\lambda\rho} = \frac{\sqrt{g}}{2\kappa}\mathrm{c}\, g^{\rho\sigma}\{g\}^\lambda_{0\sigma} \, d\sigma_{\lambda\rho} = \frac{\sqrt{g}}{4\kappa}\mathrm{c}\, g^{\rho\sigma}g^{\lambda\epsilon} \left(-d_\epsilon g_{0\sigma} + \boxed{d_0 g_{\sigma\epsilon}} + d_\sigma g_{\epsilon 0} \right) d\sigma_{\lambda\rho} = \frac{\sqrt{g}}{2\kappa}\mathrm{c}\, g^{11}g^{00}d_1 g_{00} \, d\sigma_{01} = -\frac{1}{2\kappa}\mathrm{c}r^2 \sin(\theta) \, \frac{1}{A}AA'(r) d\theta \wedge d\phi =$$
$$= -\frac{1}{2\kappa}\mathrm{c}r^2 \sin(\theta) \, A'(r) d\theta \wedge d\phi \tag{3.5.34}$$

where by $A'(r)$ we denoted the derivative with respect to r. The corresponding conserved quantity is

$$Q = \int_D \mathcal{U} = -\frac{1}{2\kappa}\mathrm{c}r^2 A'(r) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta) = \frac{2\pi}{\kappa}\mathrm{c}r^2 A'(r) = \frac{2\pi}{\kappa}\mathrm{c}r^2 \left(\frac{2Gm}{\mathrm{c}^2 r^2} + \frac{2}{3}\Lambda r \right) = \frac{m}{2}\mathrm{c}^2 + \frac{\Lambda\mathrm{c}}{6\kappa}r^3 \tag{3.5.35}$$

In our current notation, in absence of cosmological constant, the expected value for the energy of gravitational field is $Q = m\mathrm{c}^2$. Hence the integral of the Komar superpotential falls short of a factor 2, which is called the *anomalous factor*.

The value of the variation of the energy is

$$\delta Q = \frac{\mathrm{c}^2}{2}\delta m \tag{3.5.36}$$

Let us now consider the correction to the integral of the variation of the superpotential, namely the integral of $-\xi \lrcorner j^1 X \lrcorner \mathbb{F}$. This can be proven by using some package of tensor algebra to be

$$-\int_D \xi \lrcorner j^1 X \lrcorner \mathbb{F} = \frac{\mathrm{c}^2}{2}\delta m \tag{3.5.37}$$

We used Maple tensor package. We define the metric (3.5.17) and, after having checked that it is a solution of field equations, we compute the quantity

$$-\int_D \frac{\sqrt{g}}{2\kappa}\xi^\nu \delta u^\mu_{\alpha\beta}g^{\alpha\beta}\,d\sigma_{\mu\nu} = \frac{c^2}{2}\delta mc^2 \tag{3.5.38}$$

See the worksheet attached.

Thus the corrected energy is

$$\delta Q = c^2 \delta m \tag{3.5.39}$$

If we assume that the (A)dS solution (i.e. the one with $m = 0$) is zero, then the energy of the Schwarzschild (A)dS solution is mc^2 as expected.

Hence the correction does correct the anomalous factor, at least in this case. Below, we shall consider other solutions as well.

Definition of augmented variational principles

We shall hereafter take for granted that the variation of conserved quantities in gauge-natural theories is given by the corrected formula (3.5.13). We shall here prove that we can take advantage of the cohomological invariance of (3.5.13) to formally integrate it along the family of solutions defined by a deformation X obeying Dirichlet boundary conditions. More precisely, we shall show that, once the extended configuration bundle $_*\mathcal{C}$ has been introduced, in many cases there is a particular representative $_*\boldsymbol{L}$ (called the *augmented Lagrangian*) of the cohomology class of the original Lagrangian $\boldsymbol{L}$ for which the superpotential provides the formal integration for the variation of conserved quantities.

Being a formal integration means that if we consider a 1-parameter family of solutions σ_s continuously connecting the reference field $\bar\sigma = \sigma_0$ to the real field $\sigma = \sigma_1$ so that

$$\delta_X Q_D(_*\boldsymbol{L},\Xi,\sigma,\bar\sigma) := \frac{d}{ds}Q_D(_*\boldsymbol{L},\Xi,\sigma_s,\bar\sigma)\Big|_{s=0} = \int_D \delta_X \mathcal{U}(\boldsymbol{L},\Xi) - \xi\lrcorner j^{k-1}X\lrcorner\mathbb{F}(\boldsymbol{L}) =: \int_D \delta_X \mathcal{Q}(\boldsymbol{L},\Xi) \tag{3.5.40}$$

where $Q_D(_*\boldsymbol{L},\Xi,\sigma,\bar\sigma)$ is the conserved quantity associated to the augmented Lagrangian as in (3.5.9) and X is the infinitesimal generator of the deformation.

For the sake of simplicity, let us consider a first order gauge-natural Lagrangian $L = \mathcal{L}(j^1\sigma)$ on C; this is not too restrictive both because any physical theory has a first order formulation and because it can be easily extended to higher order theories (see below for the case of the second order purely metric GR).

For (3.5.40) to be the case, one has

$$
\begin{aligned}
\frac{d}{ds}Q_D(_*\boldsymbol{L},\Xi,\sigma_s,\bar\sigma)\Big|_{s=0} &= \frac{d}{ds}\int_D (j^{2k-2}\sigma_s)^*\mathcal{U}(\boldsymbol{L},\Xi) - \frac{d}{ds}\int_D \xi^{[\mu}\alpha^{\nu]}(\sigma_s,\bar\sigma)\,d\sigma_{\mu\nu} - \int_D (j^{2k-2}\bar\sigma)^*\mathcal{U}(\boldsymbol{L},\Xi) = \\
&= \int_D (j^{2k-2}\sigma)^*\delta_X\mathcal{U}(\boldsymbol{L},\Xi) - \frac{d}{ds}\int_D \xi^{[\mu}\alpha^{\nu]}(\sigma_s,\bar\sigma)\,d\sigma_{\mu\nu}
\end{aligned}
\tag{3.5.41}
$$

and (3.5.40) holds true iff one has

$$\frac{d}{ds}\xi^{[\mu}\alpha^{\nu]}(\sigma_s, \bar{\sigma})\Big|_{s=0} d\sigma_{\mu\nu} = \xi \lrcorner j^{k-1}X \lrcorner \mathbb{F}(\boldsymbol{L}) = p_i^\mu \delta \bar{y}^i \xi^\nu \, d\sigma_{\mu\nu} \qquad \Rightarrow \frac{d}{ds}\alpha^\mu(\sigma_s, \bar{\sigma})\Big|_{s=0} = -\bar{p}_i^\mu \delta \bar{y}^i \tag{3.5.42}$$

where the momentum $\bar{p}_i^\mu$ is computed on the reference field and with $\alpha(\sigma_0, \bar{\sigma}) = 0$.

For example, in first order variational principles, we see that whenever the action of the gauge group on fields comes as a restriction of a (linear) representation (which encompasses all reasonable cases of fundamental fields, namely metrics, connections, gauge fields, tetrads, tensor densities, spinors, etc.) one can set

$$\alpha^\mu(\sigma_s, \bar{\sigma}) = -\bar{p}_i^\mu(y_s^i - \bar{y}^i) \tag{3.5.43}$$

to do the job. We stress that the additional request on the transformation rules of the fields applies just if we demand that α^μ is a global divergence term by itself. If we start from a local Lagrangian $\boldsymbol{L}$ the additional term $d\boldsymbol{\alpha}$ is not expected to have a global meaning either, as for the total augmented Lagrangian $_*\boldsymbol{L}$ (see below the example about Chern–Simons and non-covariant first order gravitational Lagrangian).

In view of the discussion above, the standard superpotential of the augmented Lagrangian takes into account the correction to conserved quantities given by (3.5.13).

6. Examples

We shall here give examples of computation of the superpotential, augmented Lagrangians and corrected conserved quantities for some of the theories introduced in the previous Chapters as examples of (gauge-)natural field theories.

Hilbert Lagrangian

Let us consider the Lagrangian $\boldsymbol{L}_H = \frac{\sqrt{g}}{2\kappa}(R - 2\Lambda)\,d\sigma$ which is a second order Lagrangian for a metric g.

We already discussed the first variation formula and Noether currents in Chapter 1; see Subsection 1.10.11. The canonical stress tensors are in the form (1.10.163), i.e.

$$\begin{cases} E_\epsilon^\lambda = \frac{1}{2\kappa}\left(\frac{3}{2}R^\lambda{}_\epsilon - \delta_\epsilon^\lambda(R - \Lambda)\right) \\ E_\epsilon^{\lambda\rho} = 0 \\ E_\epsilon^{\lambda\rho\sigma} = \frac{1}{2\kappa}\left(\delta_\epsilon^\lambda g^{\rho\sigma} - g^{\lambda(\rho}\delta_\epsilon^{\sigma)}\right) \end{cases} \tag{3.6.1}$$

while the work current is

$$\mathcal{W} = -\frac{\sqrt{g}}{2\kappa}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu})\pounds_\xi g^{\mu\nu}\,d\sigma = \frac{2\sqrt{g}}{2\kappa}\left(R^\mu{}_\epsilon - \frac{1}{2}R\delta_\epsilon^\mu + \Lambda\delta_\epsilon^\mu\right)\nabla_\mu\xi^\epsilon\,d\sigma \tag{3.6.2}$$

Accordingly, the canonical work tensors are

$$
\begin{cases}
W_\epsilon = 0 \\
W_\epsilon^\alpha = \frac{1}{\kappa}\left(R^\alpha{}_\epsilon - \frac{1}{2}(R-\Lambda)\delta_\epsilon^\alpha\right) \\
W_\epsilon^{\alpha\beta} = 0 \\
W_\epsilon^{\alpha\beta\gamma} = 0
\end{cases}
\tag{3.6.3}
$$

One can check directly that the conservation identities (see (3.2.22)) are verified:

$$
\begin{cases}
\nabla_\mu E_\epsilon{}^\mu + \boxed{\frac{1}{2}E_\sigma{}^{\beta\alpha}R^\sigma{}_{\epsilon\beta\alpha}} + \frac{1}{3}E_\sigma{}^{\gamma\alpha\beta}\nabla_\alpha R^\sigma{}_{\epsilon\gamma\beta} = \boxed{W_\epsilon} = 0 \\
E_\epsilon{}^\alpha + \boxed{\nabla_\mu E_\epsilon{}^{\mu\alpha}} + E_\sigma{}^{\rho\mu\alpha}R^\sigma{}_{\epsilon\rho\mu} + \frac{2}{3}E_\epsilon{}^{\rho\mu\nu}R^\alpha{}_{\mu\nu\rho} = W_\epsilon^\alpha \\
\boxed{E_\epsilon{}^{(\beta\alpha)}} + \boxed{\nabla_\mu E_\epsilon{}^{\mu\alpha\beta}} = \boxed{W_\epsilon^{\alpha\beta}} = 0 \\
E_\epsilon{}^{(\gamma\alpha\beta)} = \boxed{W_\epsilon^{\alpha\beta\gamma}} = 0
\end{cases}
\tag{3.6.4}
$$

The last identity is trivially satisfied since $E_\epsilon{}^{(\gamma\alpha\beta)} = \delta_\epsilon^{(\lambda}g^{\rho\sigma)} - g^{(\lambda\rho}\delta_\epsilon^{\sigma)} = 0$. The second identity reads as

$$
\begin{aligned}
E_\epsilon{}^\alpha + E_\sigma{}^{\rho\mu\alpha}R^\sigma{}_{\epsilon\rho\mu} + \tfrac{2}{3}E_\epsilon{}^{\rho\mu\nu}R^\alpha{}_{\mu\nu\rho} &= \tfrac{3}{2}R^\alpha{}_\epsilon - \delta_\epsilon^\alpha(R-\Lambda) + (\delta_\sigma^\rho g^{\mu\alpha} - g^{\rho(\mu}\delta_\sigma^{\alpha)})R^\sigma{}_{\epsilon\rho\mu} + \tfrac{2}{3}(\delta_\epsilon^\rho g^{\mu\nu} - g^{\rho(\mu}\delta_\epsilon^{\nu)})R^\alpha{}_{\mu\nu\rho} = \\
&= \tfrac{3}{2}R^\alpha{}_\epsilon - \delta_\epsilon^\alpha(R-\Lambda) + R_\epsilon{}^\alpha_{.} - \boxed{\tfrac{1}{2}g^{\rho\mu}R^\alpha{}_{\epsilon\rho\mu}} + \tfrac{1}{2}R_\epsilon{}^\alpha_{.} + \tfrac{2}{3}(-R^\alpha{}_\epsilon - \tfrac{1}{2}R^\alpha{}_\epsilon - \boxed{\tfrac{1}{2}g^{\rho\nu}\delta_\epsilon^\mu R^\alpha{}_{\mu\nu\rho}}) = \\
&= \tfrac{3}{2}R^\alpha{}_\epsilon - \delta_\epsilon^\alpha(R-\Lambda) + \tfrac{1}{2}R_\epsilon{}^\alpha_{.} = 2\left(R^\alpha{}_\epsilon - \tfrac{1}{2}(R-\Lambda)\delta_\epsilon^\alpha\right) = W_\epsilon^\alpha
\end{aligned}
\tag{3.6.5}
$$

Finally, the first equation reads as

$$
\nabla_\mu E_\epsilon{}^\mu + \tfrac{1}{3}E_\sigma{}^{\gamma\alpha\beta}\nabla_\alpha R^\sigma{}_{\epsilon\gamma\beta} = \tfrac{3}{2}\nabla_\mu R^\mu{}_\epsilon - \nabla_\epsilon R + \tfrac{1}{3}\nabla_\alpha R_\epsilon{}^\alpha_{.} + \tfrac{1}{6}\nabla_\alpha R_\epsilon{}^\alpha_{.} = 2\left(\nabla_\mu R^\mu{}_\epsilon - \tfrac{1}{2}\nabla_\epsilon R\right) = 0
\tag{3.6.6}
$$

where we used the (contracted) second Bianchi identities.

In this theory, the generalised Bianchi identities specify to

$$
\nabla_\alpha W_\epsilon^\alpha = 2\left(\nabla_\mu R^\mu{}_\epsilon - \tfrac{1}{2}\nabla_\epsilon R\right) = 0
\tag{3.6.7}
$$

which in fact coincide with (contracted) second Bianchi identities for the Riemann tensor.

The reduced current is

$$
\begin{aligned}
\tilde{\mathcal{E}} &= \tfrac{\sqrt{g}}{2\kappa}\left(\left(E_\epsilon{}^\mu + \tfrac{1}{3}E_\sigma{}^{\alpha\beta\mu}R^\sigma{}_{\epsilon\alpha\beta}\right)\xi^\epsilon + \boxed{E_\epsilon{}^{(\mu\alpha\beta)}}\nabla_{\alpha\beta}\xi^\epsilon\right)\,d\sigma_\mu = \tfrac{\sqrt{g}}{2\kappa}\left(\tfrac{3}{2}R^\mu{}_\epsilon - \delta_\epsilon^\mu(R-\Lambda) + \tfrac{1}{3}(\delta_\sigma^\alpha g^{\beta\mu} - \tfrac{1}{2}g^{\alpha\mu}\delta_\sigma^\beta)R^\sigma{}_{\epsilon\alpha\beta}\right)\xi^\epsilon\,d\sigma_\mu = \\
&= \tfrac{\sqrt{g}}{2\kappa}\left(\tfrac{3}{2}R^\mu{}_\epsilon - \delta_\epsilon^\mu(R-\Lambda) + \tfrac{1}{3}(R_\epsilon{}^\mu_{.} + \tfrac{1}{2}R_\epsilon{}^\mu_{.})\right)\xi^\epsilon\,d\sigma_\mu = \tfrac{2\sqrt{g}}{2\kappa}\left(R^\mu{}_\epsilon - \tfrac{1}{2}\delta_\epsilon^\mu(R-\Lambda)\right)\xi^\epsilon\,d\sigma_\mu = \sqrt{g}W_\epsilon^\mu\xi^\epsilon\,d\sigma_\mu
\end{aligned}
\tag{3.6.8}
$$

which in fact vanishes on shell and one has

$$
d\tilde{\mathcal{E}} = \sqrt{g}\left(\boxed{\nabla_\mu W_\epsilon^\mu}\xi^\epsilon + W_\epsilon^\mu\nabla_\mu\xi^\epsilon\right)\,d\sigma = \mathcal{W}
\tag{3.6.9}
$$

Finally, the superpotential is

$$\mathcal{U} = \frac{\sqrt{g}}{\kappa}\frac{4}{3}E_{\epsilon}^{\ \mu\alpha\beta}\nabla_{\beta}\xi^{\epsilon}\,d\sigma_{\mu\alpha} = \frac{1}{3\kappa}\sqrt{g}(\delta_{\epsilon}^{\mu}g^{\alpha\beta} - \frac{1}{2}g^{\mu\alpha}\delta_{\epsilon}^{\beta} - \frac{1}{2}g^{\mu\beta}\delta_{\epsilon}^{\alpha})\nabla_{\beta}\xi^{\epsilon}\,d\sigma_{\mu\alpha} = \frac{1}{3\kappa}\sqrt{g}(\nabla^{[\alpha}\xi^{\mu]} - \boxed{\frac{1}{2}g^{\mu\alpha}\nabla_{\epsilon}\xi^{\epsilon}} - \frac{1}{2}\nabla^{[\mu}\xi^{\alpha]})\,d\sigma_{\mu\alpha} =$$
$$= \frac{1}{3\kappa}\sqrt{g}(\nabla^{[\alpha}\xi^{\mu]} + \frac{1}{2}\nabla^{[\alpha}\xi^{\mu]})\,d\sigma_{\mu\alpha} = -\frac{\sqrt{g}}{2\kappa}\nabla^{[\mu}\xi^{\alpha]}\,d\sigma_{\mu\alpha} \tag{3.6.10}$$

which agrees with what we found in (1.10.167) by suitable integration by parts, though here found by the standard procedure introduced in Section 3.2.3.

> This superpotential for the Hilbert Lagrangian is well-known since a long time ago and it is called the *Komar potential*, even though Komar actually found it for a time-like Killing vector ξ, while of course here it has been found for a general spacetime vector field.

The augmented Lagrangian for $\boldsymbol{L}_H$ is defined on the extended bundle $_*\mathrm{Lor}(M) = \mathrm{Lor}(M) \times_M \mathrm{Lor}(M)$ and it reads as

$$_*\boldsymbol{L}_H = \frac{\sqrt{g}}{2\kappa}(R - 2\Lambda)\,d\sigma - \frac{\sqrt{\bar{g}}}{2\kappa}(\bar{R} - 2\Lambda)\,d\sigma - \frac{1}{2\kappa}\nabla_{\lambda}\left(\sqrt{\bar{g}}\bar{g}^{\alpha\beta}w_{\alpha\beta}^{\lambda}\right)\,d\sigma \tag{3.6.11}$$

where we set $w_{\alpha\beta}^{\lambda} = u_{\alpha\beta}^{\lambda} - \bar{u}_{\alpha\beta}^{\lambda}$ which is a tensor.

> In fact
> $$w_{\alpha\beta}^{\lambda} = u_{\alpha\beta}^{\lambda} - \bar{u}_{\alpha\beta}^{\lambda} = \left(\Gamma_{\alpha\beta}^{\lambda} - \bar{\Gamma}_{\alpha\beta}^{\lambda}\right) - \left(\delta_{(\alpha}^{\lambda}\Gamma_{\beta)\epsilon}^{\epsilon} - \delta_{(\alpha}^{\lambda}\bar{\Gamma}_{\beta)\epsilon}^{\epsilon}\right) = K_{\alpha\beta}^{\lambda} - \delta_{(\alpha}^{\lambda}K_{\beta)\epsilon}^{\epsilon} \tag{3.6.12}$$
>
> which is a tensor since $K_{\alpha\beta}^{\lambda} := \Gamma_{\alpha\beta}^{\lambda} - \bar{\Gamma}_{\alpha\beta}^{\lambda}$ is.
> That is the augmented Lagrangian since one has
> $$\frac{d}{ds}\left(\frac{\sqrt{\bar{g}}}{2\kappa}\bar{g}^{\alpha\beta}w_{\alpha\beta}^{\lambda}\right) = \frac{d}{ds}\left(\frac{\sqrt{\bar{g}}}{2\kappa}\bar{g}^{\alpha\beta}\left((u_s)_{\alpha\beta}^{\lambda} - \bar{u}_{\alpha\beta}^{\lambda}\right)\right) = \frac{\sqrt{\bar{g}}}{2\kappa}\bar{g}^{\alpha\beta}\delta u_{\alpha\beta}^{\lambda} \quad \Rightarrow \frac{d}{ds}\boldsymbol{\alpha}(g_s, \bar{g}) = j^1 X \lrcorner \mathbb{F}(\boldsymbol{L}_H) \tag{3.6.13}$$

The superpotential of the augmented Lagrangian $_*\boldsymbol{L}_H$ is

$$_*\mathcal{U} := \mathcal{U}(_*\boldsymbol{L}_H) = -\frac{\sqrt{g}}{2\kappa}\nabla^{[\mu}\xi^{\alpha]}\,d\sigma_{\mu\alpha} + \frac{\sqrt{\bar{g}}}{2\kappa}\bar{\nabla}^{[\mu}\xi^{\alpha]}\,d\sigma_{\mu\alpha} - \frac{\sqrt{\bar{g}}}{2\kappa}\bar{g}^{\alpha\beta}\xi^{\nu}w_{\alpha\beta}^{\lambda}\,d\sigma_{\nu\lambda} \tag{3.6.14}$$

Thus the corrected superpotential receives a contribution from the real field, one from the reference field and a correction depending on both the real and the reference field, i.e.

$$\mathcal{U} = -\frac{\sqrt{g}}{2\kappa}\nabla^{[\mu}\xi^{\alpha]}\,d\sigma_{\mu\alpha} \qquad \bar{\mathcal{U}} = -\frac{\sqrt{\bar{g}}}{2\kappa}\bar{\nabla}^{[\mu}\xi^{\alpha]}\,d\sigma_{\mu\alpha} \qquad \Delta = -\frac{\sqrt{\bar{g}}}{2\kappa}\bar{g}^{\alpha\beta}\xi^{\nu}w_{\alpha\beta}^{\lambda}\,d\sigma_{\nu\lambda} \tag{3.6.15}$$

respectively, and one has

$$_*\mathcal{U} = \mathcal{U} - \bar{\mathcal{U}} + \Delta \tag{3.6.16}$$

Let us now consider the Schwarzschild deSitter (dS) solution g considered above as a real field and a different Schwarzschild dS solution $\bar{g}$ for a different value of the parameter $\bar{m}$ as a reference field in the region D at $t = 0$ and $r = +\infty$ in the case of $\Lambda \geq 0$ (when $r = +\infty$ is allowed).

We already computed the first two contributions, namely

$$Q = \int_D \mathcal{U} = \frac{m}{2}\mathrm{c}^2 - \frac{\Lambda\mathrm{c}}{6\kappa}r^3 \qquad \bar{Q} = \int_D \bar{\mathcal{U}} = \frac{\bar{m}}{2}\mathrm{c}^2 - \frac{\Lambda\mathrm{c}}{6\kappa}r^3 \tag{3.6.17}$$

For $\Lambda \leq 0$ one can compute the correction at $r \to \infty$ to be

$$\Delta Q = \lim_{r \to \infty} \int_D \Delta = \lim_{r \to \infty} \left[\frac{m - \bar{m}}{2} c^2 + \frac{3G(m - \bar{m})^2}{\Lambda r^3} - \frac{18G(m - \bar{m})^2}{\Lambda^2 r^5} + \ldots \right] = \tfrac{1}{2}(m - \bar{m})c^2 \tag{3.6.18}$$

  Let us use Maple tensor Package to evaluate the correction ΔQ. See the worksheet.

  If we had $\Lambda > 0$ one cannot take the limit to $r \to \infty$ since the radial coordinate r is valid to the upper limit r_M.

  Unfortunately, the value of r_M does depend on the mass parameter and two different metrics has different values of r_M (or if you prefer they induce two different metrics on the maximal sphere and they do not match at the boundary). For this reason, we skip this case.

Both the first two contributions diverge, though the overall corrected conserved quantity

$$_*Q = Q - \bar{Q} + \Delta := (m - \bar{m})c^2 \tag{3.6.19}$$

and gives the expected result.

If one sets $\bar{m} = 0$, the reference metric is set to the dS metric, which is considered to be the *vacuum* solution in a theory with a cosmological constant, and the corrected energy of g is computed to be $_*Q = mc^2$ as expected.

Of course, the Schwarzschild dS solution is too simple to draw a definite conclusion about augmented conserved quantities. In fact it is a 1-parameter family and, of course, it may sound easy to set the one conserved quantity to a desired value.

Hence let us consider the Kerr metric in dimension 4 as a less trivial example.

$$g = - \left(1 - \frac{2Gmr}{c^2 \rho^2} \right) (dx^0)^2 + \frac{\rho^2}{D^2} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 \left(1 + \frac{2Gmr}{c^2 \rho^2} \sin^2(\theta) \right) \right) \sin^2(\theta) d\phi^2 - 2a \frac{2Gmr}{c^2 \rho^2} \sin^2(\theta) dx^0 d\phi =$$

$$= -(dx^0)^2 + \frac{\rho^2}{D^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2(\theta) d\phi^2 + \frac{2Gmrc^2}{\rho^2} \left(dx^0 - a \sin^2(\theta) d\phi \right)^2 \tag{3.6.20}$$

where we set $mac := J$, $\rho^2 := r^2 + a^2 \cos^2(\theta)$, and $D^2 := r^2(1 - \frac{2Gm}{c^2 r}) + a^2$.

This metric becomes the Schwarzschild metric as $a \to 0$ (if not globally at least in this coordinate patch). Hereafter we are interested and restricted to the so-called *non-extremal Kerr black hole* which is obtained for $c^2 a < Gm$ in the so-called *external region*, i.e. $r \in (r_{out}, +\infty)$ with $c^2 r_{out} := Gm + \sqrt{G^2 m^2 + c^4 a^2 \cos^2(\theta)}$.

They are a 2-parameter family of solutions of Einstein equations depending on (m, a) (or on (m, J)) representing the mass and the angular velocity (or the mass and the angular momentum).

Let us choose a curve in the space of solutions represented by $(m(s), a(s))$ connecting the reference metric with $(\bar{m}, \bar{a})$ to the real metric with (m, a). All the metrics in the curve induce the same metric on the sphere at $r = +\infty$ so that Dirichlet conditions are satisfied.

We can consider spacetime vector fields $\xi = \partial_t$ and $\zeta = \partial_\phi$ generating time translations and rotations, respectively, and which should generate Noether conserved quantities to be interpreted as the energy and a component of the angular momentum, respectively.

For the vector field $\xi = \partial_t = c\partial_0$, at infinity, we obtain

$$_*Q_D = (m - \bar{m})c^2 \tag{3.6.21}$$

Let us compute the three contributions and we get

$$Q = \tfrac{1}{2}mc^2 \qquad \bar{Q} = \tfrac{1}{2}\bar{m}c^2 \qquad \Delta Q = \tfrac{1}{2}(m - \bar{m})c^2 - (m - \bar{m})\left(\frac{G^2}{c^2}m(m - \bar{m}) - \tfrac{2}{3}(a^2 - \bar{a}^2)c^2\right)\frac{1}{r^2} + O\left(\tfrac{1}{r^3}\right) \tag{3.6.22}$$

As for the Schwarzschild solution, the single contributions are affected of anomalous factors, though the total corrected conserved quantity

$$_*Q = Q - \bar{Q} + \Delta Q = (m - \bar{m})c^2 + \lim_{r \to +\infty} O\left(\tfrac{1}{r^2}\right) = (m - \bar{m})c^2 \tag{3.6.23}$$

at infinity gives the expected result.

For the vector field $\zeta = \partial_\phi$, at infinity, we obtain

$$_*Q_D = -(ma - \bar{m}\bar{a})c \tag{3.6.24}$$

Let us compute the three contributions and we get

$$Q = -mac \qquad \bar{Q} = -\bar{m}\bar{a}c \qquad \Delta Q = 0 \tag{3.6.25}$$

The total corrected conserved quantity

$$_*Q = Q - \bar{Q} + \Delta Q = -(ma - \bar{m}\bar{a})c \tag{3.6.26}$$

at infinity gives the expected result.

The result of Kerr black holes is less trivial than for Schwarzschild one. The correction, in fact, corrects the anomalous factor in the energy, while it does not mess up with the angular momentum which is not affected by anomalous factor since the beginning.

More examples will be considered in the Chapter about exact solutions.

Yang–Mills

We already discussed the dynamics of Yang–Mills theories and found Euler–Lagrange and Liouville parts; see (2.7.42). We shall here write down Noether theorem and superpotentials for the Yang–Mills Lagrangian (2.7.34)

$$\boldsymbol{L}_{YM} = -\frac{\sqrt{g}}{4g}F^A_{\mu\nu}F^{\mu\nu}_A \, d\sigma \tag{3.6.27}$$

Of course, this includes as a special case Maxwell electromagnetism which is obtained specifying the gauge group $G = U(1)$.

Let us remark that Noether currents, work currents, superpotentials, and reduced currents are all linear with respect to the Lagrangian, i.e. if one considers a total Lagrangian $\boldsymbol{L} = \alpha\boldsymbol{L}_1 + \beta\boldsymbol{L}_2$ then its Noether current $\mathcal{E}(\boldsymbol{L})$ is a linear combination of the Noether currents of the partial Lagrangians, namely one has

$$\mathcal{E}(\boldsymbol{L}) = \alpha\mathcal{E}(\boldsymbol{L}_1) + \beta\mathcal{E}(\boldsymbol{L}_2) \tag{3.6.28}$$

and similarly for the other currents.

All off-shell identities (e.g. the conservation laws $d\mathcal{E} = \mathcal{W}$, the decomposition of the Noether current $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$, and so on) hold true for any value of the coefficients α and β.

Accordingly, they hold true in particular for each single partial Lagrangian; for example one has $d\mathcal{E}_i = \mathcal{W}_i$ and $\mathcal{E}_i = \tilde{\mathcal{E}}_i + d\mathcal{U}_i$ for both $i = 1$ and $i = 2$. In other words, one has no mixing: for each term, one can trace which partial Lagrangian it comes from, and terms coming from one partial Lagrangian obey identities on their own. There is no magic cancellation between terms coming from different partial Lagrangians.

On the contrary, when considering on shell identities (e.g. $d\mathcal{E} = 0$, $\tilde{\mathcal{E}} = 0$) that refer to the space of solutions of the total Lagrangian, usually, the property of the total current does not come from analogous properties of the partial currents. For example, while it is true that the total reduced current splits as $\tilde{\mathcal{E}} = \alpha\tilde{\mathcal{E}}_1 + \beta\tilde{\mathcal{E}}_2$, one has that $\tilde{\mathcal{E}}$ vanishes on shell (i.e. along solutions of the total field equations) though the partial currents $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ do not vanish on their own (along the same solutions of the total field equations). In other words, one has cancellations between terms in the two partial currents.

Analogously, the Noether current splits $\mathcal{E} = \alpha\mathcal{E}_1 + \beta\mathcal{E}_2$, the partial currents are not conserved on shell separately but only the total current $\mathcal{E}$ is conserved on shell.

It is in view of this remark that one can study separately the contributions of Yang–Mills Lagrangian and then paste them together with the contributions from the Hilbert Lagrangian to obtain currents and superpotentials of the Hilbert–Yang–Mills theory.

The Noether current for the Lagrangian $\boldsymbol{L}_{YM}$ is

$$\mathcal{E} = \frac{\sqrt{g}}{g}\left[-F_A^{\lambda\nu}\pounds_\Xi\omega_\nu^A + \tfrac{1}{4}\xi^\lambda F_{\alpha\beta}^A F_A^{\alpha\beta}\right]d\sigma_\lambda = \frac{\sqrt{g}}{g}\left[-\left(F_A^{\lambda\nu}F_{\epsilon\nu}^A - \tfrac{1}{4}F_{\alpha\beta}^A F_A^{\alpha\beta}\delta_\epsilon^\lambda\right)\xi^\epsilon + F_A^{\lambda\nu}\nabla_\nu\xi_{(V)}^A\right]d\sigma_\lambda \tag{3.6.29}$$

and the work current is

$$\mathcal{W} = \frac{\sqrt{g}}{g}\left[\tfrac{1}{2}T_{\mu\nu}\pounds_\xi g^{\mu\nu} - \nabla_\mu F_A^{\mu\nu}\pounds_\Xi\omega_\nu^A\right]d\sigma = \frac{\sqrt{g}}{g}\left[-T^\alpha{}_\epsilon\nabla_\alpha\xi^\epsilon - \nabla_\mu F_A^{\mu\nu}(\xi^\epsilon F_{\epsilon\nu}^A - \nabla_\nu\xi_{(V)}^A)\right]d\sigma \tag{3.6.30}$$

Hence the canonical stress tensors and the canonical work tensors are

$$\begin{cases} E_\epsilon^\lambda = -\tfrac{1}{g}\left(F_A^{\lambda\nu}F_{\epsilon\nu}^A - \tfrac{1}{4}F_{\alpha\beta}^A F_A^{\alpha\beta}\delta_\epsilon^\lambda\right) =: -T^\lambda{}_\epsilon \\ E_A^{\lambda\nu} = \tfrac{1}{g}F_A^{\lambda\nu} \end{cases} \qquad \begin{cases} W_\epsilon = -\tfrac{1}{g}\nabla_\mu F_A^{\mu\nu}F_{\epsilon\nu}^A \\ W_\epsilon^\alpha = -T^\alpha{}_\epsilon \\ W_A^\nu = \tfrac{1}{g}\nabla_\mu F_A^{\mu\nu} \end{cases} \tag{3.6.31}$$

The conservation identities (3.3.7) split as

$$\begin{cases} \nabla_\mu E_\epsilon^{\ \mu} = W_\epsilon \\ E_\epsilon^{\ \alpha} = W_\epsilon^\alpha \end{cases} \qquad \begin{cases} \tfrac{1}{2}c^B{}_{CA}E_B^{\ \beta\alpha}F_{\beta\alpha}^C = 0 \\ \nabla_\mu E_A^{\ \mu\alpha} = W_A^\alpha \\ E_A^{\ (\beta\alpha)} = 0 \end{cases} \tag{3.6.32}$$

which are in fact identically satisfied.

The only non-trivial one is the first one which is verified since

$$\begin{aligned} \nabla_\lambda E_\epsilon^{\ \lambda} &= -\nabla_\lambda F_A^{\lambda\nu}F_{\epsilon\nu}^A - F_A^{\lambda\nu}\nabla_\lambda F_{\epsilon\nu}^A + \tfrac{1}{2}\nabla_\lambda F_{\alpha\beta}^A F_A^{\alpha\beta}\delta_\epsilon^\lambda = -\nabla_\alpha F_A^{\alpha\beta}F_{\epsilon\beta}^A + \tfrac{1}{2}F_A^{\alpha\beta}\left(-\nabla_\alpha F_{\epsilon\beta}^A + \nabla_\alpha F_{\beta\epsilon}^A + \nabla_\epsilon F_{\alpha\beta}^A\right) = \\ &= W_\epsilon - \tfrac{1}{2}F_A^{\alpha\beta}\left(\nabla_\alpha F_{\epsilon\beta}^A + \nabla_\beta F_{\epsilon\alpha}^A\right) = W_\epsilon + \boxed{F_A^{\alpha\beta}\nabla_{(\alpha}F_{\beta)\epsilon}^A} = W_\epsilon \end{aligned} \tag{3.6.33}$$

If we consider Yang–Mills theory coupled with Einstein–Hilbert gravity, one has the Lagrangian

$$\boldsymbol{L} = \left(\frac{\sqrt{g}}{2\kappa}(R - 2\Lambda) - \frac{\sqrt{g}}{4\mathrm{g}} F_{\alpha\beta}^A F_A^{\alpha\beta} \right) \, d\sigma \tag{3.6.34}$$

Field equations are

$$\begin{cases} G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T_{\mu\nu} \\ \nabla_\mu F_A^{\mu\nu} = 0 \end{cases} \tag{3.6.35}$$

and in particular the energy-momentum tensor $T_{\mu\nu}$ of the Yang–Mills field is not required to vanish. Accordingly, the work current (3.6.30) does not vanish along solutions of field equations.

Of course, the total work current

$$\mathcal{W}(\boldsymbol{L}) = \sqrt{g} \left[-\frac{1}{2\kappa}(G_{\mu\nu} + \Lambda g_{\mu\nu}) \pounds_\xi g^{\mu\nu} + \frac{1}{2} T_{\mu\nu} \pounds_\xi g^{\mu\nu} - \frac{1}{\mathrm{g}} \nabla_\mu F_A^{\mu\nu} \pounds_\Xi \omega_\nu^A \right] \, d\sigma \tag{3.6.36}$$

does vanish along solutions of field equations precisely because one has

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T_{\mu\nu} \tag{3.6.37}$$

Analogously, if we couple Yang–Mills to a matter Lagrangian then Yang–Mills equations become

$$\nabla_\mu F_A^{\mu\nu} = \mathrm{g} J_A^\nu \tag{3.6.38}$$

for some current J depending on the matter Lagrangian and the work current of the Yang–Mills Lagrangian alone, namely (3.6.30) does not vanish along solutions. Of course, one has contributions from the matter Lagrangian as well which are just such that the total work current does vanish on solutions (of the total Lagrangian).

For the Yang–Mills Lagrangian the only contribution to superpotential is from $E_\epsilon^{\lambda\alpha}$ and one has

$$\mathcal{U}(\boldsymbol{L}_{YM}) = \frac{\sqrt{g}}{2\mathrm{g}} F_A^{\mu\alpha} \xi_{(V)}^A \, d\sigma_{\mu\alpha} \tag{3.6.39}$$

By integration over a region D, one obtains conserved quantities

$$Q_A = \frac{1}{2\mathrm{g}} \int_D \sqrt{g} F_A^{\mu\alpha} \, d\sigma_{\mu\alpha} \tag{3.6.40}$$

These conserved quantities come as a map from the dual of Lie algebra $\mathfrak{g}$ to conserved quantities. If we change basis in the Lie algebra, the basis of right-invariant vectors ρ_A and the conserved quantities change accordingly, i.e. so that one has a well-defined element

$$Q = Q_A T^A \in \mathfrak{g}^* \tag{3.6.41}$$

where T^A is the dual basis in $\mathfrak{g}^*$.

Electroweak theory is something more involved (due to the group which is $U(1) \times \mathrm{SU}(2)$). However, one could say that the basis one chooses in $\mathfrak{su}(2)$ selects flavours (3 flavours, since $\dim(\mathrm{SU}(2)) = 3$) and the corresponding components of Q are the flavour charges.

Let us consider the augmented Lagrangian for the Yang–Mills Lagrangian $\boldsymbol{L}_{YM}$. The Liouville part is

$$X \lrcorner \mathbb{F} = -\tfrac{\sqrt{g}}{g} F_A^{\mu\nu} \delta\omega_\nu^A \, d\sigma_\mu \tag{3.6.42}$$

Hence, the correction term is

$$\alpha^\mu = \tfrac{\sqrt{\bar{g}}}{g} \bar{F}_A^{\mu\nu}(\omega_\nu^A - \bar{\omega}_\nu^A) \tag{3.6.43}$$

The augmented Lagrangian is hence

$$_*\boldsymbol{L}_{YM} = \left[-\tfrac{1}{4g}\sqrt{g}F_{\mu\nu}^A F_A^{\mu\nu} + \tfrac{1}{4g}\sqrt{\bar{g}}\bar{F}_{\mu\nu}^A\bar{F}_A^{\mu\nu} + \nabla_\mu\left(\tfrac{\sqrt{\bar{g}}}{g}\bar{F}_A^{\mu\nu}(\omega_\nu^A - \bar{\omega}_\nu^A)\right)\right] \, d\sigma \tag{3.6.44}$$

The augmented superpotential is

$$_*\mathcal{U} = \left[\tfrac{1}{2g}\sqrt{g}F_A^{\mu\alpha}\xi_{(V)}^A - \tfrac{1}{2g}\sqrt{\bar{g}}\bar{F}_A^{\mu\alpha}\bar{\xi}_{(V)}^A + \xi^\alpha \tfrac{\sqrt{\bar{g}}}{g}\bar{F}_A^{\mu\nu}(\omega_\nu^A - \bar{\omega}_\nu^A)\right] \, d\sigma_{\mu\alpha} \tag{3.6.45}$$

where we set $\xi_{(V)}^A := \xi^A + \xi^\mu\omega_\mu^A$ and $\bar{\xi}_{(V)}^A := \xi^A + \xi^\mu\bar{\omega}_\mu^A$ for the vertical parts with respect to the real and reference connection.

We stress that the correction introduced to the conserved quantities is proportional to $\bar{F}_A^{\mu\nu}$ which lives in a vector bundle and obeys linear field equations. Accordingly, $\bar{F} = 0$ is a solution and selecting such a solution as a reference field the correction vanishes.

Accordingly, when the configuration bundle is a vector bundle one has a canonical choice of the reference field (the zero section also known as the *vacuum*) for which the augmented variational principle reduces to the original one.

> All theories used in fundamental physics, except gravity, are defined on affine or vector bundles. Thus augmented variational principle appears to be necessary only when dealing with gravity.

Kerr–Newman solution

Let us consider a solution of Maxwell–Einstein equations called the *Kerr–Newman* solution representing a spinning charged black-hole, of course together with its electromagnetic field.

We are considering a gauge-natural theory associated to a principal bundle $\mathcal{P}$ for the group $U(1)$ in order to account for electromagnetic field. The configuration bundle can be chosen to be

$$\mathcal{C} = \mathrm{Lor}(M) \times_M \mathrm{Con}(\mathcal{P}) \tag{3.6.46}$$

with fibered coordinates $(x^\mu, g_{\mu\nu}, \omega_\mu)$. Using standard notation one can hence introduce the Lagrangian

$$\boldsymbol{L} = \sqrt{g}\left[\tfrac{1}{2\kappa}R - \tfrac{1}{4g}F_{\mu\nu}F^{\mu\nu}\right] \, d\sigma \tag{3.6.47}$$

which defines a gauge-natural theory.

We already computed the augmented superpotential for a symmetry generator $\Xi = \xi^\mu\partial_\mu + \zeta\rho$ to be

$$_*\mathcal{U} = \left[-\sqrt{g}\left(\tfrac{1}{2\kappa}\nabla^{[\mu}\xi^{\alpha]} - \tfrac{1}{2g}F^{\mu\alpha}\zeta_{(V)}\right) + \sqrt{\bar{g}}\left(\tfrac{1}{2\kappa}\bar{\nabla}^{[\mu}\xi^{\alpha]} - \tfrac{1}{2g}\bar{F}^{\mu\alpha}\bar{\zeta}_{(V)}\right) + \xi^\alpha\sqrt{\bar{g}}\left(\tfrac{1}{2\kappa}\bar{g}^{\nu\beta}w_{\nu\beta}^\mu + \tfrac{1}{g}\bar{F}^{\mu\nu}(\omega_\nu - \bar{\omega}_\nu)\right)\right] \, d\sigma_{\mu\alpha} \tag{3.6.48}$$

The Kerr-Newman solution is given by

$$\begin{cases} g = -\left(1 - \dfrac{2Gmr}{\mathbf{c}^2\rho^2} + \dfrac{q^2G}{4\pi\epsilon_0\mathbf{c}^4\rho^2}\right)dx^0 \otimes dx^0 + \dfrac{\rho^2}{D^2}dr \otimes dr + \rho^2 d\theta \otimes d\theta + \\[2ex] \quad + \left(r^2 + a^2 + a^2\left(\dfrac{2Gmr}{\mathbf{c}^2\rho^2} - \dfrac{q^2G\mathbf{g}}{4\pi\mathbf{c}^3\rho^2}\right)\sin^2(\theta)\right)\sin^2(\theta)d\phi \otimes d\phi - a\left(\dfrac{2Gmr}{\mathbf{c}^2} - \dfrac{q^2G\mathbf{g}}{4\pi\mathbf{c}^3}\right)\dfrac{\sin^2(\theta)}{\rho^2}\left(dx^0 \otimes d\phi + d\phi \otimes dx^0\right) \\[2ex] A = -\dfrac{q\mathbf{g}}{4\pi\mathbf{c}^2}\dfrac{r}{\rho^2}\left[\mathbf{c}dt - a\sin^2(\theta)d\phi\right] \end{cases} \tag{3.6.49}$$

where we set $ma\mathbf{c} := J$, $\rho^2 := r^2 + a^2\cos^2(\theta)$, and $D^2 := r^2 + a^2 - \frac{2Gm}{\mathbf{c}^2}r + \frac{q^2G\mathbf{g}}{4\pi\mathbf{c}^3}$.

One can check that the Kerr-Newman metric g is in fact solution of Maxwell–Einstein field equations. See Maple worksheet.

For $q = 0$, this metric reduces to Kerr solution (3.6.20) and for $a = 0$ it further reduces to Schwarzschild metric (3.5.17) (with no cosmological constant). We select the solution (3.6.49) with parameters (m, a, q) as the real field and the same solution (3.6.49) with parameters $(\bar{m}, \bar{a}, \bar{q})$ as the reference field.

Then we can set $\xi = \partial_t = \mathbf{c}\partial_0$ and compute the corrected conserved quantity to be interpreted as the energy and obtain

$$_*Q(\partial_t) = (m - \bar{m})\mathbf{c}^2 \tag{3.6.50}$$

The corrected conserved quantity associated to the vector field $\xi = \partial_\phi$ is

$$_*Q(\partial_\phi) = (ma - \bar{m}\bar{a})\mathbf{c} \tag{3.6.51}$$

to be interpreted as the angular momentum.

Finally, the corrected conserved quantity associated to the vector field $\Xi = \rho$

$$_*Q(\rho) = q - \bar{q} \tag{3.6.52}$$

is to be interpreted as the electric charge.

See the worksheet.

BTZ solution

For a further check, let us consider another solution of Einstein equations with cosmological constant in a spacetime M of dimension 3 and coordinates (t, r, θ). The cosmological constant Λ defines a scale ℓ for distances and consequently for time and mass

$$\ell := |\Lambda|^{-1/2} \qquad t_\ell := \ell\mathbf{c}^{-1} \qquad m_\ell := \dfrac{\ell\mathbf{c}^2}{G} \tag{3.6.53}$$

Dimensional analysis is further complicated by the fact that we are in dimension 3 instead of dimension 4. Accordingly the Lagrangian density needs to be of dimension $[L] = ML^{-1}T^{-1}$ so that the action has the right dimension.

This can be achieved in two different ways: one is to maintain the dimension of 2κ and modify Newton laws so that $F = G\frac{mM}{r}$. The other one is to maintain Newton laws unchanged and modify the dimension of the coupling constant to

$$2\kappa := \frac{\pi G}{c^3 \ell} \tag{3.6.54}$$

Let us hereafter follow the second strategy. However, we have to remark that there is a considerable freedom in fixing the constant in front of 2κ. Here we are only giving a further example of computation of conservation laws in an exotic context.

Let us consider the metric

$$g = -N^2 c^2 dt^2 + \frac{dr^2}{N^2} + r^2 \left(N_\theta c dt - d\theta\right)^2 \tag{3.6.55}$$

where we set

$$N^2 := -\frac{m}{m_\ell} + \Lambda r^2 + \frac{J^2}{4r^2}\frac{1}{m_\ell^2 c^2} \qquad N_\theta := -\frac{J}{2r^2}\frac{1}{m_\ell c} \tag{3.6.56}$$

The metric g is called a BTZ solution.

We can compute the contributions to augmented conserved quantities for the generator $\xi = \partial_t$

$$Q(\xi) = \frac{2l\Lambda c^4}{G} r^2 \qquad \bar{Q}(\xi) = \frac{2l\Lambda c^4}{G} r^2 \qquad \Delta Q(\xi) = l(m - \bar{m})c^2 + O(r^{-2}) \tag{3.6.57}$$

and for the generator $\zeta = \partial_\theta$

$$Q(\zeta) = -J \qquad \bar{Q}(\zeta) = -\bar{J} \qquad \Delta Q(\zeta) = 0 \tag{3.6.58}$$

The total conserved quantities at infinity are

$$_*Q(\xi) = Q(\xi) - \bar{Q}(\xi) + \Delta Q(\xi) = (m - \bar{m})c^2 \qquad _*Q(\zeta) = Q(\zeta) - \bar{Q}(\zeta) + \Delta Q(\zeta) = -(J - \bar{J}) \tag{3.6.59}$$

as expected. Let us remark that the Komar contribution to the energy in this case actually diverges and it is only by suitable cancellations that the augmented conserved quantities converges to the expected values.

To be honest, since we adjusted the constant in front of 2κ to obtain the expected results, all we proved here is that there exists one single factor in front of the coupling constant which returns the correct values of both energy and angular-momentum. Until we learn how to fix factors in front of the coupling constants, that is the best we can expect to do.

First order non-covariant Einstein Lagrangian

Let us consider the *non-covariant first order Einstein (pseudo-)Lagrangian(s)*. It is obtained by subtracting out the second derivatives of the metrics g and $\bar{g}$ under a divergence. Of course, this procedure produces a local Lagrangian depending on the coordinate patch, since the discarded divergence cannot be global if spacetime has non-trivial topology. The Lagrangian is

$$L_E = \frac{1}{2\kappa} \left(\sqrt{g}\, g^{\alpha\beta} R_{\alpha\beta} - d_\lambda(\sqrt{g}\, g^{\alpha\beta} u_{\alpha\beta}^\lambda) \right)\, d\sigma \tag{3.6.60}$$

This Lagrangian is known to produce the same equations as second order Hilbert Lagrangian (with no cosmological constant, in this case). It is not a global form, though its variation is.

The Liouville part is given by

$$j^1 X \lrcorner \mathbb{F} = \tfrac{1}{2\kappa}\left(\sqrt{g}\, g^{\alpha\beta}\delta u^\lambda_{\alpha\beta} - \delta(\sqrt{g}\, g^{\alpha\beta} u^\lambda_{\alpha\beta})\right)\ d\sigma_\lambda = -\tfrac{1}{2\kappa}\left(\delta(\sqrt{g}\, g^{\alpha\beta})u^\lambda_{\alpha\beta}\right)\ d\sigma_\lambda \tag{3.6.61}$$

so that we set

$$\alpha^\lambda = (\sqrt{g}\, g^{\alpha\beta} - \sqrt{\bar g}\,\bar g^{\alpha\beta})\bar u^\lambda_{\alpha\beta} \tag{3.6.62}$$

The augmented Lagrangian $_*\boldsymbol{L} = {_*L}\, d\sigma$ is hence given by

$$
\begin{aligned}
{_*L} =& \tfrac{1}{2\kappa}\left(\sqrt{g}\, g^{\alpha\beta}R_{\alpha\beta} - \sqrt{\bar g}\,\bar g^{\alpha\beta}\bar R_{\alpha\beta} + d_\lambda(\sqrt{\bar g}\,\bar g^{\alpha\beta}\bar u^\lambda_{\alpha\beta} - \sqrt{g}\, g^{\alpha\beta}u^\lambda_{\alpha\beta} + \sqrt{g}\, g^{\alpha\beta}\bar u^\lambda_{\alpha\beta} - \sqrt{\bar g}\,\bar g^{\alpha\beta}\bar u^\lambda_{\alpha\beta})\right) = \\
=& \tfrac{1}{2\kappa}\left(\sqrt{g}\, g^{\alpha\beta}R_{\alpha\beta} - \sqrt{\bar g}\,\bar g^{\alpha\beta}\bar R_{\alpha\beta} - d_\lambda(\sqrt{g}\, g^{\alpha\beta}(u^\lambda_{\alpha\beta} - \bar u^\lambda_{\alpha\beta}))\right) = \\
=& \tfrac{1}{2\kappa}\left(\sqrt{g}\, g^{\alpha\beta}R_{\alpha\beta} - \sqrt{\bar g}\bar g^{\alpha\beta}\bar R_{\alpha\beta} - \nabla_\lambda(\sqrt{g}\, g^{\alpha\beta}w^\lambda_{\alpha\beta})\right) = \tfrac{1}{2\kappa}\left(\sqrt{g}\, g^{\alpha\beta}R_{\alpha\beta} - \sqrt{\bar g}\bar g^{\alpha\beta}\bar R_{\alpha\beta} - \bar\nabla_\lambda(\sqrt{g}\, g^{\alpha\beta}w^\lambda_{\alpha\beta})\right)
\end{aligned}
\tag{3.6.63}
$$

which, despite we started from a different representative in cohomology class (or better, a *local* one which has not even the right to be called a Lagrangian), we still reproduce the same result already obtained for the classically equivalent Hilbert Lagrangian. This is reasonable, in view of the cohomologic invariance we proved above.

Chern–Simons

Let us now consider Chern–Simons Lagrangian as in Subsection 1.10.10.

Chern–Simons Lagrangian is a local Lagrangian (which provides global field equations, as it happens also for the non-covariant first order Einstein Lagrangian); on a 3-dimensional base M, and with obvious notation, it is given by

$$\boldsymbol{L}_{CS} = \epsilon^{\alpha\beta\lambda}\left(\delta_{ij}F^i_{\alpha\beta}A^j_\lambda - \tfrac{1}{3}\epsilon_{ijk}A^i_\alpha A^j_\beta A^k_\lambda\right)\ d\sigma \tag{3.6.64}$$

We remark that Lie algebra indices are raised and lowered by δ_{ij}.

The Liouville part is given by:

$$X\lrcorner\mathbb{F} = 2\epsilon^{\alpha\beta\lambda}\delta_{ij}\delta A^i_\beta A^j_\lambda\ d\sigma_\alpha \tag{3.6.65}$$

which defines the following correction term

$$\alpha^\mu = -2\epsilon^{\mu\beta\lambda}\delta_{ij}(A^i_\beta - \bar A^i_\beta)\bar A^j_\lambda \tag{3.6.66}$$

This correction is non-covariant, but we stress that the original Lagrangian was non-covariant either; what has to be covariant is the final augmented Lagrangian while the single terms can be non-covariant. We also remark that the contribution of $\epsilon^{\mu\beta\lambda}\delta_{ij}\bar{A}^i{}_\beta\bar{A}^j{}_\lambda$ vanishes due to symmetry properties. The augmented Lagrangian $_*\boldsymbol{L} = {}_*L\,d\sigma$ is given by

$$
\begin{aligned}
{}_*L =& \epsilon^{\alpha\beta\lambda}\left(\delta_{ij}F^i{}_{\alpha\beta}A^j{}_\lambda - \tfrac{1}{3}\epsilon_{ijk}A^i{}_\alpha A^j{}_\beta A^k_\lambda\right) - \epsilon^{\alpha\beta\lambda}\left(\delta_{ij}\bar{F}^i_{\alpha\beta}\bar{A}^j{}_\lambda - \tfrac{1}{3}\epsilon_{ijk}\bar{A}^i{}_\alpha\bar{A}^j{}_\beta\bar{A}^k{}_\lambda\right) - d_\mu(2\epsilon^{\mu\beta\lambda}\delta_{ij}A^i{}_\beta\bar{A}^j{}_\lambda) = \\
=& \epsilon^{\mu\nu\rho}\left(\delta_{ij}\bar{F}^i_{\mu\nu}B^j_\rho + \delta_{ij}\bar{\nabla}_\mu B^i_\nu B^j_\rho + \tfrac{1}{3}\epsilon_{ijk}B^i{}_\alpha B^j{}_\beta B^k_\lambda\right)
\end{aligned}
\tag{3.6.67}
$$

where we set $B^i_\mu := A^i_\mu - \bar{A}^i_\mu$ and $\bar{\nabla}_\mu$ denotes the covariant derivative induced by the vacuum connection $\bar{A}^i_\mu$.

The second expression given for the Lagrangian $_*\boldsymbol{L}$ shows that it is in fact gauge covariant. This Lagrangian is already known as the *Chern–Simons covariant Lagrangian*.

Spinor theory

Spinors have been introduced together with spin frames in Subsection 2.7.6 where we computed the variation of the Lagrangian (2.7.107) and the relevant forms for the first variation formula; see equations (2.7.114).

Symmetry generators are right-invariant vector fields on the structure bundle, namely $\Xi = \xi^\mu(x)\partial_\mu + \xi^{ab}(x)\sigma_{ab}$.

Using Lie derivatives (2.7.126), the Noether current for the gravitational part can be written as

$$
\begin{aligned}
\mathcal{E} =& \tfrac{|e|}{2\kappa}\left[2\left(e^a_\beta g^{\alpha\rho} - e^{a(\rho}\delta^{\alpha)}_\beta\right)\nabla_\rho \pounds_\Xi e^\beta_a - \xi^\alpha\left(R - 2\Lambda\right)\right]d\sigma_\alpha = \\
=& \tfrac{|e|}{2\kappa}\left[2\left(e^a_\beta g^{\alpha\rho} - e^{a(\rho}\delta^{\alpha)}_\beta\right)\left(e^\beta_c\nabla_\rho(\xi_{(V)})^c{}_a - \nabla_{\rho\lambda}\xi^\beta e^\lambda_a - \tfrac{1}{2}[\nabla_\rho,\nabla_\lambda]\xi^\beta e^\lambda_a\right) - \xi^\alpha\left(R - 2\Lambda\right)\right]d\sigma_\alpha = \\
=& \tfrac{|e|}{2\kappa}\left[-\boxed{\left(e^\rho_a e^\alpha_c + e^\alpha_a e^\rho_c\right)\nabla_\rho(\xi_{(V)})^{ca}} - \left(\delta^\lambda_\beta g^{\alpha\rho} - g^{\rho\lambda}\delta^\alpha_\beta\right)\nabla_{\rho\lambda}\xi^\beta + \left(\tfrac{3}{2}R_\epsilon{}^\alpha - \delta^\alpha_\epsilon\left(R - 2\Lambda\right)\right)\xi^\epsilon\right]d\sigma_\alpha
\end{aligned}
\tag{3.6.68}
$$

The (non-vanishing) canonical stress tensors are then

$$
\begin{cases}
E^\alpha_\epsilon = \tfrac{1}{2\kappa}\left(\tfrac{3}{2}R_\epsilon{}^\alpha - \delta^\alpha_\epsilon R + 2\delta^\alpha_\epsilon\Lambda\right) \\
E^{\alpha\rho\lambda}_\epsilon = -\tfrac{1}{2\kappa}\left(\delta^{(\lambda}_\beta g^{\rho)\alpha} - g^{\rho\lambda}\delta^\alpha_\beta\right)
\end{cases}
\tag{3.6.69}
$$

The work current for the gravitational part is

$$
\mathcal{W} = -\tfrac{|e|}{2\kappa}\left(2R^a_\mu - Re^a_\mu - 2\Lambda e^a_\mu\right)\pounds_\Xi e^\mu_a\,d\sigma = -\tfrac{|e|}{2\kappa}\left[\boxed{\left(2e^\alpha_a R_{\alpha\mu}e^\mu_b - R\eta_{ab} - 2\Lambda\eta_{ab}\right)(\xi_{(V)})^{ba}} - \left(2R^\alpha_\mu - R\delta^\alpha_\mu - 2\Lambda\delta^\alpha_\mu\right)\nabla_\alpha\xi^\mu\right]d\sigma =
\tag{3.6.70}
$$

The only (non-vanishing) work tensor is then

$$
W^\alpha_\mu = \tfrac{1}{\kappa}\left(R^\alpha_\mu - \tfrac{1}{2}R\delta^\alpha_\mu - \Lambda\delta^\alpha_\mu\right)
\tag{3.6.71}
$$

Conservation of the Noether current reduces to the following identities which are in fact identically satisfied.

$$
\begin{cases}
\nabla_\mu E_\epsilon^{\ \mu} + \frac{1}{3} E_\sigma^{\ \gamma\alpha\beta} \nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} = 0 \\
E_\epsilon^{\ \alpha} + E_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \frac{2}{3} E_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho} = W_\epsilon^\alpha \\
\nabla_\mu E_\epsilon^{\ \mu\alpha\beta} = 0 \\
E_\epsilon^{\ (\gamma\alpha\beta)} = 0
\end{cases}
\tag{3.6.72}
$$

The last two identities are trivially satisfied. The second one expands to

$$
\begin{aligned}
E_\epsilon^{\ \alpha} + E_\sigma^{\ \rho\mu\alpha} R^\sigma_{\ \epsilon\rho\mu} + \tfrac{2}{3} E_\epsilon^{\ \rho\mu\nu} R^\alpha_{\ \mu\nu\rho} &= \tfrac{1}{2\kappa}\left(\tfrac{3}{2} R_\epsilon^{\ \alpha} - \delta_\epsilon^\alpha R + 2\delta_\epsilon^\alpha \Lambda\right) - \tfrac{3}{4\kappa} g^{\rho\alpha} \delta_\sigma^\mu R^\sigma_{\ \epsilon\rho\mu} - \tfrac{1}{2\kappa} g^{\mu\rho} \delta_\epsilon^\nu R^\alpha_{\ \mu\nu\rho} = \\
&= \tfrac{1}{\kappa}\left(\tfrac{3}{4} R_\mu^{\ \alpha} - \tfrac{1}{2} \delta_\epsilon^\alpha R + \delta_\epsilon^\alpha \Lambda\right) + \tfrac{3}{4\kappa} R_\mu^{\ \alpha} - \tfrac{1}{2\kappa} R^\alpha_{\ \epsilon} = \tfrac{1}{\kappa}\left(R_\mu^\alpha - \tfrac{1}{2} R\delta_\mu^\alpha - \Lambda\delta_\epsilon^\alpha\right) = W_\epsilon^\alpha
\end{aligned}
\tag{3.6.73}
$$

Finally, the first one is

$$
\nabla_\mu E_\epsilon^{\ \mu} + \tfrac{1}{3} E_\sigma^{\ \gamma\alpha\beta} \nabla_\alpha R^\sigma_{\ \epsilon\gamma\beta} = \tfrac{1}{2\kappa}\left(\tfrac{3}{2}\nabla_\mu R_\epsilon^{\ \mu} - \nabla_\epsilon R\right) + \tfrac{1}{2\kappa}\tfrac{1}{2}\nabla_\alpha R_\epsilon^{\ \alpha} = \tfrac{1}{\kappa}\left(\nabla_\mu R_\epsilon^{\ \mu} - \tfrac{1}{2}\nabla_\epsilon R\right) = 0
\tag{3.6.74}
$$

Then, we can write down the superpotential

$$
\mathcal{U} = \frac{|e|}{3}\left(2 E_\epsilon^{\ \mu\alpha\beta} \nabla_\beta \xi^\epsilon - \boxed{\nabla_\beta E_\epsilon^{\ \mu\alpha\beta}}\xi^\epsilon\right)\, d\sigma_{\mu\alpha} = -\frac{|e|}{2\kappa}\nabla^{[\mu}\xi^{\alpha]}\, d\sigma_{\mu\alpha}
\tag{3.6.75}
$$

and the reduced current

$$
\begin{aligned}
\tilde{\mathcal{E}} &= |e|\left(\left(E_\epsilon^{\ \mu} + \tfrac{1}{3} E_\sigma^{\ \alpha\beta\mu} R^\sigma_{\ \epsilon\alpha\beta}\right)\xi^\epsilon + \boxed{E_\epsilon^{\ (\mu\alpha\beta)}}\nabla_{\alpha\beta}\xi^\epsilon\right)\, d\sigma_\mu = \tfrac{|e|}{2\kappa}\left(\tfrac{3}{2} R_\epsilon^{\ \mu} - \delta_\epsilon^\mu R + 2\delta_\epsilon^\mu \Lambda + \tfrac{1}{2} R_\epsilon^{\ \mu}\right)\xi^\epsilon\, d\sigma_\mu = \\
&= \tfrac{|e|}{\kappa}\left(R_\epsilon^{\ \mu} - \tfrac{1}{2}\delta_\epsilon^\mu R + \delta_\epsilon^\mu \Lambda\right)\xi^\epsilon\, d\sigma_\mu
\end{aligned}
\tag{3.6.76}
$$

All these results perfectly reproduce what we found in purely metric formalism, though using spin frames.

The Noether current for the spinor part can be written as $-\hbar$ davanti a DIrac

$$
\begin{aligned}
\mathcal{E} = &-\hbar|e|\left[\tfrac{i}{2}\left(\bar{\psi}\gamma^\alpha \pounds_\Xi \psi - \pounds_\Xi \bar{\psi}\gamma^\alpha \psi\right) - \tfrac{i}{4}\left(\bar{\psi}\gamma_{bcd}\psi\right)\eta^{ca} e^{d\alpha} e_\mu^b \pounds_\Xi e_a^\mu - \xi^\alpha\left(\tfrac{i}{2}\bar{\psi}\gamma^a \nabla_a\psi - \tfrac{i}{2}\nabla_a\bar{\psi}\gamma^a\psi + \mu\bar{\psi}\psi\right)\right]\, d\sigma_\alpha = \\
= &-\hbar|e|\left[\tfrac{i}{2}\xi^a\left(\bar{\psi}\gamma^\alpha\nabla_a\psi - \nabla_a\bar{\psi}\gamma^\alpha\psi\right) + \tfrac{i}{8}\xi_{(V)}^{cd}\bar{\psi}\left(\gamma_a\gamma_c\gamma_d + \gamma_c\gamma_d\gamma_a - 2\gamma_{cda}\right)\psi e^{a\alpha} + \tfrac{i}{4}\left(\bar{\psi}\gamma_{bca}\psi\right)e^{a\alpha} e_\mu^b \nabla_\beta\xi^\mu e^{c\beta} + \right. \\
&\left. - \xi^\alpha\left(\tfrac{i}{2}\bar{\psi}\gamma^a \nabla_a\psi - \tfrac{i}{2}\nabla_a\bar{\psi}\gamma^a\psi + \mu\bar{\psi}\psi\right)\right]\, d\sigma_\alpha = \\
= &-\hbar|e|\left[\xi^\lambda\left(\tfrac{i}{2}\bar{\psi}\gamma_e \nabla_\lambda\psi - \tfrac{i}{2}\nabla_\lambda\bar{\psi}\gamma_e\psi - \left(\tfrac{1}{2}\bar{\psi}\mathbb{D}\psi + \tfrac{1}{2}\mathbb{D}\bar{\psi}\psi\right)e_{e\lambda}\right) + \tfrac{i}{4}\left(\bar{\psi}\gamma_{bce}\psi\right)e_\mu^b e^{c\beta}\nabla_\beta\xi^\mu\right]e^{e\alpha}\, d\sigma_\alpha
\end{aligned}
\tag{3.6.77}
$$

We used the identity

$$
\xi_{(V)}^{cd}\left(\gamma_a\gamma_c\gamma_d + \gamma_c\gamma_d\gamma_a\right) = 2\xi_{(V)}^{cd}\gamma_{cda}
\tag{3.6.78}
$$

which one can prove essentially as in (2.7.112).

The non-vanishing stress tensors are

$$\begin{cases} E_\epsilon^\alpha = -\frac{\hbar}{2}\left(\left(i\bar\psi\gamma_e\nabla_\epsilon\psi - i\nabla_\epsilon\bar\psi\gamma_e\psi\right)e^{e\alpha} - \delta_\epsilon^\alpha\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right)\right) \\ E_\epsilon^{\alpha\mu} = -\frac{i\hbar}{4}\left(\bar\psi\gamma_{bce}\psi\right)e_\epsilon^b e^{c\mu}e^{e\alpha} \end{cases} \tag{3.6.79}$$

The work current for the spinor Lagrangian is

$$\begin{aligned} \mathcal{W} =& |e|\left(T_\mu^a \pounds_\Xi e_a^\mu + \hbar\pounds_\Xi\bar\psi\,\mathbb{D}\psi + \hbar\mathbb{D}\bar\psi\,\pounds_\Xi\psi\right)\,d\sigma = \\ =& |e|\left(\eta_{a[d}e_{c]}^\mu T_\mu^a(\xi_{(V)})^{cd} - T_\mu^a e_a^\alpha\nabla_\alpha\xi^\mu + \hbar\xi^\alpha\left(\nabla_\alpha\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\nabla_\alpha\psi\right) - \frac{\hbar}{4}\xi_{(V)}^{cd}\left(\bar\psi\gamma_c\gamma_d\mathbb{D}\psi - \mathbb{D}\bar\psi\gamma_c\gamma_d\psi\right)\right)\,d\sigma \end{aligned} \tag{3.6.80}$$

so that the non-vanishing work tensors are

$$\begin{cases} W_\epsilon = \hbar\left(\nabla_\epsilon\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\nabla_\epsilon\psi\right) = -i\hbar\nabla_a\bar\psi\gamma^a\nabla_\epsilon\psi + i\hbar\nabla_\epsilon\bar\psi\gamma^a\nabla_a\psi + \hbar\mu\nabla_\epsilon\left(\bar\psi\psi\right) \\ W_\mu^\alpha = -T_\mu^a e_a^\alpha = \frac{\hbar}{2}\delta_\mu^\alpha\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) - \frac{i\hbar}{2}\left(\bar\psi\gamma^a\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^a\psi\right)e_a^\alpha - \frac{i\hbar}{4}\nabla^d\left(\bar\psi\gamma_{bcd}\psi\right)e^{c\alpha}e_\mu^b \end{cases} \tag{3.6.81}$$

We used

$$\gamma_{[c}\gamma_{d]}\gamma_a = \tfrac{1}{2}(cda - dca) = \tfrac{1}{6}(cda - dca - cad + dac + acd - adc - 4(ad)c + 4(ca)d + 2(ca)d - 2(ad)c) = \gamma_{cda} - 2\eta_{a[d}\gamma_{c]} \tag{3.6.82}$$

together with the analogous

$$\gamma_a\gamma_{[c}\gamma_{d]} = \gamma_{acd} + 2\eta_{a[d}\gamma_{c]} \tag{3.6.83}$$

to show that

$$\begin{aligned} W_{cd} =& -\frac{\hbar}{4}\left(\bar\psi\gamma_{[c}\gamma_{d]}\mathbb{D}\psi - \mathbb{D}\bar\psi\gamma_{[c}\gamma_{d]}\psi\right) + \eta_{a[d}e_{c]}^\mu T_\mu^a = -\frac{i\hbar}{4}\left(\bar\psi\gamma_{[c}\gamma_{d]}\gamma_a\nabla^a\psi + \nabla^a\bar\psi\gamma_a\gamma_{[c}\gamma_{d]}\psi\right) - \frac{\hbar}{2}\boxed{\eta_{[cd]}\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right)} + \frac{i\hbar}{2}\left(\bar\psi\gamma_{[d}\nabla_{c]}\psi - \nabla_{[c}\bar\psi\gamma_{d]}\psi\right) + \frac{i\hbar}{4}\nabla^a\left(\bar\psi\gamma_{cda}\psi\right) = \\ =& -\frac{i\hbar}{4}\left(\bar\psi\gamma_{cda}\nabla^a\psi - 2\bar\psi\gamma_{[c}\nabla_{d]}\psi + \nabla^a\bar\psi\gamma_{acd}\psi + 2\nabla_{[d}\bar\psi\gamma_{c]}\psi\right) + \frac{i\hbar}{2}\left(\bar\psi\gamma_{[d}\nabla_{c]}\psi - \nabla_{[c}\bar\psi\gamma_{d]}\psi\right) + \frac{i\hbar}{4}\nabla^a\left(\bar\psi\gamma_{cda}\psi\right) = 0 \end{aligned} \tag{3.6.84}$$

The conservation of Noether current is then equivalent to the following identities

$$\begin{cases} \nabla_\mu E_\epsilon^{\ \mu} + \frac{1}{2}E_\sigma^{\ \beta\alpha}R^\sigma_{\ \epsilon\beta\alpha} = W_\epsilon \\ E_\epsilon^{\ \alpha} + \nabla_\mu E_\epsilon^{\ \mu\alpha} = W_\epsilon^\alpha \\ E_\epsilon^{\ (\beta\alpha)} = 0 \end{cases} \tag{3.6.85}$$

The last one is trivially satisfied. The second equation expands as

$$E_\epsilon^{\ \alpha} + \nabla_\mu E_\epsilon^{\ \mu\alpha} = -\frac{i\hbar}{2}\left(\bar\psi\gamma_e\nabla_\epsilon\psi - \nabla_\epsilon\bar\psi\gamma_e\psi\right)e^{e\alpha} + \delta_\epsilon^\alpha\frac{\hbar}{2}\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) - \frac{i\hbar}{4}\nabla^e\left(\bar\psi\gamma_{bce}\psi\right)e_\epsilon^b e^{c\alpha} \equiv W_\epsilon^\alpha \tag{3.6.86}$$

The first identity holds true since

$$\begin{aligned} \nabla_\mu E_\epsilon^{\ \mu} + \frac{1}{2}E_\sigma^{\ \beta\alpha}R^\sigma_{\ \epsilon\beta\alpha} =& -\frac{i\hbar}{2}\nabla^e\left(\bar\psi\gamma_e\nabla_\epsilon\psi - \nabla_\epsilon\bar\psi\gamma_e\psi\right) + \frac{\hbar}{2}\nabla_\epsilon\left(\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\psi\right) - \frac{i\hbar}{8}\boxed{\left(\bar\psi\gamma_{\sigma\alpha\beta}\psi\right)R_\epsilon^{\ [\sigma\alpha\beta]}} = \\ =& -\frac{i\hbar}{2}\nabla^e\left(\bar\psi\gamma_e\nabla_\epsilon\psi - \nabla_\epsilon\bar\psi\gamma_e\psi\right) + \frac{i\hbar}{2}\nabla_\epsilon\left(\bar\psi\gamma^a\nabla_a\psi - \nabla_a\bar\psi\gamma^a\psi\right) + \hbar\mu\nabla_\epsilon\left(\bar\psi\psi\right) = \\ =& -\frac{i\hbar}{2}\left(\underline{\nabla^e\bar\psi\gamma_e\nabla_\epsilon\psi} + \bar\psi\gamma_e\nabla^e\nabla_\epsilon\psi - \nabla^e\nabla_\epsilon\bar\psi\gamma_e\psi - \underline{\nabla_\epsilon\bar\psi\gamma_e\nabla^e\psi} - \underline{\nabla_\epsilon\bar\psi\gamma^a\nabla_a\psi} - \bar\psi\gamma^a\nabla_\epsilon\nabla_a\psi + \nabla_\epsilon\nabla_a\bar\psi\gamma^a\psi + \underline{\nabla_a\bar\psi\gamma^a\nabla_\epsilon\psi}\right) + \hbar\mu\nabla_\epsilon\left(\bar\psi\psi\right) = \\ =& \hbar\left(\mathbb{D}\bar\psi\nabla_\epsilon\psi + \nabla_\epsilon\bar\psi\mathbb{D}\psi\right) - \frac{i\hbar}{2}\left(\bar\psi\gamma^a[\nabla_a,\nabla_\epsilon]\psi - [\nabla_a,\nabla_\epsilon]\bar\psi\gamma^a\psi\right) =^{(2.7.103)} = \hbar\left(\mathbb{D}\bar\psi\nabla_\epsilon\psi + \nabla_\epsilon\bar\psi\mathbb{D}\psi\right) + \frac{i\hbar}{8}\bar\psi\left(\gamma_a\gamma_c\gamma_d - \gamma_d\gamma_c\gamma_a\right)\psi R^{cd}_{\ \ \alpha\epsilon}e^{a\alpha} = \\ =^{(2.7.112)} =& \hbar\left(\mathbb{D}\bar\psi\nabla_\epsilon\psi + \nabla_\epsilon\bar\psi\mathbb{D}\psi\right) - \boxed{\frac{i\hbar}{4}\left(\bar\psi\gamma_{\alpha\beta\gamma}\psi\right)R_\epsilon^{\ [\alpha\beta\gamma]}} = \hbar\left(\nabla_\epsilon\bar\psi\mathbb{D}\psi + \mathbb{D}\bar\psi\nabla_\epsilon\psi\right) \equiv W_\epsilon \end{aligned} \tag{3.6.87}$$

Generalised Bianchi identities reads as

$$
\begin{aligned}
W_\epsilon - \nabla_\alpha W_\epsilon^\alpha =& \hbar \left(\nabla_\epsilon \bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \nabla_\epsilon \psi \right) + \nabla_a T_\epsilon^a = \\
=& \hbar \left(\nabla_\epsilon \bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \nabla_\epsilon \psi \right) - \hbar \left(\nabla_\epsilon \bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \nabla_\epsilon \psi \right) - \tfrac{i\hbar}{2}\bar\psi \gamma^a [\nabla_\epsilon, \nabla_a]\psi - \tfrac{i\hbar}{2}[\nabla_a, \nabla_\epsilon]\bar\psi \gamma^a \psi + \tfrac{i\hbar}{8}[\nabla^c, \nabla^d] \left(\bar\psi \gamma_{bcd}\psi \right) e_\epsilon^b = \\
=& -\tfrac{i\hbar}{8} \bar\psi \left(\gamma_a \gamma_c \gamma_d - \gamma_d \gamma_c \gamma_a \right) \psi R^{cda}{}_b e_\epsilon^b - \tfrac{i\hbar}{4} \left(\boxed{R_{ec}\bar\psi \gamma^{ecd}\psi} - \tfrac{1}{2}\boxed{R^d{}_{[ecb]}\bar\psi \gamma^{bce}\psi} \right) e_{b\epsilon} = -\tfrac{i\hbar}{4}\boxed{\bar\psi \left(\gamma_{acd} \right) \psi R^{[cda]}{}_b} e_\epsilon^b = 0
\end{aligned}
\tag{3.6.88}
$$

As a consequence of these generalised Bianchi identities, one has

$$
\nabla_a T_\epsilon^a = -\hbar \left(\nabla_\epsilon \bar\psi \mathbb{D}\psi + \mathbb{D}\bar\psi \nabla_\epsilon \psi \right)
\tag{3.6.89}
$$

i.e. $\nabla_a T_\epsilon^a = 0$ vanishes on shell, i.e. the energy-momentum stress tensor T_μ^a is conserved on shell.

Finally, the superpotential is

$$
\mathcal{U} = -\tfrac{i\hbar}{8}|e| \left(\bar\psi \gamma_{bce}\psi \right) e_\epsilon^b e^{c\alpha} e^{e\mu} \xi^\epsilon \, d\sigma_{\mu\alpha}
\tag{3.6.90}
$$

and the reduced current is

$$
\tilde{\mathcal{E}} = |e| \left(E_\epsilon^{\ \mu} + \nabla_\alpha E_\epsilon^{\ \alpha\mu} \right) \xi^\epsilon \, d\sigma_\mu = |e| W_\epsilon^{\ \mu} \xi^\epsilon \, d\sigma_\mu = -|e| T_\epsilon^a e_a^\mu \xi^\epsilon \, d\sigma_\mu
\tag{3.6.91}
$$

Let us notice that the total reduced current vanishes on shell.

For the Dirac Lagrangian we can compute the augmented Lagrangian.

Warning: notiational conflict!
When we extend the configuration bundle we need another notation for the reference spinor since $\bar\psi$ already denote the conjugate spinor. Since this is the only conflict and it will be solved in few lines (when we shall set $\phi = 0$), for spinor only, the reference spinor will be denoted by ϕ. Accordingly, the extended configuration is $(e_a^\mu, \psi, \bar{e}_a^\mu, \phi)$

The Liouville part is

$$
X \lrcorner \mathbb{F} = -\hbar \tfrac{|e|}{2} \left[i \left(\bar\psi \gamma^\alpha \delta\psi - \delta\bar\psi \gamma^\alpha \psi \right) - \tfrac{i}{2} \left(\bar\psi \gamma_{bcd}\psi \right) \eta^{ca} e^{d\alpha} e_\mu^b \delta e_a^\mu \right] \, d\sigma_\alpha
\tag{3.6.92}
$$

thus we can define the correction

$$
\begin{aligned}
\boldsymbol{\alpha} =& -\tfrac{\hbar|\bar{e}|}{2} \left[i \left(\bar\phi \gamma^\lambda \left(\psi - \phi \right) - \left(\bar\psi - \bar\phi \right) \gamma^\lambda \phi \right) - \tfrac{i}{2} \left(\bar\phi \gamma_{bcd}\phi \right) \bar{e}^{d\lambda} \left(\bar{e}_\mu^b e^{c\mu} - \eta^{bc} \right) \right] \, d\sigma_\lambda = \\
=& -\tfrac{\hbar|\bar{e}|}{2} \left[i \left(\bar\phi \gamma^\lambda \left(\psi - \phi \right) - \left(\bar\psi - \bar\phi \right) \gamma^\lambda \phi \right) - \tfrac{i}{2} \left(\bar\phi \gamma_{bcd}\phi \right) \bar{e}^{d\lambda} \bar{e}_\mu^b e^{c\mu} \right] \, d\sigma_\lambda
\end{aligned}
\tag{3.6.93}
$$

Accordingly, the augmented Lagrangian is

$$
{}_*\boldsymbol{L}_D = -\tfrac{i\hbar}{2}|e| \left[\left(\bar\psi \gamma^a \nabla_a \psi - \tfrac{i}{2}\nabla_a \bar\psi \gamma^a \psi + \mu\bar\psi\psi \right) - \left(\bar\phi \gamma^a \bar\nabla_a \phi - \bar\nabla_a \bar\phi \gamma^a \phi + \mu\bar\phi\phi \right) + \nabla_\lambda \left(\bar\phi \gamma^\lambda \left(\psi - \phi \right) - \left(\bar\psi - \bar\phi \right) \gamma^\lambda \phi - \tfrac{1}{2} \left(\bar\phi \gamma_{bcd}\phi \right) \bar{e}^{d\lambda}\bar{e}_\mu^b e^{c\mu} \right) \right] \, d\sigma
\tag{3.6.94}
$$

where $\bar\nabla_a$ denotes the covariant derivative with respect to the spin connection induced by the reference frame $\bar{e}_a^\mu$.

Let us stress that spinors are sections of a vector bundle and in vector bundles one has a canonical choice for the reference field, namely the zero section, i.e. $\phi = 0$. By this choice, the correction vanishes and the augmented Lagrangian coincide with the standard Lagrangian

$$_*\boldsymbol{L}_D = -\hbar|e|\left(\tfrac{i}{2}\bar{\psi}\gamma^a\nabla_a\psi - \tfrac{i}{2}\nabla_a\bar{\psi}\gamma^a\psi + \mu\bar{\psi}\psi\right)\ d\sigma \tag{3.6.95}$$

This is true in any linear theory, i.e. a theory with a configuration bundle which is a vector bundle and linear field equations.

> Usually, matter obeys these requirements. Gauge fields *almost* obey them. Their configuration bundle is affine, not linear, although the field strength is a section in a vector bundle. Then, one can choose a configuration with a vanishing field strength. The choice is not unique due to gauge invariance. However, due precisely to gauge invariance, the current is not affected by gauge transformations. Moreover, field equations are not linear with respect to the potential (unless the gauge group is commutative), though they are in terms of the field strength, so that $\bar{F} = 0$ is a solution (actually, a family of solutions, when written in terms of $\bar{A}$).
>
> When considering gravity, either in purely metric formulation or in terms of spin frames, the configuration bundle is not a vector bundle and field equations are not linear. The solution corresponding to vanishes field strength (i.e. vanishing Riemann tensor, i.e. Minkowski) is not generically a solution of field equations (for example, when cosmological constant is present).
>
> Under this viewpoint, gravity is really peculiar in the family of field theories of interest for fundamental physics. Also for this reason, one should not trust too much physical intuition when it comes from field theories on Minkowski space and extend techniques uncritically to curved background.

References

add

The KBL bi-metric Lagrangian, [KBL, Phys. Rev. D 55 5957 (1997)]:

Marco Mamone Capria PHYSICS BEFORE AND AFTER EINSTEIN Un po' di storia su Einstein. Chap 5 critiche alle CQ in relativit e pseudotensore.

7. Exercises

Exercise 1: Expand $\nabla_{\mu\nu}\nabla_\rho\xi^{\mathbf{a}}$ and $\nabla_\rho\nabla_{\mu\nu\sigma}\xi^{\mathbf{a}}$ in the base of symmetrised covariant derivatives.

Exercise 2: Use the decomposition of the top term (3.2.47) to define the superpotential for a general Noether current of order 3.

Exercise 3: Chase the mistake. Why one cannot compute the correction (3.5.38) for the Schwarzschild metric as follows?

$$-\xi \lrcorner j^1 X \lrcorner \mathbb{F} = \tfrac{\sqrt{g}}{2\kappa} g^{\alpha\beta} \xi^{[\mu} \delta u_{\alpha\beta}^{\nu]} \, d\sigma_{\mu\nu} = \tfrac{\sqrt{g}}{2\kappa} g^{\alpha\beta} \xi^0 (\delta\Gamma_{\alpha\beta}^1 - \delta_\alpha^1 \delta\Gamma_{\beta\epsilon}^\epsilon) \, d\sigma_{01} = \tfrac{\sqrt{g}c}{4\kappa} (g^{\alpha\beta} g^{1\epsilon}(-\nabla_\epsilon \delta g_{\alpha\beta} + 2\nabla_\alpha \delta g_{\beta\epsilon}) - g^{1\beta} g^{\epsilon\sigma}(-\nabla_\sigma \delta g_{\beta\epsilon} + \nabla_\beta \delta g_{\epsilon\sigma} + \nabla_\epsilon \delta g_{\sigma\beta})) d\theta \wedge d\phi =$$

$$= \tfrac{\sqrt{g}c}{4\kappa} (-\underline{g^{11} g^{\alpha\beta} \nabla_1 \delta g_{\alpha\beta}} + 2g^{11} g^{11} \nabla_1 \delta g_{11} - \underline{g^{11} g^{\epsilon\sigma} \nabla_1 \delta g_{\epsilon\sigma}}) d\theta \wedge d\phi = \tfrac{\sqrt{g}c}{2\kappa} (-g^{11} g^{00} \nabla_1 \delta g_{00} - g^{11} g^{11} \nabla_1 \delta g_{11} + g^{11} g^{11} \nabla_1 \delta g_{11}) d\theta \wedge d\phi = \tfrac{\sqrt{g}c}{2\kappa} \nabla_1 \delta g_{00} d\theta \wedge d\phi =$$

$$= \tfrac{\sqrt{g}c}{2\kappa} (\partial_1 \delta g_{00} - 2\Gamma_{01}^\alpha \delta g_{\alpha 0}) d\theta \wedge d\phi = \tfrac{\sqrt{g}c}{4\kappa} (-\partial_1 \delta A + g^{00}(-\boxed{\partial_0 g_{01}} + \boxed{\partial_0 g_{10}} + \partial_1 g_{00}) \delta A) d\theta \wedge d\phi =$$

$$= -\tfrac{\delta m}{8\pi} (1 + \tfrac{A'}{A} r) \sin(\theta) d\theta \wedge d\phi = -\tfrac{\delta m}{8\pi} \left(\frac{1 - \tfrac{2Gm}{c^2 r} + \tfrac{1}{3}\Lambda r^2 + \tfrac{2Gm}{c^2 r} + \tfrac{2}{3}\Lambda r^2}{A} \right) \sin(\theta) d\theta \wedge d\phi =$$

$$= -\tfrac{\delta m}{8\pi} \left(\frac{1 + \Lambda r^2}{1 - \tfrac{2Gm}{c^2 r} + \tfrac{1}{3}\Lambda r^2} \right) \sin(\theta) d\theta \wedge d\phi \qquad \Rightarrow -\int_D \xi \lrcorner j^1 X \lrcorner \mathbb{F} = \tfrac{\delta m}{2} \left(\frac{1 + \Lambda r^2}{1 - \tfrac{2Gm}{c^2 r} + \tfrac{1}{3}\Lambda r^2} \right) \neq \tfrac{\delta m}{2}$$

Exercise 4: Compute the superpotential for the Lagrangian $\boldsymbol{L} = \tfrac{\sqrt{g}}{2\kappa} R_{\alpha\beta} R^{\alpha\beta} \, d\sigma$.

Chapter 4. Space of solutions and Hamiltonian formalism

And I go by many names also
Some people know me as "Hope"
Some people know me as the voice that you hear
when you loosen the noose on the rope
And you know how I know that I'll prosper?
'cause I stand here beside you today.

(Ren, *Hi Ren!*).

1. Introduction

The Hamiltonian formalism has been originally proposed as an equivalent formulation to the Lagrangian formalism to solve (and connect) problems in optics and mechanics, it is based on symplectic geometry which is canonically defined on the phase space of the system, which is the cotangent bundle T^*Q.

Extending to field theories, in particular to relativistic theories, a lot of issues have been pointed out. Firstly, the relativistic Lagrangians are almost always degenerate. As Utiyama arguments have shown, generally covariant dynamics depend on geometric objects which are complicated combinations of derivatives of fields which happen to have simple transformation laws. Secondly, in field theories, fields have many momenta since dynamics depends on partial derivatives, as many as the dimension of spacetime. That is intrinsic with the fact that field thories produce PDE, not ODE as in mechanics.

In field theories, dynamics are also usually first order, with the not irrelevant exception of purely metric GR which is second order although often there are equivalent first order formulations. At higher orders, there are extra structures. For example, besides naive momenta, we also have formal momenta to be discussed. Even if we could restrict Hamiltonian formalism to first order theories and still be able to cover most of the applications to physics, since GR can be formulated as a first order theory by relying on a Palatini formulation, we still believe that trying to adjust the formalism to higher order theories, at least to second order theories, may help us in singling out the correct structures.

Moreover, over time, many physical concepts have been entangled with the Hamiltonian formalism. The notions of *state* of a system, of *observables*, of *degrees of freedom* are just the first examples.

The structure of Hamiltonian field theories itself has split in different paths depending on the theory one considers. On one hand, one often forces on a field theory a structure to deal with it as it was mechanics. This is done by fixing a foliation of spacetime, and regarding *fields at a (spatial) position* $\varphi(\cdot, x)$ as (a continuous infinity of) Lagrangian coordinates. In this setting, one is trying to determine the evolution of fields at a spatial position along the parameter of the foliation, *time*, exactly as in mechanics, one eventually finds the evolution of $q^\lambda(t)$. Unfortunately, the foliation is a conventional structure. This procedure is not natural, it produces a description which depends on the foliation (which is part of the laboratory specification) and, in the case of relativistic theories, one has to eventually prove that observables of the theory are independent of the choice of the foliation. Moreover,

the fact that one has infinitely many degrees of freedom makes the discussion considerably more complex than in mechanics, which is considered as the price to be paid in field theories.

On the other hand, one mostly interested in relativistic theories (or gauge theories) should try the hard way to discuss Hamiltonian issues without introducing extra structures, i.e. without breaking manifest covariance along the way. Unfortunately, in these directions, geometric structures have proven to be much less canonical than in mechanics. There are many different communities which focus on different issues and structures producing a number of different formulations which are more or less equivalent, but not quite in details. The results of the different approaches can be mostly mapped one into the other even if the comparison is always tedious and time consuming.

The covariant approach to Hamiltonian systems is often presented as shifting the focus from symplectic structures, which in mechanics are characteristic of Hamiltonian formalism, to studying the space of solutions of a system. In mechanics, for a first order, non-degenerate dynamics, one has Cauchy theorem, hence exactly one solution for each initial condition which happens to be points on TQ. Accordingly, the manifold TQ *is*, in that case, a realization of the space of solutions and, at the same time, it gives to it a differential structure by promoting the correspondence to be a diffeomorphism. Then, always in first order, non-degenerate mechanics, one can define a *Legendre transform* $\Phi_L : TQ \to T^*Q$, so that also T^*Q is a representation of the space of solutions. When the Lagrangian is *hyper-regular* the Legendre map is a diffeomorphism, thus the differential structure induced on solution space by TQ or T^*Q is the same.

The solution space with this differential structure is a manifold (although its points are solutions of the system, namely curves) and, in fact, one can use TQ or T^*Q as *phase space*. However, on T^*Q one has a canonical symplectic structure $\boldsymbol{\omega} = d\boldsymbol{\theta}$ induced by the Liouville 1-form $\boldsymbol{\theta} = p_\lambda dq^\lambda$. This symplectic structure in T^*Q appeared to be more relevant for mechanics than the bundle structure. Transformations which preserve the symplectic structure, which strictly contain fibered transformations, preserves Hamilton equations.

> Unfortunately, this is an awkward twist. In Hamiltonian formulation for mechanics, T^*Q is both the phase space, i.e. the space of solutions of the system, and the arena to write Hamilton equations or more generally the dynamics and one can easily switch back and forth between these two viewpoints.
> As a matter of fact, the Hamilton equations are obtained as ODE on the phase space, just as Euler-Lagrange equations are written on TQ which is also a representation of the phase space.

With a bigger group of transformations which preserves the Hamilton equations, it is no wonder one can hope to find a problem-specific coordinate system in which the equations of motion are so simple that they can be easily integrated. Action-angle coordinates as well as, in some sense, Hamilton-Jacobi method are ways to exploit this commodity to actually solve the system. These methods can also be expressed quite intrinsically in a symplectic geometry and Hamiltonian systems can be expressed entirely forgetting about the bundle structure the phase space T^*Q originally had.

On the other hand, since (as long as) the Legendre transform is a diffeomorphism, one can pull-back the symplectic geometry from T^*Q to TQ. On the tangent bundle, the symplectic structure is not canonical (since the Legendre map depends on the dynamics), however, it is still true that symplectomorphisms of a structure associated with a specific dynamics preserve the equations of *that* dynamics even when they are not fibred transformations. They are, in fact, called *Bäcklund transformations* and they are also routinely used to analyze and solve Lagrangian systems.

> Let us stress that the Legendre transform reads as

$$\Phi_L : TQ \to T^*Q : (q, u) \mapsto (q, p = \pi(q, u)) \tag{4.1.1}$$

Accordingly, the Liouville form $\boldsymbol{\theta} = p_\lambda dq^\lambda$ on T^*Q can be pulled-back to $\boldsymbol{\theta}_L = \pi_\lambda(q, u)dq^\lambda$ on TQ which depends on the Lagrangian through the local functions $\pi_\lambda = \frac{\partial L}{\partial u^\lambda}$. It nevertheless defines a symplectic structure $\boldsymbol{\omega}_L = d\boldsymbol{\theta}_L$ on TQ, which is given by

$$\boldsymbol{\omega}_L = \frac{\partial L}{\partial q^\alpha \partial u^\lambda} dq^\alpha \wedge dq^\lambda + \frac{\partial L}{\partial u^\alpha \partial u^\lambda} du^\alpha \wedge dq^\lambda \tag{4.1.2}$$

which may be not simple, still it is a symplectic structure on TQ. It is not canonical since it depends on the Lagrangian. Different systems on the same TQ may define different symplectic forms.

Moreover, notice that, generically, fibered coordinates (q, u) may be non-canonical since one has not $\boldsymbol{\omega}_L = du \wedge dq$.

Now go back to what we said above and check if one should be surprised that when one goes to degenerate field theories, all this needs to be reviewed and adapted to the new situation!

Canonical transformations in Lagrangian mechanics

Let us consider a Lagrangian harmonic oscillator described on the configuration space $Q = \mathbb{R}$ or, equivalently, on the configuration bundle $[\mathbb{R} \times Q \to \mathbb{R}]$ with fibered coordinates (t, x). The dynamics is described by a Lagrangian on TQ, namely

$$L = \tfrac{1}{2}\left(u^2 - \omega^2 x^2\right) dt \tag{4.1.3}$$

its equation of motion being

$$\ddot{x} = -\omega^2 x \qquad \Longleftrightarrow \qquad \begin{cases} \dot{x} = u \\ \dot{u} = -\omega^2 x \end{cases} \tag{4.1.4}$$

The Legendre transform is

$$\Phi_L : TQ \to T^*Q : (x, u) \mapsto (x, p = u) \tag{4.1.5}$$

the corresponding Hamiltonian is $H = \tfrac{1}{2}\left(p^2 + \omega^2 x^2\right)$ which induces the Hamilton equations

$$\dot{x} = p \qquad\qquad \dot{p} = -\omega^2 x \tag{4.1.6}$$

which, in view of the Legendre transform, are of course equivalent to Euler-Lagrange equations.

Hamilton equations are hence interpreted as a dynamical system in $\mathbb{R}^2 = T^*Q$ given by the Hamiltonian vector field

$$X_H = p\frac{\partial}{\partial x} - \omega^2 x \frac{\partial}{\partial p} \tag{4.1.7}$$

On T^*Q we have a canonical Liouville form $\boldsymbol{\theta} = p\, dx$, which defines a symplectic form $\boldsymbol{\omega} = dp \wedge dx$ on T^*Q. When we interpret T^*Q as a symplectic manifold rather than as a cotangent bundle, we have much more freedom in choosing canonical coordinates.

For example, we know solutions evolve on a level surface (curves) of the Hamiltonian (which is trivially conserved). Level curves of H are ellipses $p^2 + \omega^2 x^2 = 2E$ and we can decide to find a canonical coordinate system (E, θ) in which one of the coordinate E labels the ellipse on which motion evolves so that $E = \text{const}$ along evolution.

Actually, we have a simple solution, and we *hope* we shall have a simple equation for it and Hamilton equations for the new coordinate to be easy to solve. Let us remark that, of course, we chose a simple example and we are perfectly able to solve the original equations in this case, as well.

Consider a new coordinate system (E, θ) in which θ is the angle of the point in polar coordinates. We need to show that (E, θ) are canonical coordinates.

Since, for a given value of E, it must be $2E = p^2 + \omega^2 x^2$, let us parameterize the level curve as

$$x = \tfrac{h(E)}{\omega} \cos(\theta) \qquad\qquad p = h(E) \sin(\theta) \tag{4.1.8}$$

and let us determine $h(E)$ so that this is a canonical transformation. We have

$$dx = \tfrac{h'}{\omega} \cos(\theta)\, dE - \tfrac{h}{\omega} \sin(\theta) d\theta \qquad\qquad dp = h' \sin(\theta)\, dE + h \cos(\theta) d\theta \tag{4.1.9}$$

Then we have

$$dp \wedge dx = (h' \sin(\theta)\, dE + h \cos(\theta) d\theta) \wedge \left(\tfrac{h'}{\omega} \cos(\theta)\, dE - \tfrac{h}{\omega} \sin(\theta) d\theta \right) = -\tfrac{h'h}{\omega} (\sin^2(\theta) + \cos^2(\theta))\, dE \wedge d\theta = -\tfrac{h'h}{\omega}\, dE \wedge d\theta \tag{4.1.10}$$

Accordingly we must have $h'h = \omega$ hence the transformation $(x, p) \mapsto (E, \theta)$ is canonical iff

$$hdh = \omega dE \quad \Rightarrow \tfrac{1}{2} h^2 = \omega E \quad \Rightarrow h(E) = \sqrt{2\omega E} \qquad \Rightarrow \quad \begin{cases} x = \sqrt{\tfrac{2E}{\omega}} \cos(\theta) \\[2mm] p = \sqrt{2\omega E} \sin(\theta) \end{cases} \tag{4.1.11}$$

The Hamiltonian in the new canonical coordinates reads as

$$H(E, \theta) = \tfrac{1}{2} \left(2\omega E \sin^2(\theta) + \omega^2 \tfrac{2E}{\omega} \cos^2(\theta) \right) = \omega E \left(\sin^2(\theta) + \cos^2(\theta) \right) = \omega E \tag{4.1.12}$$

In other words, the new coordinate θ is cyclic and the Hamiltonian is linear in E. Hence, Hamilton equations read as

$$\dot{\theta} = \frac{\partial H}{\partial E} = \omega \qquad\qquad \dot{E} = 0 \tag{4.1.13}$$

which are trivially integrated to

$$\theta = \theta_0 + \omega t \qquad\qquad E = E_0 \tag{4.1.14}$$

This generalizes to completely integrable systems (i.e. Hamiltonian systems with the maximal number of first integrals in involution). However, that is not the point we are after here.

Here, we wish to show that the transformation (4.1.11) can be done in the Lagrangian framework as well. It is there a pretty strange transformation since it mixes positions and velocities which is not quite what one usually does in Lagrangian formalism. If we had not derived it as a canonical transformation that would appear as pure magic.

Transformation (4.1.11) induces on TQ the transformation $(x, u) \mapsto (\theta, \epsilon)$ given by

$$x = \sqrt{\tfrac{2\epsilon}{\omega}} \cos(\theta) \qquad\qquad u = \sqrt{2\omega\epsilon} \sin(\theta) \tag{4.1.15}$$

The field $X_L = u\frac{\partial}{\partial x} - \omega^2 x\frac{\partial}{\partial u}$ accounts for Euler-Lagrange equations. For a generic function $F(x, u) = F(\theta, \epsilon)$ on TM we have

$$\partial_\theta = -\sqrt{\frac{2\epsilon}{\omega}}\sin(\theta)\partial_x + \sqrt{2\omega\epsilon}\cos(\theta)\partial_u \qquad \partial_\epsilon = \sqrt{\frac{1}{2\epsilon\omega}}\cos(\theta)\partial_x + \sqrt{\frac{\omega}{2\epsilon}}\sin(\theta)\partial_u \tag{4.1.16}$$

which can be inverted to

$$\partial_x = \sqrt{2\omega\epsilon}\cos(\theta)\partial_\epsilon - \sqrt{\frac{\omega}{2\epsilon}}\sin(\theta)\partial_\theta \qquad \partial_u = \sqrt{\frac{2\epsilon}{\omega}}\sin(\theta)\partial_\epsilon + \sqrt{\frac{1}{2\omega\epsilon}}\cos(\theta)\partial_\theta \tag{4.1.17}$$

Then the vector field X_L reads in the new coordinates

$$X_L = -\omega\left(\cos^2(\theta) + \sin^2(\theta)\right)\partial_\theta = -\omega\partial_\theta \tag{4.1.18}$$

Notice that this is not a Lagrangian field any longer, however, the integral curves are solutions of the original system.

When we pull-back the Liouville form $\boldsymbol{\theta} = pdq$ along the Legendre map we obtain

$$\boldsymbol{\theta}_L = \frac{\partial L}{\partial u}dx = u\,dq \qquad\qquad \Rightarrow \boldsymbol{\omega}_L = du \wedge dx \tag{4.1.19}$$

The symplectic form in the new coordinates reads as

$$\boldsymbol{\omega}_L = du \wedge dx = d\epsilon \wedge d\theta \tag{4.1.20}$$

In other words, the transformation (4.1.15) is a canonical transformation in the symplectic structure induced on TQ.

Hamiltonian viewpoint

 We subscribe the view that the original Hamiltonian formalism has to be interpreted has the study of solution space of the system (which perfectly translates to field theory or degenerate systems). Geometric structure can need to be modified but whatever geometric structure the phase space will turn out to have, it has to be presented intrinsically and exploited taking inspiration from what one does in mechanics. Finally, we need to compare and ensure translation from and to the formulation in terms of foliations and canonical analysis of field theories to discuss how equivalent this approach is to the usual constraint analysis.

 We also have to account for the notion of *state* of a system, of *phase space*, of physical *observables*, possibly without resorting to non-canonical structures, or at least using them only as computational tools. Possibly, we also wish to make the framework friendly to the generalization from mechanics to field theory, as well as possibly to higher order field theories.

2. Covariant Hamiltonian phase space

Already in first order mechanics, the Liouville part of a Lagrangian is a 1-form $\mathbb{F} = \frac{\partial L}{\partial u^\lambda}(t, q, u) dq^\lambda$, which is, in fact, a map $\hat{\Phi}_L : \mathbb{R} \times TQ \to \mathbb{R} \times T^*Q$,
it is precisely the Legendre map. We can take inspiration from this simple case to generalize it to (possibly higher order) field theories.

In a general field theory, the Liouville part for a first order Lagrangian is

$$\mathbb{F} = p_i^\mu \, \omega^i \wedge d\sigma_\mu = p_i^\mu \, dy^i \wedge d\sigma_\mu - p_i^\mu y_\mu^i d\sigma \tag{4.2.1}$$

i.e. an m-form of contact order 1. We can regard it as a map $\mathbb{F} : J^1 B \to Z(B)$.

For a second order Lagrangian, the Liouville part is

$$\mathbb{F} = (p_i^\mu - d_\alpha p_i^{\alpha\mu}) \, \omega^i \wedge d\sigma_\mu + p_i^{\mu\nu} \, \omega_\nu^i \wedge d\sigma_\mu = f_i^\mu \, dy^i \wedge d\sigma_\mu + f_i^{\mu\nu} \, dy_\nu^i \wedge d\sigma_\mu - \left(f_i^\mu y_\mu^i + f_i^{\mu\nu} y_{\mu\nu}^i \right) \, d\sigma \tag{4.2.2}$$

i.e. an m-form of contact order 1, which can be considered as a map $\mathbb{F} : J^3 B \to Z(J^1 B)$ which, in particular, specializes to the first order case when $p_i^{\mu\nu} = f_i^{\mu\nu} = 0$.

Contact forms are well suited to show that the Liouville part is a global, familiar object, as well as that the action gets unmodified by contact terms in view of the evaluation along the prolongation of a section. On the other hand, when we work on a given jet prolongation contact forms become awkward. The form $\mathbb{F}$ is a contact form, if one looks at the basis, it is on $J^1 B$, because of ω^i, but it is not the most general contact form on $J^1 B$ since $d\omega^i = -dy_\alpha^i \wedge dx^\alpha$ is also a contact form on $J^1 B$.

Accordingly, the most general m-form on $J^1 B$ of contact order 1 is

$$p_i^\mu \, \omega^i \wedge d\sigma_\mu + p_i^{\mu\alpha} \, d\omega^i \wedge d\sigma_{\mu\alpha} = p_i^\mu \, \omega^i \wedge d\sigma_\mu - 2p_i^{[\mu\alpha]} \, dy_\alpha^i \wedge d\sigma_\mu \tag{4.2.3}$$

In other words, the Liouville part is not simply a form of contact order 1, it has no component along $d\omega^i$, i.e. it has $p_i^{[\mu\alpha]} = 0$, namely $p_i^{\mu\alpha}$ is symmetric. In the notation introduced above we have that the Liouville part, lifted to $J^2 B$, is *reduced*.

For the second order Lagrangians, the form $\mathbb{F}$ is a contact form, if one looks at the basis, it is on $J^2 B$, because of ω_ν^i, but it is not the most general contact form on $J^2 B$ since $d\omega_\nu^i = -dy_{\nu\alpha}^i \wedge dx^\alpha$ is also a contact form on $J^2 B$. Accordingly, the most general m-form on $J^2 B$ of contact order 1 is

$$f_i^\mu \, \omega^i \wedge d\sigma_\mu + f_i^{\mu\nu} \, \omega_\nu^i \wedge d\sigma_\mu + f_i^{\mu\nu\alpha} \, d\omega_\nu^i \wedge d\sigma_{\mu\alpha} = f_i^\mu \, \omega^i \wedge d\sigma_\mu + f_i^{\mu\nu} \, \omega_\nu^i \wedge d\sigma_\mu + 2f_i^{[\mu\alpha]\nu} \, dy_{\nu\alpha}^i \wedge d\sigma_\mu \tag{4.2.4}$$

Again the Liouville part, lifted to $J^3 B$, is a *reduced* form.

Since, in Hamiltonian formalism, we need to discuss the space of all possible momenta, it is convenient to introduce a different representation for $\mathbb{F}$. Let us remark that originally the Liouville part of a Lagrangian of order $k = 2$ appeared as

$$\mathbb{F} = \left(f_i^\mu X^i + f_i^{\mu\alpha} d_\alpha X^i \right) \otimes d\sigma_\mu \tag{4.2.5}$$

to be evaluated along a holonomic section $J^{2k-1}\sigma$ and integrated on a bundary ∂D. Accordingly, the Liouville part is a functional on vertical vectors on $[J^{k-1}B \to M]$ to $(m-1)$-forms on M, i.e. a point in $V^*(J^{k-1}B) \otimes A_{m-1}(M)$, where we understand a pull-back of the bundle $A_{m-1}(M)$ to B along the projection $\pi^{k-1} : J^{k-1}B \to M$.

As far as local representations are concerned, a basis of vertical vectors on $[B \to M]$ is ∂_i and we denote by $\bar{d}y^i$ the dual basis of $V^*(B)$. Accordingly, a point in $V^*(B) \otimes A_{m-1}(M)$ is denoted precisely by

$$p_i^\mu \, \bar{d}y^i \otimes d\sigma_\mu \tag{4.2.6}$$

Let us remark that vertical vectors are injected into vectors by $i : V(B) \to TB$, which means, by duality, that $p : T^*B \to V^*(B)$ is a projection. We do not have a canonical embedding from $V^*(B)$ to T^*B. Accordingly, $\bar{d}y^i$ are not forms on B, in particular they cannot be confused with dy^i in T^*B. In fact, the first ones transform as $\bar{d}y'^i = J_k^i \bar{d}y^k$, the second as $dy'^i = J_\mu^i dx^\mu + J_k^i dy^k$.

We can see that by considering transformation laws with respect to a change of coordinates in B, hence

$$x'^\mu = x'^\mu(x) \qquad\qquad y'^i = y'^i(x,y) \tag{4.2.7}$$

In fact, we have $d\sigma'_\mu = J\bar{J}_\mu^\alpha d\sigma_\alpha$ and $X'^i = J_k^i X^k$ for vertical vectors on B. That corresponds to

$$p_i'^\mu X'^i \otimes d\sigma'_\mu = J\bar{J}_\mu^\nu p_i'^\mu J_k^i X^k \otimes d\sigma_\nu \tag{4.2.8}$$

and to transformation laws on $V^*(B) \otimes A_{m-1}(M)$ in the form

$$p_k^\nu = J\bar{J}_\mu^\nu p_i'^\mu J_k^i \tag{4.2.9}$$

which, in fact, agree with transformation laws of momenta.

At second order, the situation is similar. A basis of vertical vectors on $[J^1B \to M]$ is $(\partial_i, \partial_i^\mu)$ and we denote by $(\bar{d}y^i, \bar{d}y_\mu^i)$ the dual basis of $V^*(J^1B)$. Accordingly, a point in $V^*(J^1B) \otimes A_{m-1}(M)$ is denoted precisely by

$$\left(f_i^\mu \bar{d}y^i + f_i^{\mu\alpha} \bar{d}y_\alpha^i\right) \otimes d\sigma_\mu \tag{4.2.10}$$

We can consider transformation laws with respect to a change of coordinates in B, hence in J^1B

$$x'^\mu = x'^\mu(x) \qquad y'^i = y'^i(x,y) \qquad \Rightarrow \qquad y_\mu'^i = \bar{J}_\mu^\nu(J_\nu^i + y_\nu^k J_k^i) \tag{4.2.11}$$

In fact, we have $d\sigma'_\mu = J\bar{J}_\mu^\alpha d\sigma_\alpha$ and

$$X'^i = J_k^i X^k \qquad\qquad X_\mu'^i = \bar{J}_\mu^\alpha d_\alpha(J_k^i X^k) = \bar{J}_\mu^\alpha \left(d_\alpha J_k^i X^k + J_k^i X_\alpha^k\right) \tag{4.2.12}$$

for vertical vectors on J^1B. That corresponds to

$$\begin{aligned}
\left(f_i'^\mu X'^i + f_i'^{\mu\alpha} X_\alpha'^i\right) \otimes d\sigma'_\mu &= J\bar{J}_\mu^\nu \left(f_i'^\mu J_k^i X^k + f_i'^{\mu\alpha} \bar{J}_\mu^\alpha \left(d_\alpha J_k^i X^k + J_k^i X_\alpha^k\right)\right) \otimes d\sigma_\nu = \\
&= J\bar{J}_\mu^\nu \left(\left(f_i'^\mu J_k^i + f_i'^{\mu\alpha} \bar{J}_\mu^\alpha d_\alpha J_k^i\right) X^k + f_i'^{\mu\alpha} \bar{J}_\mu^\alpha J_k^i X_\alpha^k\right) \otimes d\sigma_\nu
\end{aligned} \tag{4.2.13}$$

and to transformation laws on $V^*(J^1B) \otimes A_{m-1}(M)$ in the form

$$f_k^\nu = J\bar{J}_\mu^\nu \left(f_i'^\mu J_k^i + f_i'^{\mu\alpha} \bar{J}_\mu^\alpha d_\alpha J_k^i\right) \qquad\qquad f_k^{\nu\alpha} = J\bar{J}_\mu^\nu f_i'^{\mu\alpha} \bar{J}_\mu^\alpha J_k^i \tag{4.2.14}$$

which, again, agree with transformation laws of formal momenta.

Some of that generalizes to any order. The Liouville part of a Lagrangian of order k can always be seen as a map $\hat{\Phi}_L : J^{2k-1}B \to V^*(J^{k-1}B) \otimes A_{m-1}(M)$, which defines (formal) momenta as well as the arena $P(J^{k-1}B) := V^*(J^{k-1}B) \otimes A_{m-1}(M)$ for Hamiltonian formalism.

In first order mechanics, one recovers $P(B) := \mathbb{R} \times T^*Q$ as well as the usual definitions of momenta and Legendre map.

As a consequence of these transformation laws, we can regard $\mathbb{F}$ as a global map

$$\hat{\Phi}_L : J^{2k-1}B \to P(J^{k-1}B) \tag{4.2.15}$$

which, in general, takes the place of Legendre transform and implicitly defines (formal) momenta.

At first order, the bundle $P(B) = V^*(B) \otimes A_{m-1}(M)$ has coordinates $(x^\mu, y^i; p_i^\mu)$ while, at second order, $P(J^1B) = V^*(J^1B) \otimes A_{m-1}(M)$ has coordinates $(x^\mu, y^i, y_\mu^i; f_i^\mu, f_i^{\mu\alpha})$.

The coordinates $(f_i^\mu, f_i^{\mu\alpha})$ transforms as formal momenta, i.e. as the corresponding coordinates of $Z(B)$ and $Z(J^1B)$. Accordingly, we have a global, intrinsic projection which makes a bundle $[Z(B) \to V^*(B) \otimes A_{m-1}(M) \to B \to M]$ (or, more generally, $[Z(J^1B) \to V^*(J^1B) \otimes A_{m-1}(M) \to J^1B \to B \to M]$).

That is analogous to the projection $[T^*(\mathbb{R} \times Q) \to \mathbb{R} \times T^*Q \to \mathbb{R} \times Q \to \mathbb{R}]$ one has in mechanics.

The bundle $P(J^{k-1}B)$ is called the *Hamiltonian phase space*, as opposed to the *Lagrangian phase phase* $J^{2k-1}B$.

Both these names are abuses of language. They are *candidates* to phase spaces if the dynamics are non-degenerate. In most cases, they contain (or they project onto) the actual phase space, where states of the system are defined as points or sections. That will become transparent by discussing Cauchy problems, below.

Let us also remark that, at first order, we have $\dim(J^1B) = m + k(m+1) = \dim(V^*(B) \otimes A_{m-1}(M))$.

At second order, instead, we have $\dim(J^2B) = m + \frac{1}{2}k(m+1)(m+2)$, $\dim(J^3B) = m + \frac{1}{6}k(m+1)(m+2)(m+3)$ and $\dim(V^*(J^1B) \otimes A_{m-1}(M)) = m + k(m+1)^2$. In general, dimensions do not match. Unless, as a particular case, we set $m = 1$, thus we have $\dim(J^2B) = 1+3k$, $\dim(J^3B) = 1+4k$ and $\dim(V^*(J^1B) \otimes A_0(M)) = 1+4k$ so that, in this case, we have $\dim(J^3B) = \dim(V^*(J^1B) \otimes A_0(M))$.

That is the magic numerology of mechanics. In mechanics, we can have that Legendre map is (can be) an isomorphism between the Lagrangian and the Hamiltonian phase spaces, at any order, while in field theory that is true at first order, while at higher orders there is no hope to have isomorphisms.

By defining momenta in this way, in the first order case, one has more momenta p_i^μ than field configurations y^i, still as many momenta as field derivatives y_μ^i. Unfortunately, in the second order case, if field equations are 4th order, the Lagrangian phase space is J^3B, hence with coordinates $(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i y_{\mu\nu\sigma}^i)$, while one has less formal momenta $(x^\mu, y^i, y_\mu^i, f_i^\mu, f_i^{\mu\nu})$ in the Hamiltonian phase space $P(J^1B)$.

Poincaré-Cartan form in a Lagrangian field theory

If we want to define a map connecting the Lagrangian world and the Hamiltonian one, we do not need too much imagination.

The form $\boldsymbol{\Theta}_L = L\,d\sigma + f_k^\alpha\,\omega^k \wedge d\sigma_\alpha + f_i^{\alpha\beta}\,\omega_\beta^k \wedge d\sigma_\alpha$ is global

$$
\begin{aligned}
\boldsymbol{\Theta}_L' =& L'\,d\sigma' + f_i'^\mu\,\omega'^i \wedge d\sigma_\mu' + f_i'^{\mu\alpha}\,\omega_\alpha'^i \wedge d\sigma_\mu' = \\
=& Ld\sigma + J\bar{J}_\mu^\alpha\left(f_i'^\mu J_k^i + f_i'^{\mu\alpha}d_\alpha' J_k^i\right)\omega^k \wedge d\sigma_\alpha + J\bar{J}_\mu^\alpha f_i'^{\mu\alpha}J_k^i\,\bar{J}_\alpha^\beta\,\omega_\beta^k \wedge d\sigma_\alpha = \\
=& L\,d\sigma + f_k^\alpha\,\omega^k \wedge d\sigma_\alpha + f_i^{\alpha\beta}\,\omega_\beta^k \wedge d\sigma_\alpha = \boldsymbol{\Theta}_L
\end{aligned}
\tag{4.2.16}
$$

On the other hand, we can write it as

$$\boldsymbol{\Theta}_L = -\mathcal{H}\,d\sigma + f_k^\alpha\,dy^k \wedge d\sigma_\alpha + f_i^{\alpha\beta}\,dy_\beta^k \wedge d\sigma_\alpha \tag{4.2.17}$$

where we set $\mathcal{H} = f_i^\alpha y_\alpha^i + f_i^{\alpha\beta}y_{\alpha\beta}^i - L$.

The quantity $-\mathcal{H}$ needs to transform as f transforms as a coordinate on $Z(J^1B)$ so that also in this form $\boldsymbol{\Theta}_L$ is global. That is a proof itself, however, let us check it directly (to be compared with (18.8.36)):

$$
\begin{aligned}
-\mathcal{H}' =& -f_i'^\alpha y_\alpha'^i - f_i'^{\alpha\beta} y_{\alpha\beta}'^i + L' = -\bar{J}\left(J\bar{J}_\alpha^\sigma\left(f_i'^\alpha J_k^i + f_i'^{\alpha\beta}\bar{J}_\beta^\rho d_\rho J_k^i\right)y_\sigma^k + J\bar{J}_\alpha^\sigma\bar{J}_\beta^\rho f_i'^{\alpha\beta}J_k^i y_{\sigma\rho}^k - L\right) + \\
& -f_i'^\alpha\bar{J}_\alpha^\sigma J_\sigma^i - f_i'^{\alpha\beta}\bar{J}_{\alpha\beta}^\sigma\left(J_\sigma^i + J_k^i y_\sigma^k\right) - \bar{J}_\alpha^\sigma f_i'^\alpha\bar{J}_\beta^\rho d_\rho J_\sigma^i = -\bar{J}\left(f_k^\sigma y_\sigma^k + f_k^{\sigma\rho}y_{\sigma\rho}^k - L\right) + \\
& -\bar{J}_\alpha^\sigma f_i'^\alpha J_\sigma^i - f_i'^{\alpha\beta}\bar{J}_{\alpha\beta}^\sigma\left(J_\sigma^i + J_k^i y_\sigma^k\right) - \bar{J}_\alpha^\sigma\bar{J}_\beta^\rho f_i'^{\alpha\beta}d_\rho J_\sigma^i = \\
=& -\bar{J}\mathcal{H} - \bar{J}_\alpha^\sigma f_i'^\alpha J_\sigma^i - f_i'^{\alpha\beta}\bar{J}_{\alpha\beta}^\sigma\left(J_\sigma^i + J_k^i y_\sigma^k\right) - \bar{J}_\alpha^\sigma\bar{J}_\beta^\rho f_i'^{\alpha\beta}d_\rho J_\sigma^i
\end{aligned}
\tag{4.2.18}
$$

Accordingly, we can define a global map $\psi_L : J^3B \to Z(J^1B)$ which is locally expressed as

$$
f = -\mathcal{H} = L - f_i^\mu y_\mu^i - f_i^{\mu\nu}y_{\mu\nu}^i \qquad f_i^\mu = p_i^\mu - d_\alpha p_i^{\mu\alpha} = \frac{\partial L}{\partial y_\mu^i} - d_\alpha\frac{\partial L}{\partial y_{\mu\alpha}^i} \qquad f_i^{\mu\nu} = p_i^{\mu\nu} = \frac{\partial L}{\partial y_{\mu\nu}^i}
\tag{4.2.19}
$$

and define the *Lagrangian Poincaré-Cartan form* $\boldsymbol{\Theta}_L = (\psi_L)^*\boldsymbol{\Theta}$ to be the global form on J^3B locally expressed as

$$
\boldsymbol{\Theta}_L = -\mathcal{H}\,d\sigma + f_i^\mu\,dy^i\wedge d\sigma_\mu + f_i^{\mu\nu}\,dy_\nu^i\wedge d\sigma_\mu = L\,d\sigma + f_i^\mu\,\omega^i\wedge d\sigma_\mu + f_i^{\mu\nu}\,\omega_\nu^i\wedge d\sigma_\mu
\tag{4.2.20}
$$

At first order, we have $f_i^{\mu\alpha} = 0$ and the map drop in order to $\psi_L : J^1B \to Z(B)$ and we simply get the *Lagrangian Poincaré-Cartan form* $\boldsymbol{\Theta}_L = (\psi_L)^*\boldsymbol{\Theta}$ to be a global form on J^1B locally expressed as

$$
\boldsymbol{\Theta}_L = -\mathcal{H}\,d\sigma + p_i^\mu\,dy^i\wedge d\sigma_\mu = L\,d\sigma + p_i^\mu\,\omega^i\wedge d\sigma_\mu
\tag{4.2.21}
$$

where we set $\mathcal{H} := p_i^\alpha y_\alpha^i - L$.

For a general second order theory, we have $\dim(J^3B) \neq \dim(P(J^1B))$. Accordingly, the map $\hat{\Phi}_L$ cannot be an isomorphism. Even at first order, where $\dim(J^1B) = \dim(P(B)) = m + k(m+1)$, the Lagrangian is very often degenerate so that, even in first order cases, the Legendre map $\hat{\Phi}_L : J^1B \to P(M)$ cannot be safely taken to be an isomorphism, anyway.

For example in electromagnetism, the Maxwell Lagrangian is first order. The Legendre map is defines as

$$
\hat{\Phi}_L : J^1B \to P(M) : (x^\mu, g_{\mu\nu}, A_\mu, d_\nu A_\mu) \mapsto (x^\mu, g_{\mu\nu}, A_\mu, -\tfrac{\sqrt{g}}{2}F^{\mu\nu})
\tag{4.2.22}
$$

However, the momenta $\pi^{\mu\nu} = -\frac{\sqrt{g}}{2}F^{\mu\nu}$ happen to be skew and one cannot hope to solve them for the field derivatives $d_\nu A_\mu$, at most for their skew part.

For the sake of generality and reasonable simplicity, the next to optimal situation, when $\hat{\Phi}_L$ is not an isomorphism, is that it is at least of constant rank.

The rank of the map refers to its tangent map. In this case, if $T\hat{\Phi}_L : TJ^1B \to TP(B)$ being of rank k defines an involutive (hence integrable) distribution Δ of rank k on $P(B)$. That means that the image $\Lambda_1 := \operatorname{Im}(\hat{\Phi}_L) \subset P(J^{k-1}B)$ is a sub-manifold such that $\Delta = T\Lambda_1$, which is called the *primary constraint*.

In some particular first order cases, e.g. for Klein-Gordon field, one can have $\Lambda_1 = P(B)$ and $\hat{\Phi}_L$ an isomorphism. However, this is the exception rather than the norm.

To deal with the primary constraint, we can assume coordinates (x^μ, y^i, z^A) on Λ_1 and consider it canonically (vertically) embedded in $P(B)$ as

$$\iota : \Lambda_1 \to P(B) : (x^\mu, y^i, z^A) \mapsto (x^\mu, y^i, p_i^\mu) \qquad\qquad p_i^\mu = \iota_i^\mu(x^\mu, y^i, z^A) \tag{4.2.23}$$

Then the Legendre map is *redefined* as a map

$$\Phi_L : J^1B \to \Lambda_1 : (x^\mu, y^i, y_\mu^i) \mapsto (x^\mu, y^i, z^A) \qquad\qquad z^A = z^A(x^\mu, y^i, y_\mu^i) \tag{4.2.24}$$

from the Lagrangian phase space J^1B to the primary constraint Λ_1, so that the previous map valued in the Hamiltonian phase space $P(B)$ is now obtained as $\hat\Phi_L = \iota \circ \Phi_L : J^1B \to P(B)$.

To make things easier, we wish to deal at least with second order theories to be able to deal with metric GR. In second order theories one has the primary constraint $\iota : \Lambda_1 \to P(J^1B)$ defined by

$$f_i^\mu = \iota_i^\mu(x^\mu, y^i, y_\mu^i, z^A) \qquad\qquad f_i^{\mu\alpha} = \iota_i^{\mu\alpha}(x^\mu, y^i, y_\mu^i, z^A) \tag{4.2.25}$$

and the Legendre map $\Phi_L : J^3B \to \Lambda$ is defined by $z^A = z^A(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, y_{\mu\nu\alpha}^i)$.

Let us also, stress that when we define the momenta as

$$f_i^\mu(j^3y) = \frac{\partial L}{\partial y_\mu^i} - d_\alpha \frac{\partial L}{\partial y_{\mu\alpha}^i} \qquad\qquad f_i^{\mu\alpha}(j^2y) = \frac{\partial L}{\partial y_{\mu\alpha}^i} \tag{4.2.26}$$

we obtain $f_i^{\mu\alpha}$ symmetric in the upper indices, while the corresponding coordinates on $P(B)$ are not assumed symmetric.

The Legendre map $\Phi_L : J^1B \to \Lambda_1$ (or $\Phi_L : J^3B \to \Lambda_1$) is surjective by construction of the primary constraint and maximal rank. It then defines a bundle projection $[J^1B \to \Lambda_1]$. We wish to show that ψ_L is constant on the fibres of such a fibration.

Given a first order Lagrangian, it defines a bilinear form $\Delta = \partial_k^\sigma \partial_i^\mu L \, \bar d y_\sigma^k \otimes \bar d y_\mu^i$, which can be seen as a linear map $\Delta : V(J^1B) \to V^*(J^1B)$.

Let us consider the curve $\gamma : \mathbb{R} \to J^1B : s \mapsto (x, y, y_\mu^i(s))$ and a function F on $P(B)$, so that it can be pulled-back on J^1B, and $(\hat\Phi_L)^*F(x, y, \pi_i^\mu(x, y, y_\mu^i(s))) = F(x, y, \iota_i^\mu(x, y, z))$. We want to determine the curve so that it projects on a fixed point. Hence, we have

$$\dot y_\sigma^k \partial_k^\sigma \pi_i^\mu \partial_\mu^i = 0 \qquad \Rightarrow \qquad \dot y_\sigma^k \frac{\partial L}{\partial y_\sigma^k \partial y_\mu^i} = 0 \tag{4.2.27}$$

namely $\dot\gamma \in \ker(\Delta)$.

We defined a distribution $\Delta \subset TJ^1B$ of such vertical vectors $v = \dot y_\mu^i \partial_i^\mu \in \Delta$ in $[J^1B \to B]$. A local vector field X of such a distribution is called the *vertical generators* l for the fibration $[J^1B \to \Lambda_1]$ and it is in fact tangent to the fibers of such a bundle. That shows the distribution is integrable, hence also involutive.

If we find a set X_A of local vertical generators, then, for involution, there exist local *structure functions* $f_{AB}{}^C$ such that

$$[X_A, X_B] = f_{AB}{}^C X_C \tag{4.2.28}$$

If we consider now $\psi_L : J^1B \to Z(B)$ which is given by $f = -\mathcal{H}$, $f_i^\mu = \pi_i^\mu(j^1y)$ we can show that ψ_L is constant along the directions in $\ker(\Delta)$. In fact, we have

$$\frac{df_i^\mu}{ds} = \partial_k^\sigma \partial_i^\mu L \, \dot y_\sigma^k = 0 \qquad\qquad \frac{d\mathcal{H}}{ds} = \frac{d}{ds}\left(\pi_i^\mu y_\mu^i - L\right) = \left(\partial_k^\sigma \partial_i^\mu L \, y_\mu^i + \pi_k^\sigma - \partial_k^\sigma L\right) \dot y_\sigma^k = 0 \tag{4.2.29}$$

For second order theories, let us consider the curve $\gamma : \mathbb{R} \to J^3 B : s \mapsto (x, y, y^i_\mu, y^i_{\mu\nu}(s), y^i_{\mu\nu\sigma}(s))$, the naive momenta along the curve

$$p^\mu_i = \pi^\mu_i(x, y, y^k_\alpha, y^k_{\alpha\beta}(s)) \qquad\qquad p^{\mu\nu}_i = \pi^{\mu\nu}_i(x, y, y^k_\alpha, y^k_{\alpha\beta}(s)) \tag{4.2.30}$$

the formal momenta along the curve

$$f^\mu_i = p^\mu_i - d_\nu p^{\mu\nu}_i \qquad\qquad f^{\mu\nu}_i = p^{\mu\nu}_i \tag{4.2.31}$$

That defines a curve $\gamma_L : \mathbb{R} \to P(J^1 B)$ which has a tangent vector with

$$\dot{f}^\mu_i = \partial^{\alpha\beta}_k p^\mu_i \, \dot{y}^k_{\alpha\beta} - d_\nu(\partial^{\alpha\beta}_k p^{\mu\nu}_i)\dot{y}^k_{\alpha\beta} - \dot{y}^k_{\alpha\nu}\partial^\alpha_k p^{\mu\nu}_i - \dot{y}^k_{\alpha\beta\nu}\partial^{\alpha\beta}_k p^{\mu\nu}_i \qquad\qquad \dot{f}^{\mu\nu}_i = \partial^{\alpha\beta}_k p^{\mu\nu}_i \, \dot{y}^k_{\alpha\beta} \tag{4.2.32}$$

We want to determine the curve so that it projects on a fixed point, i.e. $\dot{f}^\mu_i = 0$ and $\dot{f}^{\mu\nu}_i = 0$. Hence, we have

$$\dot{y}^k_{\alpha\beta}\left(\partial^{\alpha\beta}_k p^\mu_i - d_\nu(\partial^{\alpha\beta}_k p^{\mu\nu}_i) - \partial^\alpha_k p^{\mu\beta}_i\right) - \dot{y}^k_{\alpha\beta\nu}\partial^{\alpha\beta}_k p^{\mu\nu}_i = 0 \qquad\qquad \dot{y}^k_{\alpha\beta}\frac{\partial L}{\partial y^k_{\alpha\beta}\partial y^i_{\mu\nu}} = 0 \tag{4.2.33}$$

If we consider now $\psi_L : J^3 B \to Z(J^1 B)$ which is given by $f = -\mathcal{H}$, $f^\mu_i = p^\mu_i - d_\nu p^{\mu\nu}_i$, and $f^{\mu\nu}_i = p^{\mu\nu}_i$ we can show that ψ_L is constant along the directions of $\dot\gamma$. In fact, we have

$$\frac{df^\mu_i}{ds} = \dot{y}^k_{\alpha\beta}\partial^{\alpha\beta}_k f^\mu_i - \dot{y}^k_{\alpha\beta\nu}\partial^{\alpha\beta}_k f^{\mu\nu}_i = 0 \qquad\qquad \frac{df^{\mu\nu}_i}{ds} = \partial^{\alpha\beta}_k \partial^{\mu\nu}_i L \, \dot{y}^k_{\alpha\beta} = 0$$
$$\frac{d\mathcal{H}}{ds} = \frac{d}{ds}\left(f^\mu_i y^i_\mu + f^{\mu\nu}_i y^i_{\mu\nu} - L\right) = \left(f^{\mu\nu}_i \dot{y}^i_{\mu\nu} - p^{\alpha\beta}_k \dot{y}^k_{\alpha\beta}\right) = 0 \tag{4.2.34}$$

Again, we can define a distribution $\Delta \subset TJ^3 B$ of vertical vectors of $[J^3 B \to \Lambda_1]$. Its generators X_A are also called *vertical generators*, they are tangent to the fibers of $[J^3 B \to \Lambda_1]$, so that the distribution Δ is integrable, hence involutive. Accordingly, there exists local structure functions such that

$$[X_A, X_B] = f_{AB}{}^C X_C \tag{4.2.35}$$

Accordingly, the map ψ_L is compatible with the projection $[J^{2k-1} B \to \Lambda_1]$ and it defines a map $\psi_H : \Lambda_1 \to Z(J^{k-1} B)$. Let us summarize what we have at first and second order, as the situation represented by the following diagrams:
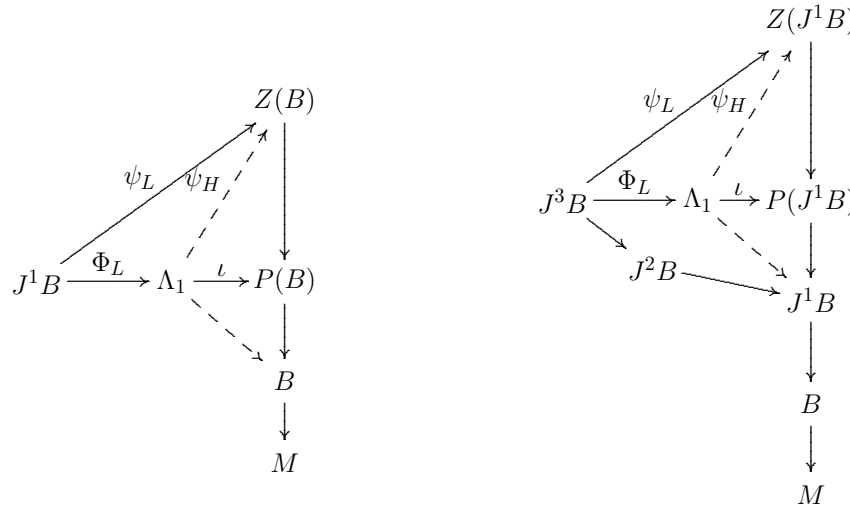
In particular, although Φ_L is not invertible and we cannot push forward $\boldsymbol{\Theta}_L$ onto Λ_1, still we can define $\boldsymbol{\Theta}_H := (\psi_H)^*\boldsymbol{\Theta}$ on Λ_1 and we have

$$(\Phi_L)^*\boldsymbol{\Theta}_H = (\Phi_L)^*(\psi_H)^*\boldsymbol{\Theta} = (\psi_H \circ \Phi_L)^*\boldsymbol{\Theta} = (\psi_L)^*\boldsymbol{\Theta} = \boldsymbol{\Theta}_L \tag{4.2.36}$$

The m-form $\boldsymbol{\Theta}_H$ we defined on the primary constraint $\Lambda_1 \subset P(J^{k-1}B)$ is called the *(Hamiltonian) Poincaré-Cartan form*.

For first order field theories, the map $\psi_H : \Lambda_1 \to Z(B) : (x, y, z) \mapsto (x, y, f_i^\mu, f)$ is defined by giving (f_i^μ, f) as functions of (x, y, z). Considering that $\psi_H \circ \Phi_L = \psi_L$ and that we know the map ψ_L, we have

$$f_i^\mu = \iota_i^\mu(x, y, z) \qquad\qquad f = -H(x, y, z) \tag{4.2.37}$$

with

$$\mathcal{H}(x, y, dy) = H(x, y, z(x, y, dy)) \qquad\qquad \pi_i^\mu(x, y, dy) = \iota_i^\mu(x, y, z(x, y, dy)) \tag{4.2.38}$$

Hence the Poincaré-Cartan form $\boldsymbol{\Theta}_H$ is written as

$$\boldsymbol{\Theta}_H = -H(x, y, z)\, d\sigma + \iota_i^\mu(x, y, z)\, dy^i \wedge d\sigma_\mu \tag{4.2.39}$$

For second order field theories, the map $\psi_H : \Lambda_1 \to Z(J^1B) : (x, y, dy, z) \mapsto (x, y, dy, f_i^\mu, f_i^{\mu\alpha}, f)$ is defined by giving $(f_i^\mu, f_i^{\mu\alpha}, f)$ as functions of (x, y, dy, z). Considering that $\psi_H \circ \Phi_L = \psi_L$ and that we know the map ψ_L, we have

$$f_i^\mu = \iota_i^\mu(x, y, dy, z) \qquad\qquad f_i^{\mu\alpha} = \iota_i^{\mu\alpha}(x, y, dy, z) \qquad\qquad f = -H(x, y, dy, z) \tag{4.2.40}$$

with

$$\mathcal{H}(x, y, dy, d^2y, d^3y) = H(x, y, dy, z(x, y, dy, d^2y, d^3y))$$
$$f_i^\mu(x, y, dy, d^2y, d^3y) = \iota_i^\mu(x, y, dy, z(x, y, dy, d^2y, d^3y)) \tag{4.2.41}$$
$$f_i^{\mu\alpha}(x, y, dy, d^2y) = \iota_i^{\mu\alpha}(x, y, dy, z(x, y, dy, d^2y, d^3y))$$

Hence the Poincaré-Cartan form $\boldsymbol{\Theta}_H$ is written as

$$\boldsymbol{\Theta}_H = -H(x, y, dy, z)\, d\sigma + \iota_i^\mu(x, y, dy, z)\, dy^i \wedge d\sigma_\mu + \iota_i^{\mu\alpha}(x, y, dy, z)\, dy_\alpha^i \wedge d\sigma_\mu \tag{4.2.42}$$

Dynamics from the Poincaré-Cartan form

Let us consider field theory of order $k = 2$ described by a Lagrangian $\boldsymbol{L} = L(j^k y)\, d\sigma$. If the Lagrangian is first order, the first order case is obtained as a special case by setting $f_i^{\mu\nu} = 0$ everywhere, as well by setting $k = 1$ instead.

By the Lagrangian Poincaré-Cartan form, we express the action as $A_D[\sigma] = \int_D (J^{2k-1}\sigma)^*\boldsymbol{\Theta}_L$ since $\boldsymbol{\Theta}_L$ differs from the Lagrangian by contact forms only. The variation of the action in this form reads as

$$\delta_X A_D[\sigma] = -\int_D (J^{2k-1}\sigma)^* \pounds_{J^{2k-1}X}\boldsymbol{\Theta}_L = -\int_D (J^{2k}\sigma)^* i_{J^{2k}X} d\boldsymbol{\Theta}_L - \int_{\partial D} (J^{2k-1}\sigma)^* i_{J^{2k-1}X}\boldsymbol{\Theta}_L \tag{4.2.43}$$

which is the first variation formula. A section $\sigma : M \to B$ is a solution iff for all deformations $X = X^i \partial_i$ we have that

$$(J^{2k}\sigma)^* i_{J^{2k}X} d\boldsymbol{\Theta}_L = 0 \tag{4.2.44}$$

while, when $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$ is a symmetry, the boundary part $(J^{2k-1}\sigma)^* i_{J^{2k-1}\Xi}\boldsymbol{\Theta}_L$ accounts for conservation laws.

First, we have

$$
\begin{aligned}
d\boldsymbol{\Theta}_L = &- d\mathcal{H} \wedge d\sigma + df_i^\mu \wedge dy^i \wedge d\sigma_\mu + df_i^{\mu\alpha} \wedge dy_\alpha^i \wedge d\sigma_\mu = \\
= &- \partial_k \mathcal{H}\, dy^k \wedge d\sigma - \partial_k^\alpha \mathcal{H}\, dy_\alpha^k \wedge d\sigma - \partial_k^{\alpha\beta}\mathcal{H}\, dy_{\alpha\beta}^k \wedge d\sigma - \partial_k^{\alpha\beta\epsilon}\mathcal{H}\, dy_{\alpha\beta\epsilon}^k \wedge d\sigma + \\
&- \left(d_\mu f_i^\mu - y_\mu^k \partial_k f_i^\mu - y_{\alpha\mu}^k \partial_k^\alpha f_i^\mu - y_{\alpha\beta\mu}^k \partial_k^{\alpha\beta} f_i^\mu - y_{\alpha\beta\epsilon\mu}^k \partial_k^{\alpha\beta\epsilon} f_i^\mu \right) dy^i \wedge d\sigma + \partial_k f_i^\mu\, dy^k \wedge dy^i \wedge d\sigma_\mu + \\
&+ \partial_k^\alpha f_i^\mu\, dy_\alpha^k \wedge dy^i \wedge d\sigma_\mu + \partial_k^{\alpha\beta} f_i^\mu\, dy_{\alpha\beta}^k \wedge dy^i \wedge d\sigma_\mu + \partial_k^{\alpha\beta\epsilon} f_i^\mu\, dy_{\alpha\beta\epsilon}^k \wedge dy^i \wedge d\sigma_\mu + \\
&- \left(d_\mu f_i^{\mu\alpha} - y_\mu^k \partial_k f_i^{\mu\alpha} - y_{\beta\mu}^k \partial_k^\beta f_i^{\mu\alpha} - y_{\beta\epsilon\mu}^k \partial_k^{\beta\epsilon} f_i^{\mu\alpha} \right) dy_\alpha^i \wedge d\sigma + \partial_k f_i^{\mu\alpha}\, dy^k \wedge dy_\alpha^i \wedge d\sigma_\mu + \\
&+ \partial_k^\beta f_i^{\mu\alpha}\, dy_\beta^k \wedge dy_\alpha^i \wedge d\sigma_\mu + \partial_k^{\beta\epsilon} f_i^{\mu\alpha}\, dy_{\beta\epsilon}^k \wedge dy_\alpha^i \wedge d\sigma_\mu
\end{aligned}
\tag{4.2.45}
$$

Let us then consider, by now, a vertical vector $\hat{X} = X^i \partial_i + X_\alpha^i \partial_i^\alpha + X_{\alpha\beta}^i \partial_i^{\alpha\beta} + X_{\alpha\beta\epsilon}^i \partial_i^{\alpha\beta\epsilon}$ and we get

$$
\begin{aligned}
\hat{X} \lrcorner d\boldsymbol{\Theta}_L = &\left(X^k \left(p_k - \underline{\partial_k f_i^\sigma y_\sigma^i} - \underline{\underline{\partial_k f_i^{\rho\sigma} y_{\rho\sigma}^i}} \right) + X_\alpha^k \left(\boxed{p_k^\alpha - p_k^\alpha} + d_\sigma p_k^{\sigma\alpha} - \partial_k^\alpha f_i^\sigma y_\sigma^i - \underline{\underline{\partial_k^\alpha f_i^{\rho\sigma} y_{\rho\sigma}^i}} \right) \right) d\sigma + \\
&+ \left(X_{\alpha\beta}^k \left(\boxed{p_k^{\alpha\beta} - p_k^{\alpha\beta}} - \partial_k^{\alpha\beta} f_i^\sigma y_\sigma^i - \underline{\underline{\partial_k^{\alpha\beta} f_i^{\rho\sigma} y_{\rho\sigma}^i}} \right) - X_{\alpha\beta\epsilon}^k \partial_k^{\alpha\beta\epsilon} f_i^\sigma y_\sigma^i \right) d\sigma + \\
&- X^i \left(d_\mu f_i^\mu - y_\mu^k \partial_k f_i^\mu - y_{\alpha\mu}^k \partial_k^\alpha f_i^\mu - \underline{y_{\alpha\beta\mu}^k \partial_k^{\alpha\beta} f_i^\mu} - \underline{y_{\alpha\beta\epsilon\mu}^k \partial_k^{\alpha\beta\epsilon} f_i^\mu} \right) d\sigma + \\
&+ X^i \left(\partial_i f_k^\mu - \partial_k f_i^\mu \right) \omega^k \wedge d\sigma_\mu + X^i \left(\underline{y_\mu^k \partial_i f_k^\mu} - y_\mu^k \partial_k f_i^\mu \right) d\sigma + \\
&+ X_\alpha^k \partial_k^\alpha f_i^\mu\, \omega^i \wedge d\sigma_\mu + X_\alpha^i y_\mu^k \partial_i^\alpha f_k^\mu\, d\sigma - X^i \partial_k^\alpha f_i^\mu\, \omega_\alpha^k \wedge d\sigma_\mu - {\color{red}X^i y_{\mu\alpha}^k \partial_k^\alpha f_i^\mu}\, d\sigma + \\
&+ X_{\alpha\beta}^k \partial_k^{\alpha\beta} f_i^\mu\, \omega^i \wedge d\sigma_\mu + X_{\alpha\beta}^k y_\mu^i \partial_k^{\alpha\beta} f_i^\mu\, d\sigma - X^i \partial_k^{\alpha\beta} f_i^\mu\, \omega_{\alpha\beta}^k \wedge d\sigma_\mu - {\color{red}X^i y_{\alpha\beta\mu}^k \partial_k^{\alpha\beta} f_i^\mu}\, d\sigma + \\
&+ X_{\alpha\beta\epsilon}^k \partial_k^{\alpha\beta\epsilon} f_i^\mu\, \omega^i \wedge d\sigma_\mu + X_{\alpha\beta\epsilon}^k y_\mu^i \partial_k^{\alpha\beta\epsilon} f_i^\mu\, d\sigma - X^i \partial_k^{\alpha\beta\epsilon} f_i^\mu\, \omega_{\alpha\beta\epsilon}^k \wedge d\sigma_\mu - {\color{blue}X^i y_{\alpha\beta\epsilon\mu}^k \partial_k^{\alpha\beta\epsilon} f_i^\mu}\, d\sigma + \\
&- X_\alpha^i \left(d_\mu f_i^{\mu\alpha} - \underline{{\color{red}y_\mu^k \partial_k f_i^{\mu\alpha}}} - y_{\beta\mu}^k \partial_k^\beta f_i^{\mu\alpha} - \underline{y_{\beta\epsilon\mu}^k \partial_k^{\beta\epsilon} f_i^{\mu\alpha}} \right) d\sigma + \\
&+ X^k \partial_k f_i^{\mu\alpha}\, \omega_\alpha^i \wedge d\sigma_\mu + {\color{red}\underline{X^i y_{\mu\alpha}^k \partial_i f_k^{\mu\alpha}}}\, d\sigma - X_\alpha^i \partial_k f_i^{\mu\alpha}\, \omega^k \wedge d\sigma_\mu - {\color{red}\underline{X_\alpha^i y_\mu^k \partial_k f_i^{\mu\alpha}}}\, d\sigma + \\
&+ X_\beta^k \partial_k^\beta f_i^{\mu\alpha}\, \omega_\alpha^i \wedge d\sigma_\mu + {\color{blue}\underline{\underline{X_\beta^k y_{\mu\alpha}^i \partial_k^\beta f_i^{\mu\alpha}}}}\, d\sigma - X_\alpha^i \partial_k^\beta f_i^{\mu\alpha}\, \omega_\beta^k \wedge d\sigma_\mu - {\color{green}X_\alpha^i y_{\mu\beta}^k \partial_k^\beta f_i^{\mu\alpha}}\, d\sigma + \\
&+ X_{\beta\epsilon}^k \partial_k^{\beta\epsilon} f_i^{\mu\alpha}\, \omega_\alpha^i \wedge d\sigma_\mu + {\color{green}\underline{\underline{X_{\beta\epsilon}^k y_{\mu\alpha}^i \partial_k^{\beta\epsilon} f_i^{\mu\alpha}}}}\, d\sigma - X_\alpha^i \partial_k^{\beta\epsilon} f_i^{\mu\alpha}\, \omega_{\beta\epsilon}^k \wedge d\sigma_\mu - {\color{green}\underline{X_\alpha^i y_{\beta\epsilon\mu}^k \partial_k^{\beta\epsilon} f_i^{\mu\alpha}}}\, d\sigma
\end{aligned}
\tag{4.2.46}
$$

which further simplifies to a form on $J^{2k}B$ (since we split out the contact part, which requires to lift the form one order of jet higher)

$$
\begin{aligned}
\hat{X} \lrcorner d\boldsymbol{\Theta}_L =& \left(X^k p_k + \underline{X^i_\alpha d_\mu p^{\mu\alpha}_i} - X^i d_\mu f^\mu_i - \underline{X^i_\alpha d_\mu f^{\mu\alpha}_i} - X^i_\alpha \partial^\alpha_k f^\sigma_i y^i_\sigma - X^k_{\alpha\beta} \partial^{\alpha\beta}_k f^\sigma_i y^i_\sigma - X^k_{\alpha\beta\epsilon} \partial^{\alpha\beta\epsilon}_k f^\sigma_i y^i_\sigma \right) d\sigma + \\
&+ \left(X^i_\alpha y^k_\mu \partial^\alpha_i f^\mu_k + X^k_{\alpha\beta} y^i_\mu \partial^{\alpha\beta}_k f^\mu_i + X^k_{\alpha\beta\epsilon} y^i_\mu \partial^{\alpha\beta\epsilon}_k f^\mu_i \right) d\sigma + \\
&+ \left(X^i \left(\partial_i f^\mu_k - \partial_k f^\mu_i \right) + X^i_\alpha \left(\partial^\alpha_i f^\mu_k - \partial_k f^{\mu\alpha}_i \right) + X^i_{\alpha\beta} \partial^{\alpha\beta}_i f^\mu_k + X^i_{\alpha\beta\epsilon} \partial^{\alpha\beta\epsilon}_i f^\mu_k \right) \omega^k \wedge d\sigma_\mu + \\
&+ \left(X^i \left(\partial_i f^{\mu\alpha}_k - \partial^\alpha_k f^\mu_i \right) + X^i_\beta \left(\partial^\beta_i f^{\mu\alpha}_k - \partial^\alpha_k f^{\mu\beta}_i \right) + X^i_{\beta\epsilon} \partial^{\beta\epsilon}_i f^{\mu\alpha}_k \right) \omega^k_\alpha \wedge d\sigma_\mu + \\
&- \left(X^i \partial^{\alpha\beta}_k f^\mu_i + X^i_\epsilon \partial^{\alpha\beta}_k f^{\mu\epsilon}_i \right) \omega^k_{\alpha\beta} \wedge d\sigma_\mu - X^i \partial^{\alpha\beta\epsilon}_k f^\mu_i \, \omega^k_{\alpha\beta\epsilon} \wedge d\sigma_\mu = \\
=& X^k \left(p_k - d_\mu p^\mu_k + d_{\mu\sigma} p^{\mu\sigma}_k \right) d\sigma + \\
&+ \left(X^i \left(\partial_i f^\mu_k - \partial_k f^\mu_i \right) + X^i_\alpha \left(\partial^\alpha_i f^\mu_k - \partial_k f^{\mu\alpha}_i \right) + X^i_{\alpha\beta} \partial^{\alpha\beta}_i f^\mu_k + X^i_{\alpha\beta\epsilon} \partial^{\alpha\beta\epsilon}_i f^\mu_k \right) \omega^k \wedge d\sigma_\mu + \\
&+ \left(X^i \left(\partial_i f^{\mu\alpha}_k - \partial^\alpha_k f^\mu_i \right) + X^i_\beta \left(\partial^\beta_i f^{\mu\alpha}_k - \partial^\alpha_k f^{\mu\beta}_i \right) + X^i_{\beta\epsilon} \partial^{\beta\epsilon}_i f^{\mu\alpha}_k \right) \omega^k_\alpha \wedge d\sigma_\mu + \\
&- \left(X^i \partial^{\alpha\beta}_k f^\mu_i + X^i_\epsilon \partial^{\alpha\beta}_k f^{\mu\epsilon}_i \right) \omega^k_{\alpha\beta} \wedge d\sigma_\mu - X^i \partial^{\alpha\beta\epsilon}_k f^\mu_i \, \omega^k_{\alpha\beta\epsilon} \wedge d\sigma_\mu
\end{aligned}
\tag{4.2.47}
$$

If we pull-back on the prolongation of a section $\sigma : M \to B$, the contact part vanishes and we get

$$
(J^{2k}\sigma)^* \hat{X} \lrcorner d\boldsymbol{\Theta}_L = X^k \left(p_k - d_\mu p^\mu_k + d_{\mu\sigma} p^{\mu\sigma}_k \right) d\sigma
\tag{4.2.48}
$$

which vanishes, for all vector fields X, if and only if σ is a solution of field equations. We do not even need to require $\hat{X}$ is the prolongation of a vertical vector field on B. We can, in the sense that we can restrict to prolongations only to enforce field equations, but we do not need to, in the sense that the quantity $(J^{2k}\sigma)^* \hat{X} \lrcorner d\boldsymbol{\Theta}_L = 0$ vanishes along solutions, for any vertical vector field $\hat{X}$.

There are a number of other variations of the way one can write field equations from the Poincaré-Cartan form. They will be discussed below.

As long as $i_{J^2\Xi}\boldsymbol{\Theta}_L$ is concerned, we have

$$
i_{J^2\Xi}\boldsymbol{\Theta}_L = - \left(f^\mu_i \left(y^i_\beta \xi^\beta - \xi^i \right) + f^{\mu\nu}_i \left(y^i_{\nu\beta} \xi^\beta - \xi^i_\nu \right) - \xi^\mu L \right) d\sigma_\mu - \xi^\alpha f^\mu_i \, \omega^i \wedge d\sigma_{\mu\alpha} - \xi^\alpha f^{\mu\nu}_i \, \omega^i_\nu \wedge d\sigma_{\mu\alpha}
\tag{4.2.49}
$$

and by pull-back on a holonomic section $\sigma : M \to B$ we get

$$
\begin{aligned}
-(J^3\sigma)^* i_{J^2\Xi}\Theta_L &= \left(f^\mu_i \left(y^i_\beta \xi^\beta - \xi^i \right) + f^{\mu\nu}_i \left(y^i_{\nu\beta} \xi^\beta - \xi^i_\nu \right) - \xi^\mu L \right) d\sigma_\mu = \\
&= \left(f^\mu_i \left(y^i_\beta \xi^\beta - \xi^i \right) + f^{\mu\nu}_i d_\nu \left(y^i_\beta \xi^\beta - \xi^i \right) - \xi^\mu L \right) d\sigma_\mu
\end{aligned}
\tag{4.2.50}
$$

which in fact accounts for the corresponding Noether current.

Variations on variational principles

Before going to Hamiltonian framework, we can use Poincaré-Cartan form to describe dynamics with different assumptions.

If we allow non-vertical vector fields $\hat{X}$ we obtain extra contributions to the quantity $\hat{X} \lrcorner d\boldsymbol{\Theta}_L$. These contributions factorize the compotent X^μ.

We have extra contributions to $\hat{X} \lrcorner d\boldsymbol{\Theta}_L$

$$X^\mu \Big(- \Big(p_k - d_\rho p_k^\rho + d_{\rho\sigma} f_k^{\rho\sigma} + y_\rho^i \partial_i f_k^\rho - \partial_k f_i^\epsilon y_\epsilon^i - \partial_k f_i^{\epsilon\sigma} y_{\epsilon\sigma}^i + y_{\alpha\rho}^i \partial_i^\alpha f_k^\rho + y_{\alpha\beta\rho}^i \partial_i^{\alpha\beta} f_k^\rho + y_{\alpha\beta\epsilon\rho}^i \partial_i^{\alpha\beta\epsilon} f_k^\rho \Big) \, dy^k \wedge d\sigma_\mu +$$

$$- \Big(d_\beta p_k^{\alpha\beta} - d_\rho f_k^{\rho\alpha} + y_\rho^i \partial_i f_k^{\rho\alpha} + y_{\beta\rho}^i \partial_i^\beta f_k^{\rho\alpha} + y_{\beta\epsilon\rho}^i \partial_i^{\beta\epsilon} f_k^{\rho\alpha} - \partial_k^\alpha f_i^\epsilon y_\epsilon^i - \partial_k^\alpha f_i^{\epsilon\sigma} y_{\epsilon\sigma}^i \Big) \, dy_\alpha^k \wedge d\sigma_\mu +$$

$$+ \Big(\partial_k^{\alpha\beta} f_i^\epsilon y_\epsilon^i + \partial_k^{\alpha\beta} f_i^{\epsilon\sigma} y_{\epsilon\sigma}^i \Big) \, dy_{\alpha\beta}^k \wedge d\sigma_\mu + \partial_k^{\alpha\beta\epsilon} f_i^\sigma y_\sigma^i \, dy_{\alpha\beta\epsilon}^k \wedge d\sigma_\mu +$$

$$+ \partial_k f_i^\rho \, dy^k \wedge dy^i \wedge d\sigma_{\rho\mu} + (\partial_i^\alpha f_k^\rho - \partial_k f_i^{\rho\alpha}) \, dy_\alpha^i \wedge dy^k \wedge d\sigma_{\rho\mu} +$$

$$+ \partial_k^{\alpha\beta} f_i^\rho \, dy_{\alpha\beta}^k \wedge dy^i \wedge d\sigma_{\rho\mu} + \partial_k^{\alpha\beta\epsilon} f_i^\rho \, dy_{\alpha\beta\epsilon}^k \wedge dy^i \wedge d\sigma_{\rho\mu} +$$

$$+ \partial_k^\beta f_i^{\rho\alpha} \, dy_\beta^k \wedge dy_\alpha^i \wedge d\sigma_{\rho\mu} + \partial_k^{\beta\epsilon} f_i^{\rho\alpha} \, dy_{\beta\epsilon}^k \wedge dy_\alpha^i \wedge d\sigma_{\rho\mu} \Big) \tag{4.2.51}$$

We can decide to pull-back this along a general section of $\hat{\sigma} : M \to J^{2k}B$ or along the prolongation of a section $\sigma : M \to B$. In the first case, the pull-back of contact forms does not vanish, however we have

$$X^\mu \Big[- \Big(p_k - d_\rho p_k^\rho + d_{\rho\sigma} p_k^{\rho\sigma} + \underline{y_\rho^i \partial_i f_k^\rho} - \partial_k f_i^\epsilon y_\epsilon^i + \Big(\underline{\partial_i^\alpha f_k^\rho} - \underline{\partial_k f_i^{\alpha\rho}} \Big) y_{\alpha\rho}^i + y_{\alpha\beta\rho}^i \partial_i^{\alpha\beta} f_k^\rho + \underline{y_{\alpha\beta\epsilon\rho}^i \partial_i^{\alpha\beta\epsilon} f_k^\rho} \Big) y_\mu^k +$$

$$- \Big(\underline{y_\rho^i \partial_i f_k^{\rho\alpha}} + \underline{\underline{y_{\beta\rho}^i \partial_i^\beta f_k^{\rho\alpha}}} + y_{\beta\epsilon\rho}^i \partial_i^{\beta\epsilon} f_k^{\rho\alpha} - \underline{\underline{\partial_k^\alpha f_i^\epsilon y_\epsilon^i}} - \underline{\underline{\partial_k^\alpha f_i^{\epsilon\sigma} y_{\epsilon\sigma}^i}} \Big) y_{\alpha\mu}^k +$$

$$+ \Big(\underline{\partial_k^{\alpha\beta} f_i^\epsilon y_\epsilon^i} + \underline{\partial_k^{\alpha\beta} f_i^{\epsilon\sigma} y_{\epsilon\sigma}^i} \Big) y_{\alpha\beta\mu}^k + \underline{\partial_k^{\alpha\beta\epsilon} f_i^\sigma y_\sigma^i y_{\alpha\beta\epsilon\mu}^k} +$$

$$+ \Big(\partial_k f_i^\rho y_\rho^k y_\mu^i - \partial_k f_i^\rho y_\mu^k y_\rho^i \Big) + \Big(\underline{\partial_i^\alpha f_k^\rho} - \underline{\partial_k f_i^{\rho\alpha}} \Big) y_{\alpha\rho}^i y_\mu^k - \Big(\underline{\underline{\partial_i^\alpha f_k^\rho}} - \underline{\underline{\partial_k f_i^{\rho\alpha}}} \Big) y_{\alpha\mu}^i y_\rho^k +$$

$$+ \partial_k^{\alpha\beta} f_i^\rho y_{\alpha\beta\rho}^k y_\mu^i - \underline{\partial_k^{\alpha\beta} f_i^\rho y_{\alpha\beta\mu}^k y_\rho^i} + \underline{\partial_k^{\alpha\beta\epsilon} f_i^\rho y_{\alpha\beta\epsilon\rho}^k y_\mu^i} - \underline{\partial_k^{\alpha\beta\epsilon} f_i^\rho y_{\alpha\beta\epsilon\mu}^k y_\rho^i} +$$

$$+ \underline{\partial_k^\beta f_i^{\rho\alpha} y_{\beta\rho}^k y_{\alpha\mu}^i} - \underline{\underline{\partial_k^\beta f_i^{\rho\alpha} y_{\beta\mu}^k y_{\alpha\rho}^i}} + \underline{\partial_k^{\beta\epsilon} f_i^{\rho\alpha} y_{\beta\epsilon\rho}^k y_{\alpha\mu}^i} - \underline{\partial_k^{\beta\epsilon} f_i^{\rho\alpha} y_{\beta\epsilon\mu}^k y_{\alpha\rho}^i} \Big] d\sigma =$$

$$= -X^\mu \left(p_k - d_\rho p_k^\rho + d_{\rho\sigma} p_k^{\rho\sigma} \right) y_\mu^k d\sigma \tag{4.2.52}$$

Thus, the horizontal part does vanish on-shell. In the second case, both the horizontal and contact part vanishes on-shell.

In other words, we have the following results:

Theorem: *$\sigma : M \to B$ is a solution iff $(J^{2k}\sigma)^* \hat{X} \lrcorner d\boldsymbol{\Theta}_L = 0$ for any vector field $\hat{X}$ on $J^{2k}B$.*

For enforcing field equations, however, one needs much less than that.

Theorem: *$\sigma : M \to B$ is a solution iff $(J^{2k}\sigma)^* (J^{2k}X) \lrcorner d\boldsymbol{\Theta}_L = 0$ for any prolongation of a vertical vector field X on B.*

Let us also check what happens if one does not want to impose holonomy on sections, i.e. if we consider a general section $\hat{\sigma} : M \to J^{2k}B$. Since the section $\hat{\sigma}$ is not holonomic, we have

$$(\hat{\sigma})^* \omega^i = \Big(\partial_\mu y^i(x) - y_\mu^i(x) \Big) dx^\mu \qquad\qquad (\hat{\sigma})^* \omega_\alpha^i = \Big(\partial_\mu y_\alpha^i(x) - y_{\alpha\mu}^i(x) \Big) dx^\mu$$

$$(\hat{\sigma})^* \omega_{\alpha\beta}^i = \Big(\partial_\mu y_{\alpha\beta}^i(x) - y_{\alpha\beta\mu}^i(x) \Big) dx^\mu \qquad\qquad (\hat{\sigma})^* \omega_{\alpha\beta\epsilon}^i = \Big(\partial_\mu y_{\alpha\beta\epsilon}^i(x) - y_{\alpha\beta\epsilon\mu}^i(x) \Big) dx^\mu \tag{4.2.53}$$

The horizontal contributions are unchanged, they keep vanishing along solutions.

We need to check what happen to contact forms, which are

$$
\begin{aligned}
(\hat{\sigma})^* \hat{X} \lrcorner d\boldsymbol{\Theta}_L = & -\left(X^\mu y^k_\mu - X^k\right)\left(p_k - d_\rho p^\rho_k + d_{\rho\sigma} p^{\rho\sigma}_k\right)\, d\sigma + \\
& +\left(X^i\left(\partial_i f^\mu_k - \partial_k f^\mu_i\right) + X^i_\alpha\left(\partial^\alpha_i f^\mu_k - \partial_k f^{\mu\alpha}_i\right) + X^i_{\alpha\beta}\partial^{\alpha\beta}_i f^\mu_k + X^i_{\alpha\beta\epsilon}\partial^{\alpha\beta\epsilon}_i f^\mu_k\right)\left(\partial_\mu y^k(x) - y^k_\mu(x)\right)\, d\sigma + \\
& +\left(X^i\left(\partial_i f^{\mu\alpha}_k - \partial^\alpha_k f^\mu_i\right) + X^i_\beta\left(\partial^\beta_i f^{\mu\alpha}_k - \partial^\alpha_k f^{\mu\beta}_i\right) + X^i_{\beta\epsilon}\partial^{\beta\epsilon}_i f^{\mu\alpha}_k\right)\left(\partial_\mu y^k_\alpha(x) - y^k_{\alpha\mu}(x)\right)\, d\sigma + \\
& -\left(X^i\partial^{\alpha\beta}_k f^\mu_i + X^i_\epsilon\partial^{\alpha\beta}_k f^{\mu\epsilon}_i\right)\left(\partial_\mu y^k_{\alpha\beta}(x) - y^k_{\alpha\beta\mu}(x)\right)\, d\sigma - X^i\partial^{\alpha\beta\epsilon}_k f^\mu_i\left(\partial_\mu y^k_{\alpha\beta\epsilon}(x) - y^k_{\alpha\beta\epsilon\mu}(x)\right)\, d\sigma
\end{aligned}
\tag{4.2.54}
$$

Hence, if the momenta are generic enough, the vanishing of $(\hat{\sigma})^*\hat{X}\lrcorner d\boldsymbol{\Theta}_L = 0$ along any vector field and any section $\hat{\sigma}: M \to J^{2k}B$ imposes field equations *as well as* holonomy, i.e. $\hat{\sigma} = j^{2k}\sigma$ for some configuration $\sigma: M \to B$.

3. Hamiltonian framework

Let us hereafter restrict to a first order Lagrangian L which has only momenta $p^\mu_i = f^\mu_i$ and no difference between naive and formal momenta.

Let σ be a configuration, i.e. a section of the bundle $[B \to M]$. Let us define $\hat{\sigma} = \Phi_L \circ J^1\sigma$, which is a section $\hat{\sigma}: M \to \Lambda_1: x \mapsto \left(x^\mu, y^i(x), z^A\left(x, y(x), \partial y(x)\right)\right)$, and $\hat{X} = X^i\partial_i + X^A\partial_A$ a vector tangent to Λ_1, vertical on $[\Lambda_1 \to M]$.

The vector field $\hat{X}$ is, by construction, tangent to Λ_1 and it projects on a generic vector X of B, just as it happens for the jet prolongations J^1X. More than that, for any vector $\hat{X}$ there exists a vector field $X^{(1)}$ on J^1B which projects as $(\Phi_L)_*X^{(1)} = \hat{X}$. The lift $X^{(1)}$ is not unique and the flow of vertical generators drag a lift into lifts.

One can check directly that for any form θ on Λ and for all $X^{(1)}$ on J^1B lift of $\hat{X}$ one has

$$
X^{(1)}\lrcorner\left((\Phi_L)^*\theta\right) = (\Phi_L)^*\left(\left((\Phi_L)_*X^{(1)}\right)\lrcorner\theta\right) = (\Phi_L)^*\left(\hat{X}\lrcorner\theta\right)
\tag{4.3.1}
$$

That has not much to do with the specific case. For any bundle $[C \to M]$, for any vector field $\hat{X}$ on M there exists a lift X_C on C, which is a projectable vector field such that $\pi_*X_C = \hat{X}$. Lifts always exist; for example, one can consider the horizontal lift with respect to a connection of the bundle $[C \to M]$, which always exists since connections are sections of an affine bundle. Moreover, any vertical field drags a lift into the others.

Also, in general, for any form θ on M and projectable vector field X_C on C, which projects on $\hat{X}$ on M, one can prove directly

$$
X_C\lrcorner\left(\pi^*\theta\right) = \pi^*\left((\pi_*X_C)\lrcorner\theta\right) = \pi^*\left(\hat{X}\lrcorner\theta\right)
\tag{4.3.2}
$$

Then, for any projectable $X^{(2)}$ on J^2B, projecting on $\hat{X}$ on Λ_1, we have

$$
\begin{aligned}
0 = & (J^2\sigma)^*X^{(2)}\lrcorner d\Theta_L = (J^2\sigma)^*X^{(2)}\lrcorner d\left((\Phi_L)^*\Theta_H\right) = (J^2\sigma)^*X^{(2)}\lrcorner(\Phi_L \circ \pi^2_1)^*d\Theta_H = \\
= & (J^2\sigma)^*(\Phi_L \circ \pi^2_1)^*\hat{X}\lrcorner d\Theta_H = (\Phi_L \circ \pi^2_1 \circ J^2\sigma)^*\hat{X}\lrcorner d\Theta_H = (\Phi_L \circ J^1\sigma)^*\hat{X}\lrcorner d\Theta_H = (\hat{\sigma})^*\hat{X}\lrcorner d\Theta_H
\end{aligned}
\tag{4.3.3}
$$

Let us define a *Hamiltonian section* to be a section $\hat{\sigma} : M \to \Lambda_1$ such that, $(\hat{\sigma})^* \hat{X} \lrcorner d\Theta_H = 0$, for all vector fields $\hat{X}$ tangent to Λ_1.

The Hamiltonian sections satisfy $(\hat{\sigma})^* \hat{X} \lrcorner d\Theta_H = 0$. We consider a vector field $\hat{X} = X^i \partial_i + X^A \partial_A$ tangent to Λ_1 and a section $\hat{\sigma} : M \to \Lambda_1 : x^\mu \mapsto (x^\mu, y^i(x), z^A(x))$.

Then we have

$$
\begin{aligned}
(\hat{\sigma})^* i_{\hat{X}} d\boldsymbol{\Theta}_H &= (\hat{\sigma})^* i_{\hat{X}} \left(-dH(x,y,z) \wedge d\sigma + d\iota_i^\mu(x,y,z) \wedge dy^i \wedge d\sigma_\mu \right) = \\
&= (\hat{\sigma})^* i_{\hat{X}} \left(-\partial_i H dy^i \wedge d\sigma - \partial_A H dz^A \wedge d\sigma - \partial_\mu \iota_i^\mu dy^i \wedge d\sigma + \partial_k \iota_i^\mu dy^k \wedge dy^i \wedge d\sigma_\mu + \partial_A \iota_i^\mu dz^A \wedge dy^i \wedge d\sigma_\mu \right) = \\
&= (\hat{\sigma})^* \left(X^i \left(-(\partial_\mu \iota_i^\mu + \partial_i H) d\sigma + (\partial_i \iota_k^\mu - \partial_k \iota_i^\mu) dy^k \wedge d\sigma_\mu - \partial_A \iota_i^\mu dz^A \wedge d\sigma_\mu \right) + X^A \left(-\partial_A H d\sigma + \partial_A \iota_i^\mu dy^i \wedge d\sigma_\mu \right) \right) = \\
&= \left(X^i \left(-d_\mu \iota_i^\mu - \partial_i H + \partial_i \iota_k^\mu y_\mu^k \right) + X^A \left(-\partial_A H + \partial_A \iota_i^\mu y_\mu^i \right) \right) d\sigma = 0
\end{aligned}
\tag{4.3.4}
$$

Hence we have the conditions for Hamiltonian sections

$$
d_\mu \iota_i^\mu = \partial_i \iota_k^\mu y_\mu^k - \partial_i H \qquad\qquad \partial_A \iota_i^\mu y_\mu^i = \partial_A H
\tag{4.3.5}
$$

which are called *Hamilton equations*.

Let us stress, if we started from a non-vertical vector field, the extra term we get would be

$$
\left(d_\mu \iota_i^\mu + \partial_i H - \partial_i \iota_k^\mu y_\mu^k \right) y_\alpha^i + \left(\partial_A H - \partial_A \iota_i^\mu y_\mu^i \right) z_\alpha^A = 0
$$

which vanishes identically in view of Hamilton equations, anyway.

Theorem: *a section $\hat{\sigma} : M \to \Lambda_1$ is a Hamiltonian section iff $\hat{\sigma} = \Phi_L \circ J^1\sigma$ for a solution $\sigma : M \to B$ of field equations.*

Proof: in view of equation (4.3.3), a solution σ of field equation satisfies $(J^2\sigma)^* X^{(2)} \lrcorner d\boldsymbol{\Theta}_L = 0$, hence it also satisfies $(\Phi_L \circ J^1\sigma)^* \hat{X} \lrcorner d\boldsymbol{\Theta}_H = 0$. Accordingly, $\hat{\sigma} = \Phi_L \circ J^1\sigma$ is a Hamilton section.

For the other half of the proof, we simply show that Hamilton equations for $\hat{\sigma} : M \to \Lambda_1 : x^\mu \mapsto (x^\mu, y^i(x), z^A(x))$ imply field equations for the projection $\sigma = \pi_B^\Lambda \circ \hat{\sigma} : M \to B : x^\mu \mapsto (x^\mu, y^i(x)))$ on B.

Hamiltonian equations are

$$
d_\mu \iota_i^\mu = \partial_i \iota_k^\mu y_\mu^k - \partial_i H \qquad\qquad \partial_A \iota_i^\mu y_\mu^i = \partial_A H
\tag{4.3.6}
$$

We need to show that Hamilton equations are equivalent, by construction, to Euler-Lagrange equations.

We can check the equivalence directly. We have $H(x, y, z(x, y, dy)) = \pi_i^\mu(x, y, dy) y_\mu^i - L(x, y, dy)$ from which we get

$$
\partial_i H + \partial_A H \partial_i z^A = \partial_i p_k^\mu y_\mu^k - p_i \qquad \partial_A H \partial_i^\mu z^A = \partial_i^\mu p_k^\alpha y_\alpha^k + {\color{red}p_i^\mu} - {\color{red}p_i^\mu}
\tag{4.3.7}
$$

and $p_i^\mu(x, y, dy) = \iota_i^\mu(x, y, z(x, y, dy))$ from which we get

$$
\partial_\alpha p_i^\mu = \partial_\alpha \iota_i^\mu + \partial_A \iota_i^\mu \partial_\alpha z^A \qquad \partial_k p_i^\mu = \partial_k \iota_i^\mu + \partial_A \iota_i^\mu \partial_k z^A \qquad \partial_k^\alpha p_i^\mu = \partial_A \iota_i^\mu \partial_k^\alpha z^A
\tag{4.3.8}
$$

from which we get

$$
\begin{aligned}
d_\mu p_i^\mu &= \partial_\mu p_i^\mu + y_\mu^k \partial_k p_i^\mu + y_{\alpha\mu}^k \partial_k^\alpha p_i^\mu = \partial_\mu \iota_i^\mu + \partial_A \iota_i^\mu \partial_\mu z^A + y_\mu^k \partial_k \iota_i^\mu + y_\mu^k \partial_A \iota_i^\mu \partial_k z^A + y_{\alpha\mu}^k \partial_A \iota_i^\mu \partial_k^\alpha z^A = \\
&= \partial_\mu \iota_i^\mu + y_\mu^k \partial_k \iota_i^\mu + \partial_A \iota_i^\mu \left(\partial_\mu z^A + y_\mu^k \partial_k z^A + y_{\alpha\mu}^k \partial_k^\alpha z^A \right) = d_\mu \iota_i^\mu
\end{aligned}
\tag{4.3.9}
$$

Then we can rewrite the first Hamilton equation

$$d_\mu p_i^\mu = \partial_i p_k^\mu y_\mu^k - \partial_A \iota_k^\mu \partial_i z^A y_\mu^k + \partial_A \iota_k^\mu y_\mu^k \partial_i z^A - \partial_i p_k^\mu y_\mu^k + p_i \tag{4.3.10}$$

which, in fact, is the Euler-Lagrange equation $p_i - d_\mu p_i^\mu = 0$

By using the second Hamilton equation into the definition of Hamiltonian

$$\partial_A \iota_k^\alpha y_\alpha^k \partial_i^\mu z^A = \partial_i^\mu p_k^\alpha y_\alpha^k \qquad \Rightarrow \qquad \left(\partial_A \iota_k^\alpha \partial_i^\mu z^A - \partial_i^\mu p_k^\alpha \right) y_\alpha^k = 0$$

and it is identically satisfied in view of the Legendre transform. ∎

Alternatively, we can consider the constraint described as a level set $\chi^a(x,y,p) = 0$. It compares with the method we used above as

$$\chi^a(x,y,\iota(x,y,z)) = 0 \qquad \Rightarrow \qquad \frac{\partial \chi^a}{\partial x} + \frac{\partial \chi^a}{\partial p}\frac{\partial \iota}{\partial x} = 0 \qquad \frac{\partial \chi^a}{\partial y} + \frac{\partial \chi^a}{\partial p}\frac{\partial \iota}{\partial y} = 0 \qquad \frac{\partial \chi^a}{\partial p}\frac{\partial \iota}{\partial z} = 0 \tag{4.3.11}$$

We also consider an extension of the Hamiltonian to the whole $P(B)$, i.e. $\hat{H}(x,y,p)$ such that

$$\hat{H}(x,y,\iota(x,y,z)) = H(x,y,z) \tag{4.3.12}$$

Then the Poincaré-Cartan form defines the Helmholtz Lagrangian

$$\boldsymbol{L}_{Hel} = \left(-\hat{H} + p_i^\mu y_\mu^i \right) d\sigma \tag{4.3.13}$$

which defines Hamilton equations by variation with deformations tangent to the constraint. The variation is

$$\delta L_{Hel} = \left(\left(-\frac{\partial \hat{H}}{\partial y^i} - d_\mu p_i^\mu \right) \delta y^i + \left(-\frac{\partial \hat{H}}{\partial p_i^\mu} + y_\mu^i \right) \delta p_i^\mu \right) d\sigma + d \left(p_i^\mu y^i d\sigma_\mu \right) \tag{4.3.14}$$

However, deformations are not independent since they are tangent to the constraint. Then Hamilton equations can be also written as

$$d_\mu p_i^\mu = -\frac{\partial \hat{H}}{\partial y^i} - \lambda_a \frac{\partial \chi^a}{\partial y^i} \qquad\qquad y_\mu^i = \frac{\partial \hat{H}}{\partial p_i^\mu} + \lambda_a \frac{\partial \chi^a}{\partial p_i^\mu} \tag{4.3.15}$$

These forms of Hamilton equations, in fact, are equivalent since we have the identities

$$\begin{aligned} \frac{\partial \hat{H}}{\partial x} + \frac{\partial \hat{H}}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial x} = \frac{\partial H}{\partial x} \qquad & \frac{\partial \hat{H}}{\partial y} + \frac{\partial \hat{H}}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial y} = \frac{\partial H}{\partial y} \qquad & \frac{\partial \hat{H}}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial z} = \frac{\partial H}{\partial z} \\ \frac{\partial \chi^a}{\partial x} + \frac{\partial \chi^a}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial x} = 0 \qquad & \frac{\partial \chi^a}{\partial y} + \frac{\partial \chi^a}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial y} = 0 \qquad & \frac{\partial \chi^a}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial z} = 0 \end{aligned} \tag{4.3.16}$$

Consequently, we have

$$y_\mu^i \frac{\partial \iota_i^\mu}{\partial z} = \frac{\partial \hat{H}}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial z} + \lambda_a \frac{\partial \chi^a}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial z} = \frac{\partial \hat{H}}{\partial p_i^\mu}\frac{\partial \iota_i^\mu}{\partial z} = \frac{\partial H}{\partial z} \tag{4.3.17}$$

as well as

$$d_\mu p_i^\mu = -\frac{\partial \hat{H}}{\partial y^i} + \frac{\partial \hat{H}}{\partial p_k^\mu}\frac{\partial \iota_k^\mu}{\partial y^i} - \frac{\partial \hat{H}}{\partial p_k^\mu}\frac{\partial \iota_k^\mu}{\partial y^i} - \lambda_a\frac{\partial \chi^a}{\partial y^i} = -\frac{\partial H}{\partial y^i} + \left(\frac{\partial \hat{H}}{\partial p_k^\mu} + \lambda_a\frac{\partial \chi^a}{\partial p_k^\mu}\right)\frac{\partial \iota_k^\mu}{\partial y^i} = -\frac{\partial H}{\partial y^i} + y_\mu^k\frac{\partial \iota_k^\mu}{\partial y^i} \tag{4.3.18}$$

Second order dynamics

Let us check that what we said about first order Hamiltonian formalism extends to second order Lagrangians, when Euler-Lagrange equations are expected to be up to $4th$ order. Let us consider a second order Lagrangian for a field theory on $[B \to M]$ and a configuration $\sigma : M \to B$. We already introduced $P(J^1B) = V^*(J^1B) \otimes A_{m-1}(M)$, with coordinates $(x^\mu, y^i, y_\mu^i, f_i^\mu, f_i^{\mu\nu})$, $[Z(J^1B) \to P(J^1B)]$ with coordinates $(x^\mu, y^i, y_\mu^i, f_i^\mu, f_i^{\mu\nu}, f)$.

For any second order Lagrangian $L : J^2B \to A_m(M)$ we can define the map $\Phi_L := \mathbb{F} : J^3B \to P(J^1B)$ given by

$$\hat{\Phi}_L := \mathbb{F} : J^3B \to P(J^1B) : (x, y, dy, d^2y, d^3y) \mapsto \left(x, y, dy, f_i^\mu = \frac{\partial L}{\partial y_\mu^i} - d_\nu\frac{\partial L}{\partial y_{\mu\nu}^i}, f_i^{\mu\nu} = \frac{\partial L}{\partial y_{\mu\nu}^i}\right) \tag{4.3.19}$$

and $\psi_L : J^3B \to Z(J^1B)$ given by

$$\psi_L : J^3B \to Z(J^1B) : \left(x, y, dy, f_i^\mu = \frac{\partial L}{\partial y_\mu^i} - d_\nu\frac{\partial L}{\partial y_{\mu\nu}^i}, f_i^{\mu\nu} = \frac{\partial L}{\partial y_{\mu\nu}^i}, f = -\mathcal{H}\right) \qquad \mathcal{H} = p_i^\mu y_\mu^i - L \tag{4.3.20}$$

and pull-back the Poincar-Cartan form $\boldsymbol{\Theta}_L$ from the multi-symplectic potential on $Z(J^1B)$ to obtain

$$\boldsymbol{\Theta}_L = -\mathcal{H}d\sigma + f_i^\mu dy^i \wedge d\sigma_\mu + f_i^{\alpha\mu} dy_\alpha^i \wedge d\sigma_\mu = Ld\sigma + f_i^\mu \omega^i \wedge d\sigma_\mu + f_i^{\alpha\mu}\omega_\alpha^i \wedge d\sigma_\mu \tag{4.3.21}$$

The Legendre transform $\hat{\Phi}_L$ defines the primary constraint $\Lambda_1 \subset P(J^1B)$. We can fix a coordinate system (x, y, dy, z) on the primary constraint so that we have the canonical embedding

$$\iota : \Lambda_1 \to P(J^1B) : (x, y, dy, z) \mapsto (x, y, dy, \iota_i^\mu(x, y, dy, z), \iota_i^{\alpha\mu}(x, y, dy, z)) \tag{4.3.22}$$

and the Legendre transform factorizes through the embedding to define

$$\Phi_L : J^3B \to \Lambda_1 : (x, y, dy, d^2y, d^3y) \mapsto (x, y, dy, z(x, y, dy, d^2y, d^3y)) \tag{4.3.23}$$

so that we have $\hat{\Phi}_L = \iota \circ \Phi_L$.

The map ψ_L factorizes through Φ_L, i.e. we can define a map

$$\psi_H : \Lambda_1 \to Z(J^1B) : (x, y, dy, z) \mapsto (x, y, dy, f_i^\mu = i_i^\mu, f_\iota^{\mu\nu} = \iota_i^{\mu\nu}, -H(x, y, dy, z)) \tag{4.3.24}$$

such that $\psi_L = \psi_H \circ \Phi_L$. Again we can pull-back the multi-symplectic potential to define the Hamiltonian Poincar-Cartan form

$$\boldsymbol{\Theta}_H = -Hd\sigma + \iota_i^\mu dy^i \wedge d\sigma_\mu + \iota_i^{\alpha\mu} dy_\alpha^i \wedge d\sigma_\mu \tag{4.3.25}$$

Consequently, we have $\boldsymbol{\Theta}_L = (\psi_L)^* \boldsymbol{\Theta} = (\psi_H \circ \Phi_L)^* \boldsymbol{\Theta} = (\Phi_L)^* (\psi_H)^* \boldsymbol{\Theta} = (\Phi_L)^* \boldsymbol{\Theta}_H$.

We we define the action we get the Helmholtz Lagrangian

$$\boldsymbol{L}_{Hel} = \left(-H + \iota_i^\mu y_\mu^i + \iota_i^{\alpha\mu} y_{\alpha\mu}^i \right) d\sigma \tag{4.3.26}$$

that defines Hamilton equations. We have

$$\delta \boldsymbol{L}_{Hel} = \left(-\partial_i H \delta y^i - \partial_i^\mu H \delta y_\mu^i - \partial_A H \delta z^A \right) d\sigma \tag{4.3.27}$$

Second order mechanics

Let us specialize to second order mechanics.

Compatibility between constraints and dynamics

We have to discuss compatibility between constraints and dynamics, to see how close field theories are to mechanics. That should lead to classification of constraints into first and second class constraints.

4. Cauchy problems in field theory

Usually, we define Cauchy problems in field theories as *global* Cauchy problems. We fix a hyper-surface $S \subset M$ called the *Cauchy surface*, and we choose it so that for any given initial conditions on S, namely the value of fields and their derivative in a transverse direction up to order $k - 1$ for equations of order k, there exists one and only one *global* section $\sigma : M \to C$ obeying field equations *and* initial conditions.

On one hand, we see some features emerging. We need for example Cauchy surface as well as a transverse direction to it, we shall call it the *evolution direction*. We also shall discuss that initial conditions on S together with field equations actually define the value of fields along curves (the *characteristics*). Hence if the characteristics happen to intersect the Cauchy surfaces in more than one point, it happens that the initial conditions at one point constrain the value of fields at the other intersection, with the effect of spoiling the Cauchy theorem in the part which requires the existence of one solution for *any* initial conditions.

This is usually avoided by requiring some topological condition on M so that one can choose Cauchy surface which intersects the characteristics once and only once. In this situation, initial conditions define one and only one global solution everywhere on M.

> That means in a relativistic theory that I want to know the fields and their evolution derivatives *everywhere* in space and then field equations give me a solution *everywhere* on spacetime.
>
> In view of the hole argument, we know this is not possible if we are discussing about configurations. It is not possible mainly for the uniqueness part of Cauchy theorem, so that we can still hope the existence part is well defined, but we shall see, in many cases, it is not.

In any event, this kind of requests sounds somehow unreasonable also physically. Maybe we could be satisfied to solve Cauchy problems in a region, so that we know fields everywhere in a building and we are able to determine fields in a room for a period of time. That is quite close to determinism, not as general to require to measure fields in a black hole on the other side of the universe in order to compute fields here and forever.

We shall argue that by doing that one cuts down many topological conditions obtain an evolution which is closer to an absolute description.

As a matter of fact, in view of general covariance of field equations, we have a sheaf of equations, meaning we can restrict equations to opensets D and discuss the Cauchy problem in D only. Let then $\bar{D} \subset M$ be a compact in spacetime, ζ a vector field supported on $\bar{D}$, namely non-zero in the interior $D \subset \bar{D}$ and vanishing on the boundary ∂D.

The pair $(\bar{D}, \zeta)$ defines a foliation of D in lines tangent to the *evolution vector* ζ, since any rank 1 distribution is involutive and hence integrable. We require the distribution to be regular so to have a bundle $[D \to S]$ on a *space manifold* S.

The bundle $[D \to S]$ is trivial. The flow of ζ acts vertically on $[D \to S]$. In fibered coordinates (k, t), which are adapted coordinates to the fibers, the flow reads as

$$\Phi_s : D \to D : (k, t) \mapsto (k, t + s) \tag{4.4.1}$$

In other words, $[D \to S]$ is a principal bundle for the group $(\mathbb{R}, +)$ and at the same time it is an affine bundle since the transformation $t' = t + s$ is also an affine transformation.

Being it affine, it allows global sections, being it principal with a global section, it is trivial. Since the bundle $[D \to S]$ is trivial, it is globally diffeomorphic to $\mathbb{R} \times S \simeq D$ and hence it has another bundle structure $[D \to \mathbb{R}]$ which has fibers S.

We call the bundle $[D \to \mathbb{R}]$ as the *ADM bundle*, which corresponds to the foliation of D into hypersurfaces $S_s \subset D$ which are called *isochronous surfaces*. The bundle $[D \to S]$ is called the *rest bundle*. Its fibers are lines which represent rest motions, which connects the position k at a time s to the same position k at the time s'. Obviously, rest motions as well as isochronous hypersurfaces depend on observes, they are not absolute structure in spacetime. Thus, in some sense, both the bundle structures, the ADM bundle and the rest bundle, are part of the specification of the observer.

Then we fix a section $\sigma : S \to D$ so that we can embed $S_0 := \sigma(S) \subset D$ as a Cauchy surface in D. Let us call the triple $(\bar{D}, \zeta, \sigma)$ a *Cauchy bubble*.

The evolution field ζ defines a flow $\Phi_s : D \to D$ which drags the Cauchy surface S_0 into a foliation $S_s = \Phi_s(S_0)$ of D. The flow is well defines since the evolution field is compactly supported.

If the vector field is not complete, it is well known that the flow is not well defined. For example, let us consider $M = \mathbb{R}^2$ and the vector field $\xi = f(x)y^2 \vec{\jmath}$, where we set $f(x) := \frac{1}{2(x^2+1)}$; see Fig.1 for the graph of $y = f(x)$. The vector field ξ as defined is not complete.
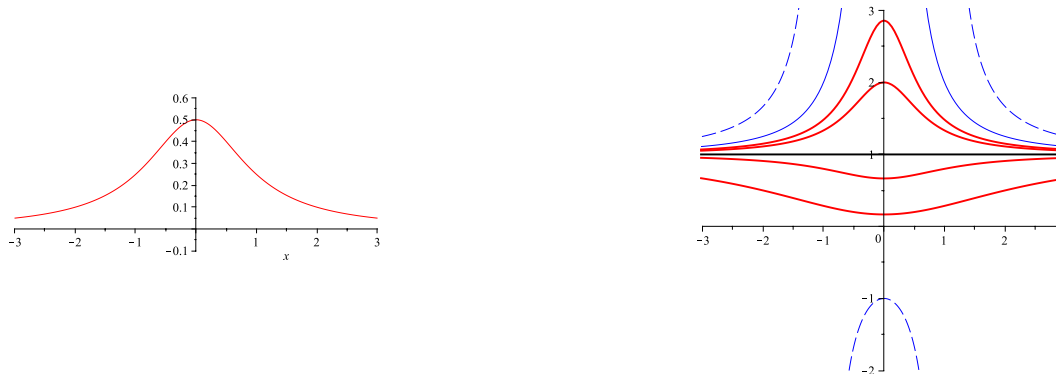
a: the function $y = f(x)$

If we consider the submanifold $S : \mathbb{R} \to \mathbb{R}^2 : k \mapsto (1, k)$ (thick, solid, black line) is dragged along the local flow of ξ. For $s < 2$, the flow (and the dragged submanifold) is everywhere well defined. For $s \geq 2$, singularities are introduced. In Fig.2 we show the dragged submanifolds for $s = -10$, $s = -1$, $s = 1$, $s = 1.3$ (thick, red, solid lines) as well as the lines for $s = 2$ (thin, blue, solid line) and $s = 4$ (thin, blue, dash line).

That defines on D a product structure $D \simeq \mathbb{R} \times S$, i.e. both an ADM bundle $[D \to \mathbb{R}]$ and a rest bundle $[D \to S]$.

Whatever topology of M, the support $D \simeq \mathbb{R} \times S$ of a Cauchy bubble is *globally hyperbolic*. In adapted coordinates, the rest motions are $\gamma_k : \mathbb{R} \to D : s \mapsto (s, k)$, the isochronous hypersurfaces are $S_s = \{(s, k) : k \in S\}$. Each rest motion γ, defines two points on the boundary ∂D, namely

$$x_- = \lim_{s \to -\infty} \gamma(s) \qquad\qquad x_+ = \lim_{s \to +\infty} \gamma(s) \tag{4.4.2}$$

so that the boundary splits into two parts $\partial D = D_- \cup D_+$ up to points where the distribution $\langle \zeta \rangle$ is tangent to ∂D.

Cauchy boubles exist, in fact we have in mind a specific prototypal example.
Let us consider an open disc $D = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and the vector field $\xi = (1 - x^2 - y^2)\partial_y$ (on $\bar{D}$) which has support in the closure $\bar{D}$ and it is non-zero everywhere in D.
Integral curves are $\gamma_{(x,y)} : \mathbb{R} \to D : s \mapsto (x, y(s; x, y))$ where

$$y(s; x, y) = \alpha \frac{\alpha \sinh(\alpha s) + y \cosh(\alpha s)}{\alpha \cosh(\alpha s) + y \sinh(\alpha s)} = \alpha \frac{\alpha \tanh(\alpha s) + y}{\alpha + y \tanh(\alpha s)} \tag{4.4.3}$$

and where we set $\alpha = \sqrt{1 - x^2}$ which is real since in D we have $x \in (-1, 1)$.
Let us remark the limits

$$\lim_{s \to +\infty} y(s; x, y) = \alpha = \sqrt{1 - x^2} \qquad\qquad \lim_{s \to -\infty} y(s; x, y) = -\alpha = -\sqrt{1 - x^2} \tag{4.4.4}$$

show that integral curves have asymptotic limits on the boundary of the disc.

In this scenario, for a *Cauchy problem*, we give initial conditions on a Cauchy surface (either S_0 or D_-) and we use equations to obtaining fields in D, actually in $\bar{D}$. The Cauchy problem is *well-posed* iff for any initial condition one finds one and only one solution.

More precisely, one should specify regularity of initial conditions and regularity of the solution. Here for us, we assume always smooth regularity for both the initial conditions and solutions. We are discussing about global sections, thus smooth an the whole M, in particular on the compact $\bar{D}$.

What we mean by initial conditions, it depends on the equations. Roughly speaking, if we consider equations of order k, we wish to give initial conditions fields, and their t-derivatives up to order $(k-1)$, on the Cauchy surface.

We already discuss that covariant equations cannot produce well-posed Cauchy problems. In view of the hole argument, uniqueness of solutions is always lost, one can produce infinitely many solutions for a given initial condition. Simply put, general covariance (or gauge covariance) is incompatible with Cauchy problems. Covariant equations, which are the only equations one can write on classes of diffeomorphisms of manifolds, are necessarily underdetermined.

We shall discuss what happens in case of variational covariant equations. As a general scenario, as we discuss about the hole argument, if we get stuck to some sort o0f determinism we have to assume that sections of the configuration bundle are not *physical states*, that are instead defined as

equivalence classes of sections up to diffeomorphisms (or gauge symmetries) so that covariant equations for sections can induce equations for physical states, hoping these equations on the quotient can be deterministic, since diffeomorphisms (or gauge symmetries) act trivially on equivalence classes.

Hereafter, we want to give conditions for these equations to be deterministic, for the corresponding Cauchy problem to be well-posed. We shall first restrict to quasilinear equations, we define an algebraic structure called the *principal symbol*, and we discuss characteristics, dependence regions, and well-posedness of the Cauchy problem.

Principal symbols of quasilinear PDE

Let us consider a system of partial differential equations (PDE) on a bundle $[C \to M]$ without fixing structures on the base manifold M. In our notation, PDE are local expressions of a (s^{th}-order) operator $D : J^sC \to E$ on some vector bundle $[E \to J^{s-1}C \to M]$. In the special case of a PDE which comes from a variational principle, assuming it is quasi-linear Euler-Lagrange operator of a Lagrangian $\boldsymbol{L}$, it reads as

$$\mathbb{E}(L) : J^sC \to V^*(C) \otimes_{J^{s-1}C} A_m(M) \tag{4.4.5}$$

If the Lagrangian were non-degenerate and order k, we would have $s = 2k$. In general, for example for Einstein-Hilbert gravity, we only know $s \le 2k$.

Given the operator $D : J^sC \to E$, we can define a PDE for (global) sections to be the preimage of the zero section $0 : J^{s-1}C \to E$ which is naturally a sub-bundle $S \subset J^sC$.

As usual, we fix fibered coordinates (x^μ, y^i) on $[C \to M]$. They define coordinates $(x^\mu, y^i, y^i_\mu, \dots, y^i_{\mu_1\dots\mu_s})$ on J^sC. Let us stress that $[J^sC \to J^{s-1}C]$ is an affine bundle, regardless to any restriction posed on $[C \to M]$. In particular we are not requiring a linear or affine structure on global sections of $[C \to M]$. In view of this affine structure on the jet bundles, we have transformation laws of derivatives of order s in the form

$$y'^i_{\mu_1\dots\mu_s} = J^i_k \bar{J}^{\alpha_1}_{\mu_1} \dots \bar{J}^{\alpha_s}_{\mu_s} y^k_{\alpha_1\dots\alpha_s} + b^i_\mu(j^{k-1}y) = a^{i\alpha_1\dots\alpha_s}_{k\mu_1\dots\mu_s}(x,y)\, y^k_{\alpha_1\dots\alpha_s} + b^i_{\mu_1\dots\mu_s}(j^{k-1}y) \tag{4.4.6}$$

A point in $E = V^*(C) \otimes_{J^{s-1}C} A_m(M)$ is in the form

$$v = v_i\, \bar{d}y^k \otimes d\sigma \tag{4.4.7}$$

Let us then fix coordinates $(x, y^i_\mu, y^i_\mu, \dots, y^i_{\mu_1\dots\mu_{s-1}}; v_i)$ on the vector bundle $[E \to J^{s-1}C]$, so that one expects transformation laws of the coordinates v_i in the form

$$v'_i = \bar{J}\bar{J}^k_i\, v_k \tag{4.4.8}$$

The most general quasi-linear variational operator $D : J^sC \to E$ then is locally expressed by $D : j^sy \mapsto (j^{s-1}y, v_i(j^sy))$ and the corresponding PDE are expressed as

$$v_i(j^sy) = e^{\alpha_1\dots\alpha_s}_{ik}(j^{s-1}y)\, y^k_{\alpha_1\dots\alpha_s} + e_i(j^{s-1}y) = 0 \tag{4.4.9}$$

This form of preserved by changes of coordinates on C, which also induces a coordinate change on $E = V^*(C) \otimes_{J^{s-1}C} A_m(M)$ in the form (4.4.8). More than that, if $e^{\alpha_1\dots\alpha_k}_{ik}(j^hy)$ depend on derivatives up to order $0 \le h \le s - 1$ in one chart they depend on derivatives up to order h in any chart. Then that is an intrinsic property, let us call h the *rank* of the operator D.

If we consider an operator with $e^{\alpha_1\dots\alpha_k}_{ik}(x)$ in one chart, and if the configuration bundle $[C \to M]$ is not assumed to be a vector (or an affine) bundle, when we change coordinates we obtain a local expression which has not that property any longer. Hence we cannot assume such a property in general.

However, if the configuration bundle is an affine or a vector bundle, fibered coordinates transform as $\left(x'^\mu = x'^\mu(x), y'^i = A^i_k(x)y^k + B^i(x)\right)$ (in the linear case $B^i = 0$) hence we have $J^i_k = A^i_k(x)$ that depends on x only. Taking that under consideration, we have new local expression as

$$
\begin{aligned}
J^j_i e'_j =& J^j_i \left(e'^{\alpha_1\ldots\alpha_s}_{jk} y'^k_{\alpha_1\ldots\alpha_s} + e'_j\right) = J^j_i \left(e'^{\alpha_1\ldots\alpha_s}_{jk} \left(J^k_l \bar{J}^{\beta_1}_{\alpha_1}\ldots\bar{J}^{\beta_s}_{\alpha_s} y^l_{\beta_1\ldots\beta_s} + b'^k_{\alpha_1\ldots\alpha_s}\right) + e'_j\right) = \\
=& \left(J^j_i J^k_l e'^{\alpha_1\ldots\alpha_s}_{jk} \bar{J}^{\beta_1}_{\alpha_1}\ldots\bar{J}^{\beta_s}_{\alpha_s}\right) y^l_{\beta_1\ldots\beta_s} + J^j_i \left(e'^{\alpha_1\ldots\alpha_s}_{jk} b'^k_{\alpha_1\ldots\alpha_s} + e'_j\right) = \bar{J} e_i
\end{aligned}
\tag{4.4.10}
$$

i.e.

$$
e^{\beta_1\ldots\beta_s}_{il} = J J^j_i J^k_l e'^{\alpha_1\ldots\alpha_k}_{jk} \bar{J}^{\beta_1}_{\alpha_1}\ldots\bar{J}^{\beta_s}_{\alpha_s} \qquad\qquad e_i = J J^j_i \left(e'^{\alpha_1\ldots\alpha_s}_{jk} b'^k_{\alpha_1\ldots\alpha_s} + e'_j\right)
\tag{4.4.11}
$$

This shows that the top oder coefficients transform tensorially as

$$
e'^{\alpha_1\ldots\alpha_s}_{ik} = \bar{J} J^{\alpha_1}_{\beta_1}\ldots J^{\alpha_s}_{\beta_s} e^{\beta_1\ldots\beta_s}_{jl} \bar{A}^j_i \bar{A}^l_k
\tag{4.4.12}
$$

In this very special case, we can require that top order coefficients depend on x only since when they do in one chart they do it in any chart. We say, accordingly, that D is a *quasi-vertical operator*.

Let us stress, however, that quasi-vertical operators are defined only on linear and affine configuration bundles, while quasi-linear operators of rank $0 \le h \le s-1$ are generally defined.

Since, in general, the top order coefficients transform as in (4.4.12), namely as points in $(S^s(T^*M))^* \otimes V^*(C) \otimes V^*(C) \otimes A_m(M)$ we can use them to define a global bundle map

$$
\sigma_D : J^s C \times S^s(T^*M) \to V^*(C) \otimes V^*(C) \otimes A_m(M) : (j^s_x\sigma; \xi^1, \ldots, \xi^s) \mapsto \xi^1_{\alpha_1}\ldots\xi^s_{\alpha_s} e^{\alpha_1\ldots\alpha_s}_{ik}(j^s_x\sigma)\,\bar{d}y^i \otimes \bar{d}y^k \otimes d\sigma
\tag{4.4.13}
$$

A point $p \in (S^s(T^*M))^* \otimes V^*(C) \otimes V^*(C) \otimes A_m(M)$ is something in the form $p(\xi^1, \ldots, \xi^s) = \epsilon_{ij}\,\bar{d}y^i \otimes \bar{d}y^k \otimes d\sigma$. If we change coordinates, it transforms as (4.4.12) precisely to make the map a global one.

The map σ_D is called the *principal symbol* of the operator D and it is an intrinsic property of the operator and of the PDE it defines.

Locally, the principal symbol $\sigma_D(\xi)$ is a matrix $e_{ij} = \xi_{\alpha_1}\ldots\xi_{\alpha_s} e^{\alpha_1\ldots\alpha_s}_{ik}$ of homogeneous polynomials in ξ, which is like the quadratic form associated to the bilinear form.

The first thing one can define out of the principal symbol is a rough classification of equations. First of all, we say an operator D is *elliptic* if the matrix e_{ij} takes values in $\mathrm{GL}(k)$, i.e. it is non-degenerate for any (non-zero) covector ξ.

Characteristic distributions

When an operator is not elliptic, we can further define different kind of operators. One of lines to classification is by considering the structure of the so-called *characteristic distributions* at a point $b = (x, y)$ in the configuration bundle B (or some jet prolongation $J^s B$ of it).

Usually, the whole of this story is told by using eigenvalues of the principal symbol. Unfortunately, that is viable when considering fields in $\mathbb{R}^k$. For us, the are sections of a bundle which locally can be considered as fields valued in the standard fibre F of the configuration bundle B, and further locally, in a chart as field components in $\mathbb{R}^k$. We cannot really go that way since in a relativistic context we have no fixed structure on spacetime or configuration bundle, in particular we have no metric on field space, i.e. on the standard fibre F, to compute eigenvalues and the corresponding eigenvectors. Principal symbols are symmetric matrices, thus one needs an

inner product to compute their canonical form. Because of that, as well as for other reasons, we need to be more canonical and repeat the discussion with no metric assumed.

On a quasi-linear first order operator, the principal symbol is generally of order zero. The principal symbol is a $k \times k$ matrix of homogeneous first order polynomials with coefficients in $b = (x, y) \in B$. If the configuration bundle is linear or affine, we can consider semi-linear operators and the polynomial are in $x \in M$.

For general operators of second order, we can consider principal symbols which are $k \times k$ matrices of homogeneous polynomials of order 2 in ξ valued in $b \in B$ or on $J^1 B$. For general operators of order s, we can consider principal symbols which are $k \times k$ matrices of homogeneous polynomials of order s in ξ valued in $J^h B$ with any $0 \le h \le s - 1$. Because of this differences, one has many cases to cover different situations. The main idea is to fix a point and to look for pairs (w, ξ) where w is a vertical vector and ξ is a covector such that

$$w^i \xi_{\alpha_1} \dots \xi_{\alpha_s} e_{ik}^{\alpha_1 \dots \alpha_s} = 0 \tag{4.4.14}$$

Such a pair (w, ξ) is called a *characteristic family* at a point. However, details about the exact spaces involved depend on the order of he operator and its rank. In any event, let us remark that the equation (4.4.14) is homogeneous both in w and ξ.

As a consequence if w is a solution, then all λw are solutions and, analogously, if ξ is a solution, then all $\lambda \xi$ are solutions as well. In other words, characteristic families are *cones* and, even if we are taking at first about vectors and covectors, we are in fact after subspaces as $\langle w \rangle$ and $\langle \xi \rangle$.

Let us consider the simplest case first. If the configuration bundle is linear or affine, we can consider an operator of order one with a principal symbol that depends on x only. If it does, the equation (4.4.14) reads as

$$w^i \xi_\alpha e_{ik}^\alpha(x) = 0 \tag{4.4.15}$$

and ξ is a covector on M (at the point x) and w is a vertical vector on B.

A non-zero w exists as a solution iff $\det(\xi_\alpha e_{ik}^\alpha) = 0$, since the equation (4.4.15) is linear and homogeneous in w. The $k \times k$ matrix $\xi_\alpha e_{ik}^\alpha$ is made of homogeneous first order polynomials in ξ, the determinant is a homogeneous polynomials in ξ of order k. This homogeneous polynomial $Q(x, \xi) := \det(\xi_\alpha e_{ik}^\alpha)$ plays a major role in what follows, let us call it the *characteristic polynomial*.

The equation $Q(x, \xi) = 0$ at a fixed $x \in M$ determines the *dual characteristic cone* $\mathcal{C}_x^* = \{\xi \in T_x^* M : Q(x, \xi) = 0\} \subset T_x^* M$. And going back, for any $\xi \in \mathcal{C}_x^*$ there exists a non-zero vertical vector $w \in V(B)$, in fact a (non-trivial) subspace $\langle w \rangle \in V(B)$.

> Let us stress that the linear or affine case has extra structures, and it makes sense in those cases and those cases only, to have one subspace $\langle w \rangle = \langle w^i \partial_i \rangle \in V_b(B)$ for any point in the fiber $b \in \pi^{-1}(x)$.
>
> Once again, the next case of general first order operators is more natural and the linear or affine case is the special case.

The shape of the dual characteristic cone depends on the number of fields k. The equation of the dual characteristic cone $Q(x, \xi) = 0$ is an homogeneous polynomial of degree k, therefore the dual characteristic cone is some sort of algebraic manifold in $T_x^* M$. For $k = 2$ it is a quadric.

> Let us remark that in the algebra of real polynomials not all polynomials can be factorized into first degree polynomials. For example, $(\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2$ cannot be factorized. Its zeros are a cone; if it could be factorized, the zeros would be the union of two planes.

Let us now consider a first order operator in the general case, the equation for characteristic families is

$$w^i \xi_\alpha e^\alpha_{ik}(x,y) = 0 \tag{4.4.16}$$

Here we fix $b = (x,y) \in B$ and, in general, both w^i and ξ_α depends on b. Accordingly, ξ is not a point in T^*M but in the pull-back bundle $[\pi^*(T^*M) \to B \to M]$. Analogously, w is a point in $V_b(B)$.

For second order operators, the principal symbols can depend on (x,y) or on (x,y,dy). The equation for characteristic families reads either as

$$w^i \xi_\alpha \xi_\beta e^{\alpha\beta}_{ik}(x,y) = 0 \qquad\qquad w^i \xi_\alpha \xi_\beta e^{\alpha\beta}_{ik}(x,y,dy) = 0 \tag{4.4.17}$$

In the first case, $\xi \in [\pi^*(T^*M) \to B \to M]$ and $w \in V_b(B)$ as in the first order case.

In the second case, we have $\xi \in [(\pi^1)^*(T^*M) \to J^1 B \to B \to M]$ and $w \in [(\pi^1)^*V(B) \to J^1 B \to B \to M]$.

Analogously, we can deal with higher order cases by pulling back on the relevant jet prolongation.

In the general case, when we have a characteristic family (w,ξ), as a matter of fact we have a subspace $\langle\xi\rangle \subset T^*M$ at $x \in M$, $\langle\xi\rangle \subset \pi^*(T^*M)$ (at $b \in B$), $\langle\xi\rangle \subset (\pi^h)^*(T^*M)$ (at $jb \in J^h B$, with h depending on the rank of the operator).

For any subspace $\langle\xi\rangle$, one can define the polar $\langle\xi\rangle^\circ$ (in $T_x M$, $\pi^*(TM)$, $(\pi^h)^*(TM)$ depending on the rank) which is a subspace of dimension $m-1$. The collection of these subspaces is called the *characteristic distribution* (which will be tangent to a hypersurface $\mathcal{C}_x$ in $T_x M$, $\mathcal{C}_b$ in $\pi^*(TM)$, $\mathcal{C}_{jb}$ in $(\pi^h)^*(TM)$ depending on the rank) which will be called the *characteristic cone*. Before talking about the characteristic cone we need however to prove that the characteristic distribution is involutive, then integrable, and the characteristic cone are the integral surface of the characteristic distribution.

The structure of the construction is quite complicated. That is why we need to proceed showly and be precise. When we consider the dual characteristic cone $\mathcal{C}^*$, which is a hypersurface in $T_x M$, we also have its tangent space $T\mathcal{C}^*$ (which of course are of dimension $m-1$) and we can consider their polar $(T\mathcal{C}^*)^\circ$ which are one-dimensional subspaces. Since $\langle\xi\rangle \subset T\mathcal{C}^*$, we have $(T\mathcal{C}^*)^\circ = \langle\xi\rangle^\circ$.

A vector $v \in (T\mathcal{C}^*)^\circ$ is called a *characteristic vector*. Characteristic vectors form a distribution of rank 1. It is automatically involutive, hence integrable. Its integral curves are well-defined, are called the *characteristics* of the differential operator

We need to show that v is characteristic iff

$$v^\mu = \frac{\partial Q}{\partial \xi_\mu} \tag{4.4.18}$$

so that we know how to get the characteristic vector directly.

Let us consider a curve $\xi(s)$ in $\mathcal{C}^*$, i.e. such that $Q(x,\xi) = 0$. Tangent vectors to $\mathcal{C}^*$ are tangent vectors to such curves, i.e.

$$\dot\xi_\alpha \frac{\partial Q}{\partial \xi_\alpha} = 0 \tag{4.4.19}$$

For example, for first order operators we have $Q = \det(\xi_\alpha e^\alpha_{ik})$ and

$$\dot\xi_\alpha \frac{\partial Q}{\partial \xi_\alpha} = \dot\xi_\alpha e^\alpha_{ik} \Delta^{ik} = 0 \tag{4.4.20}$$

In other words, tangent vectors to $\mathcal{C}^*$ are polar to $v = \frac{\partial Q}{\partial \xi_\alpha} \partial_\alpha = e^\alpha_{ik} \Delta^{ik} \partial_\alpha$ which is, by definition, the characteristic vector. Since Q is homogeneous of degree k, we have

$$\xi_\alpha v^\alpha = \xi_\alpha \frac{\partial Q}{\partial \xi_\alpha} = kQ = 0 \tag{4.4.21}$$

Let us remark that we are tacitly using swap map to identify tangent vectors to $^* \subset T^*M$ with the characteristic vector in TM.

Let us finally summarize the construction with the following Figure
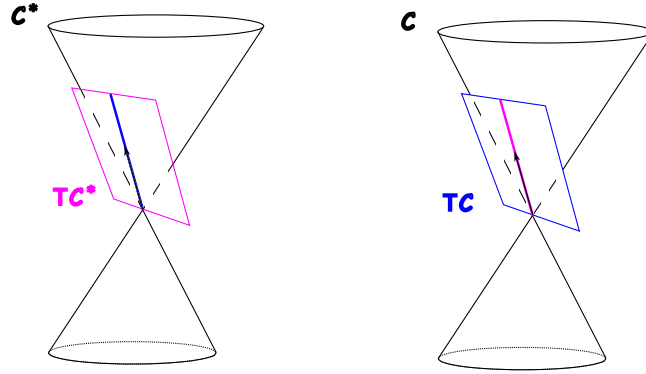
Dual characteristic and characteristic cones with characteristic vectors and their polar (purple).

Characteristics

Given a quasilinear differential operator $\mathbb{E}$, we can define its principal symbol $\sigma(\mathbb{E})$. By analyzing the algebraic structure of the principal symbols, we are able to define the characteristic and dual characteristic distributions (in some space which depends on the rank of the operator).

The distribution $T\mathcal{C}^*$ is defined as an integrable distribution, hence it is involutive by construction. Its integral surfaces are the dual characteristic cones.

The dual characteristic cone also defines the characteristic vector v, that define the characteristic cone

$$\mathcal{C}_x := \left\{ v = \frac{\partial Q}{\partial \xi_\alpha}(x, \xi)\partial_\alpha : \forall \xi \in \mathcal{C}^* \right\} \subset T_x M \tag{4.4.22}$$

The distribution $\Delta = \langle v \rangle$ is rank 1, it is involutive, it defines the *(lifted) characteristics curves*, in B, or in $J^h B$ depending on the rank of the operator. The *characteristic curve* is a lifted characteristic curve projected on M at a point x and defines a surface C_x, which is (locally around x) a conoid, called the *characteristic surface* (or the *wavefront*, or the *singular support* of the operator). We have $y \in C_x$ iff there is a characteristic curve through x and y.

The wavefront is also defined by $C_x := \{y : Q(y, \xi) = 0, \xi \in \mathcal{C}_x^*\}$. The equation for the characteristic curve $\dot{y} = \frac{\partial Q}{\partial \xi}(y, \xi)$ is the solution of the Hamiltonian system with Q as Hamiltonian. At x, since we take $\xi \in \mathcal{C}_x^*$, the value of the Hamiltonian Q is zero. Since the Hamiltonian is a first integral, the whole characteristic surface is on the level surface $Q = 0$ which is then foliated in characteristic curves.

Let us remark that an elliptic operator has no characteristic curves since the only covector for which $Q(x, \xi) = 0$ is $\xi = 0$.

So then what? Why do we care about characteristic curves. Let us consider a Cauchy bubble $(\bar{D}, \zeta, \sigma)$ and fix coordinates (t, k) on D adapted to the foliation. In these coordinates, the evolution vector reads as $\zeta = \partial_t$.

Then we consider a characteristic covector $\xi = \alpha dt + \xi_a dk^a \in \mathcal{C}_x^*$. It defines a subspace $\langle \xi \rangle^\circ$ spanned by the vectors $e_a = \xi_a \partial_0 - \alpha \partial_a \in \langle \xi \rangle^\circ$. Hence the characteristic vector $v = v^a e_a$ can be written as a linear combination of the basis e_a of the subspace $\langle \xi \rangle^\circ$.

For any ξ we can find a characteristic family, namely we can find the corresponding non-zero vertical vector w. In adapted coordinates, we have

$$w^i \xi_\alpha e_{ik}^\alpha = 0 \qquad \Longleftrightarrow \qquad \alpha w^i e_{ik}^0 = -w^i \xi_a e_{ik}^a \tag{4.4.23}$$

We can use the components of αw to create a linear combination of the equations, namely

$$\alpha w^i e_{ik}^\alpha y_\alpha^k = w^i e_i =: e \qquad \Rightarrow \qquad \alpha w^i e_{ik}^0 y_0^k + \alpha w^i e_{ik}^a y_a^k = -w^i e_{ik}^a \left(\xi_a y_0^k - \alpha y_a^k \right) = -w^i e_{ik}^a e_a(y^k) = e \tag{4.4.24}$$

In other words, this equation depends only on the derivatives of fields along the characteristic surface C_x.

Now, on the tangent space of the characteristic surface C_x, we can use two bases, i.e. (e_a) or (v, e_A) where e_A are tangent to C_x and to the Cauchy foliation. In fact, in the basis adapted to the foliation equation reads as

$$-w^i \tilde{e}_{ik}^a v(y^k) = e + w^i \tilde{e}_{ik}^A e_A(y^k) \tag{4.4.25}$$

Of course, the derivatives of fields along the directions tangent to the Cauchy foliation, i.e. $e_A(y^k)$, just depend on the value of fields on the Cauchy leaves. Accordingly, the equation is an evolution ODE on the characteristic curve.

Now, if we have that the characteristic curve intersects a Cauchy surface S_0 more than once, this equation constraints initial conditions on S_0, i.e. the value of fields at an intersection together with this ODE determines (at least partially) the value of fields at the other intersection. Consequently, if we want Cauchy problem to be well-posed we need characteristics to be transverse to the foliation (as well as to intersect the leaves only once). Since the characteristics are on the characteristic surface, the Cauchy surface needs to be chosen to be transverse to the characteristic surface. Since the characteristics depends on the fields, when we fix a Cauchy surface that selects the fields we can represent in that Cauchy problem, namely the fields which produce characteristics transverse to S_0.

We shall discuss what happens with standard GR. In that case when we fix a Cauchy surface S_0 we can see only metrics that makes that surface spacelike. If we want to see other metrics we need to change the Cauchy surface.

Let us remark that requiring that characteristics are transverse to a global Cauchy surface amounts to a topological condition for M to be satisfied. In fact, *a posteriori*, one obtains $M \simeq \mathbb{R} \times S_0$, namely that M is *globally hyperbolic*. We restricted Cauchy problems to Cauchy bubbles so that these conditions are not too restrictive.

In general, we have a characteristic family (w, ξ) for each different choice of $\langle \xi \rangle \subset \mathcal{C}_x^*$. If we can select k characteristic families $(w_{(i)}, \xi^{(i)})$ so that $w_{(i)}$ form a basis of $V_b^*(B)$ we can form a transformation matrix $P_k^i = w_{(k)}^i$ to manipulate the equations so that they depend on derivatives along characteristics curves only, one equation along a characteristics congruence. See below for an example.

That corresponds to write the equations in the frame determined by the basis $w_{(i)}$. It corresponds in matrix formalism to write the equations as

$$e_{ik}^\alpha y_\alpha^k = e_i \qquad \Rightarrow \qquad \left(\alpha\, {}^t P_i^j e_{jk}^\alpha P_i^j \right) \left(\bar{P}_i^j y_\alpha^k \right) = \alpha\, {}^t P_i^j e_j = \hat{e}_i \tag{4.4.26}$$

This contains derivatives in the directions tangent to C_x. We can change basis of such a space to adapt to the basis adapted to the foliation. In this basis all derivatives either can be written in terms of initial conditions (since they are tangent to Cauchy surface) or along the characteristic vector tangent to the characteristic curve. In some sense, the value of fields along characteristics is partially determined by initial conditions, only each field along a different family of characteristics. Because of that, in order to have a well posed Cauchy problem, we need the Cauchy surface to be transverse to characteristics as well as transverse to the evolution vector.

Finally, consider a Cauchy surfaces S_0, give initial conditions on it, and fix compact $\bar{\Omega} \subset S_0$.

We can define a subset

$$\mathcal{D}(\bar{\Omega}) = \{x \in M : \text{all characteristics through } p \text{ intersect } S_0 \text{ in } \bar{\Omega}\} \tag{4.4.27}$$

This region $\mathcal{D}(\bar{\Omega}) \subset M$ is called the *dependence region* on $\bar{\Omega}$. If we change initial conditions in $S_0 - \bar{\Omega}$ fields has to satisfy the same ODE at all points $x \in \mathcal{D}(\Omega)$. Accordingly, the value of fields in $\mathcal{D}(\bar{\Omega})$ just depend on initial conditions within $\bar{\Omega}$ and are unaffected by everything that happens outside $\mathcal{D}(\bar{\Omega})$.

This is a further motivation for Cauchy bubbles. We can fix $\bar{D}$ taking the Cauchy bubble on $\bar{D} = \mathcal{D}(\bar{\Omega})$ and we can restrict the Cauchy problem to that Cauchy bubble in view of general covariance and so that it is unaffected by anything may happen outside the dependence domain.

Symmetric hyperbolic operators

Given a quasi-linear, first order, operator $\mathbb{E}$ with a symbol $\sigma(\mathbb{E})$, we can select a Cauchy bubble $(\bar{D}, \zeta, \sigma)$, adapted coordinates (t, k^a) to the bubble. We write locally the symbol as

$$\sigma(\mathbb{E})_{ij} = \xi_\alpha e_{ij}^\alpha = \xi_0 e_{ij} + \xi_a e_{ij}^a \tag{4.4.28}$$

Definition: *we say the operator $\mathbb{E}$ is symmetric-hyperbolic iff $\xi_\alpha e_{ij}^\alpha$ is a symmetric bilinear form for any covector ξ and the symmetric bilinear form e_{ij} non-degenerate, definite (positive or negative, it does not really matter since we can multiply the whole operator by -1).*

Let us consider fibered coordinates to the ADM bundle $[D \to \mathbb{R}]$, which are adapted to the foliation. A general change of these ADM coordinates reads as

$$t' = t'(t) \qquad k'^a = k'^a(t, k) \qquad \Rightarrow \qquad J_\alpha^\mu = \begin{pmatrix} \frac{dt'}{dt} & 0 \\ \frac{\partial k'^a}{\partial t} & \frac{\partial k'^a}{\partial k^b} \end{pmatrix} = \begin{pmatrix} J_0^0 & 0 \\ J_0^a & J_b^a \end{pmatrix} \tag{4.4.29}$$

and it preserves the product structure iff $k'^a = k'^a(k)$, i.e. $\frac{\partial k'^a}{\partial t} = 0$.

In these coordinates we have derivatives

$$J_0^0 y_0'^i = J_k^i y_0^k - J_k^i y_a^k \bar{J}_b^a J_0^b + J_0^i - J_a^i \bar{J}_b^a J_0^b \qquad y_b'^i = J_a^i \bar{J}_b^a + J_k^i y_a^k \bar{J}_b^a \tag{4.4.30}$$

Accordingly, we can determine transformation laws of the split symbols

$$
\begin{aligned}
e'^{\alpha}_{ik}y'^{k}_{\alpha} =& e'_{ik}y'^{k}_0 + e'^{a}_{ik}y'^{k}_a = e'_{ik}\bar{J}^0_0(J^k_j y^j_0 - J^k_j y^j_a \bar{J}^a_b J^b_0 + J^k_0 - J^k_a \bar{J}^a_b J^b_0) + e'^{a}_{ik}(J^k_c \bar{J}^c_a + J^k_j y^j_c \bar{J}^c_a) = \\
=& e'_{ik}\bar{J}^0_0 J^k_j y^j_0 + \left(e'^{a}_{ik}\bar{J}^c_a - e'_{ik}\bar{J}^0_0 \bar{J}^c_b J^b_0\right) J^k_j y^j_c + e'_{ik}\bar{J}^0_0 J^k_0 - e'_{ik}\bar{J}^0_0 J^k_a \bar{J}^a_b J^b_0 + e'^{a}_{ik}J^k_c \bar{J}^c_a
\end{aligned}
\tag{4.4.31}
$$
$$
\bar{J}\bar{J}^k_i e^{\alpha}_{kj}y^j_\alpha = \bar{J}\bar{J}^k_i e_{kj}y^j_0 + \bar{J}\bar{J}^k_i e^c_{kj}y^j_c
$$

Consider that terms not depending on the derivatives of fields contribute to transformation laws of the tail of the operators. By comparing the term with the derivatives we get transformation laws

$$
e'_{ik} = \bar{J}J^0_0 e_{ln}\bar{J}^l_i \bar{J}^n_k \qquad\qquad e'^{a}_{ik} = \bar{J}\left(J^a_c e^c_{ln} + J^a_0 e_{ln}\right)\bar{J}^l_i \bar{J}^n_k
\tag{4.4.32}
$$

We then see that symmetric hyperbolic symbols are preserved under changes of ADM coordinates, in particular under transformations preserving the product structure $D \simeq \mathbb{R} \times S$. The main result about symmetric hyperbolic operators for us is (see Taylor)

Theorem: *if an operator is symmetric-hyperbolic, then the Cauchy problem it defines is well-posed. For any (smooth) initial condition given on S_0, there exists a unique solution.*

 For what we discussed about the hole argument, we already know that in a typical relativistic theory we cannot expect field equations to be given by a symmetric hyperbolic operator. If they were symmetric hyperbolic, then Cauchy problems will be well posed, and for any initial conditions one would have a unique solution, while in view of the hole argument we know uniqueness cannot happen, just because relativistic theories have diffeomorphisms (or automorphisms in case of gauge-natural theories) as symmetries that can clearly be localized on compact regions.

 We discussed that in relativistic theories, one has to learn that the physical states are to be understood as equivalence classes of configurations, i.e. as classes of global sections of a configuration bundle. Accordingly, field equations induce on equivalence classes physical equations that are instead expected to be deterministic. In this scenario (to be checked in details on examples), we need to have some fields which parameterize the physical states (called the *evolution fields*) and some fields that parameterize different representatives of a given physical state (called the *gauge fields*). Since in variational calculus one has as many equations as fields, we need to split field equations into two blocks, too. One block of equations (called the *constraint equations*) cannot determine gauge fields that only parameterize representatives of physical states and thus can be chosen at will, hence they need to select allowed initial conditions (or to be identically satisfied). The other block of equations (called the *evolution equations*) can determine the evolution of the evolution fields and we can hope that this evolution is given by a well-posed Cauchy problem.

 Since the representation of physical state depends on the observers, each one giving a different representation (otherwise we could simply give a simpler theory directly for the physical states), the split is not covariant, it is usually quite ugly. Usually, one uses Hamiltonian formalism to obtain the splitting, which is the so-called *canonical analysis*. However, Hamiltonian formalism usually breaks manifest covariance, it is not canonical, it artificially introduces classes of preferred observers. The same result can be obtain in a Lagrangian setting by analyzing Cauchy problems.

 In other words, we expect field equation to be a system of PDE which is neither elliptic or hyperbolic. However, we expect it to be eventually splitt in blocks (the principal symbol to be reduced in block-diagonal form) with one elliptic block (the constraint equations) and one block which is symmetric hyperbolic, which produces a well posed Cauchy problem for evolution fields only.

One can then, choose gauge fields at will, use constrain equations to select allowed initial conditions, the Cauchy problem to attach to any allowed initial condition a unique solution. Eventually, gauge fields together with the evolution fields which are solution of the Cauchy problem produce a set of covariant fields which are solutions of the original covariant field equations. Moreover, the general solution of covariant equations are obtained by applying an arbitrary symmetry transformation (including the ones used in the hole argument to show that uniqueness is spoiled) to the solutions built as above.

This accounts for over-determination (by selecting allowed initial conditions) and under-determination (by allowing symmetries to produce classes of solutions attached to the same initial conditions). If this scenario turns out to be reliable, relativistic theories are the result of a trade between adding fields to write dynamics in an absolute way (independent of the observers) and writing deterministic dynamics (which can be done in a relative way only).

Second order Cauchy problems

Even if, for the sake of simplicity, we discussed first order quasi-linear operators, the theory extends to higher order quasi-linear operators. Similarly to what happens for ODE, a quasi-linear operator of order s on $[B \to M]$ can be equivalently viewed as a first order operator on $[J^{s-1}B \to M]$ at the price of introducing derivatives up to order $(s-1)$ as independent fields.

Let us discuss what happens to a quasi-linear operator of order 2 in the form

$$e_{ik}^{\alpha\beta}(p)y_{\alpha\beta}^k = e_i(p) \tag{4.4.33}$$

where $p = x \in M$ for semi-linear equations, which are well-defined on linear or affine configuration bundles only, $p = (x, y) \in B$ for operators or rank 0, $p = j_x^1 y \in J^1B$ for operators of rank 1. We select a Cauchy bubble $(\bar{D}, \zeta, \sigma)$, we introduce adapted coordinates (t, k^a) and we split equations as

$$e_{ik}y_{00}^k + 2e_{ik}^a y_{0a}^k + e_{ik}^{ab} y_{ab}^k = e_i \tag{4.4.34}$$

Finally, we set initial conditions on the leaf S_0 as

$$y^i(0, k) = \bar{y}^i(k) \qquad\qquad y_0^i(0, k) = \bar{v}^i(k) \tag{4.4.35}$$

and altogether they set up a *Cauchy problem of order 2*.

As we do for ODE equations, we introduce new fields to represent derivatives

$$v^i := y_0^i \qquad\qquad v_a^i := y_a^i \qquad\Rightarrow\qquad d_a v^i = y_{0a}^i = d_0 v_a^i \qquad\qquad d_b v_a^i = d_a v_b^i \tag{4.4.36}$$

We cannot use these definitions as equations directly since they do not live in the dual of fields. We need to multiply them by some (invertible) coefficients to map them into the dual space $V^*(J^1B) \simeq J^1V^*(B)$.

Also we cannot use directly $y_0^i := v^i$ and $y_a^i = v_a^i$, since we obtain a first order formulation which is not symmetric hyperbolic.

$$\begin{cases} \alpha_{ik}d_0 y^k = \alpha_{ik}v^k =: \alpha_i \\ \alpha_{ik}^{ab}d_0 v_b^k - \alpha_{ik}^{ab}d_b v^k = 0 \\ e_{ik}d_0 v^k + 2e_{ik}^a d_a v^k + e_{ik}^{ab}d_b v_a^k = e_i \end{cases} \tag{4.4.37}$$

Since we can choose coefficients, provided they are invertible, we can set $\alpha_{ik} := e_{ij}$ and $\alpha_{ik}^{ab} := -e_{ik}^{ab}$ and assume that e_{ik}^{ab} is invertible as a $k(m-1) \times k(m-1)$ matrix. The equation $d_b v_a^i = d_a v_b^i$ does not contain derivatives in the direction of ζ. Accordingly, it has to added as a constraint to select initial conditions. General initial conditions for the first order system (4.4.37) would be $y^k(0,k) = \bar{y}^k(k)$, $v^k(0,k) = \bar{v}^k(k)$, and $v_a^k(0,k) = \bar{v}_a^k(k)$. The *allowed* initial conditions (i.e. the ones obeying the constraint) are with $\bar{v}_a^k(k) = d_a \bar{y}^k(k)$.

By construction, the second order Cauchy problem (4.4.34) and the corresponding (constrained) first order Cauchy problem (4.4.37) are equivalent.

Step 1: let $y^i(t,k)$ be a solution of of the second order problem and let us define $v^i(t,k) := \partial_0 y^i(t,k)$ and $v_a^i(t,k) := \partial_a y^i(t,k)$. Then (y^i, v^i, v_a^i) satisfies the corresponding first order problem as well as the constraint. First equation is satisfied because of the definition $v^i(t,k) = \partial_0 y^i(t,k)$. The second equation is time-derivative of the definition $v_a^i(t,k) = \partial_a y^i(t,k)$ and Schwarz lemma so that $\partial_0 v_a^i = \partial_{0a} y^i = \partial_a v^i$. The constraint is satisfied at any t just because we defined $v_a^i := \partial_a y^i$ (and Schwarz again). The third equation is satisfied just because it is rewritten in the new variables in the first order problem. Finally, allowed initial conditions are met because of the definition $v_a^i(t,k) = \partial_a y^i(t,k)$ restricted to the Cauchy surface $t = 0$.

Step 2: viceversa, if $(y^i(t,k), v^i(t,k), v_a^i(t,k))$ is a solution of the first order problem (plus the constraint and with allowed initial conditions), then $y^i(t,k)$ is a solution of the corresponding second order problem. From the first equation, in view of invertibility of α_{ik}, we have $d_0 y^k = v^k$. From the second equation, and in view of the invertibility of α_{ik}^{ab}, we have $d_0 v_b^k = d_b v^k = d_{0b} y^k$. From $d_0 y^k = v^k$, by derivation we obtain $d_{0a} y^k = d_a v^k = d_0 v_b^k$, then $v_b^k = d_a y^k + c_b^k(k)$. By evaluation on the Cauchy surface S_0, we see that $c_b^k = 0$. Now that we know $v^k = d_0 y^k$ and $v_b^k = d_a y^k$, we can replace these into the last equation and obtain that $y^k = y^k(t,k)$ is a solution of the second order problem.

In matrix form the first order problem reads as

$$\begin{pmatrix} e_{ik} & 0 & 0 \\ 0 & -e_{ik}^{ab} & 0 \\ 0 & 0 & e_{ik} \end{pmatrix} d_0 \begin{pmatrix} y^k \\ v_b^k \\ v^k \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{ik}^{ab}d_b \\ 0 & e_{ik}^{ab}d_b & 2e_{ik}^a d_a \end{pmatrix} \begin{pmatrix} y^k \\ v_b^k \\ v^k \end{pmatrix} = \begin{pmatrix} \alpha_i \\ 0 \\ e_i \end{pmatrix} \tag{4.4.38}$$

The first leftmost matrix is block diagonal, the blocks are squared $k \times k$, $k(m-1) \times k(m-1)$, and $k \times k$, respectively. To be non-degenerate, both e_{ik} and e_{ik}^{ab} need to be non-degenerate. For the block diagonal matrix to be definite (say positive), we need to have e_{ik} definite (positive) and e_{ik}^{ab} definite (negative).
In the second block matrix, blocks are operators and the blocks $e_{ik}^{ab}d_b$ are $k \times k(m-1)$, while the block $2e_{ik}^a d_a$ is $k \times k$. We originally chose the sign of $\alpha_{ik}^{ab} = -e_{ik}^{ab}$ to make the second block diagonal matrix symmetric.

In other words, the second order operator (4.4.34) is called *symmetric hyperbolic* iff the matrix e_{ik} is symmetric, non-degenerate, definite positive, e_{ik}^a is symmetric, and e_{ik}^{ab} is symmetric, non-degenerate, definite negative.

Because of this definition, it is still true that if a symmetric operator of order 2 is symmetric hyperbolic, then its Cauchy problem is well-defined.

Hence the scenario we presented above seems a viable way. We just have to consider a theory, write down field equations anche check that they are quasi-linear, write down the principal symbol and check whether it is elliptic or hyperbolic. If it is not either of the two, one needs to check which fields are determined by the evolutionary equations, which split fields in gauge and evolutionary fields. Finally, we need to split equations into evolution equations for the evolutionary fields and constraint equations and check that the evolution part is symmetric hyperbolic.

Examples of hyperbolic operators

Before going into analysing some relativistic theory, let us argue we can use the tools we set up for that task also to consider less structured examples of PDE.

Example 1. Let us consider the system

$$\begin{cases} \partial_0 u + \partial_1 u + \partial_2 v = b_1 \\ \partial_2 u + \partial_0 v - \partial_1 v = b_2 \end{cases} \tag{4.4.39}$$

One can consider u and v as scalar fields, thus sections of $[M \times \mathbb{R}^2 \to M]$. We do not really need to know whether these equations come from a variational principle, just that this system is a quasi-linear, first order operator.

The principal symbol is

$$\sigma = \begin{pmatrix} \xi_0 + \xi_1 & \xi_2 \\ \xi_2 & \xi_0 - \xi_1 \end{pmatrix} \qquad Q = (\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 \tag{4.4.40}$$

We see the operator is not elliptic since $\xi = dt + dx$ make the symbol degenerate. The dual characteristic cone is the cone in $T_x^* M$ of equation $Q = 0$. That is not the union of two planes, hence the characteristic polynomial Q cannot be factorized into 2 first-order polynomials.

Characteristic covectors are parameterized (up to the norm) as

$$\xi = dt + \cos(\theta)dx + \sin(\theta)dy \tag{4.4.41}$$

The polar space $\langle \xi \rangle^\circ$ is spanned by the vectors $v = v^\mu \partial_\mu$ such that

$$v^0 + \cos(\theta)v^1 + \sin(\theta)v^2 = 0 \quad \Rightarrow \quad v = v^1 \left(-\cos(\theta)\partial_0 + \partial_1 \right) + v^2 \left(-\sin(\theta)\partial_0 + \partial_2 \right) \tag{4.4.42}$$

Let us then set

$$e_1 = -\cos(\theta)\partial_0 + \partial_1 \qquad e_2 = -\sin(\theta)\partial_0 + \partial_2 \tag{4.4.43}$$

so that we have $\langle e_1, e_2 \rangle = \langle \xi \rangle^\circ$.

The characteristic vector is obtained by the equations

$$\dot{t} = \frac{\partial Q}{\partial \xi_0} = 2\xi_0 = 2 \qquad \dot{x} = \frac{\partial Q}{\partial \xi_1} = -2\xi_1 = -2\cos(\theta) \qquad \dot{y} = \frac{\partial Q}{\partial \xi_2} = -2\xi_2 = -2\sin(\theta) \tag{4.4.44}$$

as the vector $v = \partial_0 - \cos(\theta)\partial_1 - \sin(\theta)\partial_2 = -\cos(\theta)e_1 - \sin(\theta)e_2 \in \langle e_1, e_2 \rangle$. Also $u = -\sin(\theta)e_1 + \cos(\theta)e_2 = -\sin(\theta)\partial_1 + \cos(\theta)\partial_2 \in \langle e_1, e_2 \rangle$.

The vectors v, u give us a basis of $\langle v, u \rangle = \langle e_1, e_2 \rangle$ which is also adapted to the foliation since v is characteristic and u is tangent to the Cauchy surfaces $t = t_0$. Also, for any choice of θ we can find w as

$$\begin{pmatrix} 1 + \cos(\theta) & \sin(\theta) \\ \sin(\theta) & 1 - \cos(\theta) \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad w \propto -\sin(\theta)\partial_u + (1 + \cos(\theta))\partial_v \tag{4.4.45}$$

Hence we have one characteristic family for each θ. We can choose 2 values of θ, e.g. $\theta_1 = 0$ and $\theta_2 = \pi/2$, so that the corresponding $w_1 = \partial_2$ and $w_2 = -\partial_1 + \partial_2$ are independent, i.e. a basis for vertical vectors. For each choice we have a basis of $\langle \xi \rangle^\circ$

$$e_1 = -\partial_0 + \partial_1 \qquad e_2 = \partial_2 \qquad\qquad \epsilon_1 = \partial_1 \qquad \epsilon_2 = -\partial_0 + \partial_2 \tag{4.4.46}$$

as well as the corresponding adapted bases

$$v_1 = -e_1 \qquad u_1 = e_2 \qquad\qquad v_2 = -\epsilon_2 \qquad u_2 = -\epsilon_1 \tag{4.4.47}$$

We can define the transformation matrix

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad\qquad \Rightarrow \quad {}^tP = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad\qquad \bar{P} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.4.48}$$

and transform the equations accordingly

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial_0 + \partial_1 & \partial_2 \\ \partial_2 & \partial_0 - \partial_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ -b_1 + b_2 \end{pmatrix} \tag{4.4.49}$$

$$\begin{pmatrix} \partial_0 - \partial_1 & \partial_0 - \partial_1 - \partial_2 \\ \partial_0 - \partial_1 - \partial_2 & 2(\partial_0 - \partial_2) \end{pmatrix} \begin{pmatrix} u + v \\ -u \end{pmatrix} = \begin{pmatrix} b_2 \\ -b_1 + b_2 \end{pmatrix} \tag{4.4.50}$$

Since we can write $\partial_0 - \partial_1 - \partial_2 = -e_1 - e_2 = -\epsilon_1 - \epsilon_2$, we have

$$\begin{pmatrix} -e_1 & -\epsilon_1 - \epsilon_2 \\ -e_1 - e_2 & -2\epsilon_2 \end{pmatrix} \begin{pmatrix} u + v \\ -u \end{pmatrix} = \begin{pmatrix} b_2 \\ -b_1 + b_2 \end{pmatrix} \tag{4.4.51}$$

or, by using the adapted bases and making a linear combination, we obtain

$$\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 & u_2 + v_2 \\ v_1 - u_1 & 2v_2 \end{pmatrix} \begin{pmatrix} u + v \\ -u \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_2 \\ -b_1 + b_2 \end{pmatrix} \tag{4.4.52}$$

Let us remark that, being u_1 and u_2 tangent to the Cauchy surface, the corresponding contributions can be considered as a function of initial conditions. Hence, finally, we can recast the system as

$$\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} u + v \\ -u \end{pmatrix} = \begin{pmatrix} 2u_1 & u_2 \\ u_1 & u_2 \end{pmatrix} \begin{pmatrix} u + v \\ -u \end{pmatrix} + \begin{pmatrix} b_2 - 2b_1 \\ -b_1 \end{pmatrix} \tag{4.4.53}$$

If we restrict the first equation on the first family of characteristic curves (or the second on the second family of characteristic curves) these equations become ODE along the characteristics.

Example 2.

Strictly hyperbolic operators

Strictly hyperbolic operators are special symmetric hyperbolic operators in which the characteristic polynomial $Q(p, \xi)$, which is homogeneous in ξ of order k, factorizes into k first-order polynomials, namely, for first order operators, we have

$$Q(p,\xi) = \det\left(\xi_\alpha e_{ij}^\alpha(p)\right) = \left(a_{(1)}^\alpha(p)\xi_\alpha\right)\left(a_{(2)}^\beta(p)\xi_\beta\right)\ldots\left(a_{(k)}^\epsilon(p)\xi_\epsilon\right) \tag{4.4.54}$$

where p is in M, in B, or in J^sB depending on the rank of the operator.

> For second order operators, the characteristic polynomial of homogeneous of order $2k$. It is strictly hyperbolic if Q factorizes as the product of $2k$ homogeneous polynomials of first order.

For strictly hyperbolic operators, the dual characteristic cone is union of k hyperplanes, each one given by each one factor.

> This is a rather special case where we can split the dual characteristic cone into hyperplanes and use linear algebra rather than algebraic geometry to analyze the situation. Looking for examples in the literature, one often finds strictly hyperbolic systems. The treatment looks completely different because of the use of linear algebra tools. However, we are doing the same thing, just in a different way.

For a strictly hyperbolic operator, and for each factor in Q, one can determine a sub-space of dimension $(m-1)$ of covectors for which the factor is zero. By solving the equation $a^\alpha(p)\xi_\alpha = 0$, one can find a basis ξ^a of the solutions' hyperplane $\langle \xi^1, \xi^2, \ldots, \xi^{m-1}\rangle$. The sub-space polar $\langle \xi_1, \xi_2, \ldots, \xi_{m-1}\rangle^\circ = \langle v \rangle$ is one dimensional, thus spanned by a single characteristic vector v that we assume to be transverse to the Cauchy surface, hence in the form $v = \alpha\partial_0 + v^a\partial_a$ with $\alpha \neq 0$. Since characteristic vectors are such that $\xi_a(v) = 0$ one can choose the basis $\xi^a = v^a\,dt - \alpha dk^a$ for the subspace $\langle \xi^1, \xi^2, \ldots, \xi^{m-1}\rangle$.

> If we treat this case as the case of symmetric hyperbolic case above, we should choose a 1d sub-space $\langle \lambda_a\xi^a\rangle$, we obtain a hyperspace generated by $e_a = \alpha\lambda_a\partial_0 + \lambda_b v^b\partial_a$. This hyperplane contains as a special case the characteristic vector $v \propto v^a e_a = v^b\lambda_b\,(\alpha\partial_0 + v^a\partial_a)$. That shows that the choice of $\langle \xi \rangle$ in the hyperplane, does not affect the characteristic sub-space $\langle v \rangle$, just the length of the characteristic vector. In that sense, we can say that $\langle \xi_a\rangle =: U$ selects the characteristic sub-space $\langle \xi_a\rangle^\circ = \langle v \rangle = \cap_{\xi \in U}\langle \xi\rangle^\circ$.

As for symmetric hyperbolic operators, we can use the characteristic family to manipulate the equations.

> This time we have

$$w^i v^a e_{ij} = \alpha w^i e_{ij}^a \tag{4.4.55}$$

> and we can use this to make a linear combination of the equations

$$\alpha w^i e_{ik}^\alpha y_\alpha^k = w^i e_{ik}\left(\alpha y_0^k + v^a y_a^k\right) = w^i e_{ik} v(y^k) = \alpha w^i b_i \tag{4.4.56}$$

> which, again, becomes an ODE when evaluated along the corresponding characteristics.

For strictly hyperbolic operators it is only easier to find the characteristic vector since we do not need Hamiltonians, just linear algebra. However, eventually it makes sense to ask for e_{ik} to be invertible (so that we can set the system in a sort of normal form). Moreover, eventually we diagonalize the symbol, thus it was symmetric in the beginning.

A (first order) strictly hyperbolic operator is just a special case of a symmetric hyperbolic one.

5. Constraints and conservation laws

6. Examples of Hamiltonian formalism

Standard GR

We can analyze Einstein equations in vacuum, i.e. $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$. In dimension $m \neq 2$ they are equivalent to $R_{\mu\nu} = 0$.

So, first, we have to show that $R_{\mu\nu} = 0$ is a quasi-linear equation in the second derivatives of the metric $g_{\alpha\beta}$. Let us start by considering the connection

$$\{g\}^\alpha_{\beta\mu} = \tfrac{1}{2}g^{\alpha\epsilon}(-d_\epsilon g_{\beta\mu} + d_\beta g_{\mu\epsilon} + d_\mu g_{\epsilon\beta}) \tag{4.6.1}$$

Let us introduce the symbol $\sim$ for equality up to terms not involving second derivatives. The Ricci tensor is then

$$R_{\alpha\beta} \sim d_\epsilon\{g\}^\epsilon_{\alpha\beta} - d_\alpha\{g\}^\epsilon_{\beta\epsilon} \sim g^{\epsilon\lambda}d_{\epsilon(\alpha}g_{\beta)\lambda} - \tfrac{1}{2}g^{\epsilon\lambda}d_{\epsilon\lambda}g_{\alpha\beta} + \tfrac{1}{2}g^{\epsilon\lambda}(d_{\lambda\alpha}g_{\beta\epsilon} - d_{\alpha\beta}g_{\epsilon\lambda} - d_{\alpha\epsilon}g_{\lambda\beta}) \sim \tfrac{1}{2}\left(g^{\epsilon\sigma}\delta^\lambda_\alpha\delta^\rho_\beta + g^{\epsilon\sigma}\delta^\rho_\alpha\delta^\lambda_\beta - g^{\epsilon\lambda}\delta^\rho_\alpha\delta^\sigma_\beta - g^{\rho\sigma}\delta^\epsilon_\alpha\delta^\lambda_\beta\right)d_{\epsilon\lambda}g_{\rho\sigma} \tag{4.6.2}$$

Therefore, field equations $R_{\alpha\beta} = 0$ are in fact quasi-linear second order equations or rank 0.

Second, we need to trace second time-derivatives in the equations to see which fields have their evolution determined. Let us introduce the symbol $\approx$ for equality up to terms not containing second time-derivatives. The Ricci tensor is then

$$\begin{aligned}
R_{00} &\sim \tfrac{1}{2}\left(g^{\epsilon\sigma}d_{\epsilon 0}g_{0\sigma} + g^{\epsilon\sigma}d_{\epsilon 0}g_{0\sigma} - g^{\epsilon\lambda}d_{\epsilon\lambda}g_{00} - g^{\rho\sigma}d_{00}g_{\rho\sigma}\right) \approx \\
&\approx \tfrac{1}{2}\left(g^{00}d_{00}g_{00} + g^{0a}d_{00}g_{0a} + g^{00}d_{00}g_{00} + g^{0a}d_{00}g_{0a} - g^{00}d_{00}g_{00} - g^{00}d_{00}g_{00} - 2g^{0a}d_{00}g_{0a} - g^{ab}d_{00}g_{ab}\right) \approx -\tfrac{1}{2}g^{ab}d_{00}g_{ab} \\
R_{0a} &\sim \tfrac{1}{2}\left(g^{\epsilon\sigma}d_{\epsilon 0}g_{a\sigma} + g^{\epsilon\sigma}d_{\epsilon a}g_{0\sigma} - g^{\epsilon\lambda}d_{\epsilon\lambda}g_{0a} - g^{\rho\sigma}d_{0a}g_{\rho\sigma}\right) \approx \tfrac{1}{2}\left(g^{00}d_{00}g_{0a} + g^{0e}d_{00}g_{ae} - g^{00}d_{00}g_{0a}\right) \approx \tfrac{1}{2}g^{0e}d_{00}g_{ae} \\
R_{ab} &\sim \tfrac{1}{2}\left(g^{\epsilon\sigma}d_{\epsilon a}g_{b\sigma} + g^{\epsilon\sigma}d_{\epsilon b}g_{a\sigma} - g^{\epsilon\lambda}d_{\epsilon\lambda}g_{ab} - g^{\rho\sigma}d_{ab}g_{\rho\sigma}\right) \approx -\tfrac{1}{2}g^{00}d_{00}g_{ab}
\end{aligned} \tag{4.6.3}$$

from which we see that the only second order time-derivatives appearing are $d_{00}g_{ab}$. Accordingly, equations cannot determine the evolution of g_{0a} or g_{00} which are then gauge fields.

Let us consider the vector $\vec{n} = N^{-1}(\partial_0 - \beta^a\partial_a)$. With respect to changes of adapted coordinates, N is a scalar called the *lapse*, $\beta^a\partial_a$ is a vector tangent to the leaves S of the Cauchy foliation, called the *shift*.

We can restrict the metric we are looking for to the metric for which $\vec{n}$ is a unit time-like vector.

$$-1 = g(\vec{n}, \vec{n}) = N^{-2}\left(g_{00} - 2\beta^a g_{0a} + \beta^a\beta^b g_{ab}\right) \qquad 0 = g(\vec{n}, \partial_a) = N^{-1}\left(g_{0a} - \beta^b g_{ab}\right) \tag{4.6.4}$$

Then let us consider the induced metric $\gamma_{ab} := g_{ab}$ on the leaves. The inverse of the induced metric is denoted by γ^{ab} and we set $\beta_a := \beta^c \gamma_{ac}$. We obtain $g_{0a} = \beta^b g_{ab} =: \beta_a$ and

$$g_{00} = -N^2 + |\beta|^2 \tag{4.6.5}$$

where $|\beta|^2 := \beta^a \beta^b \gamma_{ab}$. In other words, we can use $(N, \beta_a, \gamma_{ab})$ instead of $g_{\alpha\beta}$ as coordinates on $\mathrm{Lor}(M)$.

We have

$$g_{\alpha\beta} = \begin{pmatrix} -N^2 + |\beta|^2 & \beta_a \\ \beta_b & \gamma_{ab} \end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^a}{N^2} \\ \frac{\beta^b}{N^2} & \gamma^{ab} - \frac{\beta^a \beta^b}{N^2} \end{pmatrix} \tag{4.6.6}$$

In view of the fact $g^{00} = -\frac{1}{N^2} = \frac{\gamma}{g}$, we obtain $g = -N^2 \gamma$.
Since also ∂_0 has to be timelike, then $|\beta|^2 < N^2$.

Then we can use the equation $R_{ab} = 0$ to determine the evolution of the induced metric γ_{ab} and eliminate second order time-derivatives from the other equations $R_{00} = 0$ and $R_{0a} = 0$ which are then constraint equations.

Now we need to write the equation $R_{ab} = 0$ in terms of the second derivatives of the evolution field γ^{ab}. Let us introduce $\doteq$ for equalities up to terms that do not contain second derivatives of the field γ^{ab}.

Notice that we have

$$\partial_{00}\gamma_{ab} \doteq -\gamma_{ac}\gamma_{bd}\partial_{00}\gamma^{cd} \qquad \partial_{0i}\gamma_{ab} \doteq -\gamma_{ac}\gamma_{bd}\partial_{0i}\gamma^{cd} \qquad \partial_{ij}\gamma_{ab} \doteq -\gamma_{ac}\gamma_{bd}\partial_{ij}\gamma^{cd} \tag{4.6.7}$$

The evolution equation then reads as

$$\begin{aligned} R_{ab} \sim \frac{1}{2}\left(g^{\epsilon\sigma}d_{\epsilon a}g_{b\sigma} + g^{\epsilon\sigma}d_{\epsilon b}g_{a\sigma} - g^{\epsilon\lambda}d_{\epsilon\lambda}g_{ab} - g^{\rho\sigma}d_{ab}g_{\rho\sigma}\right) &\doteq -\frac{1}{2N^2}\gamma_{a(c}\gamma_{d)b}d_{00}\gamma^{cd} + \\ + \frac{\beta_e}{2N^2}\left(2\gamma^{ek}\gamma_{a(c}\gamma_{d)b} - \delta^e_{(d}\gamma_{c)b}\delta^k_a - \delta^e_{(d}\gamma_{c)a}\delta^k_b\right)d_{0k}\gamma^{cd} &+ \frac{1}{2}\left(g^{ik}\gamma_{ac}\gamma_{bd} + g^{fe}\gamma_{fc}\gamma_{ed}\delta^i_a\delta^k_b - g^{ie}\delta^k_a\gamma_{bc}\gamma_{ed} - g^{ie}\delta^k_b\gamma_{ac}\gamma_{ed}\right)d_{ik}\gamma^{cd} \end{aligned} \tag{4.6.8}$$

Since fields are labelled by pair of Latin indices let us introduce the notation $\{ab\}$ to index fields. Accordingly, the principal symbol splits as

$$\begin{aligned} e_{\{ab\}\{cd\}} = -\frac{1}{2N^2}\gamma_{a(c}\gamma_{d)b} \qquad & e^k_{\{ab\}\{cd\}} = \frac{\beta_e}{2N^2}\left(2\gamma^{ek}\gamma_{a(c}\gamma_{d)b} - \delta^e_{(d}\gamma_{c)b}\delta^k_a - \delta^e_{(d}\gamma_{c)a}\delta^k_b\right) \\ & e^{ik}_{\{ab\}\{cd\}} = \frac{1}{2}\left(g^{ik}\gamma_{a(c}\gamma_{d)b} + g^{fe}\gamma_{fc}\gamma_{ed}\delta^i_a\delta^k_b - g^{ie}\delta^k_a\gamma_{b(c}\gamma_{d)e} - g^{ie}\delta^k_b\gamma_{a(c}\gamma_{d)e}\right) \end{aligned} \tag{4.6.9}$$

Since we choose the evolution field, one can always choose $N = 1$ and $\beta^a = 0$, i.e. we evolve in the orthogonal direction to g. Let us stress this is a further constraint to the metrics we can consider in a given foliation. In this orthogonal gauge, the principal symbol splits as

$$e_{\{ab\}\{cd\}} = -\frac{1}{2}\gamma_{a(c}\gamma_{d)b} \qquad e^k_{\{ab\}\{cd\}} = 0 \qquad e^{ik}_{\{ab\}\{cd\}} = \frac{1}{2}\left(\gamma^{ik}\gamma_{a(c}\gamma_{d)b} + \gamma_{cd}\delta^i_a\delta^k_b - \delta^k_a\delta^i_{(d}\gamma_{c)b} - \delta^k_b\delta^i_{(d}\gamma_{c)a}\right) \tag{4.6.10}$$

and we *only* (!) have to show that the evolution equations are symmetric hyperbolic.

For, one needs $e_{\{ab\}\{cd\}}$ to be non-degenerate. This is (in dimension $m = 4$) a 10×10 matrix. Thus, to show it is non-degenerate it is easier to find an inverse rather than to compute the determinant.

We have

$$(\gamma_{a(e}\gamma_{f)b})(\gamma^{c(e}\gamma^{f)d}) = \tfrac{1}{4}(\gamma_{ae}\gamma_{fb} + \gamma_{af}\gamma_{eb})(\gamma^{ce}\gamma^{fd} + \gamma^{cf}\gamma^{ed}) = \tfrac{1}{2}(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c) = \delta_{(a}^c\delta_{b)}^d \tag{4.6.11}$$

which is in fact the identity on the space of symmetric tensors h^{ab}.

It is also symmetric and definite-negative.

Since we can choose coordinates so that $\gamma^{ij} = \delta^{ij}$, we have

$$e_{\{ab\}\{cd\}} = -\tfrac{1}{2}h^{ab}h_{ab} \tag{4.6.12}$$

which is negative since it is minus a sum of squares.

The symbol $e^{k}_{\{ab\}\{cd\}} = 0$ is trivially symmetric and we do not ask to it anything more to be symmetric hyperbolic.

For the symbol $e^{ik}_{\{ab\}\{cd\}}$ we have to show it is symmetric, non-degenerate (as a 30×30 matrix in dimension $m = 4$), and definite-positive. Although at this point it may seem unbelievable, it is not symmetric.

We can compute the antisymmetric part

$$4e^{[i}_{\{ab\}}{}^{k]}_{\{cd\}} = {\color{red}\gamma^{ik}\gamma_{a(c}\gamma_{d)b}} + \gamma_{cd}\delta_a^i\delta_b^k - {\color{blue}\delta_a^k\delta^i_{(d}\gamma_{c)b}} - {\color{blue}\delta_b^k\delta^i_{(d}\gamma_{c)a}} - {\color{red}\gamma^{ik}\gamma_{c(a}\gamma_{b)d}} - \gamma_{ab}\delta_c^k\delta_d^i + {\color{blue}\delta_c^i\delta^k_{(b}\gamma_{a)d}} + {\color{blue}\delta_d^i\delta^k_{(a}\gamma_{b)c}} = \gamma_{cd}\delta^i_{(a}\delta^k_{b)} - \gamma_{ab}\delta^k_{(c}\delta^i_{d)} \tag{4.6.13}$$

Remember this feeling, it means nature is malicious and she does not like you.

So Einstein equations in vacuum are not symmetric hyperbolic. How is it possible? Does it mean the evolution of gravitational field is not deterministic and we cannot solve Einstein equations even in vacuum?

As a matter of fact, that is not the case of course, although some extra work is needed.

Let us first split the may-evolution part of Einstein equations in it symmetric and anti-symmetric part

$$\begin{cases} e_{\{ab\}\{cd\}}d_{00}\gamma^{cd} + e^{ik}_{(\{ab\}\{cd\})}d_{ik}\gamma^{cd} \sim 0 \\ e^{ik}_{[\{ab\}\{cd\}]}d_{ik}\gamma^{cd} \sim 0 \quad\quad \Rightarrow \quad \gamma_{cd}d_{ab}\gamma^{cd} \sim \gamma_{ab}d_{cd}\gamma^{cd} \end{cases} \tag{4.6.14}$$

Now the first equation is symmetric. However, both equations are non-covariant since the splitting is not even covariant for transformations that preserves the Cauchy foliation. We are used to run away from non-covariance. All of this book is built on the hope that *everything is covariant*. This is here to remind us that any faith will be sooner or later ill based! Even whet it seems a very reasonable faith.

As a matter of fact we can discuss solutions of the original covariant Einstein vacuum equations *precisely* because this splitting is non-covariant!

It goes like this. Just since the second equation is not covariant, we can look for a coordinate system in which it is identically satisfied. For example, we can define *harmonic coordinates* on S, i.e. coordinates for which $\gamma^{ab}\{\gamma\}^k_{ab} = 0$. In harmonic coordinates, we have

$$\gamma^{ab}\gamma^{ki}(-d_i\gamma_{ab} + d_a\gamma_{bi} + d_b\gamma_{ia}) = 0 \quad\quad \Rightarrow \quad \gamma_{ab}\gamma^{ki}d_i\gamma^{ab} \sim 2d_b\gamma^{bk} \quad\quad \Rightarrow \quad \gamma_{ab}d_{cd}\gamma^{ab} \sim 2\gamma_{ca}d_{bd}\gamma^{ba} \tag{4.6.15}$$

This solves exactly the second equation, in fact we have

$$\gamma_{ab}d_{cd}\gamma^{cd} \sim \tfrac{1}{2}\gamma^{ck}\gamma_{ef}d_{ck}\gamma^{ef} \quad\quad \gamma_{cd}d_{ab}\gamma^{cd} \sim \gamma_{ab}d_{cd}\gamma^{cd} \sim \tfrac{1}{2}\gamma_{ab}\gamma_{cd}\gamma^{fe}d_{fe}\gamma^{cd} \quad\quad 2\gamma_{cd}d_{ab}\gamma^{cd} \sim (m-1)\gamma^{ki}d_{ik}\gamma^{ab} \tag{4.6.16}$$

Now that the second equation is solved by choosing harmonic coordinates, then we have for the first equation that it is symmetric hyperbolic. The symbol reads as

$$e^{ik}_{(\{ab\}\{cd\})} = \tfrac{1}{2}\gamma^{ik}\gamma_{a(l}\gamma_{m)b} = -\gamma^{ik}e_{\{ab\}\{cd\}} \tag{4.6.17}$$

which shows that it is non-degenerate definite positive, as it should be.

In other words, we split the covariant equation $R_{ab} = 0$ in two non-covariant equations. One of these equations is identically satisfied in harmonic coordinates on S. The other is symmetric hyperbolic, hence its Cauchy problem is well-posed.

For any initial conditions, $R_{ab} = 0$ determines a solution $\gamma^{ab}(t, k)$ in harmonic coordinates. Since the initial equation was covariant, we can change coordinates on S at will, the corresponding metric γ^{ab} will solve $R_{ab} = 0$ in any coordinate system even though it will not in general solve the two non-covariant equations separately.

Our freedom of changing coordinates at will accounts for the underdetermination of Einstein equation that we expect for the hole argument. Once we have a solution γ^{ab} however, we do not have a solution on the original covariant Einstein vacuum equation $R_{\alpha\beta} = 0$, unless we start from a set of initial conditions that also satisfies the constraint equations R_{00} and R_{0a}, that accounts for over-determination.

Then a solution is obtained by solving Cauchy non-covariant equation in harmonic coordinates, rebuilding the covariant metric $g_{\mu\nu}$ on spacetime, and possibly changing coordinates at will. Once we do this, the original Cauchy surface results as a spacelike surface, the evolution direction is timelike and characteristic surfaces are given for wavefront conoids.

check harmonic coordinates

existence of harmonic coordinates

Bianchi identities enforce constraint preservation

we should try and solve equations without setting $N = 1$ and $\beta = 0$.

This structure is quite weird, however, it is more or less the standard situation one has when dealing with a relativistic theory (with or without gauge invariance). It is quite striking (to me at least) that eventually one manages to have deterministic equations on states despite in the end the only requirement we did was to write equations on a manifold up to diffeomorphisms.

Palatini standard GR

Klein-Gordon field

Electromagnetism

References

Add references

M. Toda and M. Wadati, J. Phys. Soc. Japan 39, l204(1975).

7. Exercises

Chapter 5. Dynamical equivalence between field theories

So close no matter how far
Couldn't be much more from the heart
Forever trusting who we are
And nothing else matters

(Metallica, *Nothing else matters*).

1. Introduction

Different field theories can be equivalent to each other, of course, in various different ways.

The most trivial example, after two identical theories, is when one simply changes fibered chart on configuration bundle, transforming original fields into new fields. Since we restricted to use global variational principles, we have two local Lagrangians covering the original fields and the new ones. Since general field redefinitions are not generally symmetries, the two local Lagrangians are not invariant in form, they induce field equations which are not simply the *"same local equations"* written for different fields.

The same situation is well known in mechanics where fields are identified with Lagrangian coordinates. If one considers a harmonic oscillator in the usual elongation coordinate x, the Lagrangian is

$$\boldsymbol{L} = \tfrac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right)\,dt \tag{5.1.1}$$

Then, if for some reason, one uses $q = x^3$ as a Lagrangian coordinate, the Lagrangian can be expressed as

$$\boldsymbol{L}' = \tfrac{1}{2}\left(\tfrac{1}{9q}\dot{q}^2 - \omega^2 q\right)q^{-\frac{1}{3}}\,dt \tag{5.1.2}$$

By saying that these two Lagrangians are not invariant in form, we mean that $\boldsymbol{L}'$ is not simply $\boldsymbol{L}$ written in the new coordinates, i.e. that

$$\boldsymbol{L}' \neq \tfrac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)\,dt \tag{5.1.3}$$

as it would be is the transformation were a Lagrangian symmetry.
The Euler–Lagrange equation for $\boldsymbol{L}$ is the ordinary equation for the harmonic oscillator

$$\ddot{x} = -\omega^2 x \tag{5.1.4}$$

while, away from $q = 0$, the Euler–Lagrange equation for $\boldsymbol{L}'$ is

$$\tfrac{1}{9}\tfrac{d}{dt}\left(\dot{q}q^{-\frac{4}{3}}\right) = -\tfrac{1}{2}\tfrac{2}{3}\omega^2 q^{-\frac{1}{3}} - \tfrac{1}{2}\tfrac{1}{9}\tfrac{4}{3}\dot{q}^2 q^{-\frac{7}{3}} \qquad \Rightarrow \qquad \ddot{q} = -3\omega^2 q + \tfrac{2}{3q}\dot{q}^2 \tag{5.1.5}$$

which though it seems a different equation (and that is what we mean by not being invariant *in form*) is just pretty much the same equation written in an odd coordinate system (and that is what we mean by being a global variational principle).

This situation is quite similar to what happens in covariant theories, in which one uses global variational principles, though not all fibered morphisms are Lagrangian symmetries, just the natural lift of spacetime diffeomorphisms are.

Still, being a solution is an intrinsic property, i.e. the transformation maps solutions of the original local equation into solutions of the new local equation. (Let us stress that if the transformation were a Lagrangian symmetry, it would send solutions of the original equation into solutions of the *same* local equation.)

We know all solutions of the harmonic oscillator $x(t) = A\cos(\omega t) + B\sin(\omega t)$ and we can map them using the transformation to $q(t) = (A\cos(\omega t) + B\sin(\omega t))^3$ which in fact are the solutions of equations (5.1.5).

In fact, one has

$$\dot{q}(t) = 3\omega \left(A\cos(\omega t) + B\sin(\omega t)\right)^2 \left(B\cos(\omega t) - A\sin(\omega t)\right)$$

$$\ddot{q}(t) = 6\omega^2 \left(A\cos(\omega t) + B\sin(\omega t)\right) \left(B\cos(\omega t) - A\sin(\omega t)\right)^2 - 3\omega^2 \left(A\cos(\omega t) + B\sin(\omega t)\right)^3 \tag{5.1.6}$$

$$\Rightarrow \quad -3\omega^2 q + \frac{2}{3q}\dot{q}^2 = -3\omega^2 (A\cos(\omega t) + B\sin(\omega t))^3 + 6\omega^2 \frac{(A\cos(\omega t)+B\sin(\omega t))^4}{(A\cos(\omega t)+B\sin(\omega t))^3} \left(B\cos(\omega t) - A\sin(\omega t)\right)^2 = \ddot{q}$$

That situation is not peculiar of mechanics and it is completely general, indeed. Moreover, it can be revisited from an active viewpoint, in which the issue has the different interest of finding out when two different theories are in fact *"equivalent"*.

Before doing that, we need a notation for field theories which is a little more precise than the one initially introduced in Chapter 1. Usually, in real models, one deals with more than one field, so that the configuration bundle is the fibered product of partial bundles over the spacetime, each modelling one field as sections.

Moreover, usually, different fields may enter the Lagrangian at different orders, so that each field has its own order in a specific dynamics. Accordingly, the Lagrangian still lives on a jet prolongation of the configuration bundle, though this bundle is itself the product of jet prolongations, each partial configuration bundle prolonged to a different order.

Sometimes, this precise notation is not needed and one can pull back the Lagrangian up, prolonging all fields to the maximal order, since a second order Lagrangian is also, trivially, a k-order Lagrangian for any $k \geq 2$. Sometimes, however, one wishes to keep track of the exact order of each field.

When keeping track of the order of the each single field separately, the *prolongation order* is then represented by an integer for each partial bundle, namely something like $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, instead of being given by a single integer. Consequently, jet prolongations of a configuration bundle are not totally ordered, but only partially ordered.

For a configuration bundle C which is the product of two partial configuration bundles, i.e. $C = A \times_M B$, the $(1,2)$-prolongation $(J^{(1,2)}C = J^1A \times_M J^2B)$ and the $(2,1)$-prolongation $(J^{(2,1)}C = J^2A \times_M J^1B)$ are not ordered, meaning that one has no canonical projection between them (neither $\pi : J^{(1,2)}C \to J^{(2,1)}C$ or $\pi : J^{(2,1)}C \to J^{(1,2)}C$), while of course one still has the projections

$$\pi^{(2,2)}_{(2,1)} : J^{(2,2)}C \to J^{(2,1)}C \qquad \text{and} \qquad \pi^{(2,2)}_{(1,2)} : J^{(2,2)}C \to J^{(1,2)}C \tag{5.1.7}$$

Accordingly, we say that a prolongation is *higher* than another (and we write, e.g., $J^{(2,2)}C \geq J^{(2,1)}C$) if one has a projection (e.g. $\pi^{(2,2)}_{(2,1)}$) connecting them. Two prolongations such that neither $J^{(1,2)}C \geq J^{(2,1)}$ or $J^{(2,1)}C \geq J^{(1,2)}C$ are *incommensurable* and one has no canonical projection connecting them. It is easy to show that $J^{\mathbf{k}_2}C \geq J^{\mathbf{k}_1}C$ iff there exists a list of positive (zero included) integers $\mathbf{a} = (a_1, \ldots, a_n)$ such that $\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{a}$.

Of course, given two incommensurable prolongations, one can always find a prolongation higher than both of them, which is why one can use the simplified formalism introduced in Chapter 1.

Let us then first give a precise definition of a field theory and then consider an invertible transformation between configuration bundles to investigate under which circumstances it maps solutions into solutions.

A *field theory* is a triple $\mathbb{T} = (\mathcal{C}, \mathbf{k}, \boldsymbol{L})$, where $\mathcal{C} = \mathcal{C}^1 \times_M \ldots \times_M \mathcal{C}^n$ is the *configuration bundle*, it is assumed to have global sections, *not* assumed to be a vector or an affine bundle, and it is considered to be the product of n bundles, each of which is called a *partial configuration bundle*. The second item $\mathbf{k} = (k_1, \ldots, k_n)$ is a list of positive (zero included) integers, and it defines the prolongation $J^{\mathbf{k}}\mathcal{C} = J^{k_1}\mathcal{C}^1 \times_M \ldots \times_M J^{k_n}\mathcal{C}^n$, which is the bundle on which the Lagrangian $\boldsymbol{L}$ is defined as a horizontal form.

The set $\mathrm{Sec}(\mathcal{C})$ of global sections of the configuration bundle is a functional space which is called the *configuration space* of the theory. *Field equations* are defined as a sub-manifold $\mathbb{E}(\boldsymbol{L}) \subset J^{2\mathbf{k}}\mathcal{C} = J^{2k_1}\mathcal{C}^1 \times_M \ldots \times_M J^{2k_n}\mathcal{C}^n$, even though, depending on the Lagrangian, this sub-manifold can possibly project to some lower prolongation.

This sub-manifold is defined by the Euler–Lagrange form $\mathbb{E}(L)$ defined by (1.3.33). More precisely, a point $j_x^{2\mathbf{k}}\sigma \in J^{2\mathbf{k}}\mathcal{C}$ belongs to the sub-manifold $\mathbb{E}(\boldsymbol{L}) \subset J^{2\mathbf{k}}\mathcal{C}$ iff for any vertical vector field $\hat{X} \in V(J^{2\mathbf{k}}\mathcal{C})$ one has

$$(j^{2\mathbf{k}}\sigma)^* \left(i_{\hat{X}} \mathbb{E}(L) \right) = 0 \tag{5.1.8}$$

Notice that it is enough to check that (5.1.8) holds true for all vector fields in the form $\hat{X} \equiv j^{2\mathbf{k}}X$ for some $X = X^i \partial_i$, namely a vertical vector field on $\mathcal{C}$. Also with this restriction one, in fact, has

$$(j^{2\mathbf{k}}\sigma)^* \left(i_{\hat{X}} \mathbb{E}(L) \right) = 0 \qquad \Longleftrightarrow \quad \forall X^i : \left(\mathbb{E}_i(L)X^i \right) \circ j^{2\mathbf{k}}\sigma = 0 \qquad \Longleftrightarrow \quad \mathbb{E}_i(L) \circ j^{2\mathbf{k}}\sigma = 0 \tag{5.1.9}$$

A section $\sigma \in \mathrm{Sec}(\mathcal{C})$ is a *solution* if, when prolonged to $J^{2\mathbf{k}}\mathcal{C}$ one has $j^{2\mathbf{k}}\sigma(x) \in \mathbb{E}(\boldsymbol{L})$ for any $x \in M$ or, equivalently, if $(j^{2\mathbf{k}}\sigma)^* \left(i_{\hat{X}} \mathbb{E}(L) \right) = 0$ for any vertical vector field $\hat{X}$. Then any field theory defines a subset $\mathrm{Sol}(\mathbb{T}) \subset \mathrm{Sec}(\mathcal{C})$ of sections of the configuration bundle $\mathcal{C}$ which are solutions of field equations.

Then let us consider a field theory $\mathbb{T}_2 = (\mathcal{C}_2, \mathbf{k}_2, \boldsymbol{L}_2)$, a bundle $\mathcal{C}_1$ on the same spacetime M as the bundle $\mathcal{C}_2$, and a (vertical over the spacetime M) bundle map $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ between the configuration bundles.

First of all, let us remark that the transformation $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ also defines a map between the configuration spaces, namely

$$\Phi_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2) : \sigma \mapsto \Phi_*\sigma := \Phi \circ \sigma \tag{5.1.10}$$

which associates to any configuration σ of the first theory a configuration $\Phi_*\sigma$ of the second theory.

The transformation Φ can be prolonged to $J\Phi : J^{\max(\mathbf{k})}\mathcal{C}_1 \to J^{\mathbf{k}}\mathcal{C}_2$; then, depending on the particular map Φ the prolonged map $J\Phi$ may project to lower orders.

Let us consider $\mathcal{C}_1 = A_1 \times_M B_1$ and $\mathcal{C}_2 = A_2 \times_M B_2$ and consider $\mathbf{k}_2 = (2,1)$ for the prolongation $J^{\mathbf{k}_2}\mathcal{C}_2 = J^2A_2 \times_M J^1B_2$. If the map Φ is generic one has coordinates (x, y_1, z_1) on $\mathcal{C}_1$ and (x, y_2, z_2) on $\mathcal{C}_2$, the map Φ is locally given by

$$\begin{cases} y_2 = y_2(x, y_1, z_1) \\ z_2 = z_2(x, y_1, z_1) \end{cases}$$

and if we want to prolong the map so that the prolongation $(J\phi)_{(2,1)}$ takes values on $J^{(2,1)}\mathcal{C}_2$, generically it should start from $J^{(2,2)}\mathcal{C}_1$, since one needs the second derivatives of both fields (y_1, z_1) to write the second derivatives of y_2. Accordingly, one gets ${}^{(2,2)}(J\Phi)_{(2,1)} : J^{(2,2)}\mathcal{C}_1 \to J^{(2,1)}\mathcal{C}_2$

However, for the special maps in the form

$$
\begin{cases}
y_2 = y_2(x, y_1) \\
z_2 = z_2(x, y_1, z_1)
\end{cases}
$$

one can prolong to $^{(2,1)}(J\Phi)_{(2,1)} : J^{(2,1)}\mathcal{C}_1 \to J^{(2,1)}\mathcal{C}_2$.

Hence the details of the prolongations depend on the situation under consideration, and we denote by $(J\Phi)_{\mathbf{k}_2}$ the lower prolongation possible to take values on the bundle $J^{\mathbf{k}_2}\mathcal{C}_2$. That is denoted by $(J\Phi)_{\mathbf{k}_2} : J\mathcal{C}_1 \to J^{\mathbf{k}_2}\mathcal{C}_2$ where the order $\mathbf{k}_1$ of $J\mathcal{C}_1 = J^{\mathbf{k}_1}\mathcal{C}_1$ depends on details.

Then one can further prolong to any $J^{\mathbf{k}'_1}\mathcal{C}_1$ higher than $J\mathcal{C}_1 = J^{\mathbf{k}_1}\mathcal{C}_1$, by setting $^{\mathbf{k}'_1}(J\Phi)_{\mathbf{k}_2} := (J\Phi)_{\mathbf{k}_2} \circ \pi^{\mathbf{k}'_1}_{\mathbf{k}_1} : J^{\mathbf{k}'_1}\mathcal{C}_1 \to J^{\mathbf{k}_2}\mathcal{C}_2$.

Then the situation with prolongations is described by the diagram here on the side.

Now, we can pull back along $(J\Phi)_{\mathbf{k}_2}$ on $J^{\mathbf{k}_1}\mathcal{C}_1$ the dynamics $\boldsymbol{L}_2$ defined on $J^{\mathbf{k}_2}\mathcal{C}_2$. We have a new Lagrangian

$$
\boldsymbol{L}_1 := \left((J\Phi)_{\mathbf{k}_2}\right)^* \boldsymbol{L}_2 \tag{5.1.11}
$$

(as well as we can prolong this up to any $J^{\mathbf{k}'_1}\mathcal{C}_1$ such that $\mathbf{k}'_1 \geq \mathbf{k}_1$, by setting $\boldsymbol{L}'_1 = \left(\pi^{\mathbf{k}'_1}_{\mathbf{k}_1}\right)^* \boldsymbol{L}_1$).

Then we have a new field theory $\mathbb{T}_1 = (\mathcal{C}_1, \mathbf{k}_1, \boldsymbol{L}_1)$. When two field theory are related in this way, we say that the map Φ *preserves the Lagrangian dynamics* and we want to investigate under which conditions a map which preserves the Lagrangian dynamics does map solutions into solutions, in one direction or the other.

In order to deal with field equations, we can prolong to $(J\Phi)_{2\mathbf{k}_2} : J^{2\mathbf{k}_1}\mathcal{C}_1 \to J^{2\mathbf{k}_2}\mathcal{C}_2$; then one also has $\mathbb{E}(L_1) = (J\Phi)^*_{2\mathbf{k}_2}\mathbb{E}(L_2)$.

Let us prove this for first order Lagrangians, the proof for second order Lagrangians being just longer. Let $\Phi : (x, y) \mapsto (x, y')$ be a map between configuration bundles which preserves the Lagrangian dynamics, its local expression being

$$
\begin{cases}
x' = x \\
y'^i = Y^i(x, y)
\end{cases} \tag{5.1.12}
$$

This can be prolonged to $J^1\Phi : J^1\mathcal{C}_1 \to J^1\mathcal{C}_2$ by setting

$$
y'^i_\mu = d_\mu Y^i(x, y) = J^i_\mu + J^i_A y^A_\mu \tag{5.1.13}
$$

as well as it establishes a correspondence between deformations, i.e. vertical vectors, $T\Phi : V(\mathcal{C}_1) \to V(\mathcal{C}_2)$, then prolonged to the first jet prolongation, which is defined by

$$
\delta y'^i = J^i_A \delta y^A \qquad \Rightarrow \delta y'^i_\mu = d_\mu J^i_A \delta y^A + J^i_A \delta y^A_\mu \tag{5.1.14}
$$

The Lagrangian $\boldsymbol{L}_2 = L_2(j^1 y') \, d\sigma$ can be pulled back to a first order Lagrangian

$$
\boldsymbol{L}_1 = \left(J^1\Phi\right)^* \boldsymbol{L}_2 \qquad \Rightarrow L_1(x^\mu, y^A, y^A_\mu) = L_2(x^\mu, Y^i(x, y), d_\mu Y^A(x, y)) \tag{5.1.15}
$$

The corresponding *naive* momenta read as

$$
p_A = p'_i J^i_A + p'^\mu_i d_\mu J^i_A \qquad p^\mu_A = p'^\mu_i J^i_A \tag{5.1.16}
$$

The diagram:

$$
\begin{array}{ccc}
J^{\mathbf{k}'_1}\mathcal{C}_1 & & \\
\downarrow{\scriptstyle \pi^{\mathbf{k}'_1}_{\mathbf{k}_1}} & \searrow{\scriptstyle ^{\mathbf{k}'_1}(J\Phi)_{\mathbf{k}_2}} & \\
J^{\mathbf{k}_1}\mathcal{C}_1 & \xrightarrow{(J\Phi)_{\mathbf{k}_2}} & J^{\mathbf{k}_2}\mathcal{C}_2 \\
\downarrow & & \downarrow \\
\mathcal{C}_1 & \xrightarrow{\Phi} & \mathcal{C}_2 \\
\downarrow & & \downarrow \\
M & = & M
\end{array}
$$
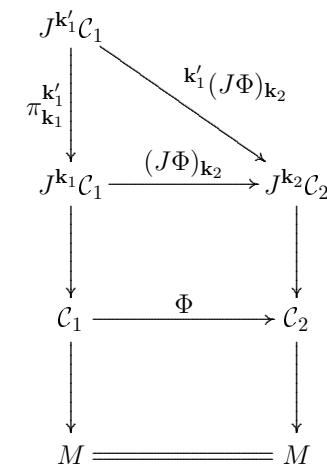
and we directly find the relation between the corresponding Euler–Lagrange operators, i.e.

$$\mathbb{E}_A = p_A - d_\mu p_A^\mu = p_i' J_A^i + p_i'^\mu d_\mu J_A^i - d_\mu \left(p_i'^\mu J_A^i \right) = \left(p_i' - d_\mu p_i'^\mu \right) J_A^i = \mathbb{E}_i' J_A^i \tag{5.1.17}$$

This already proves that $\mathbb{E}_1 = \mathbb{E}_A \omega^A \wedge d\sigma = (J^2\Phi)^* \mathbb{E}(L_2)$ as we claimed, since we know that $\omega'^i = J_A^i \omega^A$.

The same result can be obtained also by considering the variation of the Lagrangian

$$\delta L_1 = p_A \delta y^A + p_A^\mu d_\mu \delta y^A = \left(p_i' J_A^i + p_i'^\mu d_\mu J_A^i \right) \delta y^A + p_i'^\mu J_A^i d_\mu \delta y^A = p_i' J_A^i \delta y^A + p_i'^\mu \left(d_\mu J_A^i \delta y^A + J_A^i \delta y_\mu^A \right) = p_i' \delta y'^i + p_i'^\mu d_\mu \delta y'^i = \delta L_2 \circ J^1\Phi \tag{5.1.18}$$

Then, by the first variation formula, one has

$$\delta L_2 = (p_i' - d_\mu p_i'^\mu)\delta y'^i + d_\mu \left(p_i'^\mu \delta y^i \right) = \mathbb{E}_i' \delta y'^i + d_\mu \left(\mathbb{F}_i'^\mu \delta y^i \right) \tag{5.1.19}$$

and pulling it back along the transformation Φ one has

$$\delta L_1 = \delta L_2 \circ j^1\Phi = (p_i' - d_\mu p_i'^\mu)\delta y'^i + d_\mu \left(p_i'^\mu \delta y^i \right) = p_A \delta y^A + p_A^\mu d_\mu \delta y^A = (p_A - d_\mu p_A^\mu)\delta y^A + d_\mu \left(p_A^\mu \delta y^A \right) = \mathbb{E}_A \delta y'^A + d_\mu \left(\mathbb{F}_A^\mu \delta y^A \right) \tag{5.1.20}$$

where we set

$$\begin{cases} \mathbb{E}_A = p_A - d_\mu p_A^\mu = p_i' J_A^i + p_i'^\mu d_\mu J_A^i - d_\mu \left(p_i'^\mu J_A^i \right) = \left(p_i' - d_\mu p_i'^\mu \right) J_A^i = \mathbb{E}_i' J_A^i \\ \mathbb{F}_A^\mu = p_A^\mu = p_i'^\mu J_A^i = \mathbb{F}_i'^\mu J_A^i \end{cases} \tag{5.1.21}$$

Accordingly, we have for any vertical vector $X = \delta y^A \partial_A$ that

$$(j^1 X)\lrcorner d\boldsymbol{L}_1 = (j^1 X)\lrcorner d\left((j^1\Phi)^* \boldsymbol{L}_2 \right) = (j^2\Phi)^* \left(\mathbb{E}_i' \delta y'^i + d \left(\mathbb{F}_i' \delta y'^i \right) \right) = \mathbb{E}_i' J_A^i \delta y^A + d \left(\mathbb{F}_i' J_A^i \delta y^A \right) = \mathbb{E}_A \delta y^A + d \left(\mathbb{F}_A \delta y^A \right) = X\lrcorner \mathbb{E}_1 + d \left(X\lrcorner \mathbb{F}_1 \right) \tag{5.1.22}$$

where we set $\delta y'^i = \delta y^A J_A^i$.

Thus we have that, for the forms $\mathbb{E}_1$ and $\mathbb{F}_1$ the first variation formula holds true. Hence, essentially due to uniqueness and naturality of the decomposition introduced in Chapter 1 (see Subsection 1.3.3), then the form $\mathbb{E}_1 = \mathbb{E}_A \omega^A \wedge d\sigma = (J^2\Phi)^* \mathbb{E}(L_2)$ *is* the Euler–Lagrange part of the Lagrangian $\boldsymbol{L}_1$.

Let us remark that the matrix J_A^i is the local expression of the map $T\Phi : V(\mathcal{C}_1) \to V(\mathcal{C}_2) : \delta y^A \mapsto \delta y'^i = J_A^i \, \delta y^A$ between variations, as well as the local expression of the transpose map $T^*\Phi : V^*(\mathcal{C}_2) \to V^*(\mathcal{C}_1) : e_i' \mapsto e_A = e_i' J_A^i$. Of course, since the transpose map is obtained by duality form the map between variations, we have that a surjective map between variations, i.e. we have the exact sequence

$$V(\mathcal{C}_1) \xrightarrow{T\Phi} V(\mathcal{C}_2) \longrightarrow 0 \tag{5.1.23}$$

iff the corresponding transpose map is injective, since we have the exact sequence

$$0 \longrightarrow V(\mathcal{C}_2) \xrightarrow{T^*\Phi} V(\mathcal{C}_1) \tag{5.1.24}$$

If the reader does not feel confident enough to work with exact sequences, that can be also be proven directly. (Good luck.)

Let us now consider a solution $\sigma \in \mathrm{Sol}(\mathbb{T}_1)$ of the first theory and consider its image $\sigma_* := \Phi_*\sigma \in \mathrm{Sec}(\mathcal{C}_2)$ which, in general, is a configuration of the second theory. The configuration σ_* is a solution in the second theory since

$$0 = \mathbb{E}_A \circ j^{2\mathbf{k}_1}\sigma = \left(\mathbb{E}_i' J_A^i \right) \circ (J\Phi)_{2\mathbf{k}_2} \circ j^{2\mathbf{k}_1}\sigma = \left(\mathbb{E}_i' J_A^i \right) \circ j^{2\mathbf{k}_2}\sigma_* \tag{5.1.25}$$

In general, that *does not* imply that $E_i' \circ j^{2\mathbf{k}_2}\sigma_* = 0$, unless the transpose map $T^*\Phi$ is injective, i.e. unless the tangent map $T\Phi$ is surjective (which is equivalent, in turn, to require Φ to be of maximal rank). In fact, the tangent map $T\Phi$ is surjective, then the transpose map $T^*\Phi$ is injective, and hence $(T^*\Phi)(\mathbb{E}') = 0$ implies $\mathbb{E}' = 0$.

The same result can be obtained also in a different way, by considering Euler–Lagrange forms. In fact, σ_* is a solution in $\mathbb{T}_2$ iff, for all vertical vector field X on $\mathcal{C}_2$, one has $j^{2\mathbf{k}_2}(\Phi_*\sigma)^*(X \lrcorner \mathbb{E}(\boldsymbol{L}_2)) = 0$. However, for any vertical field Y on $\mathcal{C}_1$ and setting $X = T\Phi(Y)$ for the corresponding vertical field on $\mathcal{C}_2$, one has

$$j^{2\mathbf{k}_2}(\Phi_*\sigma)^*(X \lrcorner \mathbb{E}(\boldsymbol{L}_2)) = \left(j^{2\mathbf{k}_2}\sigma\right)^* \circ \left(j^{2\mathbf{k}_2}\Phi\right)^* \left(X^i \mathbb{E}_i'(\boldsymbol{L}_2)\right) = \left(j^{2\mathbf{k}_1}\sigma\right)^* \left(Y^A \mathbb{E}_A(\boldsymbol{L}_1)\right) = \left(j^{2\mathbf{k}_1}\sigma\right)^* (Y \lrcorner \mathbb{E}(\boldsymbol{L}_1)) = 0 \tag{5.1.26}$$

If the map $T\Phi : Y \mapsto X$ is surjective, and $\sigma \in \mathrm{Sol}(\mathbb{T}_1)$ then for all variation X on $\mathcal{C}_1$ one has $j^{2\mathbf{k}_2}(\Phi_*\sigma)^*(X \lrcorner \mathbb{E}(\boldsymbol{L}_2)) = 0$, i.e. $\Phi_*\sigma \in \mathrm{Sol}(\mathbb{T}_2)$ is a solution as well. Accordingly, we get that $\Phi_*(\mathrm{Sol}(\mathbb{T}_1)) \subset \mathrm{Sol}(\mathbb{T}_2)$.

Thus, if the dynamics of $\mathbb{T}_1$ is defined as a pull back of the dynamics of $\mathbb{T}_2$ along a map with a surjective tangent map, one can associate a solution of $\mathbb{T}_2$ to any solution of $\mathbb{T}_1$. However, the field theory $\mathbb{T}_2$ can have more solutions, i.e. the map $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ is not necessarily surjective. In that case, we say that $\mathbb{T}_2$ is a *dynamical extension* of the theory $\mathbb{T}_1$.

If the transformation Φ is invertible, then $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ is one-to-one as well and the two theories have solutions which are in a one-to-one correspondence. In that case we say that the map is a *dynamical equivalence*.

Dynamical equivalence is a more general and useful notion for field theories being equivalent, rather then simply having the *same* solutions, just because it applies also to field theories defined with different fields, on different configuration bundles.
Having the same solutions essentially works on theories defined on the same bundle, for example if the two Lagrangians differ by a pure divergence. Being dynamical equivalent is more general and just says that one can solve either theory, then building the solutions of the other.
Of course, dynamical equivalence is a *classical* equivalence which is not expected to extend to a quantum regime exactly, i.e. it expected to hold there only on average.

The correspondence between solutions of $\mathbb{T}_1$ into solutions of $\mathbb{T}_2$ happens only when assuming that the map $T\Phi : V(\mathcal{C}_1) \to V(\mathcal{C}_2)$ acting between variations is surjective or, equivalently, that its dual map $T^*\Phi : V^*(\mathcal{C}_2) \to V^*(\mathcal{C}_1)$ acting between the field equations is injective.

However, that is not the only correspondence that can be established between solutions. In fact, for example, if we consider a configuration $\sigma \in \mathrm{Sec}(\mathcal{C}_1)$ such that $\sigma_* = \Phi_*\sigma \in \mathrm{Sol}(\mathbb{T}_2)$ is a solution, then by considering again (5.1.25), this time we know that $\mathbb{E}_i' \circ j^{2\mathbf{k}_2}\sigma_* = 0$ and directly obtain that $\mathbb{E}_A \circ j^{2\mathbf{k}_1}\sigma = 0$, without further assumptions on the map Φ.

Though in general the preimages of solutions $\sigma_* \in \mathrm{Sol}(\mathbb{T}_2)$ are in fact solutions of $\mathbb{T}_1$, this is not enough by itself to establish that $\mathbb{T}_2$ is a dynamical extension of $\mathbb{T}_1$, nor vice versa. In fact, $\mathbb{T}_2$ can have solutions which have no preimages (unless one further assumes that $\Phi_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2)$ is a surjective map or, at least, that any solution of $\mathbb{T}_2$ has a preimage, i.e. that $\mathrm{Sol}(\mathbb{T}_2) \subset \mathrm{Im}(\Phi_*)$). On the other hand, $\mathbb{T}_1$ can have solutions such that their images are not solutions in $\mathbb{T}_2$.

Let us stress that we are not claiming that having a one-to-one map $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ is necessary to have dynamical equivalence. On the contrary, we shall show more general situations of dynamically equivalent theories. Before considering these more general cases, let us present an example of two dynamical equivalent theories.

Later in this Chapter, we shall try to extend to maps which involve derivatives (and show how awkward the situation becomes in that case) just to present a better setting based on Routh transformations, which provides us with a better setting in which the dynamics suggests both the map for doing the field transformation and the new dynamics, so that one has dynamical equivalence.

Dynamical equivalence for Brans–Dicke theories

We discussed Brans–Dicke theories in Subsection 1.10.13. Let us here consider vacuum Brans–Dicke theories which are defined on the configuration bundle $\mathcal{C} = \mathrm{Lor}(M) \times_M (M \times \mathbb{R})$, which has coordinates $(x^\mu, g^{\mu\nu}, \varphi)$, by a dynamics associated to the Lagrangian $\boldsymbol{L}_{BD}$ in (1.10.176), in the vacuum and no potential special case, i.e.

$$\boldsymbol{L}_{BD} = \tfrac{\sqrt{g}}{2\kappa} \left(\varphi^\alpha R - \tfrac{\omega}{\varphi^\beta} \nabla_\mu \varphi \nabla^\mu \varphi \right) d\sigma \tag{5.1.27}$$

Field equations for Brans–Dicke theories in this case and additionally setting $\beta := 2 - \alpha$, are

$$\begin{cases} \varphi^\alpha R_{\alpha\beta} - \tfrac{1}{2}\varphi^\alpha R g_{\alpha\beta} = \alpha\varphi^{\alpha-1}\left(\nabla_{\alpha\beta}\varphi - \Box\varphi g_{\alpha\beta}\right) + \left(\alpha(\alpha-1)+\omega\right)\varphi^{\alpha-2}\nabla_\alpha\varphi\nabla_\beta\varphi - \left(\alpha(\alpha-1)+\tfrac{\omega}{2}\right)\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} \\ \alpha\varphi^\alpha R = -2\omega\varphi^{\alpha-2}\varphi\Box\varphi - (\alpha-2)\omega\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi \end{cases} \tag{5.1.28}$$

Let us then consider a Weyl conformal map

$$\Phi : \mathrm{Lor}(M) \times_M (M \times \mathbb{R}) \to \mathrm{Lor}(M) \times_M (M \times \mathbb{R}) : (\tilde{g}, \varphi) \mapsto (g, \varphi) \qquad g_{\mu\nu} = \varphi^{-1}\tilde{g}_{\mu\nu} \tag{5.1.29}$$

which, in fact, is a one-to-one map.

We have already computed how the Levi Civita connection (see (2.7.148)), the Riemann tensor (see (2.7.150)), the Ricci tensor (see (2.7.151)) and the Ricci scalar (see (2.7.152)) transform under Weyl conformal transformations.

Then field equations can be recast as

$$\begin{cases} \tilde{R}_{\beta\alpha} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\alpha\beta} = \tfrac{m-2-2\alpha}{2\varphi}\Box\varphi g_{\alpha\beta} - \tfrac{m-2-2\alpha}{2\varphi}\nabla_{\alpha\beta}\varphi + \tfrac{3m-6+4\alpha(\alpha-1)+4\omega}{4\varphi^2}\nabla_\alpha\varphi\nabla_\beta\varphi + \tfrac{(m-7)(m-2)-8\alpha(\alpha-1)-4\omega}{8\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi g_{\alpha\beta} \\ \alpha\left(\varphi\tilde{R} + \tfrac{m-1}{\varphi}\Box\varphi + \tfrac{(m-1)(m-6)}{4\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi\right) = -\tfrac{2\omega}{\varphi}\Box\varphi - \tfrac{(\alpha-2)\omega}{\varphi^2}\nabla_\lambda\varphi\nabla^\lambda\varphi \end{cases} \tag{5.1.30}$$

so that we can further set $2\alpha := m - 2$ and $4\omega := -(m-2)(m-1)$ to get

$$\begin{cases} \tilde{R}_{\beta\alpha} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\alpha\beta} = \tfrac{(m-2)(m-1)+4\omega}{4\varphi^2}\nabla_\alpha\varphi\nabla_\beta\varphi - \tfrac{(m-2)(m-1)+4\omega}{8\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi g_{\alpha\beta} = 0 \\ \alpha\varphi\tilde{R} = -\tfrac{(m-1)(m-2)+4\omega}{2\varphi}\Box\varphi + \left(-\tfrac{(m-2)(m-1)(m-6)}{8\varphi^2} + \tfrac{(m-6)(m-2)(m-1)}{8\varphi^2}\right)\nabla_\rho\varphi\nabla^\rho\varphi = 0 \end{cases} \tag{5.1.31}$$

Since the trace of the first equation implies $\tilde{R} = 0$, the second equation is not independent and the conformal metric $\tilde{g}$ obeys vacuum Einstein equations.

In other words, we showed that there is a dynamical equivalence between vacuum Brans–Dicke theory (with $\alpha = \tfrac{m-2}{2}$, $\beta = 2 - \alpha = -\tfrac{m-4}{2}$, $\omega = -\tfrac{(m-2)(m-1)}{4}$, as well as no potential $U = 0$) for the fields (g, φ) and vacuum standard GR for the fields $(\tilde{g}, \varphi)$.

This can be also seen at the level of actions. The Brans–Dicke Lagrangian, in this case, becomes

$$\boldsymbol{L}_{BD} = \tfrac{\sqrt{g}}{2\kappa}\varphi^{\frac{m-2}{2}}\left(R + \tfrac{(m-2)(m-1)}{4\varphi^2}\nabla_\mu\varphi\nabla^\mu\varphi\right) d\sigma \tag{5.1.32}$$

which is a form on $J^{(2,1)}\mathcal{C} = J^2\text{Lor}(M) \times_M J^1(M \times \mathbb{R})$. The Weyl conformal transformation Φ can be lifted to $J\Phi : J^{(2,1)}\mathcal{C} \to J^{(2,1)}\mathcal{C}$ and the Lagrangian $\boldsymbol{L}_{BD}$ can be pulled back along the prolongation $J\Phi$ to define the form

$$
\begin{aligned}
(J\Phi)^* \boldsymbol{L}_{BD} =& \tfrac{\sqrt{\tilde{g}}}{2\kappa} \left(\tilde{R} + \tfrac{m-1}{\varphi^2}\Box\varphi + \left(\tfrac{(m-1)(m-6)}{4\varphi^3} + \tfrac{(m-2)(m-1)}{4\varphi^3} \right) \nabla_\mu\varphi\nabla^\mu\varphi \right) d\sigma = \\
=& \tfrac{\sqrt{\tilde{g}}}{2\kappa} \left(\tilde{R} + \left(-\tfrac{(m-1)(m-4)}{2\varphi^3} + \tfrac{(m-1)(m-4)}{2\varphi^3} \right) \nabla_\mu\varphi\nabla^\mu\varphi \right) d\sigma + \nabla_\rho \left(\tfrac{\sqrt{\tilde{g}}}{2\kappa} \tfrac{m-1}{\varphi^2}\nabla^\rho\varphi \right) d\sigma = \\
=& \tfrac{\sqrt{\tilde{g}}}{2\kappa} \tilde{R}\, d\sigma + d \left(\tfrac{\sqrt{\tilde{g}}}{2\kappa} \tfrac{m-1}{\varphi^2}\nabla^\rho\varphi\, d\sigma_\rho \right) = \boldsymbol{L}_H + d \left(\tfrac{\sqrt{\tilde{g}}}{2\kappa} \tfrac{m-1}{\varphi^2}\nabla^\rho\varphi\, d\sigma_\rho \right)
\end{aligned}
\tag{5.1.33}
$$

Notice that this is still a theory for the fields $(\tilde{g}, \varphi)$, i.e. the conformal factor φ scalar field turns out to be unconstrained by field equations. Accordingly, $(\tilde{g}, \varphi)$ is a solution of this theory, iff $\tilde{g}$ is a solution of vacuum Einstein equations, regardless the value of the scalar field φ.

Thus any solution of vacuum standard GR $(\tilde{g}, \varphi)$ is associated to a solution $(g = \varphi^{-1}\tilde{g}, \varphi)$ of the corresponding Brans–Dicke theory. In other words, we showed that the specific Brans–Dicke theory for (g, φ) under consideration is an extension of vacuum standard GR for $(\tilde{g}, \varphi)$ by means of a Weyl conformal transformation induced by the scalar field φ.

> Notice that we are not claiming that a solution $(\tilde{g}, \varphi)$ of vacuum Einstein equations *is a solution of that Brans–Dicke theory*, we are just saying that (g, φ) is.

On the other hand, we can consider the inverse Weyl conformal transformation

$$
\bar{\Phi} : \text{Lor}(M) \times_M (M \times \mathbb{R}) \to \text{Lor}(M) \times_M (M \times \mathbb{R}) : (g, \varphi) \mapsto (\tilde{g}, \varphi)
\tag{5.1.34}
$$

which can be easily prolonged to $J\bar{\Phi} : J^{(2,1)}\mathcal{C} \to J^{(2,1)}\mathcal{C}$. Then we can pull back the Hilbert Lagrangian to obtain the form

$$
\begin{aligned}
(J\bar{\Phi})^* \boldsymbol{L}_H =& \tfrac{\sqrt{\tilde{g}}}{2\kappa} \tilde{R}\, d\sigma = \tfrac{\sqrt{g}}{2\kappa} \varphi^{\frac{m-2}{2}} \varphi\tilde{R}\, d\sigma = \tfrac{\sqrt{g}}{2\kappa} \varphi^{\frac{m-2}{2}} \left(R - \tfrac{(m-1)}{\varphi}\Box\varphi - \tfrac{(m-1)(m-6)}{4\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi \right) d\sigma = \\
=& \tfrac{\sqrt{g}}{2\kappa} \left(\varphi^{\frac{m-2}{2}} R + \left(\tfrac{(m-1)(m-4)}{2} - \tfrac{(m-1)(m-6)}{4} \right) \varphi^{\frac{m-6}{2}}\nabla_\rho\varphi\nabla^\rho\varphi \right) d\sigma - \nabla_\mu \left(\tfrac{\sqrt{g}}{2\kappa}(m-1)\varphi^{\frac{m-4}{2}}\nabla^\mu\varphi \right) d\sigma = \\
=& \tfrac{\sqrt{g}}{2\kappa} \left(\varphi^{\frac{m-2}{2}} R + \tfrac{(m-1)(m-2)}{2}\varphi^{\frac{m-6}{2}}\nabla_\mu\varphi\nabla^\mu\varphi \right) d\sigma - d \left(\tfrac{\sqrt{g}}{2\kappa} \tfrac{(m-1)}{\varphi^2}\varphi^{\frac{m}{2}}\nabla^\mu\varphi\, d\sigma_\mu \right)
\end{aligned}
\tag{5.1.35}
$$

This Lagrangian, except for the total divergence, exactly coincides with the Lagrangian for the vacuum Brans–Dicke theory (with $\alpha = \frac{m-2}{2}$, $\beta = 2 - \alpha = -\frac{m-4}{2}$, $\omega = -\frac{(m-2)(m-1)}{4}$, and no potential $U = 0$). Thus also vacuum standard GR for $(\tilde{g}, \varphi)$ is an extension of that specific Brans–Dicke theory.

To summarise, the two theories are dynamically equivalent and there is a one-to-one correspondence between solutions of one theory and solutions of the other.

> Sorry for repeating it once again: one thing is claiming that the two theories are dynamical equivalent (i.e. that $(\tilde{g}, \varphi)$ is a solution of vacuum standard GR iff (g, φ) is a solution of (that specific) Brans–Dicke theory), another thing is claiming that they have the *same* solutions, i.e. that (g, φ) is a solution of vacuum standard GR, which is generically false.

Also, as we shall argue in the next Subsection, one thing is claiming that two theories are dynamically equivalent, a completely different thing is claiming that the two theories are completely *physically* equivalent.

In this specific case, we will use these theories as possible theories to describe gravity. In both cases, the metric field induces a connection and a geodesic equation. time-like geodesic trajectories in spacetime will be eventually identified with the physical motions of material points and massive objects. The point is here that we need to specify which metric is used to define physical motions, since two conformal metrics define different time-like geodesics, that being an independent assumption which has not much to do with field equations and the variational principle.

In the Brans–Dicke theory, one usually assumes the geodesics of g, while in the standard GR one usually assumes the geodesics of $\tilde{g}$. In view of these assumptions, even if the two theories *are* dynamically equivalent, the two theories are physically different, since they describe objects falling differently.

> This is particularly relevant, since Brans–Dicke theories are ruled out by solar system tests. In these tests, one exactly assumes that planets fall along g-geodesics and finds that they this does not match observations, e.g. about precessions of perihelia, while standard GR, in which planets orbit along $\tilde{g}$-geodesics, does fit observations.

Finding out dynamics in mechanics

> *My Lord—Tyekanik said—there could be trickery within trickery here.*
> (Frank Herbert, *Children of Dune*).

Before going to more general cases of dynamically equivalent theories, let us argue that dynamical equivalence is not necessarily to be understood as a physical complete equivalence between theories. Let us present and discuss a simple example in mechanics.

Imagine we are assigned the duty to determine a description of the dynamics of a specific mechanical system. For our safety (as we will eventually discover) we are provided with a remote access to a laboratory. In our laboratory, we can prepare a material point in an initial condition we set, release it, and track its position as time passes by. The material point is constrained to move along a line, so, as a result of an experiment, we have a list of events (t_i, x_i) in a two dimensional spacetime, where t_i is the reading of some clock we do not know in detail and x_i is the position along the line measured by a protocol we do not know in detail either.

Since we are pretty conservative students, we interpret the task to be solved as finding a Hamiltonian $H(t, x, p)$ which describes the time series that our laboratory produces. Of course, we are aware that one cannot produce in the laboratory all possible time series, as well as that in principle one could find time series that cannot be described as Hamiltonian flows, though it seems to us a good start.

> Before starting, notice the analogy with astrophysics and cosmology.
> There would be a lot of physical variables we could control, if we had direct access to the laboratory. For example, we could screen for temperature to check whether the rail is deformed in time, we could check if our test mass is rolling or sliding along the rail (which does effect conservation laws), we could test for friction directly. However, we are given a remote access to a laboratory, where we cannot control all the variables that we wish to check, and a task to be carried out with what we are given. Of course, we can assume that ordinary physics holds in the laboratory, do our job, and eventually check that there is no hint for an effect due to temperature, or friction. Or that, on the contrary, they appear as terms in the Hamiltonian we find.
> Of course, let us stress the obvious: that is not as checking the effects directly, as we wish, to the best of our capacities. That is why one could distinguish between *experiments* (when one has complete control on experimental setting) and *observations* (when we see the results and *infer* the setting; no doubt that astrophysics and cosmology are observational oriented disciplines.

Anyway, those are the conditions we are given and there is no reason to complain. We have a task to do.

Well, let us start by a first observation, just to have an idea of which direction to go for. We set up the system and get a time series. Say, we get data as shown below in Fig. 4.1.a, that we try to fit with some oscillatory function. We can repeat the experiment as many times as we want, estimate statistical errors on measurements, and consequently use some χ^2 argument to evaluate how good our fits are.

By looking at the first series (see Fig. 4.1.a), we could guess for some kind of harmonic oscillator. Say, that we find good statistical evidence for an oscillatory behaviour, well described by the motion

$$x = A\cos(\omega(t - t_0)) + \lambda \tag{5.1.36}$$

with best fit values $A = 10, \omega = 3, \lambda = 1, t_0 = 0$ for the parameters.

By the way, the data shown in the graph are fake and they have been produced exactly by adding a Gaussian noise to the exact law (5.1.36) which is shown in the Figure as a solid line. That is a reason not to bother the reader with the statistical analysis to show that the fit *is* in fact good.

Then we have a good experimental support to the law (5.1.36) as a description for the motion of our material point.

That makes our task pretty clear: we can repeat the experiment for many initial conditions and find, with some approximation, of course, the general solution of the system (for example, showing that ω and λ are always constant with respect to the change of initial conditions, while A and t_0 do parameterise initial conditions). Then we *just* have to look for a Hamiltonian which has these exact solutions, (which may be technically difficult to do but at least it is clear).

In this case, it is in fact easy to check that the Hamiltonian

$$\bar{H} = \tfrac{1}{2}p^2 + \tfrac{\omega^2}{2}x^2 - \lambda\omega^2 x \tag{5.1.37}$$

does have the correct exact solutions.

The reason why it is easy in this case is that we started from this Hamiltonian to find the motion (5.1.36) in the first place. That is not the only trick behind the curtain as one should expect and as the bar on the Hamiltonian suggests. Be patient, I shall be eventually honest about all the tricks I played.

That is a pretty neat result, considering the limited control we have on the data origin. We can figure out pretty much in detail how the system we are considering is structured: we have a harmonic force acting (associated to the parameter ω) and a constant force (associated to the parameter λ, parameterised in units of the elastic force). Probably, a good model to be proposed is a vertical spring and a weight force pulling down, x being the upward oriented vertical position and measured from the rest position of the spring (which is not an equilibrium position for the system because of the weight force action).

If we assume that the clock of the laboratory is calibrated in seconds, we fit the frequency ω, then we know the force $F_s = -\omega^2 x\,\vec{k}$ due to the spring and hence we know the weight force $F_w = -\lambda\omega^2\vec{k}$ acting. So, by repeating the experiment over and over, with increasing precision, we can also argue whether the laboratory we are using is somewhere on Earth or on some alien planet. Since the weight acceleration is not even exactly constant on Earth, being precise enough we can find *where* the laboratory is located on Earth.
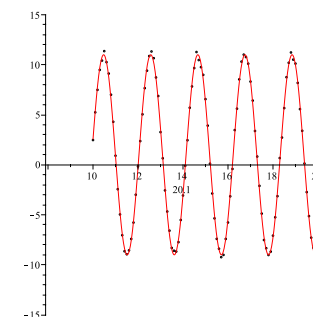
(a) The first time series produced
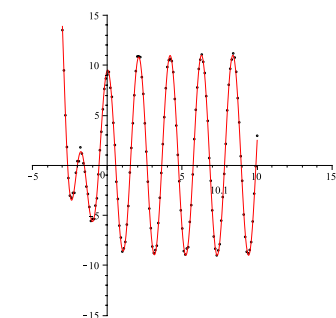
(b) A further time series

Fig 4.1: Examples of time series produced by the lab.

Of course, this last part is not much more than a speculation, because we know that the laboratory setting can fundamentally falsify data by mimiking a weight force by accelerating the whole laboratory at will. However, being *very* precise we would eventually check whether the force is really constant or it does change with x, as a real gravitational field should do.

It is amazing that we can get so much information about the laboratory without ever being there! However, we should ask how accurate this information is. So, now that we are quite confident on the laboratory setting, we get new data (see Figure 4.1.b) which do not match our previous ideas. The new observations do not match our model at early times while they agree with the model from $t \simeq 10$ on. In this situation, we would probably start over the fit analysis and discover that new and old data are well described by

$$x = \alpha(t)(A\cos(\omega(t - t_0)) + \lambda) \tag{5.1.38}$$

for a parameter function $\alpha(t) = 1 - e^{-(t+2)}$ and again with $A = 10, \omega = 3, \lambda = 1, t_0 = 0$, which is a solution of the Hamiltonian

$$H = \tfrac{\alpha^2}{2}p^2 + \tfrac{\omega^2}{2\alpha^2}x^2 - \lambda\tfrac{\omega^2}{\alpha}x + \tfrac{\dot\alpha}{\alpha}px \tag{5.1.39}$$

which, in fact, reduces to the Hamiltonian $\bar{H}$ above when $t \gg -2$ (which of course, strictly speaking, does not mean anything since the argument of the exponential is dimensional and it should be intended as to be $\nu t + 2 \gg 0$) with $\nu = 1s^{-1}$, instead).

The mechanic interpretation though becomes trickier. This Hamiltonian H is time dependent and the last term is some sort of generalised potential, so it may describe some transient force depending on velocity that eventually, at $t \gg -2$, dies out leading to the system described by $\bar{H}$. Of course, it does leave open a (we can imagine long and inconclusive) discussion about the details of how this behaviour is produced from a fundamental mechanical point of view. I also imagine that after a long discussion of the many models proposed, the transient force could be called the *dark force*, since it is responsible of a measurable acceleration of our material point, though it is quite unclear what produces it.

Of course, I am pushing a bit too far the analogy with cosmology, as it is manifest. This comes without the suggestion that cosmology of dark sources can be explained simply as we shall go on explaining our mechanical analog. And even if it did, that is something which should be proven by experiments, not something to be concluded on a theoretical level.

Thus we have a new, more accurate, though also more complex, view on the system we are studying, which is based on observations and fit them better than our first view (which, in fact, does not fit observations at all). Then, one can try to look for a canonical transformation to simplify the model. In particular, one can define new (time-dependent family of) canonical coordinates

$$\begin{cases} Q = \frac{x}{\alpha(t)} - \lambda & (\iff \quad x = \alpha(t)(Q + \lambda)) \\ P = \alpha(t)p \end{cases} \tag{5.1.40}$$

and check that the "new" Hamiltonian is simply

$$K = \tfrac{1}{2}P^2 + \tfrac{\omega^2}{2}Q^2 \tag{5.1.41}$$

Once again, I know it because I produced H by applying the inverse canonical transformation to K, in the first place.

Obviously, the canonical transformation establishes a very well founded mathematical equivalence between the two Hamiltonian systems described by H and K (which is, in fact, *one* Hamiltonian system with two different local representations).

So, at this point, what can we say about the mechanical system we are studying? Is it a harmonic oscillator (as suggested by the Hamiltonian K) or a rather complex system (as suggested by the Hamiltonian H)? Is the weight force appearing in $\bar{H}$ a real force? And, all in all, does it matter (since K and H are canonically equivalent)?

That's the first lesson we learn from this example: it does matter.

The Hamiltonian systems described by K and H *are* canonically equivalent. The canonical transformation maps solutions into solutions in a one-to-one way. Solving one system is equivalent to solving the other, observables are in one-to-one correspondence as well. And those are mathematical theorems. At the same time, we have to remember that *physically* we are talking about a time series of *positions* read in a laboratory. Now, whatever the protocol to measure the position is, the result is described by x, not by Q.

And accordingly we have two possibilities:

(i) either we argue that Q is the *real* position, the system *is* a harmonic oscillator, and, accordingly, we *have to explain* how the laboratory protocol happens to measure a *different* observable x, instead;

(ii) or we keep stick to x being the *real* position, then potentials described by H are real forces, and the system is not, physically speaking, a harmonic oscillator.

In the second case, we have to explain the origin of the dark forces since they are real forces. In the first case, we do not need to do that, *though we have to* show that the laboratory is measuring x, instead of what we should call the *real* position (if that had a precise meaning). In both cases, the two systems *are not* physically equivalent, precisely because they drive us to investigate different things.

At this point, suppose that we learn that the position in the laboratory is measured by an ultrasound device. We can hypothesise that the air pressure in the lab has been changing at early times (in relation to the exponential function appearing as a parameter, and that is why we thank to be remotely connected to the lab) so that the speed of sound changes and *real* distances are misrepresented so that x is measured for the actual position Q. If we manage to arrange the details, then the system is likely to be a simple harmonic oscillator and no dark force is there.

And here is the second lesson we learn from the example: while we can reasonably argue about it, to be honest, one may not ever be able to know without discovering a manometer in the laboratory or finding out elsewhere evidences of dark forces.

> Let me be clear once again: I am not arguing that dark sources in astrophysics and cosmology can be totally or in part described by aberrations introduced by poor observational protocols. What I am arguing, in fact, is that they may, and, just because they may, one needs to develop a framework in which these effects may be real, and eventually test for them.
>
> If they are found to be real, that is a good thing we learn about the physical world (and it is pretty amazing that we do, considering the scales we are probing). If they are not, we shall better establish that they are not, this time based on experiments rather than by an (uneducated) assumption.

Finally, we have a third point to make from this example. If we ideally had to scrutinise all possible dynamics to check which one agrees with observations, we have to consider any possible function $H(t, x, p)$ on $\mathbb{R} \times T^*\mathbb{R}$, solve the corresponding Hamilton equations and compare solutions with experimental curves obtained from the laboratory.

However, there is no need to check all functions in all possible coordinate systems other than (t, x, p). The Hamiltonian systems one obtains from $(\mathbb{R} \times T^*\mathbb{R}, H)$ by changing coordinates (as well as changing accordingly the local expression of the Hamiltonian) are the *same* Hamiltonian systems.

The two Hamiltonians H in coordinates (t, x, p) and K in coordinates (t, Q, P) are, in fact, two local expressions, in two different charts, of the same, single Hamiltonian.

> To see that one needs to work in a pre-symplectic framework, in which the Hamiltonian is not a function, but it defines a Poincaré–Cartan form, which is in fact preserved by flows of canonical transformations.

By considering all local expressions in all possible (canonical) coordinate systems on phase space, one would consider infinite times each Hamiltonian system. By fixing the coordinates, and in that fixed chart, considering all possible functions, one does already consider all possible Hamiltonian dynamics, with the additional advantage that one can fix coordinates giving, *by construction*, a physical meaning to x, which can be fixed to be directly related to what the laboratory measures (or to some other precise definition of *position*).

We shall below introduce extended theories of gravitation, and fix fields $(g, \tilde{\Gamma})$ chosen to represent precisely distances and free fall, respectively. We then let to dynamics the duty to specify their mutual relation, not excluding the case in which the dynamics could set $\tilde{\Gamma} = \{g\}$, as it is assumed in standard GR.

The representative g chosen from the conformal structure $[g]$ will be for us, by construction, the metric which realises physical distances in spacetime as geometric distances. In other words, the geometric length measured with g, called *g-length*, by definition agrees with the physical distance.

> Of course, there are a number of *a priori* different definitions of physical distances. One can use rulers (ok, not rulers which are not very well defined in relativity), parallax, radar echoes, GPS positioning, red-shift and Hubble laws, and a plethora of standard candles, each associated to its own luminosity distance. Each of these distances is *a priori* different from the others, they apply to different scales, and are usually (approximately) calibrated to (pairwise) agree on the overlap of their domains.
>
> The general tendency is to derive distances from time lapses, thus we decide to choose g to describe the time lapses measured by our best atomic clock, often ignoring that our best atomic clock is conventionally determined and it does depend on time as technologies improve.
>
> We do not need to remark here that there is no *a priori* reason to assume that this time has anything specifically to do with free falling (probably, except *a posteriori* at the scale of solar system), while we can at least expect it to have something to do with quantum theory, since atomic clocks are based on atomic systems.

2. Dynamical equivalence

Now that we have a couple of examples in mind, let us try to set up a general enough framework for dynamical extensions and equivalences. We shall give good and precise definitions of dynamical equivalence and dynamical extensions. Then we shall try to extend the example about maps between the configuration bundles to maps which depend on derivatives of fields, i.e. maps between prolongations. We shall see that, although we get some result in these more general cases, it becomes to be so awkward that it is clear that simply pulling back the dynamics is not what one needs in general.

> We shall go in a different direction in next Section. By now, we can say that a similar example is well known in mechanics. When one has a first integral and wants to write a Lagrangian for configurations at a certain value of the first integral, it is well known that in general simply replacing the first integral (which, in fact, usually depends on the velocities) into the Lagrangian gives inconsistent results.

Given two field theories, $\mathbb{T}_1$ and $\mathbb{T}_2$ defined on the same spacetime M, and an injective map $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ between solutions we say that $\hat{\Phi}_*$ is a *dynamical extension map* between the two theories $\mathbb{T}_1$ and $\mathbb{T}_2$. If a dynamical extension map is one-to-one then it is called a *dynamical equivalence map*.

> If one removes the request for an extension map to be injective, then one could always map all solutions of $\mathbb{T}_1$ into any one solution of $\mathbb{T}_2$ and, consequently, any theory would be a dynamical extension of any other theory. On the contrary, for a map to be a dynamical extension it needs to send any solution of $\mathbb{T}_1$ into a *different* solution of $\mathbb{T}_2$.

An extension map $\hat{\Phi}_*$ can come as a restriction of a map $\Phi_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2)$ to solutions of $\mathrm{Sol}(\mathbb{T}_1) \subset \mathrm{Sec}(\mathbb{T}_1)$; in general such a restriction takes values on $\mathrm{Sec}(\mathbb{T}_2)$, so, by requiring it to take values in $\mathrm{Sol}(\mathbb{T}_2) \subset \mathrm{Sec}(\mathbb{T}_2)$, we are in fact saying that it sends solutions of $\mathbb{T}_1$ into solutions of $\mathbb{T}_2$. Let us remark that in this case the restriction $\hat{\Phi}_*$ can be one-to-one, i.e. it can be a dynamical equivalence, even if the map Φ_* defined on configurations is not. In particular, it sometimes happens that one has two maps $\Phi_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2)$ and $\Psi_* : \mathrm{Sec}(\mathcal{C}_2) \to \mathrm{Sec}(\mathcal{C}_1)$ defined at the level of configurations and which are not invertible, which, however, when restricted to solutions, become two dynamical equivalences which are one the inverse of the other, and they are consequently dynamical equivalences between the two theories.

The extension map $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ can be produced by restricting to solutions a map between configurations $\Phi_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2)$ which, in turn, can be obtained as the push-forward of a bundle map $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ or, as we shall see more generally, by a map $\bar{\Phi} : J^{\mathbf{h}_1}\mathcal{C}_1 \to \mathcal{C}_2$, which, in fac,t also produces by push-forward a configuration map $\bar{\Phi}_* : \mathrm{Sec}(\mathcal{C}_1) \to \mathrm{Sec}(\mathcal{C}_2) : \sigma \mapsto \sigma_* = \bar{\Phi} \circ j^{\mathbf{h}_1}\sigma$. Although what matters for dynamical extensions and equivalences is the map $\hat{\Phi}_*$ between solutions, often, by an abuse of language, the configuration map Φ_* as well as the bundle map Φ are called *dynamical extensions* or *equivalences* as well.

We say that the theory $\mathbb{T}_2$ is a *dynamical extension* of $\mathbb{T}_1$ if there exists an extension map $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ between them. In that case one can cook a solution of $\mathbb{T}_2$ out of any solution of $\mathbb{T}_1$, while $\mathbb{T}_2$ can have other solutions which are not produced this way. We say that the two theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are *dynamically equivalent* if there exists a dynamical equivalence map $\hat{\Phi}_* : \mathrm{Sol}(\mathbb{T}_1) \to \mathrm{Sol}(\mathbb{T}_2)$ between their solutions, i.e. one can build a solution of one theory from any solution of the other.

> Of course, while it is relatively easy to show that two theories *are* dynamically equivalent (by exhibiting an equivalence map between them), it is, in general, quite hard to prove that two theories *are not*. For that, one should show that there is no map which is a dynamical equivalence, which is difficult because, in general, we know really not much about the solution sets.

We already showed that one can change the fundamental fields by a map $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$; if the map preserves the Lagrangian dynamics (and $T\Phi : V(\mathcal{C}_1) \to \mathcal{C}_2$ is surjective) then it is an extension map, if the map is one-to-one, it is an equivalence. We first attempt to generalise the result to a map which defines new fields in terms of the old ones and their derivatives.

Let us consider a field theory $\mathbb{T}_2 = (\mathcal{C}_2, \mathbf{k}_2, \boldsymbol{L}_2)$ defined on a configuration bundle $\mathcal{C}_2$ by a Lagrangian $\boldsymbol{L}_2$ of order $\mathbf{k}_2$ and a map $\bar{\Phi} : J^{\mathbf{h}_1}\mathcal{C}_1 \to \mathcal{C}_2$, which can be prolonged to a map $J\bar{\Phi} : J^{\mathbf{k}_2+\mathbf{h}_1}\mathcal{C}_1 \to J^{\mathbf{k}_2}\mathcal{C}_2$. Then we can define a dynamics on $\mathcal{C}_1$ by pulling back the form $\boldsymbol{L}_2$ on $J^{\mathbf{k}_2+\mathbf{h}_1}\mathcal{C}_1$ as

$$\boldsymbol{L}_1 = \left(J\bar{\Phi}\right)^* \boldsymbol{L}_2 \tag{5.2.1}$$

Hence we have a field theory $\mathbb{T}_1 = (\mathcal{C}_1, \mathbf{k}_2 + \mathbf{h}_1, \boldsymbol{L}_1)$ and we want to generalise the results that we already obtained for $\mathbf{h}_1 = 0$.

Let us denote by $(J\bar\Phi)_{2\mathbf{k}_2} : J^{2(\mathbf{k}_2+\mathbf{h}_1)}\mathcal{C}_1 \to J^{2\mathbf{k}_2}\mathcal{C}_2$ the prolongation to $J^{2\mathbf{k}_2}\mathcal{C}_2$ and let us denote by $E_1 = \mathbb{E}_A\,\delta y^A\,d\sigma$ and $E_2 = \mathbb{E}'_i\,\delta y'^i\,d\sigma$ the corresponding Euler–Lagrange operators as horizontal forms on $J^{2(\mathbf{k}_2+\mathbf{h}_1)}\mathcal{C}_1$ and $J^{2\mathbf{k}_2}\mathcal{C}_2$, respectively. Then we have that $(J\bar\Phi_{2\mathbf{k}_2})^* \mathbb{E}_2 - \mathbb{E}_1 = \left(\mathbb{E}'_i\delta y'^i - \mathbb{E}_A\delta y^A\right)\,d\sigma$ is an exact horizontal m-form on $J^{2(\mathbf{k}_2+\mathbf{h}_1)}\mathcal{C}_1$.

For simplicity, let us consider a first order Lagrangian $\boldsymbol{L}_2$ and a map $\bar\Phi : J^1\mathcal{C}_1 \to \mathcal{C}_2$. Then the Lagrangian $\boldsymbol{L}_1$ is generically on $J^2\mathcal{C}_1$.

Locally, the Lagrangian $\boldsymbol{L}_2$ is expressed as $L_2(y'^i, y'^i_\mu)$ and the map $\bar\Phi$ has a local expression as

$$y'^i = Y^i(x^\mu, y^A, y^A_\mu) \tag{5.2.2}$$

which is prolonged to

$$J\bar\Phi : J^2\mathcal{C}_1 \to J^1\mathcal{C}_2 \qquad y'^i_\mu = d_\mu Y^i(x^\mu, y^A, y^A_\mu) = J^i_\mu + J^i_A y^A_\mu + J^{i\alpha}_A y^A_{\alpha\mu} \tag{5.2.3}$$

The map $\bar\Phi$ also induces a map $T\bar\Phi : VJ^1\mathcal{C}_1 \to V\mathcal{C}_2$ between vertical vectors, i.e. field deformations, locally given by

$$\delta y'^i := J^i_A \delta y^A + J^{i\mu}_A d_\mu \delta y^A = T\bar\Phi(\delta y^A, d_\mu \delta y^A) \tag{5.2.4}$$

Then the Lagrangian $\boldsymbol{L}_1$ is given as

$$\boldsymbol{L}_1 = L_1(x^\mu, y^A, y^A_\mu, y^A_{\mu\nu})\,d\sigma = L_2(Y^i(x^\nu, y^B, y^B_\nu), d_\mu Y^i(x^\nu, y^B, y^B_\nu))\,d\sigma = (J\bar\Phi)^*\boldsymbol{L}_2 \tag{5.2.5}$$

and the corresponding *naive* momenta are

$$\begin{cases} p_A = p'_i J^i_A + p'^\mu_i d_\mu J^i_A \\ p^\mu_A = p'_i J^{i\mu}_A + p'^\alpha_i d_\alpha J^{i\mu}_A + p'^\mu_i J^i_A \\ p^{\mu\nu}_A = p'^{(\mu}_i J^{i\nu)}_A \end{cases} \tag{5.2.6}$$

where the momenta on the right hand side are evaluated along the prolongation of the map (5.2.2).

Then the Euler–Lagrange equations for $\boldsymbol{L}_1$ are

$$\begin{aligned}
\mathbb{E}_A &= p_A - d_\mu p^\mu_A + d_{\mu\nu} p^{\mu\nu}_A = p'_i J^i_A + p'^\mu_i d_\mu J^i_A - d_\mu \left(p'_i J^{i\mu}_A + p'^\alpha_i d_\alpha J^{i\mu}_A + p'^\mu_i J^i_A \right) + d_\mu d_\nu \left(p'^\mu_i J^{i\nu}_A \right) = \\
&= \left(p'_i - d_\mu p'^\mu_i \right) J^i_A - {\color{red}p'^\mu_i d_\mu J^i_A} + {\color{red}p'^\mu_i d_\mu J^i_A} - d_\mu \left(p'_i J^{i\mu}_A + p'^\alpha_i d_\alpha J^{i\mu}_A \right) + d_\mu \left(d_\nu p'^\nu_i J^{i\mu}_A + p'^\nu_i d_\nu J^{i\mu}_A \right) = \\
&= \left(p'_i - d_\mu p'^\mu_i \right) J^i_A + d_\mu \left(-p'_i J^{i\mu}_A - {\color{red}p'^\nu_i d_\nu J^{i\mu}_A} + d_\nu p'^\nu_i J^{i\mu}_A + {\color{red}p'^\nu_i d_\nu J^{i\mu}_A} \right) = \mathbb{E}'_i J^i_A - d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \right)
\end{aligned} \tag{5.2.7}$$

Then we have

$$\begin{aligned}
(J\bar\Phi_2)^* \mathbb{E}_2 - \mathbb{E}_1 &= \left(\mathbb{E}'_i \delta y'^i - \mathbb{E}_A \delta y^A \right)\,d\sigma = \left(E'_i \delta y'^i - \mathbb{E}'_i J^i_A \delta y^A + d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \right) \delta y^A \right)\,d\sigma = \left(E'_i \delta y'^i - \mathbb{E}'_i \left(J^i_A \delta y^A + J^{i\mu}_A d_\mu \delta y^A \right) + d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \delta y^A \right) \right)\,d\sigma = \\
&= \left({\color{red}E'_i \delta y'^i} - {\color{red}\mathbb{E}'_i \delta y'^i} + d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \delta y^A \right) \right)\,d\sigma = d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \delta y^A \right)\,d\sigma = d \left(\mathbb{E}'_i J^{i\mu}_A \delta y^A\,d\sigma_\mu \right)
\end{aligned} \tag{5.2.8}$$

which is precisely an exact horizontal m-form.

One can specify the map $\bar\Phi$ to be zero order, then $J^{i\mu}_A = 0$ and one obtains $(J\bar\Phi_2)^* \mathbb{E}_2 = \mathbb{E}_1$.

If here we assume that $\sigma \in \mathrm{Sol}(\mathbb{T}_1)$ and we consider $\sigma_* = \hat\Phi \circ j^{\mathbf{h}_1}\sigma$ we can evaluate directly the Euler–Lagrange equations by using (5.2.7)

$$0 = \mathbb{E}_A = \mathbb{E}'_i J^i_A - d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \right) \qquad \Rightarrow \mathbb{E}'_i J^i_A = d_\mu \left(\mathbb{E}'_i J^{i\mu}_A \right) \tag{5.2.9}$$

This time, unlike what happens for $\mathbf{h}_1 = 0$, i.e. when $J_A^{i\mu} = 0$, there is no clear condition to ask to $T\bar{\Phi}$ which does the job of implying that $\mathbb{E}'_i = 0$, even asking it to be surjective.

Let us now consider a configuration σ in $\mathbb{T}_1$ and its image $\sigma_* = \bar{\Phi} \circ j^{\mathbf{h}_1}\sigma$, which is a configuration in $\mathbb{T}_2$; we can directly show that if σ_* is a solution in $\text{Sol}(\mathbb{T}_2)$, then σ is a solution in $\text{Sol}(\mathbb{T}_1)$.

> We can see it directly by using the relation (5.2.7). If $\sigma_* \in \text{Sol}(\mathbb{T}_2)$, then $\mathbb{E}'_i \circ (j^2\sigma_*) = 0$, then $\mathbb{E}_A \circ (j^4\sigma) = 0$ so that $\sigma \in \text{Sol}(\mathbb{T}_1)$.
>
> Of course, σ can be a solution in $\mathbb{T}_1$ even though σ_* is not, as well as there can be solutions of $\mathbb{T}_2$ other than the ones which are images of configurations (hence solutions) in $\mathbb{T}_1$.
>
> We can also prove the same thing more intrinsically.
>
> If we consider a deformation δy^A in $\mathbb{T}_1$, by (5.2.4), it induces a deformation $\delta y'^i$ in $\mathbb{T}_2$. By (5.2.8), we have
>
> $$(j^2\sigma_*)^*\mathbb{E}_2 = (j^4\sigma)^* \circ (J\bar{\Phi})_2^* \, \mathbb{E}_2 = (j^4\sigma)^*\mathbb{E}_1 + d\left((j^4\sigma)^*\mathbb{E}'_i J_A^{i\mu} \delta y^A \, d\sigma_\mu \right) \qquad (5.2.10)$$
>
> Since σ_* is a solution, then $(j^2\sigma_*)^*\mathbb{E}_2 = 0$ for any variation on $\mathcal{C}_2$. The exact form on the right hand side also vanishes in view of boundary conditions, so for any variation on $\mathcal{C}_1$ one has $(j^4\sigma)^*\mathbb{E}_1 = 0$ and σ is a solution.
>
> Let us remark that, of course, in this proof, we did not really used any variation on $\mathcal{C}_2$. While we know that, since σ_* is a solution of $\mathbb{T}_2$, then $(j^2\sigma_*)^*\mathbb{E}_2 = 0$ vanishes along *any* variation on $\mathcal{C}_2$, eventually, we just used variations on $\mathcal{C}_2$ which are images of variation on $\mathcal{C}_1$ along the map $T\bar{\Phi}$. It is precisely because of this that one can then have extra solutions: by considering less variations, one introduces extra spurious solutions, since one has less conditions to obey.

Although this result is remarkable, it alone does not allow us to establish that $\mathbb{T}_1$ is an extension of $\mathbb{T}_2$, nor vice versa.

In fact, the correspondence $\bar{\Phi}_* : \text{Sol}(\mathbb{T}_2) \to \text{Sol}(\mathbb{T}_1)$ may not be defined on the whole $\text{Sol}(\mathbb{T}_2)$ (i.e. there may be solutions σ_2 in $\mathbb{T}_2$ which are not in the form $\sigma_2 = \bar{\Phi} \circ j^{\mathbf{h}_1}\sigma$ for some configuration $\sigma \in \text{Sec}(\mathcal{C}_1)$), so that $\mathbb{T}_1$ is not necessarily an extension of $\mathbb{T}_2$.

On the other hand, if $\mathbb{T}_2$ were an extension of $\mathbb{T}_1$ then any solution of $\mathbb{T}_1$, should correspond to a solution in $\mathbb{T}_2$, while we already noticed that $\mathbb{T}_1$ can have more solutions than the ones associated to a solution in $\mathbb{T}_2$.

Thus we need some extra property to enforce that one theory is an extension of the other. For example, if we assume that all solutions of $\mathbb{T}_2$ are in the image of $\bar{\Phi}_*$, i.e. any solution $\sigma_* \in \text{Sol}(\mathbb{T}_2)$ is in the form $\sigma_* = \bar{\Phi} \circ j^{\mathbf{h}_1}\sigma$ for some configuration (hence solution) $\sigma \in \text{Sol}(\mathbb{T}_1) \subset \text{Sec}(\mathcal{C}_1)$, then any solutions of $\mathbb{T}_2$ correspond to a section of $\mathbb{T}_1$, which is then, in fact, an extension of $\mathbb{T}_2$; (if the correspondence $\bar{\Phi} : \text{Sec}(\mathcal{C}_1) \to \text{Sec}(\mathcal{C}_2)$ is not injective just choose one preimage for any solution in $\mathbb{T}_2$).

Or, if a general deformation δy^A defines, through the map $T\bar{\Phi}$, a general deformation $\delta y'^i$, then any solution $\sigma \in \text{Sol}(\mathbb{T}_1)$ induces a solution $\sigma_* = \bar{\Phi} \circ j^{\mathbf{h}_1}\sigma \in \text{Sol}(T_2)$, i.e. now $\mathbb{T}_2$ is an extension of $\mathbb{T}_1$.

We cannot avoid to remark that the situation, which was already a bit unfit for $\mathbf{h}_1 = 0$, when prolongations are allowed becomes really awkward. We shall see in the next Section that, in fact, we need to go in a different direction.

Of course, if we somehow have two maps $\bar{\Phi} : J^{\mathbf{h}_1}\mathcal{C}_1 \to \mathcal{C}_2$ and $\bar{\Psi} : J^{\mathbf{h}_2}\mathcal{C}_2 \to \mathcal{C}_1$, such that $\boldsymbol{L}_1 = \left(J\bar{\Phi}\right)^* \boldsymbol{L}_2$ and $\boldsymbol{L}_2 = \left(J\bar{\Psi}\right)^* \boldsymbol{L}_1$, which are the inverse one of each other on solutions, then $\mathbb{T}_1$ and $\mathbb{T}_2$ are dynamically equivalent theories.

Moreover, whenever $\mathbb{T}_2$ is an extension of a theory $\mathbb{T}_1$ (by the injective correspondence $\bar{\Phi}_* : \text{Sol}(\mathbb{T}_1) \to \text{Sol}(\mathbb{T}_2)$) and we find a map $\bar{\Psi}_* : \text{Sol}(\mathbb{T}_2) \to \text{Sol}(\mathbb{T}_1)$ such that $\bar{\Phi}_* \circ \bar{\Psi}_* = \text{id}_{\text{Sol}(\mathbb{T}_2)}$, then $\mathbb{T}_1$ and $\mathbb{T}_2$ are dynamically equivalent.

Vacuum purely metric and standard Palatini GR

Let us consider a configuration bundle $\mathcal{C}_2 = \mathrm{Lor}(M) \times_M \mathrm{Con}(M)$, which is the fibered product of the bundle of Lorentzian metrics $\mathrm{Lor}(M)$ and that of torsionless connections $\mathrm{Con}(M)$ on the spacetime M. This bundle has fibered *"coordinates"* in the form $(x^\mu, g^{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu})$.

We can define a Lagrangian $\boldsymbol{L}_{Pal}$ of order $(0, 1)$ as

$$\boldsymbol{L}_{Pal} = \frac{\sqrt{g}}{2\kappa} g^{\mu\nu} \tilde{R}_{\mu\nu} \tag{5.2.11}$$

where $\tilde{R}_{\mu\nu}$ denotes the Ricci tensor of the connection $\tilde{\Gamma}$.

> This Lagrangian is called the *standard Palatini Lagrangian*, even if the original contribution of Palatini is quite marginal in that and, as a formulation of a field theory, it is due essentially to Einstein.
>
> More generally, we shall say that a field theory is in its *Palatini* (or *metric-affine*) formulation when it is a field theory for a metric g and an *independent* connection $\tilde{\Gamma}$. More precisely, when other matter fields ψ are considered, we shall reserve *Palatini field theories* for theories in which matter fields do not couple to the connection but couple only to the metric, i.e. for dynamics as
>
> $$L(g, \tilde{\Gamma}, \psi) = L(g, \tilde{\Gamma}) + L_m(g, \psi) \tag{5.2.12}$$
>
> while we reserve *metric-affine field theories* for theories in which matter fields can couple to the connection as well, i.e. for dynamics as
>
> $$L(g, \tilde{\Gamma}, \psi) = L(g, \tilde{\Gamma}) + L_m(g, \tilde{\Gamma}, \psi) \tag{5.2.13}$$
>
> In vacuum, *Palatini* and *metric-affine* are considered synonyms.
> When matter fields do not require a connection to define their dynamics, as it happens for Klein–Gordon field and Maxwell electromagnetism, then they are in their Palatini formulation straight away. For more general matter fields, as for example a Klein–Gordon-like vector field, one can define two covariant derivatives
>
> $$\nabla_\mu X^\alpha = d_\mu X^\alpha + \{g\}^\alpha_{\epsilon\mu} X^\epsilon \qquad \tilde{\nabla}_\mu X^\alpha = d_\mu X^\alpha + \tilde{\Gamma}^\alpha_{\epsilon\mu} X^\epsilon \tag{5.2.14}$$
>
> using either the Levi Civita connection of g or the independent connection $\tilde{\Gamma}$.
> Then, in this case, we have two different dynamics, namely the *Palatini* formulation, e.g.
>
> $$L_m = -\frac{\sqrt{g}}{2} \left(g^{\mu\nu} \nabla_\mu X^\alpha \nabla_\nu X^\beta + \mu^2 X^\alpha X^\beta \right) g_{\alpha\beta} \tag{5.2.15}$$
>
> or its *metric-affine* formulation
>
> $$\tilde{L}_m = -\frac{\sqrt{g}}{2} \left(g^{\mu\nu} \tilde{\nabla}_\mu X^\alpha \tilde{\nabla}_\nu X^\beta + \mu^2 X^\alpha X^\beta \right) g_{\alpha\beta} \tag{5.2.16}$$
>
> which, in principle, are two different field theories.

One can easily compute field equations for the Lagrangian $\boldsymbol{L}_{Pal}$ to be

$$\begin{cases} \tilde{R}_{(\mu\nu)} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = 0 \\ \tilde{\nabla}_\alpha \left(\sqrt{g} g^{\beta\mu} \right) = 0 \end{cases} \tag{5.2.17}$$

where we set $\mathcal{R} := g^{\alpha\beta} \tilde{R}_{\alpha\beta}$.

By variation of the Lagrangian, one has

$$\delta L_{Pal} = \frac{\sqrt{g}}{2\kappa}\left(\tilde{R}_{\mu\nu} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{\sqrt{g}}{2\kappa}g^{\beta\mu}\tilde{\nabla}_\alpha\delta\tilde{u}^\alpha_{\beta\mu} = \frac{\sqrt{g}}{2\kappa}\left(\tilde{R}_{\mu\nu} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu}\right)\delta g^{\mu\nu} - \tilde{\nabla}_\alpha\left(\frac{\sqrt{g}}{2\kappa}g^{\beta\mu}\right)\delta\tilde{u}^\alpha_{\beta\mu} + \tilde{\nabla}_\alpha\left(\frac{\sqrt{g}}{2\kappa}g^{\beta\mu}\delta\tilde{u}^\alpha_{\beta\mu}\right) \tag{5.2.18}$$

where we set $\tilde{u}^\alpha_{\beta\mu} := \tilde{\Gamma}^\alpha_{\beta\mu} - \delta^\alpha_{(\beta}\tilde{\Gamma}_{\mu)}$ so to express the variation of the Ricci tensor as $\delta\tilde{R}_{(\beta\mu)} = \tilde{\nabla}_\alpha\delta\tilde{u}^\alpha_{\beta\mu}$. The relation between $\tilde{u}^\alpha_{\beta\mu}$ and $\tilde{\Gamma}^\alpha_{\beta\mu}$ can be recast as

$$\tilde{u}^\alpha_{\beta\mu} = \tilde{\Gamma}^\alpha_{\beta\mu} - \tfrac{1}{2}\delta^\alpha_\beta\tilde{\Gamma}_\mu - \tfrac{1}{2}\delta^\alpha_\mu\tilde{\Gamma}_\beta = \left(\delta^\alpha_\rho\delta^\sigma_\beta\delta^\nu_\mu - \tfrac{1}{2}\delta^\alpha_\beta\delta^\sigma_\rho\delta^\nu_\mu - \tfrac{1}{2}\delta^\alpha_\mu\delta^\sigma_\rho\delta^\nu_\beta\right)\tilde{\Gamma}^\rho_{\sigma\nu} \tag{5.2.19}$$

so one can see it is a linear transformation, though the $\frac{m^2(m+1)}{2} \times \frac{m^2(m+1)}{2}$ matrix representing it is quite complicated.

To show that $\tilde{u}^\alpha_{\beta\mu}$ is just another set of *"coordinates"* on $\mathrm{Con}(M)$, let us show that this can be solved for $\tilde{\Gamma}$. Let us first consider the trace

$$\tilde{u}_\mu = \tilde{u}^\alpha_{\alpha\mu} = \left(1 - \tfrac{m}{2} - \tfrac{1}{2}\right)\delta^\nu_\mu\tilde{\Gamma}^\rho_{\rho\nu} = -\tfrac{m-1}{2}\tilde{\Gamma}_\mu \qquad \Longleftrightarrow \qquad \tilde{\Gamma}_\mu = -\tfrac{2}{m-1}\tilde{u}_\mu \tag{5.2.20}$$

and substitute it above in (5.2.19) to solve for $\tilde{\Gamma}^\alpha_{\beta\mu}$

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \tilde{u}^\alpha_{\beta\mu} - \tfrac{2}{m-1}\delta^\alpha_{(\beta}\tilde{u}_{\mu)} \tag{5.2.21}$$

which shows that the correspondence between $\tilde{\Gamma}^\alpha_{\beta\mu}$ and $\tilde{u}^\alpha_{\beta\mu}$ (as well as the quite complicated matrix) is invertible.

Then variations of $\tilde{u}^\alpha_{\beta\mu}$ are independent, as the variation of $\tilde{\Gamma}^\alpha_{\beta\mu}$ are, and the variation of the Lagrangian implies field equations in the form (5.2.17.)

Let us stress that the Ricci tensor $\tilde{R}_{\beta\mu}$ of a generic (non-metric) torsionless connection is not symmetric, and that the Hamilton principle only constrains its symmetric part, so that in the field equations one needs to introduce the symmetrisation $\tilde{R}_{(\mu\nu)}$.

The Levi Civita connection is associated with a bundle map $\Phi_{LC} : J^1\mathrm{Lor}(M) \to \mathrm{Lor}(M) \times_M \mathrm{Con}(M) : j^1 g^{\mu\nu} \mapsto (g^{\mu\nu}, \{g\}^\alpha_{\beta\mu})$ given by

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} = \tfrac{1}{2}g^{\alpha\lambda}\left(-d_\lambda g_{\beta\mu} + d_\beta g_{\mu\lambda} + d_\mu g_{\lambda\beta}\right) \tag{5.2.22}$$

Let us notice that Φ_{LC} is an injective map. It is not surjective since there are connections $\tilde{\Gamma}$ which are not metric (so, in particular, some $(g, \tilde{\Gamma})$ cannot be in the image of the map since $\tilde{\Gamma}$ cannot be the Levi Civita connection $\{g\}$).

This morphism is associated to the tangent map defined on field variations

$$T\Phi_{LC}(\delta g^{\mu\nu}, d_\mu\delta g^{\mu\nu}) = (\delta g^{\mu\nu}, \delta\tilde{\Gamma}^\alpha_{\beta\mu}) \qquad \delta\tilde{\Gamma}^\alpha_{\beta\mu} = \tfrac{1}{2}g^{\alpha\epsilon}\left(-\nabla_\epsilon\delta g_{\beta\mu} + \nabla_\beta\delta g_{\mu\epsilon} + \nabla_\mu\delta g_{\epsilon\beta}\right) = \tfrac{1}{2}\left(g^{\alpha\lambda}g_{\beta\rho}g_{\mu\sigma} - g_{\mu\rho}\delta^\alpha_\sigma\delta^\lambda_\beta - g_{\beta\sigma}\delta^\alpha_\rho\delta^\lambda_\mu\right)\nabla_\lambda\delta g^{\rho\sigma} \tag{5.2.23}$$

which, by the way, is the contribution due to Palatini. This correspondence between variations is invertible (hence surjective).

We can permute indices and sum

$$\begin{cases} 2g_{\alpha\gamma}\delta\tilde{\Gamma}^\gamma_{\beta\mu} = \left(g_{\beta\rho}g_{\mu\sigma}\delta^\lambda_\alpha - g_{\mu\rho}g_{\alpha\sigma}\delta^\lambda_\beta - \underline{g_{\beta\sigma}g_{\alpha\rho}\delta^\lambda_\mu}\right)\nabla_\lambda\delta g^{\rho\sigma} \\ 2g_{\beta\gamma}\delta\tilde{\Gamma}^\gamma_{\mu\alpha} = \left(g_{\mu\rho}g_{\alpha\sigma}\delta^\lambda_\beta - \underline{g_{\alpha\rho}g_{\beta\sigma}\delta^\lambda_\mu} - g_{\mu\sigma}g_{\beta\rho}\delta^\lambda_\alpha\right)\nabla_\lambda\delta g^{\rho\sigma} \end{cases} \Rightarrow g_{\alpha\gamma}\delta\tilde{\Gamma}^\gamma_{\beta\mu} + g_{\beta\gamma}\delta\tilde{\Gamma}^\gamma_{\mu\alpha} = -g_{\beta\sigma}g_{\alpha\rho}\delta^\lambda_\mu\nabla_\lambda\delta g^{\rho\sigma} \tag{5.2.24}$$

that can be eventually solved as

$$\nabla_\mu\delta g^{\rho\sigma} = -\left(g^{\beta\sigma}\delta\tilde{\Gamma}^\rho_{\beta\mu} + g^{\alpha\rho}\delta\tilde{\Gamma}^\sigma_{\mu\alpha}\right) \tag{5.2.25}$$

For later convenience, we can also transform the variation $\delta\tilde{\Gamma}^\alpha_{\beta\mu}$ of the connection into variation $\delta\tilde{u}^\alpha_{\beta\mu}$ as well, obtaining

$$\delta\tilde{u}^\alpha_{\beta\mu} = \delta\tilde{\Gamma}^\alpha_{\beta\mu} - \delta^\alpha_{(\beta}\delta\tilde{\Gamma}_{\mu)} \qquad \Longleftrightarrow \qquad \delta\tilde{\Gamma}^\alpha_{\beta\mu} = \delta\tilde{u}^\alpha_{\beta\mu} - \tfrac{2}{m-1}\delta^\alpha_{(\beta}\delta\tilde{u}_{\mu)} \tag{5.2.26}$$

and the correspondence between $\delta\tilde{\Gamma}^\alpha_{\beta\mu}$ and $\delta\tilde{u}^\alpha_{\beta\mu}$ is invertible as well. Then we have a one-to-one correspondence between $\delta\tilde{u}^\alpha_{\beta\mu}$ and $\nabla_\mu\delta g^{\rho\sigma}$.

We just have to take the composition of the correspondences above

$$\begin{aligned} \delta\tilde{u}^\alpha_{\beta\mu} =&\delta\tilde{\Gamma}^\alpha_{\beta\mu} - \delta^\alpha_{(\beta}\delta\tilde{\Gamma}_{\mu)} = \tfrac{1}{2}\left(g^{\alpha\lambda}g_{\beta\rho}g_{\mu\sigma} - \underline{g_{\mu\rho}\delta^\alpha_\sigma\delta^\lambda_\beta} - \underline{g_{\beta\rho}\delta^\alpha_\sigma\delta^\lambda_\mu} - \tfrac{1}{2}g_{\mu\rho}\delta^\lambda_\sigma\delta^\alpha_\beta - \tfrac{1}{2}g_{\beta\rho}\delta^\lambda_\sigma\delta^\alpha_\mu + \tfrac{1}{2}g_{\mu\rho}\delta^\alpha_\beta\delta^\lambda_\sigma + \tfrac{1}{2}g_{\beta\rho}\delta^\alpha_\mu\delta^\lambda_\sigma + \underline{\tfrac{1}{2}g_{\rho\sigma}\delta^\alpha_\beta\delta^\lambda_\mu} + \underline{\tfrac{1}{2}g_{\rho\sigma}\delta^\alpha_\mu\delta^\lambda_\beta} \right)\nabla_\lambda\delta g^{\rho\sigma} = \\ =& \tfrac{1}{2}\left(g^{\alpha\lambda}g_{\beta\rho}g_{\mu\sigma} - 2g_{\rho(\mu}\delta^\lambda_{\beta)}\delta^\alpha_\sigma + g_{\rho\sigma}\delta^\alpha_{(\beta}\delta^\lambda_{\mu)} \right)\nabla_\lambda\delta g^{\rho\sigma} \end{aligned} \tag{5.2.27}$$

The inverse correspondence is

$$\nabla_\nu\delta g^{\rho\sigma} = -\left(g^{\beta\sigma}\delta\tilde{\Gamma}^\rho_{\beta\nu} + g^{\alpha\rho}\delta\tilde{\Gamma}^\sigma_{\nu\alpha} \right) = \left(-\tfrac{1}{2}g^{\beta(\sigma}\delta^{\rho)}_\alpha\delta^\mu_\nu + \tfrac{2}{m-1}g^{\rho\sigma}\delta^\mu_\nu\delta^\beta_\alpha + \tfrac{2}{m-1}g^{\mu(\sigma}\delta^{\rho)}_\nu\delta^\beta_\alpha \right)\delta\tilde{u}^\alpha_{\beta\mu} \tag{5.2.28}$$

If we pull back the standard Palatini Lagrangian along the prolongation $J\Phi_{LC} : J^2\mathrm{Lor}(M) \to \mathrm{Lor}(M) \times_M J^1\mathrm{Con}(M)$ we obtain the Hilbert–Einstein Lagrangian

$$\boldsymbol{L}_H = (J\Phi)^*\boldsymbol{L}_{Pal} = \tfrac{\sqrt{g}}{2\kappa}R\,d\sigma \tag{5.2.29}$$

Then we know that any solution of the Palatini theory in the image of the map Φ_{LC} (i.e. in the form $(g, \{g\})$) comes from a solution of the metric theory.

Of course, in this case this is a pretty trivial remark. In fact, a solution $(g, \tilde{\Gamma} = \{g\})$ of the Palatini theory (in the image) obeys field equations (5.2.17), in particular, g is a solution of the first equation which is Einstein equation, since $\tilde{\Gamma} = \{g\}$.

Here it is relevant that the result follows from a construction at the level of action, that we obtain without even computing field equations.

We can also prove that there is no solution to the Palatini theory, out of the image of the map Φ_{LC}. In particular, a connection $\tilde{\Gamma}$ is a solution if it obeys the second field equation and one can show that, then, it is necessarily $\tilde{\Gamma} = \{g\}$.

Any connection $\tilde{\Gamma}^\alpha_{\beta\mu}$ can be expressed as $\tilde{\Gamma}^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + K^\alpha_{\beta\mu}$ (with a tensor $K^\alpha_{\beta\mu}$ which is symmetric in the lower indices $(\beta\mu)$).

If $\tilde{\Gamma}$ satisfies the second field equation, then one has

$$\tilde{\nabla}_\alpha\left(\sqrt{g}g^{\beta\mu}\right) = \boxed{\nabla_\alpha\left(\sqrt{g}g^{\beta\mu}\right)} + K^\beta_{\epsilon\alpha}\sqrt{g}g^{\epsilon\mu} + K^\mu_{\epsilon\alpha}\sqrt{g}g^{\beta\epsilon} - K^\epsilon_{\epsilon\alpha}\sqrt{g}g^{\beta\mu} = 0 \tag{5.2.30}$$

which is an algebraic, linear equation for the tensor K, which can be recast as

$$K_{\rho\sigma\alpha} + K_{\sigma\rho\alpha} - g_{\rho\sigma}K_\alpha = 0 \qquad \Rightarrow (2-m)K_\alpha = 0 \qquad \Rightarrow K_\alpha = 0 \qquad \Rightarrow K_{\rho\sigma\alpha} = -K_{\sigma\rho\alpha} \tag{5.2.31}$$

where we set $K_{\rho\epsilon\alpha} := g_{\beta\rho}K^\beta_{\epsilon\alpha}$ and $K_\alpha = K^\epsilon_{\epsilon\alpha}$. Then the tensor $K_{\alpha\rho\sigma}$ is symmetric in the indices $(\rho\sigma)$ and antisymmetric in the indices $[\alpha\rho]$ and we have

$$\alpha\rho\sigma = -\rho\alpha\sigma = -\rho\sigma\alpha = \sigma\rho\alpha = \sigma\alpha\rho = -\alpha\sigma\rho = -\alpha\rho\sigma \qquad \Rightarrow K_{\alpha\rho\sigma} = 0 \qquad \Rightarrow K^\alpha_{\rho\sigma} = 0 \qquad \Rightarrow \tilde{\Gamma}^\alpha_{\rho\sigma} = \{g\}^\alpha_{\rho\sigma} \tag{5.2.32}$$

where, of course, $\alpha\rho\sigma$ stands for $K_{\alpha\rho\sigma}$.

Then the metric theory is an extension of Palatini theory. The map which associates a solution in the standard metric theory to a solution of the Palatini theory is simply $\pi_* : \mathrm{Sol}(\mathbb{T}_2) \to \mathrm{Sol}(\mathbb{T}_1) : (g^{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu}) \mapsto g^{\mu\nu}$.

Once restricted to solutions, which are in fact in the form $(g^{\mu\nu}, \{g\}^\alpha_{\beta\mu})$, this map has, in fact, the property $\Phi_* \circ \pi_* = \mathrm{id}_{\mathcal{C}_2}$, so π_* is injective.

Let now consider a solution $\sigma \in \mathrm{Sol}(\mathbb{T}_1)$, namely a metric $g_{\mu\nu}$ obeying Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0 \tag{5.2.33}$$

Then $(g, \{g\})$ is a solution of Palatini theory and, of course, $\pi_*(g, \{g\}) = g$. Accordingly, the map π_* is surjective, then it is a one-to-one correspondence and thus standard metric GR and standard Palatini GR are dynamically equivalent.

Purely metric and standard Palatini GR with matter

If we allow matter *à la Palatini*, since we do not allow matter-connection couplings, the Lagrangian is

$$\boldsymbol{L}_{Pal} = \tfrac{\sqrt{g}}{2\kappa} \mathcal{R} + L_m(g, \psi) \tag{5.2.34}$$

where ψ denotes matter fields.

The field equations are

$$\begin{cases} \tilde{R}_{(\mu\nu)} - \tfrac{1}{2} \mathcal{R} g_{\mu\nu} = \kappa T_{\mu\nu} \\ \tilde{\nabla}_\alpha \left(\sqrt{g} g^{\beta\mu} \right) = 0 \\ \mathbb{E} = 0 \end{cases} \tag{5.2.35}$$

> We still have $\tilde{\Gamma} = \{g\}$ from the second field equation of the standard Palatini theory and, known that, the first equation corresponds to the equation of the metric theory. Then, as in vaccum, the Levi Civita map Φ_{LC} restricts to solutions and it is one-to-one.

If we consider the purely metric version

$$\boldsymbol{L} = \tfrac{\sqrt{g}}{2\kappa} R + L_m(g, \psi) \tag{5.2.36}$$

then it has field equations

$$\begin{cases} R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \\ \mathbb{E} = 0 \end{cases} \tag{5.2.37}$$

The Levi Civita map is a dynamical equivalence as in the vacuum case. Thus the dynamical equivalence between standard Palatini and purely metric GR is established in full generality.

> We can check that directly at the level of field equations. A metric configuration (g, ψ) is a solution of field equations (5.2.37) iff $(g, \{g\}, \psi)$ is a solution in Palatini formalism (5.2.35).
> One direction is trivial, the other relies on the fact that any solution in Palatini formulation has a connection $\tilde{\Gamma} = \{g\}$.

This same result will be obtained again as a special case of a more general result obtained in the next Section.

Vacuum metric and Palatini $f(\mathcal{R})$-theories

 The dynamical equivalence between a purely metric theory and its Palatini formulation given by the Levi Civita map, true for standard GR, is not true in general.

 To see that, let us consider the vacuum Palatini formulation of an $f(\mathcal{R})$-theory, i.e. the Lagrangian

$$\boldsymbol{L}' = \frac{\sqrt{g}}{2\kappa} f(\mathcal{R})\, d\sigma \qquad (\mathcal{R} := g^{\mu\nu}\tilde{R}_{\mu\nu}) \tag{5.2.38}$$

from which we get field equations

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = 0 \\ \tilde{\nabla}_\alpha \left(\sqrt{g}g^{\beta\mu}f'(\mathcal{R}) \right) = 0 \end{cases} \tag{5.2.39}$$

 By variation of the Lagrangian we get

$$\delta L' = \sqrt{g}\left(f'(\mathcal{R})\tilde{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} \right)\delta g^{\mu\nu} + \sqrt{g}f'(\mathcal{R})g^{\beta\mu}\tilde{\nabla}_\alpha \delta\tilde{u}^\alpha_{\beta\mu} = \sqrt{g}\left(f'(\mathcal{R})\tilde{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} \right)\delta g^{\mu\nu} - \tilde{\nabla}_\alpha \left(\sqrt{g}f'(\mathcal{R})g^{\beta\mu} \right)\delta\tilde{u}^\alpha_{\beta\mu} + \tilde{\nabla}_\alpha \left(\sqrt{g}f'(\mathcal{R})g^{\beta\mu}\delta\tilde{u}^\alpha_{\beta\mu} \right) \tag{5.2.40}$$

 Then being the variations $\delta g^{\mu\nu}$ and $\delta\tilde{u}^\alpha_{\beta\mu}$ independent, the field equations follow.

 For the purely metric $f(R)$-theories, you can refer to Subsection 1.10.12. In that case, vacuum field equations are

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \nabla_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R) \tag{5.2.41}$$

 Also in this case, we can consider the Levi Civita map $\Phi_{LC} : J^1 Lor(M) \to Lor(M) \times_M Con(M) : j^1 g^{\mu\nu} \mapsto (g^{\mu\nu}, \{g\}^\alpha_{\beta\mu})$, also in this case, it is an injective map, and also in this case one has the metric Lagrangian defined as

$$\boldsymbol{L} = (J\Phi)^* \boldsymbol{L}' = \frac{\sqrt{g}}{2\kappa} f(R)\, d\sigma \tag{5.2.42}$$

on $J^2 Lor(M)$.

 If $\sigma_* \in Sol(\mathbb{T}_2)$ is a solution of the Palatini $f(\mathcal{R})$-theory, then, in particular, by the trace of the first field equation one has

$$F(\mathcal{R}) := f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) = 0 \tag{5.2.43}$$

If the function $F(\mathcal{R})$ has isolated zeroes (as, for example, if it is analytic), then one has a discrete set $S = \{\mathcal{R}_0, \mathcal{R}_1, \ldots\} \subset \mathbb{R}$ of possible values which are dictated by the choice of the function $f(\mathcal{R})$. Then we can solve this equation for $\mathcal{R} \in S$ which is hence constant on M (any continuous function in a discrete set is locally continuous).

 If the scalar curvature $\mathcal{R}$ is constant, then $f'(\mathcal{R})$ is constant and the the second field equation is again

$$\tilde{\nabla}_\alpha \left(\sqrt{g}g^{\beta\mu} \right) = 0 \qquad \Longleftrightarrow \qquad \tilde{\Gamma} = \{g\} \tag{5.2.44}$$

as in the standard case.

Then, again in this case, the Palatini theories have no solution out of the image of the Levi Civita map and the metric $f(R)$-theory is an extension of the Palatini $f(\mathcal{R})$-theory.

In fact, a solution of Palatini $f(\mathcal{R})$-theory is in the form $(g_{\mu\nu}, \{g\}^\alpha_{\beta\mu})$ with a metric which obeys the equation

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} = 0 \tag{5.2.45}$$

so that the metric $g_{\mu\nu}$ is also a solution of the metric $f(R)$-theory since the scalar curvature is constant, then $\nabla_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R) = 0$ as well, and, accordingly, it satisfies metric field equation (5.2.41).

Of course, the metric $f(R)$-theory, generically can have solutions in which neither $f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu}$ or $\nabla_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R)$ are zero (though they are both equal). Then metric $f(R)$-theory has more solutions than the ones which come from the corresponding Palatini $f(\mathcal{R})$-theory and dynamical equivalence, which holds for the standard case, is lost in general.

Hidden relations

While the Levi Civita map is a dynamical equivalence between standard Palatini and standard purely metric GR, as well as a dynamical extension of vacuum Palatini $f(\mathcal{R})$-theory into vacuum purely metric $f(R)$-theory, it fails to extend to more general cases.

If we consider standard metric-affine GR we have the Lagrangian

$$\boldsymbol{L}_{MG} = \tfrac{\sqrt{g}}{2\kappa}\mathcal{R} + L_m(g, \psi, \tilde{\nabla}\psi) \tag{5.2.46}$$

The variation is

$$\begin{aligned}\delta L_{MG} &= \tfrac{\sqrt{g}}{2\kappa}\left(\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} + \tfrac{\sqrt{g}}{2\kappa}g^{\mu\nu}\delta\tilde{R}_{\mu\nu} + \tfrac{\sqrt{g}}{2}\tilde{T}^{\mu\nu}_\alpha\delta\tilde{\Gamma}^\alpha_{\mu\nu} + p_i\delta\psi^i + p_i^\mu\tilde{\nabla}_\mu\delta\psi^i = \\ &= \tfrac{\sqrt{g}}{2\kappa}\left(\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} - \left(\tilde{\nabla}_\alpha\left(\tfrac{\sqrt{g}}{2\kappa}g^{\mu\nu}\right) - \tfrac{\sqrt{g}}{2}T^{\mu\nu}_\alpha\right)\delta\tilde{u}^\alpha_{\mu\nu} + \left(p_i - \tilde{\nabla}_\mu p_i^\mu\right)\delta\psi^i + \tilde{\nabla}_\alpha\left(\tfrac{\sqrt{g}}{2\kappa}g^{\mu\nu}\delta\tilde{u}^\alpha_{\mu\nu} + p_i^\alpha\delta\psi^i\right)\end{aligned} \tag{5.2.47}$$

where we set

$$-\frac{\sqrt{g}}{2}T_{\mu\nu} = \frac{\partial L_m}{\partial g^{\mu\nu}} \qquad\qquad \frac{\sqrt{g}}{2}\tilde{T}^{\mu\nu}_\alpha = \frac{\partial L_m}{\partial\tilde{\Gamma}^\alpha_{\mu\nu}} \tag{5.2.48}$$

while $\tfrac{\sqrt{g}}{2}T^{\beta\mu}_\alpha$ is defined as the derivative of the matter Lagrangian with respect to $\tilde{u}^\alpha_{\beta\mu}$, i.e., by using (5.2.21), one has

$$T^{\mu\nu}_\alpha = \tilde{T}^{\rho\sigma}_\beta\frac{\partial\tilde{\Gamma}^\beta_{\rho\sigma}}{\partial\tilde{u}^\alpha_{\mu\nu}} = \tilde{T}^{\mu\nu}_\alpha - \tfrac{2}{m-1}\tilde{T}^{(\mu}\delta^{\nu)}_\alpha \tag{5.2.49}$$

Then field equations are

$$\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa T_{\mu\nu} \qquad \tilde{\nabla}_\alpha\left(\sqrt{g}g^{\beta\mu}\right) = \kappa\sqrt{g}\,T^{\beta\mu}_\alpha \qquad \tilde{\nabla}_\mu p_i^\mu = p_i \tag{5.2.50}$$

Based on the equivalence between purely metric and Palatini formulation of the standard GR, it is usually considered relevant to *freeze* the connection $\tilde{\Gamma} = \{g\}$ in the metric-affine Lagrangian to obtain a purely metric theory, in this case based on the Lagrangian

$$\boldsymbol{L} = \frac{\sqrt{g}}{2\kappa} R + L_m(g, \psi, \nabla\psi) \tag{5.2.51}$$

While there is no general variational expectation for the two theories to be dynamically equivalent, this purely metric theory is called *the* metric formulation of *its* metric-affine theory.

> Sometimes the discussion goes also the other way. One starts from the purely metric theory (5.2.51) and somehow *promotes* the Levi Civita in it to be and independent connection $\tilde{\Gamma}$, thence obtaining a metric-affine theory based on the Lagrangian (5.2.46). Also in this case, the metric-affine theory is called *the* metric-affine formulation of the original purely metric theory. Again with no theoretical support, this time with the additional issue that, in general, the two operations of *promoting* and *freezing* are not even the inverse of one another!
>
> In fact, if the metric-affine theory contains terms factorising $\tilde{\Gamma} - \{g\}$, then when one freezes the connection such terms vanish and they are cancelled form the purely metric formulation. When one then promotes the Levi Civita back to be independent and one defines *the* metric-affine formulation, a *different* metric-affine theory is obtained where the terms factorising $\tilde{\Gamma} - \{g\}$ have been removed.
>
> Thus, not only there is no reason for those corresponding theories to be equivalent, but that correspondence between purely metric and metric-affine theories is not even well-defined.
>
> For some reason, however, it is generally believed that sometimes dynamical equivalence between purely metric and metric-affine fails to hold true. We shall show that this is generally false: standard purely metric and metric-affine theories *are* dynamically equivalent, though along a map different with respect to the Levi Civita map. However, here let us point out that because of this supposed lack of equivalence some prefer the purely metric formulations, some prefer metric-affine formulations, some restrict to the cases in which equivalence holds considering it as a constraint on possible coupling between gravity and matter.
>
> In my opinion, all this discussion is ill-posed, based on unmotivated expectations and false claims, and usually considered entirely reasonable, though in the wrong direction. One should instead discuss the issue, assuming all formulations to be *a priori* inequivalent, investigating when dynamical equivalence holds true. Then one should choose among dynamically inequivalent theories, mainly based on physical motivations, and discuss how and to what extend dynamical equivalences among equivalent theories extend to full physical equivalence. Of course, it takes some time to do it and we shall spend most of the rest of this Chapter to provide results to perform this project.

For the purely metric theory (5.2.51) the field equation is

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \kappa \hat{T}_{\mu\nu} := \kappa \left(T_{\mu\nu} + \tfrac{1}{2} \nabla_\lambda T_\alpha^{\beta\epsilon} \left(g^{\alpha\lambda} g_{\beta\mu} g_{\epsilon\nu} - 2 g_{\mu\epsilon} \delta_\beta^\lambda \delta_\nu^\alpha + g_{\mu\nu} \delta_\beta^\alpha \delta_\epsilon^\lambda \right) \right) \tag{5.2.52}$$

To pass form metric-affine to purely metric formulation we have to expand $\delta\tilde{u}_{\beta\mu}^\alpha$ in the variation of the Lagrangian in terms of $\delta g^{\mu\nu}$ using equation (5.2.27). So we have

$$\frac{\sqrt{g}}{2} T_\alpha^{\beta\mu} \delta\tilde{u}_{\beta\mu}^\alpha = \frac{\sqrt{g}}{4} T_\alpha^{\beta\mu} \left(g^{\alpha\lambda} g_{\beta\rho} g_{\mu\sigma} - 2 g_{\rho(\mu} \delta_{\beta)}^\lambda \delta_\sigma^\alpha + g_{\rho\sigma} \delta_{(\beta}^\alpha \delta_{\mu)}^\lambda \right) \nabla_\lambda \delta g^{\rho\sigma} = $$
$$= -\frac{\sqrt{g}}{4} \nabla_\lambda T_\alpha^{\beta\epsilon} \left(g^{\alpha\lambda} g_{\beta\mu} g_{\epsilon\nu} - 2 g_{\mu\epsilon} \delta_\beta^\lambda \delta_\nu^\alpha + g_{\mu\nu} \delta_\beta^\alpha \delta_\epsilon^\lambda \right) \delta g^{\mu\nu} + \nabla_\lambda \left(\frac{\sqrt{g}}{4} T_\alpha^{\beta\mu} \left(g^{\alpha\lambda} g_{\beta\rho} g_{\mu\sigma} - 2 g_{\rho(\mu} \delta_{\beta)}^\lambda \delta_\sigma^\alpha + g_{\rho\sigma} \delta_{(\beta}^\alpha \delta_{\mu)}^\lambda \right) \delta g^{\rho\sigma} \right) \tag{5.2.53}$$

In the metric-affine theory, the second field equation does not imply $\tilde{\Gamma} = \{g\}$ any longer, as instead it happened in Palatini formulation. It defines more general connections, which depending on details, can even be non-metric.

Thus, simply, the Levi Civita map does not contain all solutions of the metric-affine theory in its image. Accordingly, it cannot be used to establish that the metric theory is an extension of the metric-affine theory. That is reasonable, since the connection is not defined out of the metric anymore, it contains extra information which would be lost by the projection $\pi_* : \mathrm{Sec}(\mathcal{C}_2) \to \mathrm{Sec}(\mathcal{C}_1)$.

> We still do not have enough tools to show that the two theories are in fact dynamically equivalent, though along a different map. We shall be back to this issue in due time.

This is not the only example of supposed lack of equivalence between purely metric, Palatini, and metric-affine theories. If we consider $f(\mathcal{R})$-theories the supposed lack of equivalence appears even before considering metric-affine formulations.

In fact, when metric ψ is allowed, Palatini $f(\mathcal{R})$-theories are based on the following Lagrangian

$$L = \tfrac{\sqrt{g}}{2\kappa} f(\mathcal{R}) + L_m(g, \psi, \nabla\psi) \tag{5.2.54}$$

> The variation fo the Lagrangian reads as

$$\delta L = \tfrac{\sqrt{g}}{2\kappa}\left(f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} - \tilde{\nabla}_\alpha\left(\sqrt{g}f'(\mathcal{R})g^{\mu\nu}\right)\delta\tilde{u}^\alpha_{\mu\nu} + (p_i - \nabla_\mu p^\mu_i)\,\delta\psi^i + \mathrm{Div} \tag{5.2.55}$$

> where we denoted by $-\tfrac{\sqrt{g}}{2}T_{\mu\nu}$ the derivative of the matter Lagrangian with respect to $g^{\mu\nu}$ (and possibly, if the matter Lagrangian depends on $\{g\}$, one can expand $\delta\{g\}$ in terms of $\nabla_\alpha \delta g^{\mu\nu}$ and integrate it by parts, obtaining a contribution to $T_{\mu\nu}$, p_i, and p^μ_i are the momenta of the matter Lagrangian with respect to the matter fields, and a pure divergence) and Div denotes pure divergence terms which come from integration by parts.

Field equations are

$$f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} \qquad \tilde{\nabla}_\alpha\left(\sqrt{g}f'(\mathcal{R})g^{\mu\nu}\right) = 0 \qquad \nabla_\mu p^\mu_i = p_i \tag{5.2.56}$$

Now, by tracing the first equation by $g^{\mu\nu}$, one obtains the master equation in the form

$$f'(\mathcal{R})\mathcal{R} - \tfrac{m}{2}f(\mathcal{R}) = \kappa T \tag{5.2.57}$$

This time, unlike in the vacuum case, this allows generically to solve it for $\mathcal{R} = \mathcal{R}(T)$ so that the curvature does not need to be constant on the spacetime M. It is instead expressed as a function of the matter distribution which is contained in the trace $T := g^{\mu\nu}T_{\mu\nu}$.

Then, using this information, the second equation does not imply $\tilde{\Gamma} = \{g\}$ any longer. On the contrary, one can show that it implies $\tilde{\Gamma} = \{\tilde{g}\}$ for a conformal metric $\tilde{g} = \varphi g$ for a conformal factor $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$; see Subsection 7.2.2. Accordingly, the freezing of the connection induced by the Lavi Civita map $\tilde{\Gamma} = \{g\}$ does not preserve solutions. The purely metric theory has more solutions and the Palatini theory has solutions which are not in the image of the Levi Civita map. Accordingly, neither the Levi Civita map is an extension of the metric formulation from the Palatini formulation, nor vice versa.

If we consider metric-affine $f(\mathcal{R})$-theories the situation just gets worse as we already saw, in the special case of standard GR.

Of course, that does not exclude that there are other maps (e.g. the Levi Civita map for the conformal metric, i.e. $\tilde{\Gamma} = \{\tilde{g}\}$) which can be extensions. Here the point is exactly to show that we do not have yet enough tools to deal with this issue in general.

About different formalisms

Let us summarise the results we have up to this point.

We have a couple of results about solutions transported along maps which preserve the Lagrangian dynamics. The first result establishes a dynamical equivalence when the correspondence between solutions is induced by a map between the configuration bundles. That is simple and it works properly (under a further regularity assumption on the rank of the map to be maximal), showing that one can choose fundamental fields at will aiming to simplify field equations, solve them and finally go back to the original fields. This is similar to what usually happens in mechanics.

The same result can be recast in active form, just by checking globality of the maps.

The second result is about maps which depend on field derivatives, so that they involve some jet prolongations of the configuration bundles. We still have some result about solution mapping, however, the general result is not enough to generally show equivalence unless some further property is required on the map. This is sufficient to prove equivalence in some specific cases (for example, in standard GR between purely metric and metric-affine formalism). However, it is clear that this is not the way. We rather need a way to build maps (instead choosing them suitably) and induce dynamics so that they are automatically equivalent.

When we consider the standard GR theory, we have a way of freezing the connection by using the Levi Civita map and define a purely metric theory for any Palatini or metric-affine theory. While in vacuum and in the Palatini cases, the Levi Civita map is a dynamical equivalence, in a metric-affine theory one can just show that the metric formalism is a dynamical extension of the metric-affine formulation.

When we generalise to $f(\mathcal{R})$-theories, the situation just does get worse; in vacuum, the purely metric formulation is an extension of the Palatini formulation, not a dynamical equivalence. When matter is allowed the Levi Civita map is not even a dynamical extension of the purely metric formulation with respect to the Palatini formulation, even more so for the metric-affine formulation.

The pattern here is clearly that dynamical equivalence is a coincidence in particularly simple cases, which is not preserved in more general theories. Though, as a second thought, we realise that we still do not even know when metric and metric-affine theories are, in fact, dynamically equivalent. What we know by now is that a particular map is not a dynamical equivalence, or a dynamical extension, depending on the case. We need a more general procedure to better produce dynamical equivalences.

Before going into it, let us remark that, historically, GR was developed (1916) before (or at the same time) one had a general definition of connection (1919, even if general gauge connections kept being developed until 1950s and beyond). At Einstein times, geometries were described in terms of metrics. It is hence natural that, traditionally, one introduced the Palatini formalism in a theory by starting from its purely metric formulation and *promoting* the Levi Civita connection to be an arbitrary, usually torsionless, connection. The whole standard GR is, historically, a firmly purely metric theory, until 1970 when the Palatini (and metric-affine) formulations have eventually being considered on a physical ground. (Before it, of course, Einstein and other people struggled to deal with Palatini, metric-affine and even purely affine—i.e. theories based on a connection and no metric at all—though these were honestly considered more mathematical rather than physical theories for gravitation.) Whether the torsion of the connection plays a physical role has to be discussed on a physical stance as well.

3. Routh transform and dual Lagrangians

The *Routh transform* (or *partial Legendre transformation*) is used in mechanics to select different Lagrangian coordinates and it is related to Legendre
transform which connects Lagrangian and Hamiltonian formalism.

For example, let us consider a planar motion in a central potential, described by the Lagrangian

$$L = \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \tag{5.3.1}$$

The variation of the Lagrangian reads as

$$\delta L = \dot{r}\,\delta\dot{r} + \left(r\dot{\theta}^2 + V'(r)\right)\,\delta r + r^2\dot{\theta}\,\delta\dot{\theta} + 0\,\delta\theta \tag{5.3.2}$$

where one recognises the momenta $p = \dot{r}$ and $J = r^2\dot{\theta}$.

By a standard integration by parts with respect to the time derivative, it can be written as

$$\delta L = \tfrac{d}{dt}\left(\dot{r}\delta r + r^2\dot{\theta}\,\delta\theta\right) + \left(-\ddot{r} + r\dot{\theta}^2 + V'(r)\right)\,\delta r - \tfrac{d}{dt}\left(r^2\dot{\theta}\right)\,\delta\theta \tag{5.3.3}$$

from which one sees that Lagrangian formalism corresponds to select (r, θ) as *control variables*.

Then the Euler–Lagrange equations are equivalently obtained directly from the Lagrangian variation as

$$
\begin{cases}
p = \dfrac{\partial L}{\partial \dot{r}} = \dot{r} \\[2mm]
\dot{p} = \dfrac{\partial L}{\partial r} = r\dot{\theta}^2 + V'(r)
\end{cases}
\qquad
\begin{cases}
J = \dfrac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \\[2mm]
\dot{J} = \dfrac{\partial L}{\partial \theta} = 0
\end{cases}
\tag{5.3.4}
$$

Now, we know how to solve the angular equation parametrically. The map

$$r^2\dot{\theta} = J \qquad \Longleftrightarrow \qquad \dot{\theta} = \frac{J}{r^2} \tag{5.3.5}$$

expresses the angular variable in terms of the radial one so that the angular equation is automatically satisfied.

Then it would be convenient to use (r, J) as fundamental variables instead and recast the variation as

$$\delta\left(L - J\dot{\theta}\right) = \dot{r}\,\delta\dot{r} + \left(r\dot{\theta}^2 + V'(r)\right)\,\delta r - \dot{\theta}\,\delta J \tag{5.3.6}$$

Let us then define the function

$$W(r, \dot{r}, \theta, J) = L - J\dot{\theta} = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2\tfrac{J^2}{r^4} + V(r) - \tfrac{J^2}{r^2} = \tfrac{1}{2}\dot{r}^2 + V(r) - \tfrac{J^2}{2r^2} \tag{5.3.7}$$

The function $W(r, \dot{r}, \theta, J)$ (which, of course, as it happens to the Lagrangian, is not function but rather the coefficient of a 1-form) is called the *Routh function*. The equations describing the system in terms of the Routh function are, in part (for the $(r, \dot{r})$ degree of freedom), Euler–Lagrange equations, in part (for the (θ, J) degree of freedom) are Hamilton equations.

The variation of this function is

$$\delta W = \dot{r}\delta\dot{r} + \left(V'(r) + \tfrac{J^2}{r^3}\right)\,\delta r - \tfrac{J}{r^2}\,\delta J + 0\,\delta\theta \tag{5.3.8}$$

which induces the equations in the form

$$\begin{cases} p = \dfrac{\partial W}{\partial \dot{r}} = \dot{r} \\[2mm] \dot{p} = \dfrac{\partial W}{\partial r} = V'(r) + \tfrac{J^2}{r^3} \end{cases} \qquad\qquad \begin{cases} -\dot{\theta} = \dfrac{\partial W}{\partial J} = -\tfrac{J}{r^2} \\[2mm] \dot{J} = \dfrac{\partial W}{\partial \theta} = 0 \end{cases} \tag{5.3.9}$$

Accordingly, one can obtain both the definition of J and the radial equation from the Routh function W.

Let us remark that equation (5.3.4) and (5.3.9) are in fact equivalent, by construction. Notice that the radial equation is obtained as an Euler–Lagrange equation, while the angular equation is obtained as a Hamilton equation. This can be avoided by using a procedure similar to the definition of the Helmholtz Lagrangian on the Routh function.

We can introduce coordinates $(r, \dot{r}, \theta, J, u_\theta, u_J)$ and define the Lagrangian

$$L_H(r, \dot{r}, \theta, J, u_\theta, u_J) = Ju_\theta + W(r, \dot{r}, \theta, J) = J\left(u_\theta - \dot{\theta}\right) + L(r, \dot{r}, \theta, \dot{\theta}) = Ju_\theta + \tfrac{1}{2}\dot{r}^2 + V(r) - \tfrac{J^2}{2r^2} \tag{5.3.10}$$

where $\dot{\theta}$ is expressed as a function of $(r, \dot{r}, \theta, J)$, namely in this case $\dot{\theta} = \tfrac{J}{r^2}$.

This is an ordinary Lagrangian for the 3 pairs $(r, \dot{r})$, (θ, u_θ), and (J, u_J), being, of course, degenerate with respect to J since it does not depend on u_J at all. The corresponding Euler–Lagrange equations are

$$\ddot{r} = V' + \frac{J^2}{r^3} \qquad\qquad \dot{J} = 0 \qquad\qquad \begin{cases} u_\theta = \tfrac{d\theta}{dt} \\[2mm] 0 = u_\theta - \tfrac{J}{r^2} \end{cases} \tag{5.3.11}$$

respectively. These equations, in fact, account for the radial equation, the conservation of J and the definition of J. They are trivially equivalent to (5.3.9), and, consequently, to the original equations (5.3.4), though this time they are Euler-Lagrangian equation of a Lagrangian, namely L_H.

Let us remark that passing from the Lagrangian L to the Lagrangian L_H can, in fact, be described as *promoting* the angular momentum J to be and independent variable, though this time by a precise procedure. Notice also that, in this case, the precise procedure to promote J to be an independent degrees of freedom does not amount to simply replacing $\dot{\theta} = \tfrac{J}{r^2}$ in the original Lagrangian. That would produce the Lagrangian

$$L_J = \tfrac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + V(r) \tag{5.3.12}$$

which is *not* equivalent to the original Lagrangian, since, for example, it produces an incorrect radial equation.

One can also prove in mechanics that this procedure generally leads to an equivalent Helmholtz Lagrangian.

Let us consider a Lagrangian

$$L = L(q^0, u^0, q^i, u^i) \tag{5.3.13}$$

The variation is

$$\delta L = \frac{\partial L}{\partial u^0}\delta u^0 + \frac{\partial L}{\partial q^0}\delta q^0 + \frac{\partial L}{\partial u^i}\delta u^i + \frac{\partial L}{\partial q^i}\delta q^i \tag{5.3.14}$$

which induces Euler–Lagrange equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^0}\right) = \frac{\partial L}{\partial q^0} \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial u^i}\right) = \frac{\partial L}{\partial q^i} \tag{5.3.15}$$

Then one can define $J = \frac{\partial L}{\partial u^0}$ and integrate by parts with respect to the variation δ to recast the variation of the Lagrangian as

$$\delta\left(L - Ju^0\right) = -u^0\delta J + \frac{\partial L}{\partial q^0}\delta q^0 + \frac{\partial L}{\partial u^i}\delta u^i + \frac{\partial L}{\partial q^i}\delta q^i \tag{5.3.16}$$

and define the Routh function as

$$W(q^0, J, q^i, u^i) = L(q^0, U^0, q^i, u^i) - JU^0 \tag{5.3.17}$$

where the function $U^0(q^0, J, q^i, u^i)$ is obtained by solving the definition of J for the velocity u^0. The equations associated to this Routh function are

$$\begin{cases} \dot{q}^0 = -\dfrac{\partial W}{\partial J} \\[2mm] \dot{J} = \dfrac{\partial W}{\partial q^0} \end{cases} \qquad \begin{cases} p_i = \dfrac{\partial W}{\partial u^i} = \dfrac{\partial L}{\partial u^i} + \left(\dfrac{\partial L}{\partial u^0} - J\right)\dfrac{\partial U^0}{\partial u^i} \\[3mm] \dot{p}_i = \dfrac{\partial W}{\partial q^i} = \dfrac{\partial L}{\partial q^i} + \left(\dfrac{\partial L}{\partial u^0} - J\right)\dfrac{\partial U^0}{\partial q^i} \end{cases} \tag{5.3.18}$$

The first block of equations accounts for the Routh transform and for the equation for q^0, in fact

$$\dot{q}^0 = -\frac{\partial W}{\partial J} = -\left(\frac{\partial L}{\partial u^0} - J\right)\frac{\partial U^0}{\partial J} + U^0 = U^0 \qquad \Longleftrightarrow \qquad \frac{\partial L}{\partial u^0} = J \tag{5.3.19}$$

and

$$\dot{J} = \frac{\partial W}{\partial q^0} = \frac{\partial L}{\partial q^0} + \left(\frac{\partial L}{\partial u^0} - J\right)\frac{\partial U^0}{\partial q^0} = \frac{\partial L}{\partial q^0} \tag{5.3.20}$$

Finally, let us add $\dot{q}^0$ and $\dot{J}$ as Lagrangian velocities for q^0 and J and define the Helmholtz Lagrangian as

$$L_H(q^0, \dot{q}^0, J, \dot{J}, q^i, u^i) = J\dot{q}^0 + W(q^0, J, q^i, u^i) = J\left(\dot{q}^0 - U^0(q^0, J, q^i, u^i)\right) + L(q^0, U^0(q^0, J, q^i, u^i), q^i, u^i) \tag{5.3.21}$$

hence obtaining the three sets of Euler–Lagrange equations

$$\begin{cases} p_0 = \dfrac{\partial L_H}{\partial \dot{q}^0} = J \\[3mm] \dfrac{dp_0}{dt} = \dfrac{\partial L_H}{\partial q^0} = \dfrac{\partial L}{\partial q^0} \end{cases} \quad \begin{cases} p_J = \dfrac{\partial L_H}{\partial \dot{J}} = 0 \\[3mm] \dfrac{dp_J}{dt} = \dfrac{\partial L_H}{\partial J} = \dot{q}^0 - U^0 = 0 \end{cases} \quad \begin{cases} p_i = \dfrac{\partial L_H}{\partial \dot{q}^i} = \dfrac{\partial L}{\partial u^i} + \left(\dfrac{\partial L}{\partial u^0} - J\right)\dfrac{\partial U^0}{\partial u^i} \\[3mm] \dfrac{dp_i}{dt} = \dfrac{\partial L_H}{\partial q^i} = \dfrac{\partial L}{\partial q^i} + \left(\dfrac{\partial L}{\partial u^0} - J\right)\dfrac{\partial U^0}{\partial q^i} \end{cases} \tag{5.3.22}$$

which are equivalent to the original equations, as well as the Routh transform $\dot{q}^0 = U^0(q^0, J, q^i, u^i) \iff J = \frac{\partial L}{\partial u^0}$.

Planar non-central motions

Let us here consider the more general case of a planar motion which is not central (so that the angular momentum J is not conserved) described by the Lagrangian

$$L = \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + cr^2 + 2br\theta + a\theta^2\right) \tag{5.3.23}$$

with $a, b, c \in \mathbb{R}$ constants and, for future convenience, we consider $a \neq 0$. The variation of the Lagrangian reads as

$$\delta L = \dot{r}\,\delta\dot{r} + \left(r\dot{\theta}^2 + cr + b\theta\right)\delta r + r^2\dot{\theta}\,\delta\dot{\theta} + (br + a\theta)\,\delta\theta \tag{5.3.24}$$

so that, by defining $J = r^2\dot\theta$ for the angular momentum which, however now, is not conserved. The equations of motion are

$$\ddot{r} = \frac{J^2}{r^3} + cr + b\theta \qquad\qquad \dot{J} = br + a\theta \qquad\qquad (5.3.25)$$

Then we integrate by parts (with respect to the variation δ) to obtain

$$\delta(L - J\dot\theta) = \dot{r}\delta\dot{r} + \left(r\dot\theta^2 + cr + b\theta\right)\delta r - \dot\theta\delta J + (br + a\theta)\,\delta\theta \qquad\qquad (5.3.26)$$

and define the Routh function

$$W(r, \dot{r}, \theta, J) = \tfrac{1}{2}\left(\dot{r}^2 + cr^2 + 2br\theta + a\theta^2\right) - \frac{J^2}{2r^2} \qquad\qquad (5.3.27)$$

The variation of the Routh function is

$$\delta W = \dot{r}\,\delta\dot{r} + \left(\frac{J^2}{r^3} + cr + b\theta\right)\delta r - \frac{J}{r^2}\,\delta J + (br + a\theta)\,\delta\theta \qquad\qquad (5.3.28)$$

and equations of motion are

$$\ddot{r} = \frac{J^2}{r^3} + cr + b\theta \qquad\qquad \begin{cases} \dot\theta = \frac{J}{r^2} \\[2mm] \dot{J} = br + a\theta \end{cases} \qquad\qquad (5.3.29)$$

which are obviously equivalent to the original Euler–Lagrange equations and the definition of the angular momentum J. This time these are coupled equations and we cannot write a purely radial equation since J is not constant, as it happens in central motions.

In this case the Helmholtz Lagrangian corresponds to introduce extra velocities (u_θ, u_J) and define

$$L_H(r, \dot{r}, \theta, u_\theta, J, u_J) = Ju_\theta + W(r, \dot{r}, \theta, J) = J\left(u_\theta - \frac{J}{r^2}\right) + \tfrac{1}{2}\left(\dot{r}^2 + cr^2 + 2br\theta + a\theta^2\right) + \frac{J^2}{2r^2} \qquad\qquad (5.3.30)$$

Then we can define

$$\dot{J} = br + a\theta \qquad\Longleftrightarrow\qquad \theta = \frac{\dot{J} - br}{a} \qquad\qquad (a \neq 0) \qquad\qquad (5.3.31)$$

and go to coordinates $(r, \dot{r}, J, \dot{J})$. We can go back to the variation (5.3.28) and integrate by parts $\dot{J}\delta\theta = \delta(\dot{J}\theta) - \theta\delta\dot{J}$ and recast it is

$$\delta L^* = \delta\left(W - \dot{J}\theta\right) = \dot{r}\,\delta\dot{r} + \left(\frac{J^2}{r^3} + cr + b\theta\right)\delta r - \frac{J}{r^2}\,\delta J - \theta\,\delta\dot{J} \qquad\qquad (5.3.32)$$

Then we have the Lagrangian

$$\begin{aligned} L^* &= \tfrac{1}{2}\left(\dot{r}^2 + cr^2 + 2br\frac{\dot{J} - br}{a} + a\frac{(\dot{J} - br)^2}{a^2}\right) - \frac{J^2}{2r^2} - \dot{J}\frac{\dot{J} - br}{a} = \tfrac{1}{2}\dot{r}^2 + \frac{c}{2}r^2 - \frac{J^2}{2r^2} + \frac{2br\dot{J} - 2b^2r^2 + \dot{J}^2 + b^2r^2 - 2br\dot{J} - 2\dot{J}^2 + 2br\dot{J}}{2a} = \\ &= \tfrac{1}{2}\dot{r}^2 - \frac{(\dot{J} - br)^2}{2a} + \frac{c}{2}r^2 - \frac{J^2}{2r^2} \end{aligned} \qquad\qquad (5.3.33)$$

The variation of the Lagrangian L^* is in fact

$$\delta L^* = \dot{r}\,\delta\dot{r} + \left(\frac{J^2}{r^3} + cr + b\frac{(\dot{J} - br)}{a}\right)\delta r - \frac{J}{r^2}\delta J - \frac{(\dot{J} - br)}{a}\delta\dot{J} \qquad\qquad (5.3.34)$$

and the corresponding equations of motions are

$$\ddot{r} = \frac{J^2}{r^3} + cr + b\frac{(\dot{J}-br)}{a} \qquad \ddot{J} - b\dot{r} = a\frac{J}{r^2} \qquad (\iff \ddot{J} = b\dot{r} + a\dot{\theta} = b\dot{r} + a\frac{J}{r^2} \quad \Leftarrow \dot{J} = br + a\theta) \tag{5.3.35}$$

Then the Lagrangians L and L^* are dynamically equivalent by construction. However, let us notice that the Lagrangian L^* has many features which are considered strange (e.g. it has a generalised potential dependent on velocities) if not severe inconsistencies (negative kinetic energy for the J degree of freedom, if $a > 0$). For sure, it does not seem to describe the same system as the Lagrangian L which is pretty standard, though it does.

Let us remark that the Routh function is not a Lagrangian. In this example, it is clear that one has an Euler-Lagrangian equation for $(r, \dot{r})$ while (θ, J) obey Hamilton equations and they are a pair of canonically conjugated coordinates. However, when one composes two Routh transformations with respect to a pair of variables, then a real Lagrangian L^* is obtained again, in which the momentum J is now treated as a position while the angle θ is replaced by the velocity $\dot{J}$ of J. The systems is again described in terms of Euler–Lagrange equations, this time of a different Lagrangian L^*, which is called the *dual Lagrangian.*

Geodesics in spherically symmetric geometries

Let us consider a Lorentzian manifold (M, g) of dimension 3 and consider the Lagrangian

$$\boldsymbol{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, ds \tag{5.3.36}$$

A curve $x^\mu(s)$ is *g-time-like* if its tangent vector $\dot{x}$ obeys $g(\dot{x}, \dot{x}) < 0$. The curve is called a *g-time-like g-geodesics trajectory* if it is a solution of its Euler–Lagrange equations.

The Riemannian manifold (M, g) is called *static and spherically symmetric* if there exists a coordinate system (t, r, ϕ) in which the metric has the local expression

$$g = -A(r)\, dt^2 + B(r)\, dr^2 + r^2\, d\phi^2 \tag{5.3.37}$$

Then the Lagrangian takes the form

$$\boldsymbol{L} = \sqrt{A(r)\left(\frac{dt}{ds}\right)^2 - B(r)\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2}\, ds = \sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}\, dt \tag{5.3.38}$$

where the last step corresponds to choosing $s = t$ as a parameter along a *g*-time-like curve and the dot denotes derivative with respect to t.

This Lagrangian $L(r, \dot{r}, \phi, \dot{\phi})$ is covariant with respect to diffeomorphisms *and reparameterisations.* It has two first integrals (which are associated to the fact that t and ϕ do not appear explicitly in the Lagrangian)

$$\begin{cases} K = \dfrac{-r^2\dot{\phi}}{\sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}} \qquad \iff \dot{\phi} = -\frac{K}{r^2}\sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2} \\[4mm] E = \dfrac{-B(r)\dot{r}^2 - r^2\dot{\phi}^2}{\sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}} - \sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2} = \dfrac{-B(r)\dot{r}^2 - r^2\dot{\phi}^2 - A(r) + B(r)\dot{r}^2 + r^2\dot{\phi}^2}{\sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}} \end{cases} \tag{5.3.39}$$

The intersections of the level sets of these two first integrals determine trajectories which are solutions of Euler–Lagrange equations (which in fact are all the solutions since the Lagrangian is invariant with respect to parameterisations so that a parameterisation along the solution cannot be determined).

The equations are then equivalent to

$$
\begin{cases}
r^2\dot\phi = -K\sqrt{A - B\dot r^2 - r^2\dot\phi^2} & \Rightarrow r^2\dot\phi^2 = K^2\dfrac{A - B\dot r^2}{r^2 + K^2} \\[2ex]
E\sqrt{A - B\dot r^2 - r^2\dot\phi^2} = -A & \Rightarrow \dfrac{(r^2 + K^2)B\dot r^2 + K^2 A - K^2 B\dot r^2}{r^2 + K^2} = A\left(1 - \dfrac{A}{E^2}\right)
\end{cases}
\tag{5.3.40}
$$

Then we have

$$
\dot r^2 = \frac{A}{B}\left(1 - \frac{A}{E^2}\right)\left(1 + \frac{K^2}{r^2}\right) - \frac{K^2}{r^2}\frac{A}{B} = \frac{A}{B}\left(1 - \frac{A}{E^2} - \frac{K^2}{r^2}\frac{A}{E^2}\right) = \frac{A}{B}\left(1 - \frac{A}{E^2}\frac{r^2 + K^2}{r^2}\right)
\tag{5.3.41}
$$

which depends only on r. Then

$$
\dot\phi^2 = \frac{K^2}{r^2(r^2 + K^2)}\frac{A^2}{E^2}\frac{r^2 + K^2}{r^2} \qquad \Rightarrow \dot\phi = \frac{KA}{Er^2} \qquad \Rightarrow \dot r^2 = \frac{A}{B}\frac{E^2 r^2 - A(r^2 + K^2)}{E^2 r^2}
\tag{5.3.42}
$$

Accordingly, one solves the radial equation (5.3.41) to find $r(t)$ and then uses the angular equation (5.3.42) to determine $\phi(t)$. Solutions are parameterised by the value of first integrals (K, E).

The angular equation can be identically solved parametrically by

$$
\dot\phi^2 = \frac{K^2}{r^2}\frac{A - B\dot r^2}{r^2 + K^2} \qquad \Rightarrow \dot\phi = -\frac{K}{r}\sqrt{\frac{A - B\dot r^2}{r^2 + K^2}}
\tag{5.3.43}
$$

One can then define the Routh function, using this parameterisation of the angular solution, as

$$
W(r, \dot r, \phi, K) = L - K\dot\phi = \sqrt{(r^2 + K^2 - K^2)\frac{A - B\dot r^2}{r^2 + K^2}} + \frac{K^2}{r}\sqrt{\frac{A - B\dot r^2}{r^2 + K^2}} = \frac{r^2 + K^2}{r}\sqrt{\frac{A - B\dot r^2}{r^2 + K^2}} = \frac{1}{r}\sqrt{(A - B\dot r^2)(r^2 + K^2)}
\tag{5.3.44}
$$

Since it is independent of t, one has a first integral

$$
E^* = -\left(\frac{B\dot r^2}{r} + \frac{A - B\dot r^2}{r}\right)\sqrt{\frac{r^2 + K^2}{A - B\dot r^2}} = \frac{-A}{\sqrt{A - B\dot r^2}}\sqrt{\frac{r^2 + K^2}{r^2}}
\tag{5.3.45}
$$

The first integral E^* is equivalent to E along the angular map, in fact

$$
E = \frac{-A}{\sqrt{A - B\dot r^2 - r^2\dot\phi^2}} = \frac{-A}{\sqrt{A - B\dot r^2 - K^2\frac{A - B\dot r^2}{r^2 + K^2}}} = \frac{-A}{\sqrt{(r^2 + K^2 - K^2)\frac{A - B\dot r^2}{r^2 + K^2}}} = \frac{-A}{r}\sqrt{\frac{r^2 + K^2}{A - B\dot r^2}} = E^*
\tag{5.3.46}
$$

Then we have a new way of writing equivalent equations as

$$
\begin{cases}
p_r^* = \dfrac{\partial W}{\partial \dot{r}} \\[2mm]
\dot{p}_r^* = \dfrac{\partial W}{\partial r}
\end{cases}
\qquad
\begin{cases}
-\dot{\phi} = \dfrac{\partial W}{\partial K} = \dfrac{K}{r}\sqrt{\dfrac{A - B\dot{r}^2}{r^2 + K^2}} \\[2mm]
\dot{K} = \dfrac{\partial W}{\partial \phi} = 0
\end{cases}
\tag{5.3.47}
$$

The second set of these equations accounts for the definition and conservation of K.

The radial equation $\dot{p}_r^* = \frac{\partial W}{\partial r}$ is equivalent to the conservation of E^*, i.e. to the conservation of E. In particular, the radial equation is equivalent to conservation of E^*, which is equivalent to conservation of E, proving equivalence of the two systems.

Quite interestingly, checking equivalence, which is quite trivial in view of the theory, is quite complicated directly.

For the radial equation from W, we first need to compute the momenta:

$$
\begin{cases}
p_r^* = \dfrac{\partial W}{\partial \dot{r}} = \dfrac{-B\dot{r}}{\sqrt{A - B\dot{r}^2}}\sqrt{\dfrac{r^2 + K^2}{r^2}} \\[3mm]
\dfrac{\partial W}{\partial r} = \dfrac{A' - B'\dot{r}^2}{2\sqrt{A - B\dot{r}^2}}\sqrt{\dfrac{r^2 + K^2}{r^2}} + \sqrt{A - B\dot{r}^2}\,\dfrac{\partial}{\partial r}\sqrt{\dfrac{r^2 + K^2}{r^2}} = \dfrac{(A' - B'\dot{r}^2)r(r^2 + K^2) - 2K^2(A - B\dot{r}^2)}{2r^2\sqrt{r^2 + K^2}\sqrt{A - B\dot{r}^2}}
\end{cases}
\tag{5.3.48}
$$

where we used

$$
\frac{\partial}{\partial r}\sqrt{\frac{r^2 + K^2}{r^2}} = \frac{1}{2}\frac{r}{\sqrt{r^2 + K^2}}\frac{2r^3 - 2r(r^2 + K^2)}{r^4} = \frac{-K^2}{r^2\sqrt{r^2 + K^2}}
\tag{5.3.49}
$$

Then, we can compute $\dot{p}_r^*$ and $\frac{\partial W}{\partial r}$ as

$$
\begin{aligned}
\dot{p}_r^* &= \frac{-B\dot{r}^2}{\sqrt{A - B\dot{r}^2}}\frac{\partial}{\partial r}\sqrt{\frac{r^2 + K^2}{r^2}} + \frac{-2(B'\dot{r}^2 + B\ddot{r})(A - B\dot{r}^2) + B\dot{r}(A'\dot{r} - B'\dot{r}^3 - 2B\dot{r}\ddot{r})}{2(A - B\dot{r}^2)^{3/2}}\sqrt{\frac{r^2 + K^2}{r^2}} = \\[2mm]
&= \frac{2K^2 B\dot{r}^2(A - B\dot{r}^2) + \left(-2(B'\dot{r}^2 + B\ddot{r})(A - B\dot{r}^2) + B\dot{r}(A'\dot{r} - B'\dot{r}^3 - 2B\dot{r}\ddot{r})\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}} = \\[2mm]
&= \frac{2K^2 AB\dot{r}^2 - 2K^2 B^2\dot{r}^4 + \left(-2AB'\dot{r}^2 - 2AB\ddot{r} + 2BB'\dot{r}^4 + 2B^2\dot{r}^2\ddot{r} + A'B\dot{r}^2 - BB'\,\dot{r}^4 - 2B^2\dot{r}^2\ddot{r}\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}} = \\[2mm]
&= \frac{2K^2 AB\dot{r}^2 - 2K^2 B^2\dot{r}^4 + \left(-2AB'\dot{r}^2 - 2AB\ddot{r} + BB'\dot{r}^4 + A'B\dot{r}^2\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}} \\[3mm]
\frac{\partial W}{\partial r} &= \frac{-2K^2(A^2 - 2AB\dot{r}^2 + B^2\dot{r}^4) + (A' - B'\dot{r}^2)(A - B\dot{r}^2)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}} = \\[2mm]
&= \frac{-2K^2 A^2 + 4K^2 AB\dot{r}^2 - 2K^2 B^2\dot{r}^4 + (AA' - AB'\dot{r}^2 - A'B\dot{r}^2 + BB'\dot{r}^4)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}}
\end{aligned}
\tag{5.3.50}
$$

Thus the radial equation for W is

$$
\begin{aligned}
\dot{p}_r^* - \frac{\partial W}{\partial r} &= \frac{2K^2 AB\dot{r}^2 - 2K^2 B^2\dot{r}^4 + 2K^2 A^2 - 4K^2 AB\dot{r}^2 + 2K^2 B^2\dot{r}^4 + \left(-2AB'\dot{r}^2 - 2AB\ddot{r} + BB'\dot{r}^4 + A'B\dot{r}^2 - AA' + AB'\dot{r}^2 + A'B\dot{r}^2 - BB'\dot{r}^4\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}} = \\[2mm]
&= \frac{2K^2 A^2 - 2K^2 AB\dot{r}^2 + \left(-AA' - AB'\dot{r}^2 - 2AB\ddot{r} + 2A'B\dot{r}^2\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot{r}^2)^{3/2}}
\end{aligned}
\tag{5.3.51}
$$

If one wants to compare directly with the radial equation for L, one can use the conservation of K in the momenta of W

$$
\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot r} = \frac{-2B\dot r}{2\sqrt{A - B\dot r^2 - r^2\dot\phi^2}} = \frac{-2B\dot r}{2\sqrt{(r^2 + K^2 - K^2)\frac{A - B\dot r^2}{r^2 + K^2}}} = \frac{-B\dot r}{\sqrt{A - B\dot r^2}}\sqrt{\frac{r^2 + K^2}{r^2}} = p_r^* \\
\frac{\partial L}{\partial r} &= \frac{A' - B'\dot r - 2r\dot\phi^2}{2\sqrt{A - B\dot r^2 - r^2\dot\phi^2}} = \frac{A' - B'\dot r - 2K^2\frac{A - B\dot r^2}{r(r^2 + K^2)}}{2\sqrt{(r^2 + K^2 - K^2)\frac{A - B\dot r^2}{r^2 + K^2}}} = \frac{(A' - B'\dot r)r(r^2 + K^2) - 2K^2(A - B\dot r^2)}{2r^2\sqrt{r^2 + K^2}\sqrt{A - B\dot r^2}} = \frac{\partial W}{\partial r}
\end{aligned}
\tag{5.3.52}
$$

so that the radial equations are equivalent.

The equivalence of the radial equation with the conservation of E^* is also quite involved. The conservation can be expanded as

$$
\begin{aligned}
\dot E^* &= \frac{\partial}{\partial r}\sqrt{\frac{r^2 + K^2}{r^2}}\frac{-A\dot r(A - B\dot r^2)}{(A - B\dot r^2)^{3/2}} + \sqrt{r^2 + K^2}\frac{-2A'\dot r(A - B\dot r^2) + A(A'\dot r - B'\dot r^3 - 2B\dot r\ddot r)}{2r(A - B\dot r^2)^{3/2}} = \\
&= \frac{2K^2 A(A - B\dot r^2) + \left(-2A'\dot r(A - B\dot r^2) + A(A' - B'\dot r^2 - 2B\ddot r)\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot r^2)^{3/2}}\dot r = \\
&= \frac{2K^2 A^2 - 2K^2 AB\dot r^2 + \left(-AA' - AB'\dot r^2 - 2AB\ddot r + 2A'B\dot r^2\right)r(r^2 + K^2)}{2r^2\sqrt{r^2 + K^2}(A - B\dot r^2)^{3/2}}\dot r = \left(\dot p_r^* - \frac{\partial W}{\partial r}\right)\dot r = 0
\end{aligned}
\tag{5.3.53}
$$

Of course, non-triviality of the direct computation is in recognising that radial equation (5.3.51) is, in fact, equivalent to a total time derivative, as shown by (5.3.53).

Klein–Gordon equation

Let us here discuss equivalent formulations of Klein–Gordon equation. We shall consider the metric to be eventually fixed as in special relativity. The Klein–Gordon Lagrangian for a real scalar field is

$$
L = -\frac{\sqrt g}{2}\left(g^{\mu\nu}\overset{*}{\nabla}_\mu\psi\,\overset{*}{\nabla}_\nu\psi + \mu^2\psi^2\right)
\tag{5.3.54}
$$

Its variation reads as

$$
\delta L = -\frac{\sqrt g}{2}T_{\mu\nu}\delta g^{\mu\nu} - \left(\sqrt g g^{\mu\nu}\nabla_\mu\psi\right)\delta\nabla_\nu\psi - \left(\mu^2\sqrt g\psi\right)\delta\psi
\tag{5.3.55}
$$

Set $p^\nu := -\sqrt g g^{\mu\nu}\nabla_\mu\psi$ for the momenta and integrate by parts

$$
\delta\left(L - p^\mu\nabla_\mu\psi\right) = -\frac{\sqrt g}{2}T_{\mu\nu}\,\delta g^{\mu\nu} - \nabla_\mu\psi\,\delta p^\mu - \left(\mu^2\sqrt g\psi\right)\delta\psi
\tag{5.3.56}
$$

Then the Routh function for the coordinates (g, ψ, p^μ) is defined as

$$
W = L - p^\mu\nabla_\mu\psi = \frac{1}{2\sqrt g}p^\mu p^\nu g_{\mu\nu} - \frac{\sqrt g}{2}\mu^2\psi^2
\tag{5.3.57}
$$

and one can check directly that its variation reproduces the variation as obtained after the integration by parts (5.3.56), namely we have

$$
\delta W = -\frac{\sqrt g}{2}\overset{*}{T}_{\mu\nu}\,\delta g^{\mu\nu} + \frac{1}{\sqrt g}p^\mu g_{\mu\nu}\delta p^\nu - \mu^2\sqrt g\psi\delta\psi
\tag{5.3.58}
$$

Its field equations are

$$
\begin{cases}
-\nabla_\mu \psi = \dfrac{\partial W}{\partial p^\mu} = \dfrac{1}{\sqrt{g}} p^\nu g_{\mu\nu} \\[3mm]
\nabla_\mu p^\mu = \dfrac{\partial W}{\partial \psi} = -\mu^2 \sqrt{g}\,\psi
\end{cases}
\qquad \Rightarrow \Box\psi - \mu^2\psi = 0
\tag{5.3.59}
$$

which account for the original Klein–Gordon equation as well as for the definition of the momentum p^μ.

Of course, the Routh function is not a Lagrangian, though we can introduce the Helmholtz Lagrangian

$$
L_H = \tfrac{1}{2\sqrt{g}} p^\mu p^\nu g_{\mu\nu} - \tfrac{\sqrt{g}}{2}\mu^2\psi^2 + p^\mu \psi_\mu
\tag{5.3.60}
$$

for the fundamental fields $(\psi, \psi_\mu; p^\mu, p^\mu_\nu)$, which in fact is zero order in p^μ. Its Euler–Lagrange equations are

$$
d_\mu \left(\frac{\partial L_H}{\partial p^\nu_\mu} \right) \equiv 0 = \psi_\nu + \tfrac{1}{\sqrt{g}} p^\mu g_{\mu\nu}
\qquad\qquad
d_\mu \left(\frac{\partial L_H}{\partial \psi_\mu} \right) \equiv \nabla_\mu p^\mu = -\mu^2 \sqrt{g}\,\psi
\tag{5.3.61}
$$

which are in fact equivalent to (5.3.59), though this time in a Lagrangian formalism.

For $\mu \neq 0$, we can now define $p = -\mu^2 \sqrt{g}\,\psi$ and integrate by parts again

$$
\delta\left(W - p\psi\right) = -\tfrac{\sqrt{g}}{2} T^*_{\mu\nu}\, \delta g^{\mu\nu} - \nabla_\mu \psi\, \delta p^\mu - \psi \delta p
\tag{5.3.62}
$$

to define a new Lagrangian $L^* = W - p\psi$ which is a function of $(g, p^\mu, p = \nabla_\mu p^\mu)$, i.e. it depends only on the trace $\nabla_\mu p^\mu$. We have

$$
L^* = \tfrac{1}{2\sqrt{g}} p^\mu p^\nu g_{\mu\nu} + \tfrac{1}{2} p\psi - p\psi = \tfrac{1}{2\sqrt{g}}\left(p^\mu p^\nu g_{\mu\nu} + \tfrac{1}{\mu^2}\left(\nabla_\mu p^\mu\right)^2 \right)
\tag{5.3.63}
$$

That is a Lagrangian for a vector density of weight 1, namely p^μ, which depends on the derivatives of fields only through the combination $\overset{*}{\nabla}_\mu p^\mu$. Of course, its variation matches with the variation (5.3.62), namely

$$
\delta L^* = -\tfrac{\sqrt{g}}{2} T^*_{\mu\nu}\, \delta g^{\mu\nu} + \tfrac{1}{\sqrt{g}} p^\mu g_{\mu\nu}\, \delta p^\nu + \tfrac{1}{\mu^2 \sqrt{g}} \nabla_\epsilon p^\epsilon \delta^\mu_\nu \delta \nabla_\mu p^\nu
\tag{5.3.64}
$$

The matter field equations are

$$
\tfrac{\sqrt{g}}{\mu^2} \nabla_\nu \left(\tfrac{1}{\sqrt{g}} \nabla_\epsilon p^\epsilon \right) = p^\mu g_{\mu\nu}
\qquad \Rightarrow g^{\mu\nu} \nabla_\nu \nabla_\epsilon p^\epsilon = \mu^2 p^\mu
\tag{5.3.65}
$$

which everything seems but a Klein–Gordon equation. However, one can easily show that p^μ is a solution of this equation iff $\psi = -\tfrac{1}{\mu^2 \sqrt{g}} \nabla_\mu p^\mu$ is a solution of the original Klein–Gordon equation.

If p^μ is a solution and we define $\psi = -\tfrac{1}{\mu^2 \sqrt{g}} \nabla_\mu p^\mu$ then we have

$$
\Box\psi - \mu^2\psi = -\tfrac{1}{\mu^2 \sqrt{g}}\left(\nabla_\alpha \left(g^{\alpha\beta} \nabla_\beta \nabla_\mu p^\mu \right) - \mu^2 \nabla_\mu p^\mu \right) = -\tfrac{1}{\mu^2 \sqrt{g}}\left(\nabla_\alpha \left(\mu^2 p^\alpha \right) - \mu^2 \nabla_\mu p^\mu \right) = -\tfrac{1}{\sqrt{g}}\left(\nabla_\alpha p^\alpha - \nabla_\mu p^\mu \right) = 0
\tag{5.3.66}
$$

i.e. ψ is a solution of the original Klein–Gordon equation. On the other hand, if ψ is a solution of the Klein–Gordon equation and we define $p^\mu = -\sqrt{g}g^{\mu\nu}\nabla_\mu\psi$ then we have

$$g^{\mu\nu}\nabla_\nu\nabla_\epsilon p^\epsilon - \mu^2 p^\mu = \left(g^{\mu\nu}\delta_\sigma^\epsilon\nabla_\nu\nabla_\epsilon p^\sigma - \mu^2\delta_\sigma^\mu p^\sigma\right) = -\sqrt{g}g^{\mu\nu}\nabla_\nu\left(\Box\psi - \mu^2\psi\right) = 0 \tag{5.3.67}$$

i.e. p^μ is a solution of the equation (5.3.65).

Let us now consider these maps from a bundle perspective.

Let $S = M \times \mathbb{R}$ be the bundle for the Klein–Gordon field ψ and $A_{m-1}(M)$ the bundle for p^μ.

The original Klein–Gordon theory is defined on $\mathcal{C}_1 = \mathrm{Lor}(M) \times_M S$ and it is of order $(0,1)$, so that the Lagrangian is a horizontal form over $J\mathcal{C}_1 = \mathrm{Lor}(M) \times_M J^1 S$. The dual theory is defined on $\mathcal{C}_2 = \mathrm{Lor}(M) \times A_{m-1}(M)$, it is of order $(0,1)$, i.e. the Lagrangian is a horizontal form over $J\mathcal{C}_2 = \mathrm{Lor}(M) \times_M J^1 A_{m-1}(M)$. The map $p^\nu = -\sqrt{g}g^{\mu\nu}\nabla_\mu\psi$ can be regarded as a map

$$\bar{p}: \mathrm{Lor}(M) \times_M J^1 S \to \mathrm{Lor}(M) \times_M A_{m-1}(M) \tag{5.3.68}$$

while $\psi = -\frac{1}{\mu^2\sqrt{g}}\nabla_\mu p^\mu$ defines a map

$$\bar{\psi}: \mathrm{Lor}(M) \times_M J^1 A_{m-1}(M) \to \mathrm{Lor}(M) \times_M S \tag{5.3.69}$$

Neither of these maps are one-to-one on configurations. Of course, they are not injective (all constants are sent to zero) nor surjective (for example, a $p^\mu\,d\sigma_\mu$ in the image of the map $\bar{p}$ is an $(m-1)$-form which, by Hodge duality, corresponds to a 1-form and this 1-form $\nabla_\mu\psi\,dx^\mu$ happens to be exact).

We directly proved above how these two maps can be restricted to solutions and are there the inverse one of the other, thus defining a dynamical equivalence. As far as the dynamics are concerned the maps can be prolonged to the jet where the Lagrangians live

$$J\bar{p}: J^2\mathcal{C}_1 \to J\mathcal{C}_2 \qquad\qquad J\bar{\psi}: J^2\mathcal{C}_2 \to J\mathcal{C}_1 \tag{5.3.70}$$

where we set $J^2\mathcal{C}_1 = \mathrm{Lor}(M) \times_M J^2 S$ and $J^2\mathcal{C}_2 = \mathrm{Lor}(M) \times_M J^2 A_{m-1}(M)$.

The dual Lagrangian (5.3.63) lives on $J\mathcal{C}_2$ and hence can be pulled back along the map $J\bar{p}$ on $J^2\mathcal{C}_1$. Hence one obtains

$$\begin{aligned}
(J\bar{p})^* L^* &= -\frac{\sqrt{g}}{2}\left(g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2 - 2g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi - \frac{1}{\mu^2}\left(\Box\psi\Box\psi + \mu^4\psi^2\right)\right) = \\
&= -\frac{\sqrt{g}}{2}\left(g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right) + \frac{\sqrt{g}}{2\mu^2}\left(\Box\psi - \mu^2\psi\right)^2 + \nabla_\beta\left(\sqrt{g}g^{\alpha\beta}\nabla_\alpha\psi\psi\right)
\end{aligned} \tag{5.3.71}$$

Now, the last term is a pure divergence, thus it does not affect solutions. The other term vanishes on solutions which, in general, is not clear whether it affects solutions. Solutions of the theory depends on the Lagrangian, not on its value on shell. Overall, we can say that the map does not even preserves the Lagrangian dynamics, still it induces a map which sends solutions into solutions, as we showed directly.

For the term $l = \frac{\sqrt{g}}{2\mu^2}\left(\Box\psi - \mu^2\psi\right)^2$ we have the variation

$$\delta l = \frac{\sqrt{g}}{\mu^2}\left(\Box\psi - \mu^2\psi\right)\left(\Box\delta\psi - \mu^2\delta\psi\right) - \frac{\sqrt{g}}{2}t_{\mu\nu}\delta g^{\mu\nu} + \mathrm{Div} = \frac{\sqrt{g}}{\mu^2}\left(\Box\left(\Box\psi - \mu^2\psi\right) - \mu^2\left(\Box\psi - \mu^2\psi\right)\right)\delta\psi - \frac{\sqrt{g}}{2}t_{\mu\nu}\delta g^{\mu\nu} + \mathrm{Div} \tag{5.3.72}$$
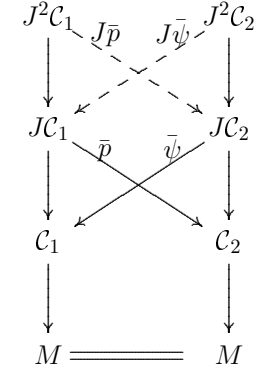
That shows two things: first, that the field equation does change, so that the modification to Lagrangian cannot be a pure divergence.

Second, that the modification of the field equation still vanishes along solutions (since $\Box\psi - \mu^2\psi = 0$). Thus any solution of the original Klein–Gordon theory is still a solution for the theory based on $(J\bar{p})^*L^*$.

The other way around one has

$$
\begin{aligned}
(J\bar{\psi})^*L =& \frac{1}{2\mu^4\sqrt{g}}\left(-g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha\nabla_\nu\nabla_\beta p^\beta - (2-1)\mu^2(\nabla_\mu p^\mu)(\nabla_\nu p^\nu)\right) = \\
=& -\frac{1}{2\mu^4\sqrt{g}}\left(g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha\nabla_\nu\nabla_\beta p^\beta + \mu^4 g_{\mu\nu}p^\mu p^\nu + 2\mu^2\nabla_\mu p^\mu\nabla_\nu p^\nu\right) + \frac{1}{2\sqrt{g}}\left(g_{\mu\nu}p^\mu p^\nu + \tfrac{1}{\mu^2}(\nabla_\mu p^\mu)(\nabla_\nu p^\nu)\right) = \\
=& -\frac{1}{2\mu^4\sqrt{g}}\left(g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha\nabla_\nu\nabla_\beta p^\beta + \mu^4 g_{\mu\nu}p^\mu p^\nu - 2\mu^2\nabla_\nu\nabla_\mu p^\mu p^\nu\right) + \frac{1}{2\sqrt{g}}\left(g_{\mu\nu}p^\mu p^\nu + \tfrac{1}{\mu^2}(\nabla_\mu p^\mu)(\nabla_\nu p^\nu)\right) - \nabla_\nu\left(\frac{1}{\mu^2\sqrt{g}}\nabla_\mu p^\mu p^\nu\right) = \\
=& -\frac{1}{2\mu^4\sqrt{g}}\left(g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha - \mu^2 p^\mu\right)g_{\nu\sigma}\left(g^{\sigma\rho}\nabla_\rho\nabla_\beta p^\beta - \mu^2 p^\sigma\right) + \frac{1}{2\sqrt{g}}\left(g_{\mu\nu}p^\mu p^\nu + \tfrac{1}{\mu^2}(\nabla_\mu p^\mu)(\nabla_\nu p^\nu)\right) - \nabla_\nu\left(\frac{1}{\mu^2\sqrt{g}}\nabla_\mu p^\mu p^\nu\right)
\end{aligned}
\tag{5.3.73}
$$

Thus, again in this case, the map which establishes the equivalence does not even preserve the Lagrangian dynamics. The pure divergence does not affect solutions while the difference $l^* = -\frac{1}{2\mu^4\sqrt{g}}\left(g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha - \mu^2 p^\mu\right)g_{\nu\sigma}\left(g^{\sigma\rho}\nabla_\rho\nabla_\beta p^\beta - \mu^2 p^\sigma\right)$ does contribute to field equation (so it is not a pure divergence) though it does not affect solutions, anyway.

The variation of the difference l^* is

$$
\delta l^* = -\frac{1}{\mu^4\sqrt{g}}\left(g^{\sigma\rho}\nabla_\rho\nabla_\beta - \mu^2\mathbb{I}\right)\left(g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha - \mu^2 p^\mu\right)g_{\nu\sigma}\delta p^\sigma - \frac{\sqrt{g}}{2}t^*_{\mu\nu}\delta g^{\mu\nu} + \text{Div}
\tag{5.3.74}
$$

Of course, the contribution to field equation does vanish along solutions of the theory based on L^*, since, in that case, one has $g^{\mu\nu}\nabla_\mu\nabla_\alpha p^\alpha = \mu^2 p^\mu$.

What we want to point out here is that dynamical equivalence can connect quite different theories, with a different number of degrees of freedom, different fundamental fields, and a different order, still being a quite precise mathematical equivalence.

When dealing with maps which depends on derivatives, one is forced to enlarge the definition of *preserving the dynamics* from *preserving the Lagrangian* (possibly modulo pure divergence) to *preserving the solutions*.

Should I go into on shell symmetries?

This example shows that two dynamics can have the *"same solutions"* but rather different field equations (hence their action differ by more than a constant due to a pure divergence). Such a transformation will be called a *symmetry* which is more general that a (generalised) Lagrangian symmetry.

Let us also notice that most of the interpretation of physics of fields (e.g. what is the kinetic energy, the minimal coupling, which interactions are described by a potential) are often not invariant by these Routh transformations. Accordingly, if we are lucky, we have to adjust the interpretation every time we perform Routh transform (or when we compose two of them, i.e. we make a duality transformation). If we are not, they are inessential guidelines for interpretation, since they cannot be essential for describing the physical world if they cannot be formulated independently of the mathematical formulation *we choose* for describing a physical system.

General equivalences among first order theories

We are now able to state and prove some general theorems out of these examples.

Let us first consider fields (y^i, z) (where z, collectively, represents a family of fields) and a first order dynamics for them, namely a covariant Lagrangian

$$\boldsymbol{L} = L(j^1 y, j^1 z)\, d\sigma \tag{5.3.75}$$

which variation is expressed as

$$\delta L = p_i \delta y^i + p_i^\mu \delta y_\mu^i + p\delta z + p^\mu \delta z \tag{5.3.76}$$

where we defined the momenta as usual, in particular

$$p_i = \frac{\partial L}{\partial y^i}(j^1 y, j^1 z) \qquad\qquad p_i^\mu = \frac{\partial L}{\partial y_\mu^i}(j^1 y, j^1 z) \tag{5.3.77}$$

Now, suppose we are able to solve the definition of p_i^μ with respect to y_μ^i, i.e.

$$y_\mu^i = Y_\mu^i(y, p, j^1 z) \tag{5.3.78}$$

and we integrate by parts the variation as

$$\delta L = p_i \delta y^i + \delta\left(p_i^\mu y_\mu^i\right) - y_\mu^i \delta p_i^\mu + p\delta z + p^\mu \delta z \qquad \Rightarrow \delta\left(L - p_i^\mu y_\mu^i\right) = p_i \delta y^i - y_\mu^i \delta p_i^\mu + p\delta z + p^\mu \delta z \tag{5.3.79}$$

Thus we can define the Routh function

$$W(y, p, j^1 z) = L(y, Y(y, p, j^1 z), j^1 z) - p_i^\mu Y_\mu^i(y, p, j^1 z) \tag{5.3.80}$$

as well as the Helmholtz Lagrangian

$$L_H(j^1 y, j^0 p, j^1 z) = p_i^\mu y_\mu^i + W(y, p, j^1 z) = p_i^\mu\left(y_\mu^i - Y_\mu^i(y, p, j^1 z)\right) + L(y, Y(y, p, j^1 z), j^1 z) \tag{5.3.81}$$

The original field equations for L are

$$d_\mu p_i^\mu = p_i \qquad\qquad d_\mu p^\mu = p \tag{5.3.82}$$

In terms of the Routh function $W(y, p, j^1 z)$, we have Hamilton equations for y and Euler–Lagrange equation for z, i.e.

$$\begin{cases} d_\mu y^i = -\dfrac{\partial W}{\partial p_i^\mu} \\[2mm] d_\mu p_i^\mu = \dfrac{\partial W}{\partial y^i} \end{cases} \qquad\qquad d_\mu\left(\frac{\partial W}{\partial z_\mu}\right) = \frac{\partial W}{\partial z} \tag{5.3.83}$$

which are in fact equivalent to the original field equations.

By the definition (5.3.80) of the Routh function, we have a relation among momenta, namely

$$\frac{\partial W}{\partial p_i^\mu} = \frac{\partial L}{\partial y_\nu^j} \frac{\partial Y_\nu^j}{\partial p_i^\mu} - Y_\mu^i - p_j^\nu \frac{\partial Y_\nu^j}{\partial p_i^\mu} \qquad \frac{\partial W}{\partial y^i} = \frac{\partial L}{\partial y^i} + \frac{\partial L}{\partial y_\nu^j} \frac{\partial Y_\nu^j}{\partial y^i} - p_j^\nu \frac{\partial Y_\nu^j}{\partial y^i}$$

$$\frac{\partial W}{\partial z_\mu} = \frac{\partial L}{\partial y_\nu^j} \frac{\partial Y_\nu^j}{\partial z_\mu} + \frac{\partial L}{\partial z_\mu} - p_j^\nu \frac{\partial Y_\nu^j}{\partial z_\mu} \qquad \frac{\partial W}{\partial z} = \frac{\partial L}{\partial y_\nu^j} \frac{\partial Y_\nu^j}{\partial z} + \frac{\partial L}{\partial z} - p_j^\nu \frac{\partial Y_\nu^j}{\partial z}$$

(5.3.84)

By substituiting these into the equations (5.3.83), we obtain directly the original field equations (5.3.82).

Thus for any solution $(y^i(x), z(x))$ of the original equations (5.3.82), we can define $p_i^\mu(x) = \frac{\partial L}{\partial y_\mu^i}(j^1 y(x), j^1 z(x))$ so that $(y^i(x), p_i^\mu(x), z(x))$ is a solution of the equation (5.3.83). Vice versa, for any solution $(y^i(x), p_i^\mu(x), z(x))$ of the equation (5.3.83), one has a solution $(y^i(x), z(x))$ of the original equations (5.3.82) and, moreover, one has $p_i^\mu(x) = \frac{\partial L}{\partial y_\mu^i}(j^1 y(x), j^1 z(x))$. Accordingly, the descriptions by L and W are equivalent.

Let us stress that in the equations (5.3.80) for y^i and p_i^μ, the first equation reproduces the Routh transformation (precisely, its inverse) while the second reproduces the original dynamics.

The Euler–Lagrange equations for the Helmholtz Lagrangian are

$$d_\mu \left(\frac{\partial L_H}{\partial y_\mu^i} \right) = \frac{\partial L_H}{\partial y^i} \qquad d_\mu \left(\frac{\partial L_H}{\partial p_{i\mu}^\alpha} \right) = 0 = \frac{\partial L_H}{\partial p_i^\alpha} \qquad d_\mu \left(\frac{\partial L_H}{\partial z_\mu} \right) = \frac{\partial L_H}{\partial z}$$

(5.3.85)

which are again equivalent to the original dynamics together with the Routh transformation.

From the definition (5.3.81) of the Helmholtz Lagrangian, we have *naive* momenta

$$\frac{\partial L_H}{\partial y_\mu^i} = p_i^\mu \qquad \frac{\partial L_H}{\partial y^i} = \frac{\partial W}{\partial y^i} = \frac{\partial L}{\partial y^i}$$

$$\frac{\partial L_H}{\partial p_{i\mu}^\alpha} = 0 \qquad \frac{\partial L_H}{\partial p_i^\alpha} = y_\alpha^i + \frac{\partial W}{\partial p_i^\alpha} = y_\alpha^i - Y_\mu^i$$

$$\frac{\partial L_H}{\partial z_\mu} = \frac{\partial W}{\partial z_\mu} = \frac{\partial L}{\partial z_\mu} \qquad \frac{\partial L_H}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial L}{\partial z}$$

(5.3.86)

so that the first equation in (5.3.85) for L_H reproduces the dynamics for y, the second equation reproduces the Routh transformation, and the third equation reproduces the dynamics for z.

Thus using Routh transform, when possible, is an actual improvement since the transformation to be done on fields is suggested by the integration by parts we made on the variation of the Lagrangian and we have a prescription to define a new Lagrangian so that dynamics is automatically equivalent to the original Lagrangian. That allows us to connect quite different formulations of Lagrangian dynamics, in a much wider context than simple redefinition of fields induced by maps between configuration bundles. In this case, contrarily to maps between configuration bundles, the corresponding field theories may be defined on configuration bundles with different dimensions, i.e. with a different number of fundamental fields. Accordingly, the number of fundamental fields looses its physical meaning and it becomes a mathematical choice.

Routh transforms, of course, need that we are able to invert part of the momenta in terms of part of the field derivatives. Lagrangians in field theories are often, almost always, degenerate but often that is possible on a subset of fields. In other words, even when the Lagrangian degeneracy prevents Legendre transformations, it often allows *partial* Legendre transformations, i.e. Routh transformations.

Although Routh transformations allow a more systematic comparison among seemingly different theories, let us stress that the general problem of excluding that two specific field theories are in fact dynamically equivalent with respect to some higher order map is still well out of reach. Even though Routh transforms provide maps along which dynamical equivalence holds true, there is no clue that *any* dynamical equivalence can be generated this way.

Finally, let us point out that one can also differently integrate by parts the original Lagrangian variation (5.3.76), e.g.

$$\delta\left(L - p_i y^i\right) = -y^i \delta p_i + p_i^\mu \delta y_\mu^i + p\delta z + p^\mu \delta z \tag{5.3.87}$$

this time solving the definition of p_i with respect to $y^i = Y^i(y_\nu^j, p_j, j^1 z)$ to obtain a different Routh function

$$W'(y, p, j^1 z) = L(Y(y, p, j^1 z), y, j^1 z) - p_i Y^i(y, p, j^1 z) \tag{5.3.88}$$

We can check directly that equations form this other Routh function W', namely

$$\begin{cases} p_i = d_\mu \dfrac{\partial W'}{\partial y_\mu^i} \\[2mm] y^i = -\dfrac{\partial W'}{\partial p_i} \end{cases} \qquad d_\mu\left(\dfrac{\partial W'}{\partial z_\mu}\right) = \dfrac{\partial W'}{\partial z} \tag{5.3.89}$$

are equivalent to the original equations (and the Routh transform) though they are not in the form of Hamilton equations.

Momenta are

$$\frac{\partial W'}{\partial y_\mu^i} = p_j \frac{\partial Y^j}{\partial y_\mu^i} + p_i^\mu - p_j \frac{\partial Y^j}{\partial y_\mu^i} \qquad \frac{\partial W'}{\partial p_i} = p_j \frac{\partial Y^j}{\partial p_i} - Y^i - p_j \frac{\partial Y^j}{\partial p_i} \qquad \frac{\partial W'}{\partial z} = p_j \frac{\partial Y^j}{\partial z} + \frac{\partial L}{\partial z} - p_j \frac{\partial Y^j}{\partial z} \qquad \frac{\partial W'}{\partial z_\mu} = p_j \frac{\partial Y^j}{\partial z_\mu} + \frac{\partial L}{\partial z_\mu} - p_j \frac{\partial Y^j}{\partial z_\mu} \tag{5.3.90}$$

from which equivalence easily follows.

All this discussion about equivalence is essentially local. However, if we control transformation rules of fields p_i^μ and we define a bundle on which they are global sections, then the Routh transform can be easily set up as a global bundle map (as we did for Klein–Gordon field allowing prolongations of the configuration bundle to be involved) and the equivalence can be established at a global level.

Also, we did not used here covariant derivatives. By using total derivatives instead, terms fail to be geometrically meaningful. Of can however, take these guidelines and repeat the computation using covariant derivatives when possible. One only needs to devote extra care to the fact that covariant derivatives and variations do not commute.

Standard purely metric and metric-affine formalism

The situation with Routh transform is greatly improved. Now we are able to consider a theory and, integrating by parts with respect to δ its variation, to define both the map and the new dynamics so that the two field theories are automatically dynamically equivalent. It may not cover the most general situation though, we shall see, that it covers a lot of cases.

We are ready to start and prove that, in standard GR, the purely metric and metric-affine formulations *are* in fact dynamically equivalent, provided one defines the map correctly. That is contrary to what is generally believed (i.e. that equivalence does not hold in general).

Before considering the most general equivalence, let us, however, prove from scratch the equivalence in the vacuum case (just for practicing and comparing with what we said above). Let us start from the standard purely metric formulation. We can choose $\hat{g}^{\mu\nu} = \sqrt{g}\, g^{\mu\nu}$ as a fundamental field (since one can easily check that its expression in terms of $g_{\mu\nu}$ is one-to-one and hence can be inverted). Thus we consider the Lagrangian

$$L = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu} R_{\mu\nu} \tag{5.3.91}$$

where $R_{\mu\nu} = d_\alpha u^\alpha_{\mu\nu} - u_{\mu\nu}$ is the Ricci tensor of g and we set $u^\alpha_{\mu\nu} := \{g\}^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\{g\}_{\nu)}$ and $u_{\mu\nu} := \{g\}^\alpha_{\beta\mu}\{g\}^\beta_{\alpha\nu} - \{g\}_\alpha \{g\}^\alpha_{\mu\nu}$. As usual, the object $u^\alpha_{\mu\nu}$ is in a one-to-one correspondence with the Levi Civita connection.

From the standard formula of Ricci tensor we have

$$R_{\mu\nu} = R_{(\mu\nu)} = d_\alpha\{g\}^\alpha_{\mu\nu} - d_{(\nu}\{g\}^\alpha_{\mu)\alpha} + \{g\}^\alpha_{\beta\alpha}\{g\}^\beta_{\mu\nu} - \{g\}^\alpha_{\beta\nu}\{g\}^\beta_{\mu\alpha} = d_\alpha\left(\{g\}^\alpha_{\mu\nu} - \delta^\alpha_{(\nu}\{g\}_{\mu)}\right) - u_{\mu\nu} = d_\alpha u^\alpha_{\mu\nu} - u_{\mu\nu} \tag{5.3.92}$$

The variation of the Lagrangian is

$$\delta L = \tfrac{1}{2\kappa}R_{\mu\nu}\delta\hat{g}^{\mu\nu} + \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\delta R_{\mu\nu} = \tfrac{1}{2\kappa}\left(d_\alpha u^\alpha_{\mu\nu} - u_{\mu\nu}\right)\delta\hat{g}^{\mu\nu} + \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\nabla_\alpha\delta u^\alpha_{\mu\nu} = \tfrac{1}{2\kappa}\left(d_\alpha u^\alpha_{\mu\nu} - u_{\mu\nu}\right)\delta\hat{g}^{\mu\nu} + \overset{*}{\nabla}_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\delta u^\alpha_{\mu\nu}\right) - \boxed{\tfrac{1}{2\kappa}\nabla_\alpha\hat{g}^{\mu\nu}\delta u^\alpha_{\mu\nu}} =$$
$$= \tfrac{1}{2\kappa}\left(d_\alpha u^\alpha_{\mu\nu} - u_{\mu\nu}\right)\delta\hat{g}^{\mu\nu} - \tfrac{1}{2\kappa}\left(d_\alpha u^\alpha_{\mu\nu}\delta\hat{g}^{\mu\nu} + u^\alpha_{\mu\nu}\delta\hat{g}^{\mu\nu}_\alpha\right) + \delta d_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u^\alpha_{\mu\nu}\right) = -\tfrac{1}{2\kappa}\left(u_{\mu\nu}\delta\hat{g}^{\mu\nu} + u^\alpha_{\mu\nu}\delta\hat{g}^{\mu\nu}_\alpha\right) + \delta d_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u^\alpha_{\mu\nu}\right) \tag{5.3.93}$$

where the covariant derivative ∇_μ is associated to the Levi Civita connection of the metric g. Let us remark that $u_{\mu\nu}$ and $u^\alpha_{\mu\nu}$, essentially (i.e. modulo a constant), are the momenta of the Lagrangian with respect to $\hat{g}^{\mu\nu}$ and $\hat{g}^{\mu\nu}_\alpha$.

Let us now prove some lemmas.

Lemma: *The following identity holds true*

$$\hat{g}^{\mu\nu}_\alpha = \{g\}_\alpha\hat{g}^{\mu\nu} - 2\hat{g}^{\lambda(\nu}\{g\}^{\mu)}_{\lambda\alpha} \tag{5.3.94}$$

Proof: *Since $\hat{g}^{\mu\nu}$ is a function of g, its covariant derivative with respect to the Levi Civita connection is vanishing.*

On the other hand, it is a tensor density of weight 1, so one has

$$0 = \nabla_\alpha\hat{g}^{\mu\nu} = \hat{g}^{\mu\nu}_\alpha + \{g\}^\mu_{\lambda\alpha}\hat{g}^{\lambda\nu} + \hat{g}^{\mu\lambda}\{g\}^\nu_{\lambda\alpha} - \{g\}_\alpha\hat{g}^{\mu\nu} \qquad \Rightarrow \hat{g}^{\mu\nu}_\alpha = \{g\}_\alpha\hat{g}^{\mu\nu} - 2\hat{g}^{\lambda(\nu}\{g\}^{\mu)}_{\lambda\alpha} \tag{5.3.95}$$

∎

Lemma: *The following identity holds true*

$$u^\alpha_{\mu\nu}\hat{g}^{\mu\nu}_\alpha = -2\hat{g}^{\mu\nu}u_{\mu\nu} \tag{5.3.96}$$

Proof: *We have*

$$u^\alpha_{\mu\nu}\hat{g}^{\mu\nu}_\alpha = \left(\{g\}^\alpha_{\mu\nu} - \delta^\alpha_\mu\{g\}_\nu\right)\left(\{g\}_\alpha\hat{g}^{\mu\nu} - \{g\}^\mu_{\lambda\alpha}\hat{g}^{\lambda\nu} - \hat{g}^{\mu\lambda}\{g\}^\nu_{\lambda\alpha}\right) = \underline{\{g\}_\alpha\{g\}^\alpha_{\mu\nu}\hat{g}^{\mu\nu}} - \underline{\underline{\{g\}^\alpha_{\mu\nu}\{g\}^\mu_{\lambda\alpha}\hat{g}^{\lambda\nu}}} - \underline{\{g\}^\alpha_{\beta\mu}\{g\}^\beta_{\alpha\nu}\hat{g}^{\mu\nu}} - \{g\}_\nu\left(\{g\}_\alpha\hat{g}^{\alpha\nu} - \{g\}_\alpha\hat{g}^{\alpha\nu} - \underline{\hat{g}^{\alpha\lambda}\{g\}^\nu_{\lambda\alpha}}\right) =$$
$$= -2\left(\{g\}^\alpha_{\beta\nu}\{g\}^\beta_{\alpha\mu} - \{g\}_\alpha\{g\}^\alpha_{\mu\nu}\right)\hat{g}^{\mu\nu} = -2\hat{g}^{\mu\nu}u_{\mu\nu} \tag{5.3.97}$$

By using these lemmas, we can easily show that the variation of the standard purely metric Lagrangian reads as

$$\delta U = -\tfrac{1}{2\kappa}\left(u_{\mu\nu}\delta\hat{g}^{\mu\nu} + u_{\mu\nu}^{\alpha}\delta\hat{g}_{\alpha}^{\mu\nu}\right) \tag{5.3.98}$$

where we set $U := \tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}R_{\mu\nu} - d_{\alpha}\left(\hat{g}^{\mu\nu}u_{\mu\nu}^{\alpha}\right)\right)$ which is, in fact, the first order non-covariant Einstein *"Lagrangian"*.

We have

$$U = \tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}\left(d_{\alpha}u_{\mu\nu}^{\alpha} - u_{\mu\nu}\right) - d_{\alpha}\hat{g}^{\mu\nu}u_{\mu\nu}^{\alpha} - \hat{g}^{\mu\nu}d_{\alpha}u_{\mu\nu}^{\alpha}\right) = -\tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}u_{\mu\nu} + \hat{g}_{\alpha}^{\mu\nu}u_{\mu\nu}^{\alpha}\right) = -\tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}u_{\mu\nu} - 2\hat{g}^{\mu\nu}u_{\mu\nu}\right) = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u_{\mu\nu} \tag{5.3.99}$$

which in fact does not depend on the second derivatives of the metric.

From the variation of the Lagrangian U (or, equivalently, L), we recover field equations easily as

$$d_{\alpha}u_{\mu\nu}^{\alpha} - u_{\mu\nu} = R_{\mu\nu} = 0 \tag{5.3.100}$$

which, in fact, agree with vacuum Einstein equations.

At this point, we can select $\hat{g}^{\mu\nu}$ and $\tilde{u}_{\mu\nu}^{\alpha}$ as new fundamental fields and apply the Routh transform accordingly. This prescribes the transformations

$$\tilde{u}_{\mu\nu}^{\alpha} = u_{\mu\nu}^{\alpha} \qquad \Longleftrightarrow \quad \tilde{\Gamma}_{\mu\nu}^{\alpha} = \{g\}_{\mu\nu}^{\alpha} \tag{5.3.101}$$

which can be inverted as

$$\hat{g}_{\alpha}^{\mu\nu} = \tilde{\Gamma}_{\alpha}\hat{g}^{\mu\nu} - 2\hat{g}^{\lambda(\nu}\tilde{\Gamma}_{\lambda\alpha}^{\mu)} \tag{5.3.102}$$

After the integration by parts, we have

$$\delta W = \tfrac{1}{2\kappa}\delta\left(\hat{g}^{\mu\nu}u_{\mu\nu} + u_{\mu\nu}^{\alpha}\hat{g}_{\alpha}^{\mu\nu}\right) = -\tfrac{1}{2\kappa}u_{\mu\nu}\delta\hat{g}^{\mu\nu} + \tfrac{1}{2\kappa}\hat{g}_{\alpha}^{\mu\nu}\delta u_{\mu\nu}^{\alpha} \tag{5.3.103}$$

and define the Routh function as

$$W(\hat{g},\tilde{u}) = U + \tfrac{1}{2\kappa}u_{\mu\nu}^{\alpha}\hat{g}_{\alpha}^{\mu\nu} = \tfrac{1}{2\kappa}u_{\mu\nu}\hat{g}^{\mu\nu} - 2\tfrac{1}{2\kappa}u_{\mu\nu}\hat{g}^{\mu\nu} = -\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{u}_{\mu\nu} \tag{5.3.104}$$

where we set $\tilde{u}_{\mu\nu} := \tilde{\Gamma}_{\beta\mu}^{\alpha}\tilde{\Gamma}_{\alpha\nu}^{\beta} - \tilde{\Gamma}_{\alpha}\tilde{\Gamma}_{\mu\nu}^{\alpha}$ for $u_{\mu\nu}$ computed along the Routh map.

The variation of W reads as

$$\delta W = -\tfrac{1}{2\kappa}\left(\tilde{u}_{\mu\nu}\delta\hat{g}^{\mu\nu} + \hat{g}^{\mu\nu}\delta\tilde{u}_{\mu\nu}\right) \tag{5.3.105}$$

and it induces Hamilton-like equations

$$\hat{g}_{\alpha}^{\mu\nu} = -\frac{\partial W}{\partial\tilde{u}_{\mu\nu}^{\alpha}} \qquad\qquad -\tfrac{1}{2\kappa}d_{\alpha}\tilde{u}_{\mu\nu}^{\alpha} = \frac{\partial W}{\partial\hat{g}^{\mu\nu}} = -\frac{1}{2\kappa}\tilde{u}_{\mu\nu} \tag{5.3.106}$$

The first equation reproduces the Routh map, the second equation the correct vacuum standard Einstein equation $\tilde{R}_{(\mu\nu)} = 0$.

To check that, we need to expand the first equation (just watch out and expand symmetrisations to do it properly)

$$\hat{g}^{\mu\nu}_{\alpha} = -\frac{\partial W}{\partial \tilde{\Gamma}^{\beta}_{\rho\sigma}} \frac{\partial \tilde{\Gamma}^{\beta}_{\rho\sigma}}{\partial \tilde{u}^{\alpha}_{\mu\nu}} = \left(-\hat{g}^{\lambda\sigma}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \hat{g}^{\lambda\rho}\tilde{\Gamma}^{\sigma}_{\beta\lambda} + \hat{g}^{\rho\sigma}\tilde{\Gamma}_{\beta} + \tfrac{1}{2}\hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{\sigma}_{\lambda\epsilon}\delta^{\rho}_{\beta} + \tfrac{1}{2}\hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{\rho}_{\lambda\epsilon}\delta^{\sigma}_{\beta}\right)\left(\delta^{\beta}_{\alpha}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \tfrac{1}{m-1}\delta^{\beta}_{\rho}\delta^{\mu}_{\alpha}\delta^{\nu}_{\sigma} - \tfrac{1}{m-1}\delta^{\beta}_{\rho}\delta^{\nu}_{\alpha}\delta^{\mu}_{\sigma}\right) =$$

$$= -2\hat{g}^{\lambda(\mu}\tilde{\Gamma}^{\nu)}_{\alpha\lambda} + \hat{g}^{\mu\nu}\tilde{\Gamma}_{\alpha} + \hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\mu}_{\lambda\epsilon}\delta^{\nu)}_{\alpha} - \tfrac{1}{m-1}\left(-\hat{g}^{\lambda(\mu}\delta^{\nu)}_{\alpha}\tilde{\Gamma}_{\lambda} - \hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha} + \hat{g}^{\lambda(\mu}\delta^{\nu)}_{\alpha}\tilde{\Gamma}_{\lambda} + \tfrac{m+1}{2}\hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha} - \hat{g}^{\lambda(\mu}\delta^{\nu)}_{\alpha}\tilde{\Gamma}_{\lambda} - \tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha} + \hat{g}^{\lambda(\mu}\delta^{\nu)}_{\alpha}\tilde{\Gamma}_{\lambda} + \tfrac{m+1}{2}\hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha}\right) =$$

$$= -2\hat{g}^{\lambda(\mu}\tilde{\Gamma}^{\nu)}_{\alpha\lambda} + \hat{g}^{\mu\nu}\tilde{\Gamma}_{\alpha} + \hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\mu}_{\lambda\epsilon}\delta^{\nu)}_{\alpha} - \tfrac{m+1-2}{m-1}\hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha} = -2\hat{g}^{\lambda(\mu}\tilde{\Gamma}^{\nu)}_{\alpha\lambda} + \hat{g}^{\mu\nu}\tilde{\Gamma}_{\alpha} + \hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\mu}_{\lambda\epsilon}\delta^{\nu)}_{\alpha} - \hat{g}^{\lambda\epsilon}\tilde{\Gamma}^{(\nu}_{\epsilon\lambda}\delta^{\mu)}_{\alpha} = -2\hat{g}^{\lambda(\mu}\tilde{\Gamma}^{\nu)}_{\alpha\lambda} + \hat{g}^{\mu\nu}\tilde{\Gamma}_{\alpha}$$

$$(5.3.107)$$

which in fact agrees with (5.3.102).

Now, even if the Routh function singles out a dynamic which is dynamically equivalent to the original standard purely metric GR dynamics, still it does not define a Lagrangian field theory. To do that we need to go to the Helmholtz Lagrangian for it, namely:

$$L_H = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}d_\alpha\tilde{u}^{\alpha}_{\mu\nu} + W = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\left(d_\alpha\tilde{u}^{\alpha}_{\mu\nu} - \tilde{u}_{\mu\nu}\right) = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{R}_{\mu\nu} \tag{5.3.108}$$

which is precisely the vacuum standard Palatini Lagrangian. That proves dynamical equivalence which has already been checked also directly.

Now we can consider a more general metric-affine theory to compare with. Let us consider a quite general standard purely metric dynamics for a metric g and a collection of (first order) matter fields ψ

$$L = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}R_{\mu\nu} + L_m(j^1 g, j^1 \psi) \tag{5.3.109}$$

We will not specify much about how matter couples to the metric, except that the relevant Lagrangians are covariant. The variation of this Lagrangian density is thence

$$\delta L = \tfrac{1}{2\kappa}\delta\hat{g}^{\mu\nu}R_{\mu\nu} + \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\nabla_\alpha\delta u^{\alpha}_{\mu\nu} - \tfrac{1}{2}T_{\mu\nu}\delta\hat{g}^{\mu\nu} - \tfrac{1}{2}T^{\alpha}_{\mu\nu}\delta\hat{g}^{\mu\nu}_{\alpha} + p\delta\psi + p^{\alpha}\delta\psi_{\alpha} =$$

$$= \tfrac{1}{2\kappa}\delta\hat{g}^{\mu\nu}R_{\mu\nu} + \overset{*}{\nabla}_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\delta u^{\alpha}_{\mu\nu}\right) + \boxed{\tfrac{1}{2\kappa}\nabla_\alpha\hat{g}^{\mu\nu}\delta u^{\alpha}_{\mu\nu}} - \tfrac{1}{2}T_{\mu\nu}\delta\hat{g}^{\mu\nu} - \tfrac{1}{2}T^{\alpha}_{\mu\nu}\delta\hat{g}^{\mu\nu}_{\alpha} + p\delta\psi + p^{\alpha}\delta\psi_{\alpha} =$$

$$= \tfrac{1}{2\kappa}\delta\hat{g}^{\mu\nu}(d_\alpha u^{\alpha}_{\mu\nu} - u_{\mu\nu}) + \delta\left(d_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u^{\alpha}_{\mu\nu}\right)\right) - \tfrac{1}{2\kappa}d_\alpha u^{\alpha}_{\mu\nu}\delta\hat{g}^{\mu\nu} - \tfrac{1}{2\kappa}u^{\alpha}_{\mu\nu}\delta\hat{g}^{\mu\nu}_{\alpha} - \tfrac{1}{2}T_{\mu\nu} - \tfrac{1}{2}T^{\alpha}_{\mu\nu}\delta\hat{g}^{\mu\nu}_{\alpha} + p\delta\psi + p^{\alpha}\delta\psi_{\alpha} =$$

$$= -\tfrac{1}{2\kappa}\left(u_{\mu\nu} + \kappa T_{\mu\nu}\right)\delta\hat{g}^{\mu\nu} - \tfrac{1}{2\kappa}\left(u^{\alpha}_{\mu\nu} + \kappa T^{\alpha}_{\mu\nu}\right)\delta\hat{g}^{\mu\nu}_{\alpha} + p\delta\psi + p^{\alpha}\delta\psi_{\alpha} + \delta\left(d_\alpha\left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u^{\alpha}_{\mu\nu}\right)\right)$$

$$(5.3.110)$$

where we set

$$-\tfrac{1}{2}T_{\mu\nu} := \frac{\partial L_m}{\partial\hat{g}^{\mu\nu}} \qquad\qquad -\tfrac{1}{2}T^{\alpha}_{\mu\nu} := \frac{\partial L_m}{\partial\hat{g}^{\mu\nu}_{\alpha}} \tag{5.3.111}$$

Since $\delta\hat{g}^{\mu\nu}_{\alpha}$ is not a tensor neither the coefficients $T_{\mu\nu}$ and $T^{\alpha}_{\mu\nu}$ *a priori* are. In particular, we can show that $T^{\alpha}_{\mu\nu}$ is in fact a tensor as well as the combination $T_{\mu\nu} - d_\alpha T^{\alpha}_{\mu\nu}$ is, while $T_{\mu\nu}$ alone is not.

If we consider two coordinate systems and we use the fact that the Lagrangian density is, in fact, a scalar density of weight 1, we have transformation rules

$$\delta\hat{g}'^{\mu\nu} = \bar{J}J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma} \qquad\qquad \delta\hat{g}'^{\mu\nu}_{\alpha} = \bar{J}J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma}_{\beta}\bar{J}^{\beta}_{\alpha} + \bar{J}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right)\delta\hat{g}^{\rho\sigma} + \bar{J}J^{\theta}_{\epsilon}\bar{J}^{\epsilon}_{\theta\alpha}J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma} \tag{5.3.112}$$

so that, for transformation rules of a linear combination, we have

$$-\tfrac{1}{2}\left(T'_{\mu\nu}\delta\hat{g}'^{\mu\nu} + T'^{\alpha}_{\mu\nu}\delta\hat{g}'^{\mu\nu}_{\alpha}\right) = -\tfrac{1}{2}\bar{J}\left(T'_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma} + T'^{\alpha}_{\mu\nu}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma}_{\beta}\bar{J}^{\beta}_{\alpha} + d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right)\delta\hat{g}^{\rho\sigma} + J^{\theta}_{\epsilon}\bar{J}^{\epsilon}_{\theta\alpha}J^{\mu}_{\rho}J^{\nu}_{\sigma}\delta\hat{g}^{\rho\sigma}\right)\right) =$$
$$= -\tfrac{1}{2}\bar{J}\left(\left(T'_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} + T'^{\alpha}_{\mu\nu}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right) + T'^{\alpha}_{\mu\nu}J^{\theta}_{\epsilon}\bar{J}^{\epsilon}_{\theta\alpha}J^{\mu}_{\rho}J^{\nu}_{\sigma}\right)\delta\hat{g}^{\rho\sigma} + T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\alpha}\delta\hat{g}^{\rho\sigma}_{\beta}\right) \tag{5.3.113}$$

The transformation rules for the coefficients hence are

$$T_{\rho\sigma} = T'_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} + T'^{\alpha}_{\mu\nu}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right) + T'^{\alpha}_{\mu\nu}J^{\theta}_{\epsilon}\bar{J}^{\epsilon}_{\theta\alpha}J^{\mu}_{\rho}J^{\nu}_{\sigma} \qquad T^{\beta}_{\rho\sigma} = T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\alpha} \tag{5.3.114}$$

Thus we see that $T^{\beta}_{\rho\sigma}$ is in fact a tensor, while $T_{\rho\sigma}$ is not. We also can take the derivative of the second one obtaining

$$d_{\beta}T^{\beta}_{\rho\sigma} = J^{\theta}_{\beta}d'_{\theta}T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\alpha} + T'^{\alpha}_{\mu\nu}d_{\beta}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\alpha}\right) = d'_{\alpha}T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} + T'^{\alpha}_{\mu\nu}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right) + T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\epsilon\alpha}J^{\epsilon}_{\beta} \tag{5.3.115}$$

Thus we can consider the combination and its transformation rules

$$T_{\rho\sigma} - d_{\alpha}T^{\alpha}_{\rho\sigma} = T'_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} + \textcolor{red}{T'^{\alpha}_{\mu\nu}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right)} + \textcolor{blue}{T'^{\alpha}_{\mu\nu}J^{\theta}_{\epsilon}\bar{J}^{\epsilon}_{\theta\alpha}J^{\mu}_{\rho}J^{\nu}_{\sigma}} - d'_{\alpha}T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} - \textcolor{red}{T'^{\alpha}_{\mu\nu}d'_{\alpha}\left(J^{\mu}_{\rho}J^{\nu}_{\sigma}\right)} - \textcolor{blue}{T'^{\alpha}_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma}\bar{J}^{\beta}_{\epsilon\alpha}J^{\epsilon}_{\beta}} = \left(T'_{\mu\nu} - d'_{\alpha}T'^{\alpha}_{\mu\nu}\right)J^{\mu}_{\rho}J^{\nu}_{\sigma} \tag{5.3.116}$$

Thus it is a tensor, as well.

The variation suggests to set $\tilde{u}^{\alpha}_{\mu\nu} := u^{\alpha}_{\mu\nu} + \kappa T^{\alpha}_{\mu\nu}$, which implies $\tilde{u}_{\nu} := u_{\nu} + \kappa T_{\nu}$ (where we set $T_{\nu} := T^{\alpha}_{\alpha\nu}$) and, accordingly, it defines a connection

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \tilde{u}^{\alpha}_{\mu\nu} - \tfrac{2}{m-1}\delta^{\alpha}_{(\mu}\tilde{u}_{\nu)} = \{g\}^{\alpha}_{\mu\nu} + \kappa\left(T^{\alpha}_{\mu\nu} - \tfrac{2}{m-1}\delta^{\alpha}_{(\mu}T_{\nu)}\right) \tag{5.3.117}$$

and, for future convenience, let us set $K^{\alpha}_{\mu\nu} := \kappa\left(T^{\alpha}_{\mu\nu} - \tfrac{2}{m-1}\delta^{\alpha}_{(\mu}T_{\nu)}\right)$ (so that then $K_{\nu} := \kappa\left(1 - \tfrac{m}{m-1} - \tfrac{1}{m-1}\right)T_{\nu} = -\tfrac{2\kappa}{m-1}T_{\nu}$) as the difference $\tilde{\Gamma} - \{g\}$.

Then we have

$$u_{\mu\nu} + \kappa T_{\mu\nu} = \{g\}^{\alpha}_{\beta\mu}\{g\}^{\beta}_{\alpha\nu} - \{g\}_{\alpha}\{g\}^{\alpha}_{\mu\nu} + \kappa T_{\mu\nu} = \left(\tilde{\Gamma}^{\alpha}_{\beta\mu} - K^{\alpha}_{\beta\mu}\right)\left(\tilde{\Gamma}^{\beta}_{\alpha\nu} - K^{\beta}_{\alpha\nu}\right) - \left(\tilde{\Gamma}_{\alpha} - K_{\alpha}\right)\left(\tilde{\Gamma}^{\alpha}_{\mu\nu} - K^{\alpha}_{\mu\nu}\right) + \kappa T_{\mu\nu} =$$
$$= \tilde{u}_{\mu\nu} - \tilde{\Gamma}^{\beta}_{\alpha\nu}K^{\alpha}_{\beta\mu} - \tilde{\Gamma}^{\alpha}_{\beta\mu}K^{\beta}_{\alpha\nu} + K^{\alpha}_{\beta\mu}K^{\beta}_{\alpha\nu} + K^{\alpha}_{\mu\nu}\tilde{\Gamma}_{\alpha} + K_{\alpha}\tilde{\Gamma}^{\alpha}_{\mu\nu} - K_{\alpha}K^{\alpha}_{\mu\nu} + \kappa T_{\mu\nu} = \tilde{u}_{\mu\nu} + H_{\mu\nu} \tag{5.3.118}$$

where we set $\tilde{u}_{\mu\nu} := \tilde{\Gamma}^{\alpha}_{\beta\mu}\tilde{\Gamma}^{\beta}_{\alpha\nu} - \tilde{\Gamma}_{\alpha}\tilde{\Gamma}^{\alpha}_{\mu\nu}$ and

$$H_{\mu\nu} := \kappa T_{\mu\nu} \pm \boxed{\kappa d_{\alpha}T^{\alpha}_{\mu\nu}} - 2\{g\}^{\beta}_{\alpha(\nu}K^{\alpha}_{\mu)\beta} - 2K^{\beta}_{\alpha(\nu}K^{\alpha}_{\mu)\beta} + K^{\alpha}_{\mu\nu}\{g\}_{\alpha} + K^{\alpha}_{\mu\nu}K_{\alpha} + K_{\alpha}\{g\}^{\alpha}_{\mu\nu} + K_{\alpha}\tilde{K}^{\alpha}_{\mu\nu} + K^{\alpha}_{\beta\mu}K^{\beta}_{\alpha\nu} - K_{\alpha}K^{\alpha}_{\mu\nu} =$$
$$= \kappa\left(T_{\mu\nu} - d_{\alpha}T^{\alpha}_{\mu\nu}\right) + \left(\textcolor{orange}{\kappa d_{\alpha}T^{\alpha}_{\mu\nu}} - \{g\}^{\beta}_{\alpha\nu}K^{\alpha}_{\mu\beta} - \{g\}^{\beta}_{\alpha\mu}K^{\alpha}_{\nu\beta} + K^{\alpha}_{\mu\nu}\{g\}_{\alpha} + K_{\alpha}\{g\}^{\alpha}_{\mu\nu}\right) - \underline{2K^{\beta}_{\alpha\nu}K^{\alpha}_{\beta\mu}} + \textcolor{red}{K_{\alpha}\tilde{K}^{\alpha}_{\mu\nu}} + K^{\alpha}_{\mu\nu}K_{\alpha} + \underline{K^{\alpha}_{\beta\mu}K^{\beta}_{\alpha\nu}} - \textcolor{red}{K_{\alpha}K^{\alpha}_{\mu\nu}} =$$
$$= \kappa\left(T_{\mu\nu} - d_{\alpha}T^{\alpha}_{\mu\nu}\right) + \kappa\left(d_{\alpha}T^{\alpha}_{\mu\nu} - \{g\}^{\beta}_{\alpha\nu}T^{\alpha}_{\mu\beta} - \{g\}^{\beta}_{\alpha\mu}T^{\alpha}_{\nu\beta} + T^{\alpha}_{\mu\nu}\{g\}_{\alpha}\right) - \tfrac{2\kappa}{m-1}T_{\alpha}\{g\}^{\alpha}_{\mu\nu} +$$
$$- \tfrac{\kappa}{m-1}\left(\textcolor{green}{-2T_{\alpha}\{g\}^{\alpha}_{\mu\nu}} - \{g\}_{\nu}T_{\mu} - \{g\}_{\mu}T_{\nu} + \{g\}_{\mu}T_{\nu} + \{g\}_{\nu}T_{\mu}\right) - K^{\beta}_{\alpha\nu}K^{\alpha}_{\beta\mu} + K_{\alpha}K^{\alpha}_{\mu\nu} =$$
$$= \kappa\left(T_{\mu\nu} - d_{\alpha}T^{\alpha}_{\mu\nu}\right) + \kappa\nabla_{\alpha}T^{\alpha}_{\mu\nu} - K^{\beta}_{\alpha\nu}K^{\alpha}_{\beta\mu} + K_{\alpha}K^{\alpha}_{\mu\nu} \tag{5.3.119}$$

Since we have

$$
\begin{aligned}
-K_{\alpha\nu}^\beta K_{\beta\mu}^\alpha + K_\alpha K_{\mu\nu}^\alpha =& \kappa \left(-K_{\alpha\nu}^\beta T_{\beta\mu}^\alpha + K_\alpha T_{\mu\nu}^\alpha \right) - \tfrac{\kappa}{m-1}\left({\color{red}-K_\nu T_\mu} + K_\mu T_\nu - K_{\mu\nu}^\beta T_\beta + {\color{red}K_\nu T_\mu} \right) = \\
=& \kappa^2 \left(-T_{\alpha\nu}^\beta T_{\beta\mu}^\alpha + \tfrac{1}{m-1}T_\nu T_\mu + {\color{red}\tfrac{1}{m-1}T_\alpha T_{\mu\nu}^\alpha} - \tfrac{2}{m-1}T_\alpha T_{\mu\nu}^\alpha - \tfrac{1}{m-1}\left({\color{red}-\tfrac{2}{m-1}T_\mu T_\nu} - T_\alpha T_{\mu\nu}^\alpha + \tfrac{1}{m-1}T_\nu T_\mu + \tfrac{1}{m-1}T_\mu T_\nu \right) \right) = \\
=& \kappa^2 \left(-T_{\beta\mu}^\alpha T_{\alpha\nu}^\beta + \tfrac{1}{m-1}T_\mu T_\nu \right)
\end{aligned}
\tag{5.3.120}
$$

it can be recast as

$$
H_{\mu\nu} := \kappa \left(\left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \nabla_\alpha T_{\mu\nu}^\alpha - \kappa T_{\beta\mu}^\alpha T_{\alpha\nu}^\beta + \tfrac{\kappa}{m-1}T_\mu T_\nu \right)
\tag{5.3.121}
$$

Thus $H_{\mu\nu}$ is in fact a symmetric tensor and it can be rewritten in term of $(\hat{g}^{\mu\nu}, \tilde{\Gamma}_{\mu\nu}^\alpha)$, namely

$$
\begin{aligned}
H_{\mu\nu} =& \kappa \left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \kappa \tilde{\nabla}_\alpha T_{\mu\nu}^\alpha + \kappa^2 \left(T_{\alpha\nu}^\beta - \tfrac{1}{m-1}\delta_\alpha^\beta T_\nu - \tfrac{1}{m-1}\delta_\nu^\beta T_\alpha \right) T_{\mu\beta}^\alpha + \kappa^2 \left(T_{\alpha\mu}^\beta - \tfrac{1}{m-1}\delta_\alpha^\beta T_\mu - \tfrac{1}{m-1}\delta_\mu^\beta T_\alpha \right) T_{\nu\beta}^\alpha + \\
& + \tfrac{2\kappa^2}{m-1}T_\alpha T_{\mu\nu}^\alpha - \kappa^2 T_{\beta\mu}^\alpha T_{\alpha\nu}^\beta + \tfrac{\kappa^2}{m-1}T_\mu T_\nu = \\
=& \kappa \left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \kappa \tilde{\nabla}_\alpha T_{\mu\nu}^\alpha + \kappa^2 \left({\color{red}T_{\alpha\nu}^\beta T_{\mu\beta}^\alpha} - \tfrac{1}{m-1}T_\nu T_\mu - \tfrac{1}{m-1}T_\alpha T_{\mu\nu}^\alpha + T_{\alpha\mu}^\beta T_{\nu\beta}^\alpha - \tfrac{1}{m-1}T_\mu T_\nu - \tfrac{1}{m-1}T_\alpha T_{\mu\nu}^\alpha \right) + \\
& + {\color{green}\tfrac{2\kappa^2}{m-1}T_\alpha T_{\mu\nu}^\alpha} - \kappa^2 T_{\beta\mu}^\alpha T_{\alpha\nu}^\beta + {\color{green}\tfrac{\kappa^2}{m-1}T_\mu T_\nu} = \kappa \left(\left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \tilde{\nabla}_\alpha T_{\mu\nu}^\alpha + \kappa T_{\alpha\mu}^\beta T_{\beta\nu}^\alpha - \tfrac{\kappa}{m-1}T_\mu T_\nu \right)
\end{aligned}
\tag{5.3.122}
$$

The variation of the Lagrangian can then be recast as

$$
\delta \left(L + \tfrac{1}{2\kappa}\tilde{u}_{\mu\nu}^\alpha \hat{g}_\alpha^{\mu\nu} \right) = -\tfrac{1}{2\kappa}\left(\tilde{u}_{\mu\nu} + H_{\mu\nu} \right)\delta \hat{g}^{\mu\nu} - \tfrac{1}{2\kappa}\hat{g}_\alpha^{\mu\nu}\delta \tilde{u}_{\mu\nu}^\alpha + p\delta\psi + p^\alpha \delta\psi_\alpha + \delta \left(d_\alpha \left(\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}u_{\mu\nu}^\alpha \right) \right)
\tag{5.3.123}
$$

and we define the Routh function

$$
\begin{aligned}
W(\hat{g}, \tilde{\Gamma}, j^1\psi) =& \tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}u_{\mu\nu} + \hat{g}_\alpha^{\mu\nu}\tilde{u}_{\mu\nu}^\alpha \right) + L_m = \tfrac{1}{2\kappa}\left(\hat{g}^{\mu\nu}u_{\mu\nu} + \hat{g}_\alpha^{\mu\nu}u_{\mu\nu}^\alpha + \kappa \hat{g}_\alpha^{\mu\nu}T_{\mu\nu}^\alpha \right) + L_m = \\
=& -\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{u}_{\mu\nu} - \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}H_{\mu\nu} + \tfrac{1}{2}\hat{g}^{\mu\nu}\left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \tfrac{1}{2}\hat{g}^{\mu\nu}\left(d_\alpha T_{\mu\nu}^\alpha + \{g\}_\alpha T_{\mu\nu}^\alpha - \{g\}_{\mu\alpha}^\epsilon T_{\epsilon\nu}^\alpha - \{g\}_{\nu\alpha}^\epsilon T_{\mu\epsilon}^\alpha \right) + L_m = \\
=& -\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{u}_{\mu\nu} - \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}H_{\mu\nu} + \tfrac{1}{2}\hat{g}^{\mu\nu}\left(T_{\mu\nu} - d_\alpha T_{\mu\nu}^\alpha \right) + \tfrac{1}{2}\hat{g}^{\mu\nu}\nabla_\alpha T_{\mu\nu}^\alpha + L_m = \\
=& -\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{u}_{\mu\nu} - \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}H_{\mu\nu} + \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}H_{\mu\nu} - \tfrac{\kappa}{2}\hat{g}^{\mu\nu}\left(T_{\alpha\mu}^\beta T_{\beta\nu}^\alpha - \tfrac{1}{m-1}T_\mu T_\nu \right) + L_m = \\
=& -\tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{u}_{\mu\nu} - \tfrac{\kappa}{2}\hat{g}^{\mu\nu}\left(T_{\alpha\mu}^\beta T_{\beta\nu}^\alpha - \tfrac{1}{m-1}T_\mu T_\nu \right) + L_m
\end{aligned}
\tag{5.3.124}
$$

and the Helmholtz Lagrangian

$$
L_H(\hat{g}, \tilde{\Gamma}, j^1\psi) = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}d_\alpha \tilde{u}_{\mu\nu}^\alpha + W = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{R}_{\mu\nu} - \tfrac{\kappa}{2}\hat{g}^{\mu\nu}\left(T_{\alpha\mu}^\beta T_{\beta\nu}^\alpha - \tfrac{1}{m-1}T_\mu T_\nu \right) + L_m = \tfrac{1}{2\kappa}\hat{g}^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{L}_m
\tag{5.3.125}
$$

where we set $\tilde{L}_m := L_m - \frac{\kappa}{2}\hat{g}^{\mu\nu}\left(T^\beta_{\alpha\mu}T^\alpha_{\beta\nu} - \frac{1}{m-1}T_\mu T_\nu\right)$ for the new matter Lagrangian in the metric-affine formulation. Of course, in Palatini theories one has $T^\alpha_{\mu\nu} = 0$ and hence the matter Lagrangian is the same in the purely metric and Palatini formulation. This is not true in general for metric-affine theories, though we still have dynamical equivalence given by the one-to-one correspondence between solutions

$$(\hat{g}^{\mu\nu}, \psi) \quad \longleftrightarrow \quad \left(\hat{g}^{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + \kappa\left(T^\alpha_{\mu\beta} - \frac{2}{m-1}\delta^\alpha_{(\mu}T_{\beta)}\right), \psi\right) \tag{5.3.126}$$

4. More formulations of geometric theories

Let us present some more geometric theories which somehow describe a dynamical geometry on spacetime, which are, at least in some cases, equivalent to gravitational theories. These will be here used as further examples of discussing equivalence among theories and later used as building blocks for defining gravitational theories.

Vielbein-affine formalism

We introduced *Vielbein* in Subsection 2.7.5 and *spin fames* in Subsection 2.7.6. We already know that, in both cases, they are described by fields e^μ_a, the difference being in gauge transformations depending on the group $\mathrm{SO}(\eta)$ or $\mathrm{Spin}(\eta)$, respectively.

Let us denote by $F(\mathcal{P})$ the bundle of *Vielbein* (or *spin fames*) on the structure bundle $\mathcal{P}$ (which a G-principal bundle with G being the gauge group, namely $G = \mathrm{SO}(\eta)$ for *Vielbein* or $G = \mathrm{Spin}(\eta)$ for spin frames).

We also have a bundle $\mathrm{Con}(\mathcal{P}) = J^1\mathcal{P}/G$ which parameterises principal connections as sections. Let us consider the product bundle $C = F(\mathcal{P}) \times_M \mathrm{Con}(\mathcal{P})$ as a configuration bundle, which has fibered coordinates $(x^\mu, e^\mu_a, \tilde{\omega}^{ab}_\mu)$.

As usual, we can define the curvature of the connection $\tilde{\omega}^{ab}_\mu$ and denote it by $\tilde{R}^{ab}{}_{\mu\nu}$ (which can be recovered as a piece of *"coordinates"* in the first prolongation, analogously to what it is done in Utiyama-like arguments).

For later convenience let us set

$$\tilde{R}^a_\mu = e^\nu_b \tilde{R}^{ab}{}_{\mu\nu} \qquad\qquad \tilde{R} = e^\mu_a e^\nu_b \tilde{R}^{ab}{}_{\mu\nu} \tag{5.4.1}$$

Let us consider the Lagrangian

$$\boldsymbol{L} = \frac{|e|}{2\kappa}\tilde{R}\,d\sigma \tag{5.4.2}$$

The endowed theory is called *standard Vielbein-affine* (or *frame-affine*) *theory.* The variation of this Lagrangian is

$$\begin{aligned}\delta L = &\frac{|e|}{2\kappa}e^a_\mu \delta e^\mu_a \tilde{R} + 2\frac{|e|}{2\kappa}\delta e^\mu_a \tilde{R}^a_\mu + 2\frac{|e|}{2\kappa}e^\mu_{[a}e^\nu_{b]}\delta\tilde{R}^{ab}{}_{\mu\nu} = 2\frac{|e|}{2\kappa}\left(\tilde{R}^a_\mu - \frac{1}{2}\tilde{R}e^a_\mu\right)\delta e^\mu_a + 2\frac{|e|}{2\kappa}e^\mu_{[a}e^\nu_{b]}\tilde{\nabla}_\mu\delta\tilde{\omega}^{ab}{}_\nu = \\ = &2\frac{|e|}{2\kappa}\left(\tilde{R}^a_\mu - \frac{1}{2}\tilde{R}e^a_\mu\right)\delta e^\mu_a - 2\tilde{\nabla}_\mu\left(\frac{|e|}{2\kappa}e^\mu_{[a}e^\nu_{b]}\right)\delta\omega^{ab}{}_\nu + 2\overset{*}{\tilde{\nabla}}_\mu\left(\frac{|e|}{2\kappa}e^\mu_{[a}e^\nu_{b]}\delta\tilde{\omega}^{ab}{}_\nu\right)\end{aligned} \tag{5.4.3}$$

Thus field equations are

$$\begin{cases} \tilde{R}^a_\mu - \frac{1}{2}\tilde{R}e^a_\mu = 0 \\ \tilde{\nabla}_\mu \left(ee^\mu_{[a}e^\nu_{b]} \right) = 0 \end{cases} \tag{5.4.4}$$

The second field equation reads as

$$\tilde{\nabla}_\mu \left(ee^\mu_a e^\nu_b \right) \epsilon^{abcd}\epsilon_{ghcd} = 0 \quad \Longleftrightarrow \quad \tilde{\nabla}_\mu \left(ee^c_\gamma e^d_\delta e^\gamma_e e^\delta_f e^\mu_a e^\nu_b \epsilon^{abef} \right) \epsilon_{ghcd} = \tilde{\nabla}_\mu \left(e^c_\gamma e^d_\delta \right) \epsilon^{\mu\nu\gamma\delta}\epsilon_{ghcd} = 0$$

$$\Longleftrightarrow \quad \left(\tilde{\nabla}_\mu e^c_\gamma \right) e^d_\delta \epsilon^{\mu\nu\gamma\delta}\epsilon_{ghcd} = 0 \quad \Longleftrightarrow \quad \tilde{\nabla}_{[\mu}e^c_{\gamma]} = 0 \tag{5.4.5}$$

Let $\Gamma^a_{b\mu}$ the the spin connection induced by the frame and $K^a_{b\mu}$ be the difference between the connections $\tilde{\omega}$ and Γ, i.e.

$$\tilde{\omega}^{ab}_\mu = \Gamma^{ab}_\mu + K^{ab}_\mu \tag{5.4.6}$$

Thus one has

$$\tilde{\nabla}_{[\mu}e^c_{\gamma]} = 0 \quad \Longleftrightarrow \quad K^c_{b[\mu}e^b_{\gamma]} = 0 \tag{5.4.7}$$

By setting $K_{abc} = \eta_{ad}K^d_{b\lambda}e^\lambda_c$, one has

$$K_{abc} = -K_{bac} \qquad\qquad K_{abc} = K_{acb} \tag{5.4.8}$$

Hence one has that

$$K_{abc} = -K_{bac} = -K_{bca} = K_{cba} = K_{cab} = -K_{acb} = -K_{abc} \quad \Longleftrightarrow \quad K_{abc} = 0 \quad \Longleftrightarrow \quad \tilde{\omega}^{ab}_\mu = \Gamma^{ab}_\mu \tag{5.4.9}$$

Once the connection $\tilde{\omega}$ is known to be induced by the frame, the first field equations is expressed in terms of the frame only

$$R^a_\mu - \frac{1}{2}Re^a_\mu = 0 \quad \Longleftrightarrow \quad R^\alpha_\mu - \frac{1}{2}R\delta^a_\mu = 0 \quad \Longleftrightarrow \quad R_{\alpha\mu} - \frac{1}{2}Rg_{\alpha\mu} = 0 \tag{5.4.10}$$

where $g_{\mu\nu}$ is the metric induced by the frame. Accordingly, frame-affine field equations impose that the induced metric satisfies vacuum Einstein equations. Thus frame-affine theories are, in vacuum, equivalent to standard GR.

Ashtekar (anti)self dual formulation

We discussed spin frames in Subsection 2.7.6 and we just provided a metric-affine dynamics for them in Subsection 5.4.1. Here we present a different equivalent dynamics due to Ashtekar (and then developed by Barbero, Immirzi, and Holst). As usual, we begin here to expand computations in order to later be ready to discuss it.

Let us fix Euclidean signature in dimension $m = 4$. The framework (in its simpler and original formulation) is based on the classical group splitting $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. The two copies of SU(2) are called the *(anti)self dual group components*. We shall denote by $p_\pm : \mathrm{Spin}(4) \to \mathrm{SU}(2)$ the two projectors onto each factor.

Let us define the even Clifford algebra $\mathcal{C}^+(4)$ which is generated by the elements

$$(\mathbb{I}, E_{01}, E_{02}, E_{03}, E_{12}, E_{13}, E_{23}, E_{0123}) = (\mathbb{I}, E_{ab}, E) \tag{5.4.11}$$

where we set $E = E_{0123}$. An element in $\mathcal{C}^+(4)$ is of the form

$$a\mathbb{I} + bE_{01} + cE_{02} + dE_{03} + eE_{12} + fE_{13} + gE_{23} + hE_{0123} = A\mathbb{I} + B^{ab}E_{ab} + CE \tag{5.4.12}$$

Let us set $i = E_{23}$, $j = -E_{13}$, $k = E_{12}$, $iE = Ei = -E_{01}$, $j\mathbb{E} = Ej = -E_{02}$, $kE = Ek = -E_{03}$. Hence an element in $\mathcal{C}^+(4)$ can be recast in the form $u + Ev$ where we set $u = a + gi - fj + ek$ and $v = h - bi - cj - dk$. By the way, notice that $E^2 = \mathbb{I}$.

Let us set $\mathrm{Spin}(4) = \{u + Ev : u, v \in \mathbb{H}, |u|^2 + |v|^2 = 1, u\bar{v} + v\bar{u} = 0\}$.

Let us define a group isomorphism

$$\varphi : \mathrm{Spin}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2) : u + Ev \mapsto (q_+, q_-) = (u - v, u + v) \tag{5.4.13}$$

We have that both q_+ and $q_- \in \mathrm{SU}(2)$. In fact,

$$|q_\pm|^2 = (u \mp v)(\bar{u} \mp \bar{v}) = |u|^2 + |v|^2 \mp (v\bar{u} + u\bar{v}) = 1 \tag{5.4.14}$$

where the isomorphism $\mathrm{SU}(2) \simeq S^3$ is understood.

The map φ is a group homomorphism In fact, $\varphi(\mathbb{I} + E0) = (\mathbb{I}, \mathbb{I})$ and

$$\varphi[(u + Ev)(r + Es)] = \varphi[(ur + vs) + E(vr + us)] = (ur + vs - vr - us, ur + vs + vr + us) = (u - v, u + v)(r - s, r + s) = \varphi(u + Ev) \cdot \varphi(r + Es) \tag{5.4.15}$$

Finally, the map φ is a group isomorphism. In fact, let us pick $(q_+, q_-) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ and define $S = \frac{1}{2}[(q_+ + q_-) - E(q_+ - q_-)]$. To prove that $S \in \mathrm{Spin}(4)$ we have to check that constraints are satisfied:

$$\frac{1}{4}(|q_+ + q_-|^2 + |q_+ - q_-|^2) = \frac{1}{4}(|q_+|^2 + |q_-|^2 + 2(q_+\bar{q}_- + q_-\bar{q}_+)) + \frac{1}{4}(|q_+|^2 + |q_-|^2 - 2(q_+\bar{q}_- + q_-\bar{q}_+)) = 1$$
$$-\frac{1}{4}((q_+ + q_-)(\bar{q}_+ - \bar{q}_-) + (q_+ - q_-)(\bar{q}_+ + \bar{q}_-)) = \frac{1}{4}(2|q_+|^2 - 2|q_-|^2) = 0 \tag{5.4.16}$$

Of course, one can prove that the maps $u + Ev \mapsto (q_+, q_-) \mapsto S$ are inverse one to each other.

We can define the group homomorphisms $p_\pm : \mathrm{Spin}(4) \to \mathrm{SU}(2) : u + Ev \mapsto q_\pm = u \mp v$ so that the group isomorphism reads as $\varphi = p_+ \times p_-$.

Let us denote by $\mathfrak{spin}(4)$ the Lie algebra of the group $\mathrm{Spin}(4)$; one has

$$\mathfrak{spin}(4) = \{\dot{u} + E\dot{v} : \dot{u} + \dot{\bar{u}} = 0, \ \dot{v} + \dot{\bar{v}} = 0\} \tag{5.4.17}$$

An element $\zeta \in \mathfrak{spin}(4)$ in the Lie algebra is in the form

$$\zeta = \dot{g}i - \dot{f}j + \dot{e}k + E(-\dot{b}i - \dot{c}j - \dot{d}k) = \dot{b}E_{01} + \dot{c}E_{02} + \dot{e}E_{12} + \dot{g}E_{23} + \dot{f}E_{13} + \dot{d}E_{03} = \frac{1}{2}\zeta^{ab}E_{ab} \tag{5.4.18}$$

Let us define the duality on $\mathfrak{spin}(4)$ by $* : \mathfrak{spin}(4) \to \mathfrak{spin}(4) : \dot{u} + E\dot{v} \mapsto -(\dot{v} + E\dot{u})$. The duality is involutive; in fact $* * (\dot{u} + E\dot{v}) = -*(\dot{v} + E\dot{u}) = (\dot{u} + E\dot{v})$. Hence possible eigenvalues are ± 1, only. The duality, as defined above, is the same as $*(E_{ab}) = \frac{1}{2}\epsilon_{ab}{}^{cd}E_{cd}$. In fact, we have

$$
\begin{aligned}
* \left(\tfrac{1}{2}\zeta^{ab}E_{ab} \right) &= * \left(\dot{b}E_{01} + \dot{c}E_{02} + \dot{d}E_{03} + \dot{e}E_{12} + \dot{f}E_{13} + \dot{g}E_{23} \right) = *(\dot{g}i - \dot{f}j + \dot{e}k + E(-\dot{b}i - \dot{c}j - \dot{d}k)) = \left(\dot{b}i + \dot{c}j + \dot{d}k - E(\dot{g}i - \dot{f}j + \dot{e}k) \right) = \\
&= \left(\dot{b}E_{23} - \dot{c}E_{13} + \dot{d}E_{12} + \dot{g}E_{01} - \dot{f}E_{02} + \dot{e}E_{03} \right) = \tfrac{1}{4}\zeta^{ab}\epsilon_{ab}{}^{cd}E_{cd}
\end{aligned}
\tag{5.4.19}
$$

The eigenvectors of the duality are elements in the form $\dot{u} \pm E\dot{u}$ (for which in fact one has $*(\dot{u} \pm E\dot{u}) = \mp(\dot{u} \pm E\dot{u})$) Notice that the elements in the form $\dot{u} + E\dot{u}$ correspond to pair $(q_+ = 0, q_-)$ and are antiself dual (i.e. they correspond to the eigenvalue -1), while elements in the form $\dot{u} - E\dot{u}$ correspond to pair $(q_+, q_- = 0)$ and are self dual (i.e. they correspond to the eigenvalue 1).

Given the projectors $p_\pm$ on the group, the tangent map at the identity $T_e p_\pm : \mathfrak{spin}(4) \to \mathfrak{su}(2)$ is the standard duality on the Lie algebra which splits the algebra $\mathfrak{spin}(4)$ in the two (anti)self dual eigenspaces.

Since we have $q_\pm = u \mp v$, we have on the algebras

$$
T_e p_\pm : \mathfrak{spin}(4) \to \mathfrak{su}(2) : \dot{u} + E\dot{v} \mapsto q_\pm = \dot{u} \mp \dot{v} : \begin{cases} E_{ij} \mapsto \epsilon_{ij}{}^k E_k \\ E_{0k} \mapsto \pm E_k \end{cases}
\tag{5.4.20}
$$

where we set $E_k = \frac{1}{2}\epsilon_k{}^{ij}E_{ij}$ as a basis for the algebra $\mathfrak{su}(2)$. For a generic element $\zeta = \frac{1}{2}\zeta^{ab}E_{ab}$ in the algebra we have

$$
Tp_\pm(\zeta) = \zeta^{0k}Tp_\pm(E_{0k}) + \tfrac{1}{2}\zeta^{ij}Tp_\pm(E_{ij}) = \left(\pm \zeta^{0k} + \tfrac{1}{2}\zeta^{ij}\epsilon_{ij}{}^k \right) E_k =: {}^\pm\zeta^k E_k \qquad {}^\pm\zeta^k := \pm \zeta^{0k} + \tfrac{1}{2}\zeta^{ij}\epsilon_{ij}{}^k =: {}^\pm p_{ab}^k \zeta^{ab}
\tag{5.4.21}
$$

where we have set

$$
{}^\pm p_{0i}^k := -{}^\pm p_{i0}^k = \pm\tfrac{1}{2}\delta_i^k \qquad {}^\pm p_{ij}^k := \tfrac{1}{2}\epsilon^k{}_{ij}
\tag{5.4.22}
$$

We define p_k^{ab} (by index raising and lowering) as

$$
{}^\pm p_k^{0i} = -{}^\pm p_k^{i0} = \pm\tfrac{1}{2}\delta_i^k \qquad {}^\pm p_k^{ij} = \tfrac{1}{2}\epsilon_k{}^{ij}
\tag{5.4.23}
$$

so that the following identities hold true

$$
{}^\pm p_{ab}^i {}^\pm p_k^{ab} = 2{}^\pm p_{0h}^i {}^\pm p_k^{0h} + {}^\pm p_{jh}^i {}^\pm p_k^{jh} = \tfrac{1}{2}\delta_h^i \delta_k^h + \tfrac{1}{4}\epsilon^i{}_{jh}\epsilon^{jh}{}_k = \left(\tfrac{1}{2}\delta_k^i + \tfrac{1}{2}\delta_k^i \right) = \delta_k^i \qquad {}^\pm p_{ab}^i {}^\pm p_i^{cd} = \tfrac{1}{2}\delta_{[a}^c \delta_{b]}^d \pm \tfrac{1}{4}\epsilon_{ab}{}^{cd}
\tag{5.4.24}
$$

In view of the second one, let us also set

$$
{}^\pm E_{ab} := \tfrac{1}{2} \left(E_{ab} \pm \tfrac{1}{2}\epsilon_{ab}{}^{cd}E_{cd} \right) \qquad \Longleftrightarrow \qquad \begin{cases} {}^\pm E_{0k} = \tfrac{1}{2} \left(E_{0k} \pm \tfrac{1}{2}\epsilon_k{}^{ij}E_{ij} \right) \\ {}^\pm E_{ij} = \pm\tfrac{1}{2}\epsilon_{ij}{}^k \left(E_{0k} \pm \tfrac{1}{2}\epsilon_k{}^{lm}E_{lm} \right) \end{cases}
\tag{5.4.25}
$$

Of course, ${}^\pm E_{0k}$ and ${}^\pm E_{ij}$ so defined are dependent, in fact, one has

$$
{}^\pm E_{ij} = \pm\epsilon_{ij}{}^k {}^\pm E_{0k} \qquad\qquad {}^\pm E_{0k} = \pm\tfrac{1}{2}\epsilon^{ij}{}_k {}^\pm E_{ij}
\tag{5.4.26}
$$

and

$$
\tfrac{1}{2}{}^\pm p_k^{ab} E_{ab} = {}^\pm p_k^{0i} E_{0i} + \tfrac{1}{2}{}^\pm p_k^{ij} E_{ij} = \pm\tfrac{1}{2} E_{0k} + \tfrac{1}{4}\epsilon_k{}^{ij} E_{ij} = \pm\tfrac{1}{2} \left(E_{0k} \pm \tfrac{1}{2}\epsilon_k{}^{ij} E_{ij} \right) = \pm {}^\pm E_{0k}
\tag{5.4.27}
$$

An element $\zeta = \frac{1}{2}\zeta^{ab}E_{ab}$ of the algebra can be recast as

$$\zeta = \frac{1}{2}\zeta^{ab}E_{ab} = \frac{1}{2}\,\zeta^{ab}\left({}^+E_{ab} + {}^-E_{ab}\right) = \left(\zeta^{0k} + \frac{1}{2}\zeta^{ij}\epsilon_{ij}{}^k\right){}^+E_{0k} + \left(\zeta^{0k} - \frac{1}{2}\zeta^{ij}\epsilon_{ij}{}^k\right){}^-E_{0k} = \left({}^+\zeta^k\,{}^+E_{0k} - {}^-\zeta^k\,{}^-E_{0k}\right) \tag{5.4.28}$$

Moreover, we have

$$\pm\zeta := \frac{1}{2}\zeta^{ab}\,{}^{\pm}E_{ab} = \frac{1}{4}\zeta^{ab}\left(E_{ab} \pm \frac{1}{2}\epsilon_{ab}{}^{cd}E_{cd}\right) = \frac{1}{4}\left(\zeta^{cd} \pm \frac{1}{2}\epsilon_{ab}{}^{cd}\zeta^{ab}\right)\,E_{cd} =: \frac{1}{2}\,{}^{\pm}\zeta^{ab}\,E_{ab} \tag{5.4.29}$$

where we set

$$\pm\zeta^{cd} := \frac{1}{2}\left(\zeta^{cd} \pm \frac{1}{2}\epsilon_{ab}{}^{cd}\zeta^{ab}\right) \qquad \Longleftrightarrow \qquad \begin{cases} \pm\zeta^{0k} := \frac{1}{2}\left(\zeta^{0k} \pm \frac{1}{2}\epsilon_{ij}{}^k\zeta^{ij}\right) = \pm\frac{1}{2}\,{}^{\pm}\zeta^k \\[2mm] \pm\zeta^{ij} := \pm\,\epsilon_k{}^{ij}\left(\frac{1}{2}\left(\zeta^{0k} \pm \frac{1}{2}\epsilon_{lm}{}^k\zeta^{lm}\right)\right) \end{cases} \tag{5.4.30}$$

Again, ${}^{\pm}\zeta^{0k}$ and ${}^{\pm}\zeta^{ij}$ are not independent and one has

$$\pm\zeta^{ij} = \pm\,\epsilon_k{}^{ij}\,{}^{\pm}\zeta^{0k} \qquad\qquad \pm\zeta^{0k} = \pm\frac{1}{2}\,\epsilon^k{}_{ij}\,{}^{\pm}\zeta^{ij} \tag{5.4.31}$$

We also have

$$\frac{1}{2}{}^{\pm}p^k_{ab}\zeta^{ab} = \pm\frac{1}{2}\zeta^{0k} + \frac{1}{4}\epsilon_{ij}{}^k\,\zeta^{ij} = \frac{1}{2}\,{}^{\pm}\zeta^k = \pm\,{}^{\pm}\zeta^{0k} \tag{5.4.32}$$

Hence the tangent map of the group duality is:

$$Tp_{\pm}({}^+E_{0k}) = \frac{1}{2}\left(\pm 1 + 1\right)E_k = \begin{cases} E_k \\ 0 \end{cases} \qquad\qquad Tp_{\pm}({}^-E_{0k}) = \frac{1}{2}\left(\pm 1 - 1\right)E_k = \begin{cases} 0 \\ -E_k \end{cases} \tag{5.4.33}$$

and

$$Tp_{\pm}({}^+E_{ij}) = \epsilon_{ij}{}^k Tp_{\pm}({}^+E_{0k}) = \begin{cases} \epsilon_{ij}{}^k\,E_k \\ 0 \end{cases} \qquad\qquad Tp_{\pm}({}^-E_{ij}) = -\epsilon_{ij}{}^k Tp_{\pm}({}^-E_{0k}) = \begin{cases} 0 \\ \epsilon_{ij}{}^k\,E_k \end{cases} \tag{5.4.34}$$

Thus the elements ${}^{\pm}E_{0k}$ are a basis adapted to the duality $Tp_{\pm}$.

Let us remark that the traditional view based on the algebra is not in general sufficient; in field theories one needs, in fact, to be precise on what happens also at the group level. This further information is needed, for example, to define the principal bundles on which the (anti)self dual connections are defined.

When we define spin frames one has a principal structure bundle P for the group $\mathrm{Spin}(4)$, on which one has "coordinates" (x^μ, S), which is determined in terms of a cocycle $\gamma_{(\alpha\beta)} : U_{\alpha\beta} \to \mathrm{Spin}(4)$. In view of the group isomorphism $\varphi : \mathrm{Spin}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ one can define two cocycles

$$\pm\gamma_{(\alpha\beta)} : U_{\alpha\beta} \to \mathrm{SU}(2) : x \mapsto p_{\pm}(\gamma_{(\alpha\beta)}(x)) \tag{5.4.35}$$

Both the cocycles ${}^{\pm}\gamma_{(\alpha\beta)}$ are cocycles, since they are images of a cocycle through a group homomorphism.

$$\pm\gamma_{\alpha\beta} \cdot {}^{\pm}\gamma_{\beta\alpha} = p_{\pm}(\gamma_{\alpha\beta} \cdot \gamma_{\beta\alpha}) = p_{\pm}(\mathbb{I}) = \mathbb{I} \qquad\qquad \pm\gamma_{\alpha\beta} \cdot {}^{\pm}\gamma_{\beta\gamma} \cdot {}^{\pm}\gamma_{\gamma\alpha} = p_{\pm}(\gamma_{\alpha\beta} \cdot \gamma_{\beta\gamma} \cdot \gamma_{\gamma\alpha}) = p_{\pm}(\mathbb{I}) = \mathbb{I} \tag{5.4.36}$$

Hence, they define (modulo isomorphisms) two principal bundles $^{\pm}P$ for the group SU(2) and one can prove a global isomorphism

$$P \simeq {}^{+}P \times_M {}^{-}P \tag{5.4.37}$$

The two projections are then denoted by $\pi_{\pm} : P \to {}^{\pm}P$.

In view of how transition functions are defined, there is not much to be checked. One has local maps

$$\iota(x, S) = (x, {}^{+}S, {}^{-}S) \in {}^{+}P \times_M {}^{-}P \tag{5.4.38}$$

and they glue together automatically, since $^{\pm}\gamma_{(\alpha\beta)} = p_{\pm} \circ \gamma_{(\alpha\beta)}$.
The local expressions of the projections $\pi_{\pm}$ are

$$\pi_{\pm} : P \to {}^{\pm}P : (x, S) \mapsto (x, p_{\pm}(S)) \tag{5.4.39}$$

Let $H_p \subset T_p P$ be a principal connection on P, $^{\pm}p = \pi_{\pm}(p) \in {}^{\pm}P$, and let us define the subspaces $^{\pm}H_{\pm_p} = T\pi_{\pm}(H_p) \subset T_{\pm_p}{}^{\pm}P$.

The map $\pi_{\pm}$ (each of them, say π_+) is surjective though not injective. Hence one has infinitely many subspaces $H_{+_p} \subset T_{+_p}{}^{+}P$, one for any preimage $(^{+}p, {}^{-}p)$, i.e. one for any $^{-}p \in {}^{-}P$.

However, let us consider two points $^{-}p_1, {}^{-}p_2 \in {}^{-}P$ (so that there exist a group element $^{-}S \in$ SU(2) such that $^{-}p_2 = {}^{-}p_1 \cdot {}^{-}S$). Then we have

$$T\pi_+ \left(H_{(^{+}p, {}^{-}p_2)} \right) = T\pi_+ \left(H_{(^{+}p, {}^{-}p_1)\cdot(e, \cdot{}^{-}S)} \right) = T\pi_+ \left(T_{(^{+}p, {}^{-}p_1)} R_{(e, {}^{-}S)} \left(H_{(^{+}p, {}^{-}p_1)} \right) \right) = T \left(\pi_+ \circ R_{(e, {}^{-}S)} \right) \left(H_{(^{+}p, {}^{-}p_1)} \right) = T\pi_+ \left(H_{(^{+}p, {}^{-}p_1)} \right) \tag{5.4.40}$$

where we used the fact that

$$\pi_+ \circ R_{(e, {}^{-}S)}({}^{+}p, {}^{-}p_1) = \pi_+({}^{+}p, {}^{-}p_1 \cdot {}^{-}S) = {}^{+}p = \pi_+({}^{+}p, {}^{-}p_1) \qquad \Rightarrow \pi_+ \circ R_{(e, {}^{-}S)} = \pi_+ \tag{5.4.41}$$

Accordinlgly, all points $p = ({}^{+}p, {}^{-}p \cdot {}^{-}S) \in P$ in the preimage of $^{+}p \in {}^{+}P$ induce the same subspace $^{+}H_{+_p} \subset T_{+_p}{}^{+}P$ at $^{+}p \in {}^{+}P$ and the family $^{+}H_{+_p}$ defines a connection on ^{+}P.
Moreover, if we consider $^{+}p' = {}^{+}p \cdot {}^{+}S$, then, for any $^{-}p \in {}^{-}P$, one has

$$\begin{aligned}{}^{+}H_{+_{p'}} = T_{+_{p'}}\pi_+ \left(H_{(^{+}p \cdot {}^{+}S, {}^{-}p)} \right) &= T_{+_{p'}}\pi_+ \left(H_{(^{+}p, {}^{-}p)\cdot(^{+}S, e)} \right) = T_{+_{p'}}\pi_+ \left(TR_{(^{+}S, e)} \left(H_{(^{+}p, {}^{-}p)} \right) \right) = T_{+_p} \left(\pi_+ \circ R_{(^{+}S, e)} \right) \left(H_{(^{+}p, {}^{-}p)} \right) = \\ &= T_{+_p} \left(R_{+S} \circ \pi_+ \right) \left(H_{(^{+}p, {}^{-}p)} \right) = T_{+_p} R_{+S}^{+} H_{+_p} \qquad \Rightarrow {}^{+}H_{+p \cdot +S} = T_{+_p} R_{+S} \left({}^{+}H_{+_p} \right)\end{aligned} \tag{5.4.42}$$

where we used equivariance of the map π_+, i.e. $\pi_+ \circ R_{(^{+}S, e)} = R_{+S} \circ \pi_+$. Thus the connection $^{+}H_{+_p}$ is principal.
An analogous argument goes with $^{-}H_{-_p}$.

Let us finally remark how beautifully intrinsic this argument is; it relies completely con functorial properties of maps and connections, and as a result the connections are global. Let us also stress that, even once we know these connections are globally defined, we still need the local expressions to do computations (for example, we define covariant derivatives, the Lagrangian, and, eventually, the variation of the Lagrangian, which, in fact, depends on the details of how covariant derivatives depends on fundamental field). However, we do not need to check transformation rules for the connections (nor for any derived structure), since we already know they are global connections. The golden rule (*"everything is in transformation rules"*) works both ways.

Golden rule

Thus any connection ω on P induces two connections $^{\pm}\omega$ on $^{\pm}P$. Let us then look for the local expression of these principal connections $^{\pm}\omega$.

Let σ_{ab} be a right invariant pointwise basis of vertical vectors of P, so that the connection ω has a local expression $\omega = dx^\mu \otimes (\partial_\mu - \omega_\mu^{ab}\sigma_{ab})$. We can get inspiration from what we did in the algebra and define $^{\pm}\sigma_{ab} := \frac{1}{2}\left(\sigma_{ab} \pm \frac{1}{2}\epsilon_{ab\cdot\cdot}{}^{cd} \sigma_{cd} \right)$. As usual, $^{\pm}\sigma_k := \pm Tp_{\pm}(^{\pm}\sigma_{0k}) =: \frac{1}{2}\sigma_k$ are a right invariant pointwise basis of vertical vectors of $^{\pm}P$.

We have (with some abuses of notation and by defining ${}^{\pm}\omega_\mu^k = \pm\omega_\mu^{0k} + \frac{1}{2}\epsilon_{ij}{}^k\omega_\mu^{ij} = {}^{\pm}p_{ab}^k\,\omega_\mu^{ab}$)

$$\omega(\xi) = \xi^\mu(\partial_\mu - \omega_\mu^{ab}\sigma_{ab}) = \xi^\mu(\partial_\mu - {}^+\omega_\mu^k\,{}^+\sigma_{0k} + {}^-\omega_\mu^{k-}\sigma_{0k})) \xrightarrow{Tp_\pm} {}^{\pm}\omega(\xi) = \xi^\mu(\partial_\mu - 2{}^{\pm}\omega_\mu^k\,{}^{\pm}\sigma_k) = \xi^\mu(\partial_\mu - {}^{\pm}\omega_\mu^k\,\sigma_k) \tag{5.4.43}$$

Let us set ${}^{\pm}\omega_\mu^{ij} := \epsilon^{ij}{}_k{}^{\pm}\omega_\mu^k$, which corresponds to define $\boldsymbol{\sigma}_{ij} := \epsilon_{ij}{}^k\,{}^{\pm}\sigma_k = \frac{1}{2}\epsilon_{ij}{}^k\,\sigma_k$ as a new basis for vertical vector fields on ${}^{\pm}P$. This defines the (anti)self dual connection on ${}^{\pm}P$:

$$^{\pm}\omega = dx^\mu \otimes \left(\partial_\mu - {}^{\pm}\omega_\mu^{ij}\,\boldsymbol{\sigma}_{ij}\right) = dx^\mu \otimes \left(\partial_\mu - \frac{1}{2}{}^{\pm}\omega_\mu^{ij}\,\epsilon_{ij}{}^k\sigma_k\right) = dx^\mu \otimes \left(\partial_\mu - {}^{\pm}\omega_\mu^k\,\sigma_k\right) \tag{5.4.44}$$

where we defined ${}^{\pm}\omega_\mu^{ij} = \epsilon^{ij}{}_k{}^{\pm}\omega_\mu^k = \epsilon^{ij}{}_k{}^{\pm}p_{ab}^k\omega_\mu^{ab}$. That expresses the coefficients of the induced (anti)self dual connections as a function of the coefficients of the original spin connection.

Then given the spin connection ω_μ^{ab} one can define the (anti)self dual connections ${}^{\pm}\omega_\mu^k$.

Vice versa, if the fix $({}^+\omega_\mu^k, {}^-\omega_\mu^k)$ two connections on ${}^+P$ and ${}^-P$, respectively, then we can solve for the original spin connection. In fact,

$$\begin{cases} {}^+\omega_\mu^k = {}^+p_{ab}^k\omega_\mu^{ab} = \omega_\mu^{0k} + \frac{1}{2}\epsilon^k{}_{ij}\omega_\mu^{ij} \\ {}^-\omega_\mu^k = {}^-p_{ab}^k\omega_\mu^{ab} = -\omega_\mu^{0k} + \frac{1}{2}\epsilon^k{}_{ij}\omega_\mu^{ij} \end{cases} \iff \begin{cases} \omega_\mu^{0k} = \frac{1}{2}\left({}^+\omega_\mu^k - {}^-\omega_\mu^k\right) \\ \omega_\mu^{ij} = \epsilon^{ij}{}_k\left({}^+\omega_\mu^k + {}^-\omega_\mu^k\right) \end{cases} \tag{5.4.45}$$

Accordingly, the information in the spin connection ($24 = 4 \times \frac{4}{2}(4-1)$ functions) is equivalently described by the information contained in both the (anti)self dual connections ($2 \times (3 \times 4) = 2 \times 12$ functions).

Let us denote by $R_{\mu\nu}^{ab}$ the curvature of the spin connection, namely

$$R_{\mu\nu}^{ab} = d_\mu\omega_\nu^{ab} - d_\nu\omega_\mu^{ab} + \omega^a{}_{c\mu}\omega_\nu^{cb} - \omega^a{}_{c\nu}\omega_\mu^{cb} = d_\mu\omega_\nu^{ab} - d_\nu\omega_\mu^{ab} + \left(\eta_{cd}\delta_e^a\delta_f^b - \eta_{cd}\delta_f^a\delta_e^b\right)\omega_\mu^{ec}\omega_\nu^{df} \tag{5.4.46}$$

and by ${}^{\pm}F_{\mu\nu}^k$ the curvature of the (anti)self dual connections, namely

$$^{\pm}F_{\mu\nu}^k = d_\mu{}^{\pm}\omega_\nu^k - d_\nu{}^{\pm}\omega_\mu^k - \epsilon^k{}_{ij}{}^{\pm}\omega_\mu^i{}^{\pm}\omega_\nu^j \tag{5.4.47}$$

We nicely have that the (anti)self dual part of the spin curvature of ω_μ^{ab} is the curvature of the (anti)self dual part of the connection.

The (anti)self dual part of the curvature is

$$^{\pm}R_{\mu\nu}^k = \pm R_{\mu\nu}^{0k} + \frac{1}{2}\epsilon^k{}_{ij}R_{\mu\nu}^{ij} \tag{5.4.48}$$

Hence we have

$$\begin{aligned} ^{\pm}R_{\mu\nu}^k &= \pm R_{\mu\nu}^{0k} + \frac{1}{2}\epsilon^k{}_{ij}R_{\mu\nu}^{ij} = \pm\left(d_\mu\omega_\nu^{0k} - d_\nu\omega_\mu^{0k} + \omega^0{}_{i\mu}\omega_\nu^{ik} - \omega^0{}_{i\nu}\omega_\mu^{ik}\right) + \frac{1}{2}\epsilon^k{}_{ij}\left(d_\mu\omega_\nu^{ij} - d_\nu\omega_\mu^{ij} + \omega^i{}_{e\mu}\omega_\nu^{ej} - \omega^i{}_{e\nu}\omega_\mu^{ej}\right) = \\ &= d_\mu{}^{\pm}\omega_\nu^k \pm \omega^0{}_{i\mu}\omega_\nu^{ik} + \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{0\mu}\omega_\nu^{0j} + \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{l\mu}\omega_\nu^{lj} - [\mu\nu] = d_\mu{}^{\pm}\omega_\nu^k \pm \omega^0{}_{i\mu}\omega_\nu^{ik} + \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{0\mu}\omega_\nu^{0j} - \frac{1}{4}\epsilon^{ki}{}_j\left(\epsilon^h{}_{li}\epsilon_{pqh}\right)\omega^{pq}{}_\mu\omega_\nu^{lj} - [\mu\nu] = \\ &= d_\mu{}^{\pm}\omega_\nu^k \pm \omega^0{}_{i\mu}\omega_\nu^{ik} + \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{0\mu}\omega_\nu^{0j} - \frac{1}{4}\left(\delta_j^h\delta_l^k - \eta_{jl}\eta^{kh}\right)\epsilon_{pqh}\omega^{pq}{}_\mu\omega_\nu^{lj} - [\mu\nu] = d_\mu{}^{\pm}\omega_\nu^k \pm \omega^0{}_{i\mu}\omega_\nu^{ik} + \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{0\mu}\omega_\nu^{0j} + \frac{1}{4}\epsilon^i{}_{lm}\omega^k{}_{i\mu}\omega^{lm}{}_\nu - [\mu\nu] \end{aligned} \tag{5.4.49}$$

Let us compute on the side

$$-\frac{1}{2}\epsilon^k{}_{ij}{}^{\pm}\omega_\mu^i{}^{\pm}\omega_\nu^j = \frac{1}{2}\epsilon^k{}_{ij}\omega^i{}_{0\mu}\omega_\mu^{0j} \pm \omega^0{}_{i\mu}\omega_\nu^{ik} + \frac{1}{4}\epsilon^i{}_{lm}\omega^k{}_{i\mu}\omega^{lm}{}_\nu \tag{5.4.50}$$

Hence we have:

$$^{\pm}R^k_{\mu\nu} = d_\mu{}^{\pm}\omega^k_\nu - \tfrac{1}{2}\epsilon^k{}_{ij}{}^{\pm}\omega^i_\mu{}^{\pm}\omega^j_\nu - [\mu\nu] = d_\mu{}^{\pm}\omega^k_\nu - d_\nu{}^{\pm}\omega^k_\mu - \epsilon^k{}_{ij}{}^{\pm}\omega^i_\mu{}^{\pm}\omega^j_\nu =: {}^{\pm}F^k_{\mu\nu} \tag{5.4.51}$$

which proves the proposition.

Ashtekar formulation relies on the observation that the self dual Lagrangian

$$^{+}\boldsymbol{L} = \tfrac{1}{2\kappa}\epsilon^{\mu\nu\rho\sigma}{}^{+}R^{ab}_{\mu\nu}e^c_\rho e^d_\sigma \epsilon_{abcd}\, d\sigma = \tfrac{1}{2\kappa}\epsilon^{\mu\nu\rho\sigma}F^k_{\mu\nu}e^c_\rho e^d_\sigma p_{kcd}\, d\sigma \tag{5.4.52}$$

where we set $F^k_{\mu\nu} := {}^{+}p^k_{ab}R^{ab}_{\mu\nu} = {}^{+}F^k_{\mu\nu}$ for the curvature of the self dual connection $A^k_\mu := {}^{+}\omega^k_\mu$. We show that this still leads to field equations equivalent to Einstein equations in vacuum. The self dual Lagrangian (5.4.52) is first order in the self dual connection and zero order in the tetrad.

Field equations are easily shown to be equivalent to

$$\nabla_{[\mu}e^a_\nu e^b_{\rho]}p_{iab} = 0 \qquad\qquad F^i_{[\mu\nu}e^a_{\rho]}p_{iab} = 0 \tag{5.4.53}$$

As usual let us consider the variation of the Lagrangian

$$\begin{aligned}\delta\, {}^{+}L &= \tfrac{1}{2\kappa}\left(\epsilon^{\mu\nu\rho\sigma}e^c_\rho e^d_\sigma p_{kcd}\delta F^k_{\mu\nu} + 2\,\epsilon^{\mu\nu\rho\sigma}F^k_{\mu\nu}e^c_\rho p_{kcd}\delta e^d_\sigma\right) = \tfrac{1}{\kappa}\left(\epsilon^{\mu\nu\rho\sigma}e^c_\rho e^d_\sigma p_{kcd}\nabla_\mu\delta A^k_\nu + \epsilon^{\mu\nu\rho\sigma}F^k_{\mu\nu}e^c_\rho p_{kcd}\delta e^d_\sigma\right) = \\ &= \tfrac{1}{\kappa}\left(-\epsilon^{\mu\nu\rho\sigma}\nabla_\mu\left(e^c_\rho e^d_\sigma p_{kcd}\right)\delta A^k_\nu + \epsilon^{\mu\nu\rho\sigma}F^k_{\mu\nu}e^c_\rho p_{kcd}\delta e^d_\sigma\right) + \tfrac{1}{\kappa}\nabla_\mu\left(\epsilon^{\mu\nu\rho\sigma}e^c_\rho e^d_\sigma p_{kcd}\delta A^k_\nu\right)\end{aligned} \tag{5.4.54}$$

In the filed equation endowed with the connection is $\epsilon^{\mu\nu\rho\sigma}\nabla_\mu\left(e^c_\rho e^d_\sigma p_{kcd}\right) = 0$ in which the covariant derivative is with respect to the self dual connection, indeed the thing to be derives, namely $e^c_{[\rho}e^d_{\sigma]}p_{kcd}$, is in fact a SU(2) object living on ^{+}P. It has a pair of antisymmetric frame indices, i.e. it transforms as something in the Lie algebra $\mathfrak{spin}(4)$, and when we multiply it by p_{kcd} it is like applying Tp_+ and going to the Lie algebra $\mathfrak{su}(2)$.

While the combination of frame fields as $e^c_{[\rho}e^d_{\sigma]}p_{kcd}$ lives in the Lie algebra $\mathfrak{su}(2)$ and, accordingly can be take its covariant derivative with respect to A^k_μ, not so for a frame along as e^c_ρ. This is an object in the Lie algebra $\mathfrak{spin}(4)$ and when is taking its covariant derivatives with respect to ω^{ab}_μ or, equivalently, one needs both the self and antiself dual connections.

When applying the Leibniz rule one has

$$\overset{A}{\nabla}_\mu\left(e^c_{[\rho}e^d_{\sigma]}p_{kcd}\right) = \overset{\omega}{\nabla}_\mu e^c_{[\rho}e^d_{\sigma]}p_{kcd} = 0 \tag{5.4.55}$$

as one can check by expanding both side and using the projection from the spin connection to the (anti)self dual connections.

The equation, anyway, just depends on the self dual connection A.

In principle, the first field equation determines the connection to be the self dual part of the spin connection induced by the spin frame; then the second field equation reduces to Einstein field equation, showing the equivalence.

The first field equation constrains the connection A^k_μ to be the self dual part of the connection induced by the frame. The antiself dual part $^{-}\omega^k_\mu$ of ω^{ab}_μ is free to be fixed arbitrarily. The second field equation depends on A^i_μ only. Hence, after the solution $^{+}\omega^{ab}_\mu = {}^{+}\Gamma^{ab}_\mu$ of the second field equation has been used, the first field equation depends just on j^2e.

As shown below, the second equation amounts to ask that the induced metric obeys Einstein equation. Hence, self-dual formulation is, in vaccum, dynamically equivalent to GR. We shall here prove the equivalence of self dual formulation with standard GR.

The first field equation is $\epsilon^{\mu\nu\rho\sigma}\nabla_\mu e^a_\rho e^b_\sigma\, p^k_{ab} = 0$ which is equivalent to

$$\nabla_{[\mu}e^a_\rho e^b_{\sigma]}\, p^k_{ab} = 0 \tag{5.4.56}$$

By multiplying by p_k^{cd}—using (5.4.24)—and introducing $k_\mu^{ab} = \omega_\mu^{ab} - \Gamma_\mu^{ab}$ as the difference between the connection ω_μ^{ab} and the connection induced by the spin frame, we obtain

$$\left(\nabla_{[\mu}e^c_\rho e^d_{\sigma]} - \nabla_{[\mu}e^d_\rho e^c_{\sigma]}\right) = -\nabla_{[\mu}e^a_\rho e^b_{\sigma]}\epsilon_{ab}{}^{cd}_{..} \qquad\Longleftrightarrow$$

$$\Longleftrightarrow \quad k^c{}_{a[\mu}e^a_\rho e^d_\sigma + k^c{}_{a[\rho}e^a_{\sigma]}e^d_\mu + k^c{}_{a[\sigma}e^a_{\mu]}e^d_\rho - k^d{}_{a[\mu}e^a_\rho e^c_\sigma - k^d{}_{a[\rho}e^a_{\sigma]}e^c_\mu - k^d{}_{a[\sigma}e^a_{\mu]}e^c_\rho = -(k^a{}_{e[\mu}e^e_\rho e^b_\sigma \epsilon_{ab}{}^{cd}_{..} + k^a{}_{e[\rho}e^e_{\sigma]}e^b_\mu \epsilon_{ab}{}^{cd}_{..} + k^a{}_{e[\sigma}e^e_{\mu]}e^b_\rho \epsilon_{ab}{}^{cd}_{..}) \qquad\Longleftrightarrow \qquad (5.4.57)$$

$$\Longleftrightarrow \quad k^c{}_{a[\rho}e^a_{\sigma]} + e^\mu_d k^d{}_{a\mu}e^a_{[\sigma}e^c_{\rho]} = -\left(k^a{}_{e\mu}e^e_{[\rho}e^b_{\sigma]}e^\mu_d + k^a{}_{d[\sigma}e^b_{\rho]}\right)\epsilon_{ab}{}^{cd}_{..}$$

Let us now multiply this by e^ρ_c so that the following is obtained

$$2e^\mu_d k^d{}_{a\mu}e^a_\sigma = \epsilon_{abc}{}^d_. \, e^\mu_d k^{bc}_\mu e^a_\sigma \qquad\Rightarrow\qquad e^\mu_d\left(k^d{}_{a\mu} + \tfrac{1}{2}\epsilon_{bc}{}^d_{.a}\, k^{bc}_\mu\right)e^a_\sigma = 0 \qquad\Rightarrow\qquad e^\mu_d{}^+k^d{}_{a\mu} = 0 \qquad (5.4.58)$$

Thus this condition partially determines the self dual part leaving the antiself dual part completely unconstrained. Now the strategy is clear; just split (5.4.57) into self dual and antiself dual part and use the condition just found.

Before doing that let us prove a couple of properties. First, let us show that

$$\pm k^{dc}_\rho \epsilon_d{}^{efg}_{...} = \pm\left(\pm k^{ef}_\rho \eta^{gc} + \pm k^{fg}_\rho \eta^{ec} + \pm k^{ge}_\rho \eta^{fc}\right) \qquad (5.4.59)$$

In fact, we have $\pm k^{ab}_\rho \epsilon_{ab}{}^{dc}_{..} = \pm 2\,{}^\pm k^{dc}_\rho$:

$$\pm 2\,{}^\pm k^{dc}_\rho \epsilon_d{}^{efg}_{...} = 2^\pm k^{ab}_\rho \epsilon_{ab}{}^{dc}_{..}\epsilon_d{}^{efg}_{...} = 2(\pm k^{ef}_\rho \eta^{gc} + \pm k^{fg}_\rho \eta^{ec} + \pm k^{ge}_\rho \eta^{fc}) \qquad (5.4.60)$$

∎

Second, the same result (that we obtained few times already), i.e. that $k^a_{\sigma\rho} = k^a_{\rho\sigma}$ and $k^{ab}_\rho = -k^{ba}_\rho$ imples $k^{ab}_\rho = 0$.

Proof: *let us low indices and transform all of them to algebra indices by using the suitable frame quantities. The hypothesis reads then as $k_{abc} = k_{acb}$ and $k_{abc} = -k_{bac}$.* *Hence we have:*

$$k_{abc} = k_{acb} = -k_{cab} = -k_{cba} = k_{bca} = k_{bac} = -k_{abc} \qquad (5.4.61)$$

Hence $k_{abc} = 0$.

∎

For the self dual part of (5.4.57) we obtain

$$(2+1)^+k^c{}_{a[\rho}e^a_{\sigma]} = -\epsilon_a{}^{bcd}_{...}e^\mu_d{}^+k^a{}_{e\mu}e^e_{[\rho}e_{b\sigma]} \qquad\Rightarrow\qquad {}^+k^c{}_{a[\rho}e^a_{\sigma]} = 0 \qquad\Rightarrow\qquad {}^+k^{ca}_\rho = 0 \qquad (5.4.62)$$

For the antiself dual part of (5.4.57) we obtain

$${}^-k^c{}_{a[\rho}e^a_{\sigma]} + e^\mu_d{}^-k^d{}_{a\mu}e^a_{[\sigma}e_{c\rho]} = e^\mu_d{}^-k^d{}_{a\mu}e^a_{[\sigma}e_{c\rho]} + (2-1)^-k^c{}_{a[\rho}e^a_{\sigma]} \qquad\Rightarrow\quad 0 = 0 \qquad (5.4.63)$$

To summarise the first field equation determines the self dual part of the connection (to be the self dual part of the connection induced by the spin frame) leaving the antiself dual part completely unconstrained.
The second field equation reads as

$${}^+p_{kab}\, F^k_{[\mu\nu}\, e^a_{\rho]} = 0 \qquad (5.4.64)$$

By remembering that $F^k_{\mu\nu} = {}^+p^k_{ab} R^{ab}_{\mu\nu}$, we get

$$
\begin{aligned}
2\epsilon^{\mu\nu\rho\sigma} R^{cd}_{\mu\nu} p^k_{cd} p_{kab} e^a_\rho =&\, R_{ab\mu\nu} e^a_\rho \epsilon^{\mu\nu\rho\sigma} \pm R^{cd}_{\mu\nu} \epsilon_{cdab} e^a_\rho \epsilon^{\mu\nu\rho\sigma} = -R_{b\rho\mu\nu}\epsilon^{\mu\nu\rho\sigma} \pm e e^\mu_e e^\nu_f e^\sigma_g R^{cd}_{\mu\nu} \epsilon_{abcd} \epsilon^{aefg} = -R_{b\rho\mu\nu}\epsilon^{\mu\nu\rho\sigma} \pm 3! e e^\mu_e e^\nu_f e^\sigma_g R^{cd}_{\mu\nu} (\delta^e_{[b} \delta^f_c \delta^g_{d]}) = \\
=&\, -R_{b\rho\mu\nu}\epsilon^{\mu\nu\rho\sigma} \pm 2e(e^\mu_b e^\nu_c e^\sigma_d + e^\mu_c e^\nu_d e^\sigma_b + e^\mu_d e^\nu_b e^\sigma_c) R^{cd}_{\mu\nu} = -R_{b\rho\mu\nu}\epsilon^{\mu\nu\rho\sigma} \mp 4e(R_{cb} - \tfrac{1}{2} R\eta_{bc}) e^{c\sigma} = \mp 4e(R_{cb} - \tfrac{1}{2} R\eta_{bc}) e^{c\sigma} = 0
\end{aligned}
\tag{5.4.65}
$$

where $R_{\alpha\rho\mu\nu}$ is the Riemann tensor of the metric induced by the frame (so that, by the first Bianchi identities, one has $R_{\alpha[\rho\mu\nu]} = 0$).

The Einstein equation for the induced metric is recovered. By multiplication by e^b_α we obtain

$$
\sqrt{g}(R_\alpha{}^\sigma - \tfrac{1}{2} R\delta^\sigma_\alpha) = 0 \qquad\qquad \Longleftrightarrow\ \ R_{\alpha\sigma} - \tfrac{1}{2} R g_{\alpha\sigma} = 0
\tag{5.4.66}
$$

Both the Ashtekar and standard GR determine as solutions, frames which induce metrics which solve (vacuum) Einstein equations.

References

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Part I debrief

In Part I, we focused on the general structure of covariant theories and we introduced some standard tools (Lagrangian formalism, Noether theorem)
which are essential for discussing classical field theories, in general.

We also introduced some not-so-standard tools (at least in a quite general introduction to GR), from augmented variational principles to Routh transforms, from Utiyama-like arguments to the hole argument(s). Their applications should still be fully uncovered but we gave some examples which should be enough motivation for introducing them.

The framework which emerges from it is a fundamental perspective on what it means to describe the physical world around us in a way which is independent of the observer, i.e. in an *absolute* way. While local expressions are observer dependent, i.e. *relative*, their geometric origin is absolute and one should learn to consider local expressions as relative representations of the geometric absolute situation. In some part of the literature, to do that in the best way, they decide to avoid completely local representations, using only intrinsic descriptions, with the result of often rendering simple things in a quite twisted formalism. For example, think about the Riemann tensor: it has four indices so one has *a priori* 6 different contractions, and 22 different (anti)symmetrisations. If one needed to use them all, a systematic intrinsic notation for them should be introduced. I just believe that using indices is a simple, effective, practice way of denoting the different operations. Accordingly, we tried to boot up a formalism in which local expressions are used to represent global objects, which is closer to the use they made in physics.

We also obtained some non-trivial result. We proved that any (gauge-)natural theory admit superpotentials, so that all Noether currents are, not only conserved, but on-shell exact. Computing superpotentials involves pretty much the whole geometric structure of field theory, from Lie derivatives to structure bundles.

We also showed that, contrary to what is usually believed, there is a dynamical equivalence between any metric-affine theory and a suitably defined purely metric theory. This is obtained at the price of getting modifications to the matter Lagrangian and to the relevant connection structure, modifications which in both cases do depend on the matter coupling, i.e. they depend on what is the matter content of the theory, and vanish, recovering the standard equivalences, for Palatini theories in which matter does not couple to the connection. As a consequence of these modifications, the metric theory to which a general metric-affine is dynamically equivalent is not the one obtained simply by setting the connection equal to the Levi Civita connection of the metric in the Lagrangian, as it happens in Palatini and vacuum theories. It is important we do not start by getting bad habits so that we do not have to quit them later.

Some of the topics are not fully developed. We should need a better understanding of what is a background structure (i.e. also what is a background-free theory). We discussed some non-trivial examples of canonical structures which are not backgrounds, though, these very same examples show that we are quite far away from a satisfactorily precise theory for them.

We should also need a better understanding of the structure of gauge groups and of the definition of physical states. We sketched a framework for it, possibly indicating that the general covariant theory could be richer than that of gauge theories, allowing some freedom to define gauge transformations and, consequently, the physical states and what are gauge invariant observables in a field theory. But we still need a concrete example of it.

We also start sketching initial value problems. More will follow. They are going to be equivalent to Hamiltonian formalism, and they will lead us to extend some mathematical object (e.g. the principal symbol of a PDE) in the non-linear case, taking advantage of the the geometric structures on which Lagrangian framework is built on. In principle, initial value problems (as Hamiltonian formalism) are about determining solutions out of initial conditions given at sometime by determining their evolution in time. As such they are not very much in general covariance spirit. On the same time,

they are important for numerical gravity and quantum gravity. Disentangling the relation between them and the hole argument is a way to obtain a covariant view on the issue, but it is a way which is only partially available.

Overall, natural theories are quite peculiar with respect to other fundamental physical theories. They do not have a linear or affine structure as the other fundamental theories have, they do not come from a simpler theory in which one has no geometric structure by allowing a dynamical evolution for it. On the contrary, they specialise to trivial geometry in some situations. This can be said also by noticing that one cannot switch off a dynamical geometry in an absolute way, as one can switch off electromagnetic interactions (by simply describing a world with no charge). If one did the same thing with gravity, one did not obtained Minkowski, one would obtain a quite unphysical vacuum theory in which there is nothing to describe.

Also physically speaking, Minkowski spacetime is much different from a vacuum state for gravitational field. It is a theory for vacuum and it fails as a model as soon as one allows any other field to exist. Any other field (or particle) should carry some energy, hence bending the spacetime geometry with its back-reaction, and spoiling the approximation which is at the basis of Minkowski framework. This is particularly evident as particles are considered. No matter how small the mass of a test particle is, if it is allowed to exist in Minkowski spacetime it will dramatically bend the spacetime geometry to a Schwarzschild geometry, in which somewhere the curvature is actually singular. One can maintain flatness approximation out of that region, away from the test particle, though there is always a region in which the curvature grows at will.

For this reason, SR is in principle an inconsistent framework or a theory of empty spacetime. Any description of matter (and physics) in SR is fundamentally inconsistent. Of course, one could argue that the approximation is good enough if we stay away from the particles, i.e. in something like the limit $r \gg 2\mu G/c^2$, which is of course true. When the mass μ of the particle is small, the region in which the approximation fails is small, though, since it is a limit of a dimensional quantity, from a fundamental viewpoint, there *small* does not mean anything. A number is *big* or *small*. A mass or a length is not *big* and *small* in any reasonable sense, since the number which represents it depends on the units we choose.

One can further argue that homogeneous units can be used so that mass, lengths, time lapses, charges, are adimensional. That is true, however, to define the adimensional mass one needs to use the Newton constant G, i.e. still allowing gravity at a fundamental level.

All hints are leaning toward a simple fact: while one can describe a world with the gravitational field with no further fields, one cannot describe other fields with no gravitational (or inertial) field. This is why usually one says that the gravitational field is what left of physical once one take away from spacetime what is only mathematical. Also LQG got to a beautiful view on this issue: there are *"no more fields on spacetime, just fields on fields"*.

> Yeah, I know it does seem to say the opposite thing. However, as a matter of fact, they show that the gravitational field is described by a spin network—or a spin foam—which does not leave in a manifold, though it represents it (*no more fields on spacetime*). Any additional field is described as more numbers leaving on this structure (*just fields on fields*). That is, as a metter of fact, at least to some extent, the other fields on the gravitational one. I do not think one could set up a coherent setting for electromagnetic field without the gravitational field (technically speaking, the formulation of electromagnetism by WIlson loop, does not deliver due to the fact that one has no general covariance, and cannot solve the constraints to the end).

The good news is that, this not too dramatic. We can (and should) learn to live in a gravitational (i.e. non-linear, curved) spacetime and still give a fundamental bootstrap for a physical, absolute, description of what is around us. We just need to introduce structures in the proper way, instead of borrowing them from a framework (e.g. a Minkowskian SR framework) which we are already familiar with, though it is pretty clear should be obtained as a limit of GR, instead of being used as a foundation for GR.

I know it sounds a bit extreme as a viewpoint, but that is what we want to do. After all, at the end of the day, it is not that different from what they argue in loop quantum gravity, when they discuss background independence and non-perturbative approaches. Of course, they start off the need for a

quantum perspective in gravity, we are here go back to foundations of *classical* general relativity. No surprise most of the structures we introduce are very similar to what they start from.

As a stating point, we discussed many frameworks for dynamics geometries: we started from (purely-metric, Palatini, metric-affine) standard GR, but the also introduced purely metric $f(R)$-theories, as well as Palatini $f(\mathcal{R})$-theories, (purely metric) Brans–Dicke theories, purely frame and frame-affine standard frameworks, Ashtekar selfdual formulation, as well as conformal gravity.

We showed that purely metric standard GR is dynamically equivalent to metric-affine standard GR, that we can transform back and forth between some Brans–Dicke theories and standard GR (at price of considering different metrics in the two theories). We also discuss equivalence between standard GR and Palatini $f(\mathcal{R})$-theories in vacuum.

As matter fields, we considered electromagnetism, Yang–Mills, Klein–Gordon scalar field, Dirac spinors.

In the next Part, we shall start and discuss the dynamical geometries of spacetime, finally connecting them to the gravitational field.

Part II

Gravitational theories

The curvature of space has to be determined on the bases of astronomical observations.

(On the Hypotheses that lie at the Foundations of Geometry, *B. Riemann*; Göttingen 1854)

II. Introduction to Part II

What we did until now is essentially mathematics. We introduced quite a deal of framework for field theories though, in a sense, the physical meaning of the theories we have been describing has been considered inessential, the tools being important on their own.

Now we have to show what those tools are made for and apply them to discuss dynamical spacetime geometries, and gravitational field in particular. On one hand, we have to characterise the relation between spacetime geometries and the gravitational field, and that is where physics begins. There are many possible things that one can mean by *spacetime geometry*: from a Lorentzian metric structure (g, as in standard purely metric GR) to metric-affine ($g, \tilde{\Gamma}$), to even purely affine ($\tilde{\Gamma}$, only), from purely frame (e_a^μ) to frame-affine (e_a^μ, ω_μ^{ab}), or conformal structures ($[g]$) to replace the metric.

On the other hand, we saw that potentially different approaches may be dynamically equivalent and one should discuss whether they are physically equivalent as well. On one hand, dynamical equivalence often depends on matter couplings, on the other hand, physical equivalence also relies on the observational content of the theory, which is still to be developed.

All our discussion about gravitational field is based on Ehlers–Pirani–Schild (EPS) framework, in which they derive spacetime geometry from the motion of light rays and particles which are potentially observable. They find out that the geometry of spacetime is allowed to be more general that usually assumed in standard, purely metric GR (i.e. a Lorentzian metric structure), though that choice is allowed in this family as well. It turns out that is dynamics that selects which is the EPS geometric structure actually realised in spacetime. While standard GR dynamics privileges a Lorentzian, purely metric structure, other dynamics single out more complex structures. Thus, at least, it is clear that one is not free to select geometries and dynamics independently, they come in pairs.

Then we need to extract the observational content of each of these theories, compare with observations which will eventually single out one dynamics (which comes together with a definition of geometry on spacetime) which better suits experiments. Hopefully, in such dynamics, we should be able to discuss weak field limit and recover Minkowskian SR (actually, a generally covariant version of SR, as it happens every time that a theory is recovered as a limit of a theoretically incommensurable, more fundamental, theory) and derive Newtonian limit from scratch.

Of course, we are not able to go that far yet, though that is the direction to be explored. It is a bit extreme position, not very pragmatic, though it is the path which we chose to have ahead.

We shall not discuss (by now) quantum gravity. Our analysis in entirely classical, it is concerned with the classical regime which itself is not fundamental. However, even if eventually some sort of more fundamental quantum framework is in order, from an observational viewpoint, the classical regime makes a damn good sense!

So good that, our current problem in quantum gravity is that, apparently, there not a single observation we are currently able to make in which both relativistic effects of the gravitational field and quantum effects of it are relevant. For that reason, quantum gravity is, currently, overpopulated of models and deserted of data. Honestly, we are barely able to observe the effect of a (Newtonian) gravitational field on quantum fields. However, time goes by fast. When I was student, we did not even ever saw a particle falling in a gravitational field, today we are on the verge of observing entanglement induced by a (Newtonian) gravitational field produced by a quantum system on another. Moreover, we (well they did it for us) just detected gravitational waves generated by merging of black holes, which, in principle, carry a lot of information about the strong field regime (as well as should be very clean signals of that regime since they do not depend on matter content, on astrophysical details, on stellar models, . . .). That opens a new observational domain which is still all to be uncovered.

However, most of the current approaches to quantum gravity rely on standard GR and try to give a quantum version of it. If classical models other than standard GR are better suited to describe gravity, that is something one would like to know before starting to even attempt to quantise the gravitational field. We already have a hint that even classically the gravitational field is somehow more fundamental than other fields, surely it has peculiar properties which can make its quantisation go along guidelines different from what happens for other fields.

Another example, is dynamical equivalence and other classical equivalences: they are *classical* equivalences which hold on shell. If we have learnt anything about the quantum world in this last century and we take it seriously, quantum world is effected by off shell behaviour and classical world, which is on shell and it emerges only on average. Consequently, two classical theories, even if they are classically equivalent, they are not necessarily still equivalent when quantisation is concerned. There is not even any guarantee that they both emerge from a quantum theory! In other words, carefully consider the foundations of classical gravitational theories can provide us with new information about quantum regimes, even in complete absence of classical effects to distinguish between them. However, we repeat it, there are classical models for gravity which are not even classically equivalent. In this case, even more, it may be important to develop quantum gravity strategies.

On the other hand, classical theories of gravitations needs to fit astrophysical and cosmological observations. These observations have grown constantly in number and accuracy. They are very well understood within standard GR, though, often, they are not as well understood within more general models. I suppose one could say that often it is hard to trace what, in an observation, depends on the model and what does not. Thus that is the issue in which we decided it to work. At least, to understand observation in the wider family of extended theories of gravitation, which contains standard GR, as well as other, less degenerate theories.

About the metric structures, one could in principle have many options to choose the signature. The signature is discrete, so it cannot change (continuously) from one point to another in spacetime (a continuous function valued in a discrete set is necessarily locally constant). Thus one should ask why Lorenztian is better than Euclidean, or any other signature? One usually has in mind that SR should eventually hold true in the weak field approximation, and so one *assumes* the Lorentzian signature in view of it.

It is quite interesting to see that it is not necessary. We shall see that EPS framework has an answer for it and the signature is necessarily Lorentzian as a consequence of the assumptions about light rays and light cones. So the signature is implicitly determined by the (topological) shape of the light cones we observe, and Minkowski framework should *then* follow in the weak field approximation. That is an example of what we mean by building gravitational theories on a fundamental level, not relying on simple, (fundamentally wrong), partial theories such as SR or Newtonian physics.

At the very least, we would like to keep track of these choices, develop a framework which is general enough, and use what we know to make our choices among possible (and well-defined, self-consistent) models, which one is physically realised (that is usually something to be decided on observations). It is quite important that we have (at least) a theoretical grasp on more general models among which we have to choose on an observational stance; if we have to design experiments to see differences among different models, it is utterly important that our experiments are not biased and we are aware of how different models affect the data analysis, which, we shall see, is quite difficult.

Another issue is about the use of approximations. Gravitational theories are difficult to be dealt with. Let us stress that there are issues still open in Newtonian gravity (for example a good framework for n-bodies interactions), imagine how bad the situation can be in a relativistic regime! Most of the times, approximations are very well justified by our physical intuition about when classical limits or Newtonian limits should approximately apply. If one is modelling something in solar system, our physical intuition has been at least well tested. When it is extrapolated to a globular cluster 10 billion light years away, it is an extrapolation, which is probably all right, though certainly an extrapolation to be checked.

The material of this part is organised as follows: In Chapter 5 we introduce EPS axiomatic approach to spacetime geometry. The main idea of it is to derive the geometry of spacetime as a byproduct of potentially observable structures, as the motion of light rays and particles. We follow quite closely the original work, with few exceptions: first we work in arbitrary dimension and, second, we introduce the geodesic equations for particles based essentially on covariance rather than on equivalence principle.

What EPS obtained as a candidate geometry of spacetime to represent the gravitational field is something more general than the classical Lorentzian geometry originally used for standard GR.

In Chapter 6 we introduce extended theories of gravitations as the relativistic theories which implement (in a specific form, as a consequence of field equations) the most general structure identified by EPS. As a prototype of extended theories of gravitation we investigate in particular Palatini $f(\mathcal{R})$-theories. We briefly review their general properties and investigate different conformal frames and their equivalence.

In Chapter 7 we consider the application of extended theories of gravitation to cosmology. This establishes a more general setting for cosmology, more general than the standard approach, in which we can discuss a model in which the evolution of the universe is (restricted to some specific aspects) reproduced without introducing dark sources at a fundamental level. We shall also discuss that this model also has various issues, though we believe it is worth studying.

In the further Part III we shall specialise to standard GR as a specific extended theory of gravitation.

Chapter 6. Spacetime geometry

The geometry of Tlön comprises two somewhat different disciplines: the visual and the tactile. The latter corresponds to our own geometry and is subordinated to the first. The basis of visual geometry is the surface, not the point. This geometry disregards parallel lines and declares that man in his movement modifies the forms which surround him.

(Jorge Luis Borges, *Tlön, Uqbar, Orbis Tertius*).

1. Introduction

In Part I we introduced spacetime and its geometric structures as primitive concepts. We postulated some sort of dynamics and discuss their observability in view of general covariance principle. Unfortunately, we discovered that even in the simplest case of a purely metric theory general covariance principle does not allow to observe all degrees of freedom encoded by the metric tensor.

Of course, field theories and Hamiltonian formalism developed tools to analyse the situation. There are difficulties in applying these tools directly, but we shall do it later on; see here.

Still, one of the problems of relativistic theories is that it is difficult to relate them in general to what astrophysicists routinely observe. Often they use special coordinates, adapted to our Newtonian habits, though corrected in view of GR effects. Whatever coordinates are fixed, this breaks explicit general covariance and most of the structure of the relativistic theory is lost.

More generally, in relativistic theories, it is not easy to trace the relation between observable quantities and field theoretical objects. For these reasons, we shall here try a different approach. Instead of defining observable objects in terms of field theoretical quantities, let us try to list observable objects and their observable properties, finding enough axioms to build geometric structures on spacetime. This axiomatic approach has various advantages. First of all, it clarifies the origin of geometric structures instead of assuming them. Moreover, in this axiomatic approach, it is easier to discuss the observability of the objects defined along the way.

We are not too interested here in a complete axiomatic analysis of gravitational theories. In particular, we are not interested in showing independence of axioms. On the contrary, we privilege discussion about physical motivations of the axioms. The axioms are very closed to the one proposed by Ehlers, Pirani and Schild (EPS; see[18]). We just adapted the axiom about coordinates in order to generalise the discussion to a generic dimension, we sometimes group them differently, and we added some details. We strongly recommend to read the original paper, which is definitely worth the time.

Before starting off, we spend some time to define some very basic structures (spacetime, worldlines, light rays, particles, clocks). In the rest of the Chapter, terms are used strictly as defined. In particular, clocks are simply parameterised curves, with *any* parameterisation. Since we aim to provide a fundamental framework, there is no possibility of defining something like *uniform clocks*, or *proper time*. The first is, in principle, untenable (if it

were one should declare *uniform* with respect to what, since, of course, there is no absolute definition of what uniform means). The second depends on a metric structure, which we still have to define.

2. Events and spacetime

Events are primitive objects. Physically, they represent something which can happen and can be observed, not having any spatial extension or time duration. A lamp switching on at some time, a supernova explosion, a train leaving from a station can be all modelled by events.

> Of course, none of them is really an event (e.g. a supernova has a spatial extension and duration) but one can, in some situation, think about them as events. No difference between geometric points, lines, and triangles and their physical counterparts; they are different but I never meet anyone arguing that geometric points are irrelevant since they have no physical actual realisation. They are just a model for many interesting situations and they make easy to describe physics in mathematical terms.

The collection of all possible events is called *spacetime*. One usually assumes to have a notion of continuity on spacetime. Mathematically, this directly corresponds to have a topological structure on spacetime. Geometrically speaking, one can define spacetime to be a topological space, and then events to be points of spacetime.

An event usually needs 4 numbers to be specified, usually one for time and 3 for the spatial position (though, as we shall see, one could equivalently use 4 times or any other combination); for this reason, one usually assumes that spacetime has dimension 4. Mathematically, this corresponds to a notion of charts, i.e. to assume spacetime to be a topological manifold. On topological manifolds, one can define a notion of *dimension* which can thence be set to 4.

Then, we well know that we can measure velocities of moving points, which are tangent vectors (or tangent directions, as we shall see more precisely) to spacetime. This requires spacetime to be a smooth manifold (well, at least a $\mathcal{C}^1$ manifold).

Hence, one should say that spacetime is a smooth manifold of dimension 4; however, there are different models in which a spacetime of different dimension is introduced. Moreover, when nothing changes in some spacetime directions, one often understands the trivial directions using lower dimensional descriptions of spacetime.

Finally, not much of what follows will actually depend on the specific dimension of spacetime and when it does it will be interesting to know it. For these reasons, we shall drop specific requests on the spacetime dimension, simply assuming that spacetime is a smooth manifold M of dimension m.

Hereafter, some low dimension (e.g. $m = 2$) will often behave peculiarly and will often be discussed on its own (or excluded). However, in most situations dimension will not be relevant *provided it is high enough*.

We shall see below that the differential structure of the spacetime manifold will be suitably required to be *compatible* with particles and light rays.

> Let us point out that *initially* we claim that spacetime is a manifold, *any* manifold. However, at some point (here) we shall required that it allows an atlas made of some special charts. And fixing an atlas on a manifold, is just as fixing its differential structure. Accordingly, the differential structure on spacetime, will be fixed by the axioms, spacetime will not be *a* manifold, it will be *that* manifold.

Topology of spacetime

At this point it is important to understand that the assumptions about topology of spacetime are mathematical and (but) are not without physical consequences. On the contrary, it is likely that such assumptions will turn out to be questionable on the physical ground. There are in fact hints that a quantum description of spacetime leads, somehow essentially, to a discontinuous structure on spacetime, similarly to when a quantum description of particles leads to the loss of the notion of a continuous trajectory.

There are models in quantum gravity in which spacetime is described by more general structures (e.g. CW-complexes) in which the dimension, as well as the topological and geometric structure, just emerge at a classical level.

Although these criticisms are reasonable and well motivated, we are not currently able to develop a quantum gravity approach from scratch (and we have a lot of problems with it also not starting from scratch!). On the other hand, there are certainly regimes for which continuity (and differential structure) in spacetime is very well established. To be true, there is currently no observable situation in which direct evidences of quantum framework appear. Classical general relativity is a very good model, which today seems to be able to explain everything we see in the physics of gravitation and model it with a precision which goes far bejond the experimental accuracy we can obtain. It may be that when we will be able to resolve events at the Planck scale we shall be forced to give up continuity. However, the continuous framework is very good at the scales we are able to observe today. This is exactly why we have problems in developping a quantum gravity paradigm! We have no observation to lead us into this new, more fundamental, scale of description of the physical world and we grope in the dark with not much more than just mathematics and logic leading us.

Anyway, let us forget about all these problems in quantum gravity for now and let us face the problem of formulating a classical model for gravity.

3. Worldlines

When we observe something (e.g. the planet P) moving around, we can regard it as a continuous family of events, each event recording that P passed by space location x at time t. The motion of P is described by a *trajectory* in M.

A *trajectory* of spacetime, i.e. a 1-dimensional sub-manifold of M, namely the subset $w(P) = \{(t,x)\}$. Such a trajectory is assumed to be continuous (the planet P will not jump from one place to another) and smooth (we want to be able to define tangent vectors).

The first thing to be clarified is the difference between *trajectory* and *curve*. A trajectory is a subset of M, while a curve is a (continuous—or smooth depending on the regularity required) map $\gamma : I \to M$, where $I \subset M$ is an interval in $\mathbb{R}$. There is no loss of generality in assuming that $0 \in I$.

A motion in spacetime is a (continuous) sequence of events, as such it is a trajectory. As we shall see, not all trajectories can be considered as physical motions. For example, if we had a 2-dimensional spacetime with coordinates (t,x), t being the time of an event and x the position both measured by a given observer, then the trajectory $\{(t = t_0, x = s) : s \in \mathbb{R}\}$ is not a motion since the moving point would be at infinitely many different positions at the same time $t = t_0$. We shall see later on other examples of trajectories which cannot represent physical motions. Trajectories which can be regarded as the motion of a physical particle (or photons) will be called *worldlines*.

If one has an extended object moving in space, its motion can be represented by a sub-manifold in spacetime of one dimension higher than the object. For example, the motion of a 1 dimensional object, namely a line, can be represented by a 2 dimensional surface in spacetime, called a *worldsheet*.

A curve is *simple* when it is injective. Of course, given a (simple) curve γ its image $\gamma(\mathbb{R})$ is a trajectory, though a curve contains more information than its trajectory: a curve is a trajectory covered in a specific way.

On the contrary, any (piece of a) trajectory can be considered as the image of a curve, e.g. one can fix a metric on M and then use the arc-length along the trajectory as a parameter.

For example, let us consider an Euclidean plane $(\mathbb{R}^2, \delta)$, where δ denotes the standard inner product of $\mathbb{R}^2$. Let us fix Cartesian coordinates (x, y), with respect to a δ-orthonormal frame.

A straight line $S = \{(x, y) : ax + by = c\}$ is simply the subset of all points (x, y) satisfying the equation $ax + by = c$. Notice that one cannot have at the same time $a = b = 0$ otherwise the solution space is not a line (it is empty for $c \neq 0$ or the whole $\mathbb{R}^2$ for $c = 0$). Except that, there is one straight line for each triple (a, b, c) (while, of course, the triple $(\lambda a, \lambda b, \lambda c)$ defines the same straight line for any non-zero $\lambda \in \mathbb{R}$).

One can provide a global parameterisation of the straight line S by choosing a parameter s long the trajectory, e.g. assuming $a \neq 0$ one can choose:

$$\gamma : \mathbb{R} \to \mathbb{R}^2 : s \mapsto \left(x = \frac{c - bs}{a}, y = s \right) \tag{6.3.1}$$

The straight line with $a = 0$ cannot be parameterised in the same way; however one has in this case $b \neq 0$ and can define another global parameterisation

$$\sigma : \mathbb{R} \to \mathbb{R}^2 : s \mapsto \left(x = s, y = \frac{c - as}{b} \right) \tag{6.3.2}$$

In the generic case in which both $a \neq 0$ and $b \neq 0$, one can use both parameterisations of the same straight line. Both these parameterisations happen to be *global* parameterisations, since the parameter s is free to range over the whole $\mathbb{R}$ corresponding to covering the whole straight line.

Of course, then one can choose a different parameter s', with $s = \phi(s')$, and define infinitely many parameterisations of the same straight line in $\mathbb{R}^2$, namely

$$\rho : \mathbb{R} \to \mathbb{R}^2 : s' \mapsto \left(x = \frac{c - b\phi(s')}{a}, y = \phi(s') \right) \tag{6.3.3}$$

In this case, the parameterisation can be local when the map ϕ is not globally invertible.

For example, one can set $\phi(s') = s'^2 + 1$ so that $s' \in (0, +\infty)$ is mapped into $s \in (1, +\infty)$. The reparameterisation corresponding to (6.3.3) reads as

$$\rho : \mathbb{R} \to \mathbb{R}^2 : s' \mapsto \left(x = \frac{c - bs'^2 - b}{a}, y = s'^2 + 1 \right) \tag{6.3.4}$$

By specializing the parameters of the straight line to be $(a = 1, b = 0, c = 0)$, we obtain the straight line coinciding with the y-axis and two parameterisations of that trajectory, namely:

$$\gamma : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (x = 0, y = s) \qquad \rho : \mathbb{R} \to \mathbb{R}^2 : s' \mapsto \left(x = 0, y = s'^2 + 1 \right) \tag{6.3.5}$$

If we think to $\mathbb{R}^2$ as being the space, and the curves γ, ρ being the description of a motion of a point in space (not the trajectory to be the worldline of a motion in $\mathbb{R}$), then the first is representing a uniform motion along the y-axis, while the second is representing an accelerated motion along the y-axis. (When interpreted as worldlines, they both represent a uniform motion in $\mathbb{R}$.)

These two motions represent locally the same trajectory. Hence a *curve* is a trajectory with a specific parameterisation set on it.

As we can regard a *curve* as a trajectory with some extra information on it, we can also do the other way around. Two (simple) curves $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ such that there exists a (global) reparameterisation (i.e. a global diffeomorphism $\phi : I_1 \to I_2$) of γ_1, leading to γ_2, i.e. $\gamma_2 = \gamma_1 \circ \phi$, can be said to *share the same trajectory* and we write $\gamma_1 \sim \gamma_2$.

Check that this defines an equivalence relation.

Since we defined an equivalence relation on the set of all (simple) curves $\gamma(M)$ on spacetime, we can define the quotient set $\boldsymbol{\tau}(M) = \gamma(M)/\sim$. An element $[\gamma] \in \boldsymbol{\tau}(M)$ is an equivalence class of curves which can be all obtained one from another by reparameterisations. These curves represent all possible parameterisation along the same trajectory. All these curves share exactly the trajectory which is a (the) characteristic of the whole class. In this way, we can define a trajectory to be an equivalence class (modulo reparameterisations) of curves.

In what follows, trajectories on a spacetime M will be denoted by bold Greek letters, while curves by the corresponding Greek letter. In other words, we might write:

$$\boldsymbol{\gamma} = [\gamma] \in \boldsymbol{\tau}(M) \tag{6.3.6}$$

To be precise, in the previous example, we made some abuses of language. In fact, the two curves γ and ρ given in (6.3.5) do not define the same trajectory. In fact, γ defines the y-axis, while ρ defines half line of the y-axis, namely $\{(0, y) : y \geq 1\}$, which is a proper subset of the y-axis.

One should say that a trajectory is a line which is *locally* parameterised by a curve.

Notice that a (local) parameterisation along a curve is (the inverse map of) a chart of the trajectory thought as a 1-dimensional manifold. The fact that we defined reparameterisations by diffeomorphisms, means that we are defining a differentiable structure on trajectories.

Oriented worldlines

Reparameterisations are defined to be diffeomorphisms $\phi : I_1 \rightarrow I_2$; *global* diffeomorphisms. As a consequence, their derivative at a point cannot be $\dot{\phi}(s) = 0$. Consequently, if ϕ has a positive derivative at a point $s \in I_1$ it has positive derivative everywhere in I_1. Or, analogously, for negative derivatives. We have hence two classes of reparameterisations: the ones with (everywhere) positive derivative (which will be called *orientation preserving*) and the ones with (everywhere) negative derivative, which will be called *orientation reversing*.

One can define an equivalence relation among curves weaker than the one to define trajectories. Two curves γ_1 and γ_2 such that there exists an orientation preserving reparameterisation (i.e. a global diffeomorphism $\phi_+ : I_1 \rightarrow I_2$ with everywhere positive derivative) of γ_1 leading to γ_2, i.e. $\gamma_2 = \gamma_1 \circ \phi_+$, can be said to be *weakly equivalent* and we write $\gamma_1 \sim_+ \gamma_2$.

Check that this defines an equivalence relation.

Since we defined an equivalence relation on the set of all curves $\gamma(M)$, we can define the quotient set $\boldsymbol{\tau}_+(M) = \gamma(M)/\sim_+$. A class in $\boldsymbol{\tau}_+(M)$ will be called an *oriented trajectory*.

One can define an orientation reversing reparameterisation $\phi_- : s \mapsto -s$; two oriented trajectories $\boldsymbol{\gamma}_1 = [\gamma_1]$ and $\boldsymbol{\gamma}_2 = [\gamma_2]$ are said to be equivalent if for any representative γ_1 of $\boldsymbol{\gamma}_1$ then $\gamma_1 \circ \phi_-$ is a representative of $\boldsymbol{\gamma}_2$. This defines an equivalence relation (denoted by $\sim_\pm$) on $\boldsymbol{\tau}_+(M)$ and by the quotient set $\boldsymbol{\tau}(M) = \boldsymbol{\tau}_+(M)/\sim_\pm$ we recover trajectories.

$\gamma(M)$

$\downarrow{\scriptstyle p_+}$

$\boldsymbol{\tau}_+(M)$

$\downarrow{\scriptstyle p_\pm}$

$\boldsymbol{\tau}(M)$

To summarise, one has *curves* $\gamma : I \rightarrow M$ on spacetime, oriented trajectories $\boldsymbol{\gamma}_+$ which are equivalence classes of curves, and trajectories $\boldsymbol{\gamma}$ which are equivalence classes of oriented trajectories. Each trajectory, as a class of oriented trajectories, contains two oriented trajectories which are the two possible orientations of the same trajectory.

Clocks

It is important to notice that a trajectory contains all physical information about the motion in a space Q. The trick is that we are in spacetime. The trajectory in spacetime projects onto space and the information along the time direction encodes the way such a spatial trajectory is covered.

> The procedure is quite simple and well known: imagine one wants to describe a motion in 1 spatial dimension (say along a straight line with coordinate q). One obvious choice is to use a map $q : \mathbb{R} \to \mathbb{R} : t \mapsto q(t)$, as one does usually in mechanics. That is a curve in $\mathbb{R}$.
> However, one can also consider the graph of such a curve $L := \{(t, q(t)) : t \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$ which is, in fact, a trajectory in the "spacetime" $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The graph L is a trajectory in $\mathbb{R}^2$.

Of course, such a trajectory in spacetime contains all information about the motion. For example, the velocity of the motion $q : \mathbb{R} \to Q : t \mapsto q^i(t)$ is encoded by the vectors tangent to the space

$$v = \dot{q}^i(t)\frac{\partial}{\partial q^i} \tag{6.3.7}$$

but also by the direction of the tangent line to the trajectory in spacetime, given by the linear subspace in the tangent of spacetime which is generated by the vector

$$u = \frac{\partial}{\partial t} + \dot{q}^i(t)\frac{\partial}{\partial q^i} \tag{6.3.8}$$

Let us call v the *spatial velocity*, or the *Newtonian velocity*.

If $\sigma : I \to M : s \mapsto x^\mu(s)$ is a curve in a spacetime M of dimension $m = \dim(M)$ representing a worldline, then the tangent vector

$$u = \dot{x}^\mu(s)\partial_\mu \tag{6.3.9}$$

is called the *spacetime velocity* or *covariant velocity*.

> We have to stress that the covariant velocity has no physical meaning: if one selects a different representative σ' for the worldline $\boldsymbol{\sigma} = [\sigma]$, its covariant velocity u' differs from u. To be precise, the direction of u and u' is the same (i.e. they generate the same linear subspace in $T_x M$), i.e. u' is parallel to u, i.e. $u' = \lambda u$ with $\lambda \in \mathbb{R}$. Hence, the direction of u has a physical meaning (it is a characteristic of the worldline, not only of a representative curve), while the "length" of u does not, since it depends on the representative.

Given the curve σ on spacetime, it defines a covariant velocity $u = u^\mu\partial_\mu$; one can define the quantities

$$w^i = \frac{u^i}{u^0} \tag{6.3.10}$$

which in fact is independent of the representative and depends only on the worldline. If coordinates on spacetime are referred to an observer and x^0 has to do with the time measured by that observer, while x^i have to do with the positions, then one can define a spatial vector

$$w = w^i\partial_i \tag{6.3.11}$$

which is the Newtonian velocity measured by that observer.

All these remarks are, in fact, quite trivial. At any basic diving course, graphs like these are shown.

They are called *square profile dive*. They, in fact, represent a worldline in a 2 dimensional spacetime. Here, the only relevant spatial coordinate is depth h.

Once the conventions of the observer are specified (depth is measured in meters, starting from sea surface, pointing downward, time measured in minutes, starting when dive starts), then one can figure out everything about the diving which is represented. The diver goes down to 12 meters with a constant velocity in 3 minutes, then he stays at 12 meters for 6 minutes (looking at oysters), then he starts going up again at a constant velocity until he reaches depth 5 meters in 2.5 minutes and stays there for 2 minutes, and so on.

Notice how the slope of curves informs us about velocity and horizontal information informs us about times.
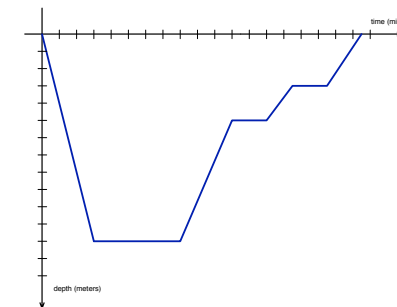
Fig 5.1: A square profile dive

Despite worldlines are endowed with a covariant meaning, also curves has a physical interpretation, though not a covariant one. Consider a curve $\sigma : \mathbb{R} \to M$ in spacetime: as we said above, it represents a trajectory (and we require it to be a worldline) with some extra information on it, namely a parameterisation. A parameterisation is assigning a name $s \in \mathbb{R}$ to each event along the worldline. Well, this is essentially what a *clock* does!

A clock is a moving object which is described by a worldline that tics. Although ticking is a discrete process, one can idealise the situation by assuming that a clock stores a continuous register which monotonically increases along the worldline. Such a register defines a parameter along the worldline, i.e. it selects a representative curve for it.

For these reasons, we define a *clock* to be a curve representative of a worldline. Of course, general covariance principle, in particular, claims that there is no *a priori* preferred parameterisation to be selected. Choosing a clock is part of the specification of an observer. As such, clocks have a physical meaning, though not an *absolute* physical meaning.

4. Light rays and particles

In spacetime, we can trace the worldlines of photons or short laser pulses moving under the influence of the gravitational field only, subject to no other force. The worldlines representing their motion in spacetime will be called *light rays*. They have not to be confused with the lines dual to the motions of wavefronts used in optic approximation, which are trajectories in *space* (not in spacetime) and contain no information about the way the light moves along its path. For us, a light ray is a trajectory in spacetime which represents a particle moving at the speed of light.

Let us denote by $\mathcal{L}$ the set of all possible light rays in M.

Let us stress that we here want to *define* a light ray to be an element in $\mathcal{L}$. This is in contrast with what it is usually done in the standard approach to GR, where one usually defines light rays as light-like geodesics, but we cannot do it since for us the spacetime comes without a metric on it. Instead, we want to provide the sets $\mathcal{L}$ and $\mathcal{P}$ as primitive notions, ask some properties for them by axioms, and then discuss how one can build a metric (or, precisely, a conformal structure) on spacetime out of them.

And this is purely mathematics and it does not claim anything about the physical spacetime, except that *if* one had two sets $\mathcal{L}$ and $\mathcal{P}$ which obey the properties expressed by the axioms, *then* a metric or a conformal structure would be defined out of them.

Since we do have set up experimental protocols, we *measure* physical light rays and then we build with them the set $\mathcal{L}$ (as well as the set $\mathcal{P}$ measuring particles), since the sets $\mathcal{L}$ and $\mathcal{P}$ so-defined in terms of *observations* happen to obey the properties required by axioms, then the metric (or the conformal structure) on spacetime does follow and it is linked to potential observations.

Notice that $\mathcal{L}$ should eventually contain quite a lot of worldlines; physical experience shows us that, at any event, one has a different light ray for any spatial direction set as initial condition. We shall guarantee that one has enough light rays in $\mathcal{L}$ by an axiom we shall require in a while.

In the meanwhile, let us consider what happens at an event $x \in M$: first of all, we can define $\mathcal{L}_x \subset \mathcal{L}$ the set of all light rays passing through x. For each light ray $\gamma \in \mathcal{L}_x$, let us consider a parameterisation $\gamma \in \boldsymbol{\gamma}$ such that $\gamma(0) = x$. Being the light ray a smooth manifold, one can define its tangent vector $\dot{\gamma}$.

A spacetime vector $\dot{\gamma} \in T_x M$ tangent to a representative γ of a light ray $\boldsymbol{\gamma} = [\gamma] \in \mathcal{L}_x$ is called a *light-like vector*. Once a vector $\dot{\gamma}$ is shown to be light-like, by changing the representative for the light ray $\boldsymbol{\gamma}$ (e.g. just by rescaling the parameter), it is easy to show that any multiple $\dot{\gamma}' := \lambda \dot{\gamma}$ is again light-like. Hence, the set of all light-like vectors is a cone in the topological sense. We shall see below that it is also a *cone* in the sense of primary schools and ice creams (though in higher dimension). Such a cone $\mathcal{C}_x \subset T_x M$ is called the *light cone* at x. Of course, the light cones are made of straight lines (in the tangent space TM) each determining a tangent direction to the spacetime which is tangent to a light ray's worldline. As such, the light cone is defined just in terms of worldlines and is thence endowed with an absolute physical meaning.

Similarly, let us denote by $\mathcal{P}$ the set of all possible worldlines of freely falling (i.e. subjected to gravitational field , only) particles in M.

As for light rays, we *define* a free falling particle to be an element in $\mathcal{P}$ and $\mathcal{P}$ is a given set of trajectories in spacetime obeying the axioms we shall require below. Axioms come with consequences, i.e. all theorems we are able to prove from them.

Then, in view of the observational protocols, we can trace in M the worldlines of all freely falling physical particles and then check that they define a physical realisation of the set $\mathcal{P}$ obeying the axioms. Accordingly, we will be able to deduce for them the properties proven in the theorems.

Also particles are expected to be quite a number; one for any initial condition, i.e. an event x and a spatial velocity not exceeding the speed of light. Again, this expectation is about *physical particles*, while the forthcoming axioms are needed to guarantee that there are as many *theoretic counterparts* of them in $\mathcal{P}$. Also, as above, $\mathcal{P}_x \subset \mathcal{P}$ contains all particles through an event $x \in M$; each such particle can be parameterised as a representative curve $\sigma^\mu(s)$ such that $\sigma^\mu(0) = x^\mu$.

The tangent vector $\dot{\sigma}$ to a curve σ representing a free falling particle is called a *particle-like* vector. A tangent vector which is not light-like or particle-like will be said to be *ghost-like*.

'Ghost-like' has to be understood as a name only. We just temporarily need a name for what will eventually be called *space-like vectors*. Similarly, 'particle-like' vectors will eventually be identified with *time-like vectors*.

Here the issue is originated again by the fact that, in the beginning, we do not have a metric on spacetime. Once we shall eventually define a Lorentzian metric g on spacetime, then we will define *time-like* vectors, *space-like* vectors or ^{0}vectors, depending on the sign of $g(v, v)$ being negative, positive or null, respectively. It is only at that point that we can argue that, in fact, time-like vectors agree with particle-like vectors, ^{0}vectors agree with light-like vectors, and space-like vectors agree with ghost-like vectors.
Only at that point the standard naming of standard GR will be recovered.

As usual, let us denote by $T_x M$ the tangent space to spacetime at an event $x \in M$. That is a vector space and we can define a projective space out of it, let it be $\mathbf{P}_x(M) = \mathbf{P}(T_x M)$.

Two non-zero vectors $u, v \in V$ in a vector space V are equivalent and we write $u \sim v$ iff there exists a $\lambda \in \mathbb{R}$ (of course, $\lambda \neq 0$) such that $u = \lambda v$. This defines an equivalence relation and the projective space $\mathbf{P}_x(V) = (V - \{0\})/\sim$ is defined to be the quotient space. Each point in $\mathbf{P}_x(V)$ represents a direction in V. In the special case $V \equiv T_xM$, we write $\mathbf{P}_x(M) := \mathbf{P}_x(T_xM)$; by an abuse of language, a point in $\mathbf{P}_x(M)$ is also called a *direction* in M at x.

Let us also denote by $(TL)_x \subset T_xM$ the set of all light-like vectors at $x \in M$, which are tangent to some light rays. The subset $(TL)_x =: \mathcal{C}_x \subset T_xM$ of light-like vectors at x is also called the *light cone*. Analogously, let us denote by $(TP)_x \subset T_xM$ the set of all particle-like vectors, tangent to some particle.

Notice that neither $(TL)_x$ nor $(TP)_x$ are vector spaces on their own; however, if $u \in (TL)_x$ then $\lambda u \in (TL)_x$ for all $\lambda \in \mathbb{R}$. Analogously, if $v \in (TP)_x$ then $\lambda v \in (TP)_x$ for all $\lambda \in \mathbb{R}$. In other words, both $(TL)_x$ and $(TP)_x$ are compatible with the quotient to the projective space $\mathbf{P}_x(M)$ and define two subsets $\mathbb{L}_x \subset \mathbf{P}_x(M)$ and $\mathbb{P}_x \subset \mathbf{P}_x(M)$ which are called the *light-like* and *particle-like* directions, respectively.

Experience shows that there are (freely falling or not) particles with any (spatial) velocity slower than the speed of light. Since spatial velocity is encoded into spacetime directions, particle-like vectors $(TP)_x$ induce particle-like directions $\mathbb{P}_x$ which fills in an $(m-1)$ dimensional region in the projective space $\mathbf{P}_x(M)$. Also, the light-like directions $\mathbb{L}_x$ define a region (though of dimension $(m-2)$) in $\mathbf{P}_x(M)$. Thanks to one of the axioms below (Axiom P), we shall see that light-like directions form the boundary of particle-like directions which together form a compact set in $\mathbf{P}_x(M)$ and they are called *causal directions* in spacetime.

5. Axioms: part I

We shall here list and discuss the axioms required in EPS (see[18]). The list will be split in two parts, since, with the axioms presented here in the first part, the *conformal structure* of spacetime can be defined.

After we shall have discussed such a structure, we shall present the other axioms (see here) for the *projective structure* and compatibility. Here for reference, we shall mention as a side note to which axiom in EPS we refer since often we split or group EPS axioms for the sake of discussing their physical meaning.

The differential structure will be constrained to be compatible with particles and light rays later on (see Axiom DPR). The fist axiom has already been discussed above when introducing spacetime and events. We repeat it here to be complete.

Axiom STC (Spacetime continuity):
Spacetime is a smooth manifold M of dimension m.
Events are points of spacetime.

EPS: D_3

Then, EPS (see Axiom D_1) required a smooth structure on particles and light rays. Here, this has been obtained by requiring the changes of parameterisations to be diffeomorphisms. Each local parameterisations is in fact a chart (actually, the inverse of a chart) on the worldline as a 1-dimensional sub-manifold of M. Changes of parameters are transition functions of the atlas of such charts; requiring changes of parameterisations to be smooth does in fact define a smooth structure on the abstract manifold representing the worldline.

> **Axiom AE (Almost emptyness):**
> *For any event $e \in M$ and for any particle $P \in \mathcal{P}$*
> *there exists a neighbourhood $V_e \subset M$ of e such that*
> *for each $x \in V_e$ there are* at most *two light rays $R_1, R_2 \in \mathcal{L}_x$ which hits P in V.*

This axiom claims that there is no lensing (see Link) at small scales.

> In fact, we shall see below that, in normal situations, a massive body in between x and P can bend light rays so that, e.g. there are two distinct light rays travelling from x to P (the observer placed in x would see two distinct images of P at the same time). The axiom AE just required that this does not occur when e and x are sufficiently closed to each other, i.e. in a sense that going to small scales there are no bodies massive enough to produce lensing.

Then, we need to be sure that locally we can always send a light ray from one particle to a nearby one. This will be also the basis for constructing parallax charts and state the compatibility condition between the differential structure of spacetime, particles and light rays.

Fig 5.2: Axioms AE and ECHO.

> **Axiom ECHO (Echo signals):**
> *For any event $e \in M$ and for any particle $P \in \mathcal{P}_e$*
> *there exists a neighbourhood $U_e \subset V_e \subset M$ of e such that*
> *for each $p \in U_e$ there are* exactly *two light rays $R_1, R_2 \in \mathcal{L}_p$ which hits P at two events $e_1, e_2 \in V_e$ (the neighbourhood V_e being defined above in axiom AE). If $p \notin P$ then $e_1 \neq e_2$.*

Of course, one has $U_e \subset V_e$. This axiom claims that, if the event p is closed enough to a particle P, then one can send a light ray from P to p and then back to P. This is called an *echo* of the event p on the particle P.

A trivial consequence of Axiom ECHO is that, given an event x closed enough to a particle P, one can always find a light ray starting from P (say at e_1) and hitting x and a light ray from x hitting P (say at e_2). In other words, there are enough light rays to hit any nearby target.

Fig 5.3: Message and echo maps.

Given two particles P and Q, we say we have *message* from P to Q if there exists an open set $I \subset P$, an open set $J \subset Q$, and a smooth map $m : I \to J$ such that, for any $p \in I$, there exists a light ray from p to $q = \phi(p) \in J$. In other words, messages from P to Q are local maps, which, by an abuse of language, will be denoted by $m : P \to Q$.

We say to have an *echo* of Q on P if we have a message $m : I \to J$ from P to Q followed by a message $m' : J \to I'$ from Q to P. Again echoes $e : I \to I'$ are local maps from an open set $I \subset P$ into a different open set $I' \subset P$; an echo, by an abuse of language, is denoted by $e : P \to P$.

Axiom ECHO just claims that, if the particles are *closed enough*, then messages (and thence echoes) do in fact exist.

Let us consider two particles P and Q with a parameterisation fixed on them, i.e. two *clocks*. A message assigns an event $q \in Q$ to an event $p \in P$, i.e. it is a map between the two abstract 1-manifolds.

> **Axiom DPR (Differential structure on light rays and particles):**
>
> a) a message between two particles P and Q is a smooth map.
>
> b) the echo on P of Q is a map from P to itself; it is ovbiously smooth (being the composition of two messages); it is smoothly invertible.
>
> c) any light ray and particle is a smooth sub-manifold on M.
>
> d) for a message from $P \to Q : p \mapsto q$ then the direction of the light ray depends smoothly on the initial event $p \in P$.

At an event $p_1 \in M$, one can find a neighbourhood $U_1 \subset V_1$ such that any $x \in U_1$ can exchange and echo with a particle P_1 though p_1. Now fix a point $p_2 \in U_1$ (not on P_1) and a particle P_2 through p_2; one can find a neighbourhood U_2 such that any point of it can exchange an echo with P_2. Any point in $U_{12} = U_1 \cap U_2 \neq \varnothing$ can exchange an echo both with P_1 and P_2. The procedure can be repeated m times so that one can find an open set U and m particles P_i such that for any $x \in U$ there is an echo of x on all particles P_i. In particular, one has m messages from x to P_i. Let us fix a parameterisation on P_i choosing a parameter s_i. Then the message from x to P_i hits an event $p_i \in P_i$ corresponding to the parameter s_i.

> This is similar to GPS. The particles P_i are satellites (which, of course, are free falling in the Earth's gravitational field) which host an atomic clock. The satellites broadcast their time. When one receives a signal from a satellite, it means that is on a hypersurface (the wavefront) determined by the satellite's position, which is supposed to be known. When one has received signals from m satellites, it means it stands at the intersection of m hypersurfaces which identify a point in the spacetime.
>
> Actually, the current GPS works in space (not in spacetime), i.e. it assumes a canonical local time and a Newtonian approximation is assumed in view of the weak gravitational field. Moreover, in order to keep the receiving apparatuses at $x \in U$ simple, instead of receiving the time broadcasted by the satellites the point x is able to measure the *difference* of times received by pairs of satellites. In this way, instead of staying at the intersection of wavefornts, the receiver knows to stay at the intersection of hyperboloids depending on the positions of the satellites; in fact, the difference of time signals received is correlated directly to the difference of distances of the satellites from x and the locus of points in which such a difference of distances is constant is in fact an hyperboloid with *foci* in the satellites considered. But these technical details do not alter the fact that the coordinates introduced are quite similar to a more general and simpler GPS.
>
> Alternatively, one can consider that P_i are looking at the event x from different perspectives and reconstruct its position as astronomers do measuring the parallax of an object, being it a planet or a star. For this reason, the coordinates defined in this way will be called *parallax coordinates*.
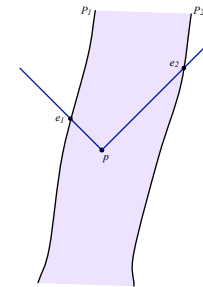
Fig 5.4: Axiom DST.

Then, at any event p there exist a neighbourhood U and m clocks P_i such that we can associate to any $x \in U$ a point $(s_i) \in \mathbb{R}^m$, i.e. a map $\Phi_{(U,P_i)} : U \to \mathbb{R}^m$.

> **Axiom DST (Differential structure on spacetime):**
>
> *There is a set of charts $\Phi_{(U,P_i)}$ chosen as above that form an atlas of M;*
> *a) Such an atlas is a smooth atlas.*
> *b) Any other similar chart $\Phi_{(U',P_i')}$ is smooth compatible with such an atlas.*

 Hence the differential structure of spacetime given at the beginning must be compatible with particles and light rays so that the parallax coordinates are part of the atlas. Of course, one is allowed to use any chart which is smoothly compatible with the atlas defined by parallax charts.

 Then we have to be sure there are enough particles and light rays. Within experimental accuracy, we know a number of pretty well established facts about particles and light rays. For example, we know that all known particles (being them freely falling or not) travel slower than light and there is nothing we can do to accelerate them faster than light. We know that, locally in spacetime, we should have two kinds of light-like vectors: the ones pointing to the future and the ones pointing to the past. The same for particle-like vectors. There should be two kinds: the ones pointing to the future and the ones pointing to the past.

> **Axiom CS (Causal structure on spacetime):**
>
> *a) the set $\mathbf{P}_x(M) - \mathbb{L}_x$ disconnects into two connected components,*
>   *named* inner-cone *and* outer-cone *directions.*
> *b) The set $(TL)_x - \{\vec{0}\}$ of all non-zero light-like vectors disconnects into two connected components*
>   *(one representing the* past *and one representing the* future, *though we do not fix which is which).*
> *c) The set of non-zero inner-cone vectors disconnects into two connected components*
>   *(one representing the* past *and one representing the* future, *though we do not fix which is which).*
>   *This is not the case for non-zero outer-cone vectors which instead are connected*
>   *(and this is how the inner and outer vectors are named).*

 In view of this axiom, while future and past components are symmetrical and can be exchanged, not the same happens for inner-cone and outer-cone vectors. In fact, inner-cone vectors are the one with a direction which lies in the component of $\mathbf{P}_x(M) - \mathbb{L}_x$ which disconnects when the zero vector is removed. That is a topological property which singles out which component is inner and which is outer.

 Moreover, we know that the velocities of particles can be any velocity up to the speed of light.

> **Axiom P (Particles):**
> *a) At any given event $x \in M$ there is one and only one (free falling) particle for any inner-cone vector.*
> *b) The direction tangent to particles (freely falling or not) are inner-cone.*

Let us remark that the Axiom P claims that particle-like vectors fill in one of the connected components (the inner-cone) in the tangent space T_xM determined by the light-like vectors $(TL)_x$.

Further axioms about particles and their compatibility with light rays will be added later on. Before of that let us draw some consequences from what we already know.

6. Conformal structure of spacetime

We have already been able to build a family of cones $\mathcal{C}_x = (TL)_x \subset T_xM$ out of light rays.

Here, *'cone'* refers simply to the fact that, when $v \in \mathcal{C}_x$, then $\lambda v \in \mathcal{C}_x$ as well. The cone $\mathcal{C}_x$ is called the *light cone* at $x \in M$. We can thence say that we showed how light rays define the sheaf of light cones, i.e. a light cone at each point.

We shall see later how light cones are an important structure in gravitational theories; gravitational field will be shown to deform the light cones. A sheaf of light cones are also called a *conformal structure* on spacetime.

Let us stress that the light cones $\mathcal{C}_x$ are cones in the tangent spaces of the spacetime M. We can also define *wavefronts* in the spacetime as

$$\nu_p = \{x \in M : \text{there exists a light ray passing through } x \text{ and } p\}$$

Since light-like vectors split into two connected components (past and future), also light rays through p can be split into future and past demi-rays. Accordingly, also the wavefront ν_p splits into two parts, denoted by ν_p^+ and ν_p^-. These two (possibly singular) hypersurfaces of M are the worldsheets of a light wavefront emitted from (respectively, focusing to) the event p.

The set of spacetime vectors tangent to ν_p at p is by definition $(TL)_p$; since $(TL)_p$ is not a vector subspace, then $\nu_p^\pm$ is not a regular hypersurface, at least at the point p, meaning that regularity is lost at focusing (emission).

> In a while, we shall characterise the wavefronts $\nu_p^\pm$ in M in terms of observations of times of messages exchanged by clocks. We shall also prove that the cones $\nu_p^\pm$ are in fact locally regular hypersurfaces except at $p \in M$, i.e. they are regular in a neighbourhood of p except at p itself.

Keep that in mind. It is the first thing we learn about wavefronts and it will be used in few pages.

We shall show hereafter that the conformal structure is described by a conformal class of Lorentzian metrics on spacetime. Actually, deriving a metric structure (together with its signature) out of observations about particles and light rays is a reason (for us the main reason) to prefer EPS axiomatic approach to the usual geometric approach which *postulates* directly a Lorentzian metric on spacetime.

An interesting side effect is that, in the traditional approach, nothing guarantees that we shall be eventually able to observe the metric field, precisely to determine a unique metric out of observations to represent a physical situation. On the contrary, by the axiomatic approach we shall build structures out of observations and will be easier to discuss physical interpretations of structures along the way; of course, one can always discuss new axioms to force new/less observables in the model, but that will need new axioms.

There are a number of equivalent descriptions of conformal structures on spacetimes, all of them important and useful to disclose some aspects of the theory.

The spacetime metric

Let us consider a particle P through an event $p \in P \subset M$; we know there is a neighbourhood U of p such that any event $x \in U$ can reflect an echo to P; in other words, one has two light rays L_1, L_2 passing through x and hitting the particle P at two events $p_1, p_2 \in P$ both in a suitable neighbourhood V which contains U.

Let us fix a parameterisation along the particle P; this corresponds to fix a function $s : P \to \mathbb{R}$ (actually, the inverse of the parameterisation, i.e. a coordinate chart on P) so that $s(p) = 0$. The (smooth) map $s : P \to \mathbb{R}$ associates to any point $x \in P$ the value of its parameter $s(x) \in \mathbb{R}$. In particular, let us denote by $s_1 = s(p_1)$ and $s_2 = s(p_2)$ the values of the parameter in the events $p_1, p_2 \in P$.

We can hence define a (smooth) function $G : U \to \mathbb{R} : x \mapsto -s_1 s_2$ in the neighbourhood U; this function of course depends on the choice of the particle P through the event p and from the parameterisation fixed along it. We shall be back soon on these conventions.

Let us now fix a coordinate chart x^μ around the point p; as usual, let us assume, without loss of generality, that the point p corresponds to the origin of coordinates, i.e. $x^\mu(p) = 0$. The local expression of the scalar function G has the following 2nd order Taylor polynomial around the point p

$$\mathbb{T}_p^2(G) = G(0) + \partial_\mu G(0) x^\mu + \tfrac{1}{2} \partial_{\mu\nu} G(0) x^\mu x^\nu \tag{6.6.1}$$

One obviously has $\lim_{x \to p} G(x) = 0$, since, for $x = p$, it is $p = p_1 = p_2$ as well; hence $s_1 = s_2 = 0$.

Then let us suppose that one has $\partial_\mu G(0) \neq 0$; then the level set $G_0 = \{x \in U : G(x) = 0\}$ would be a regular hypersurface in M. However, we can prove that $G_0 \equiv \nu_p$, i.e. the wavefront in the spacetime.

> First of all, let us consider a light ray L passing through p and let $x \in L$ be an event on it. If $x \in U$ then it can echo on P; however, at least one of p_1 or p_2 coincide with p and hence $G(x) = 0$. In other words, G vanishes on all light rays passing through p. Hence we showed that $\nu_p \subset G_0$.
> Vice versa, if $G(x) = 0$ that means that either $s_1 = 0$ or $s_2 = 0$, i.e. either $p_1 = p$ or $p_2 = p$. Let us suppose, for simplicity and with no loss of generality, that $p = p_1$; that means there is light ray passing through p and x, hence $G_0 \subset \nu_p$.
> Being both $\nu_p \subset G_0$ and $G_0 \subset \nu_p$, one obviously has $G_0 \equiv \nu_p$.

But we already know that the surface ν_p, thus G_0, is not-regular at p, which contradicts the hypothesis. Then one necessarily has $\partial_\mu G(0) = 0$.   ▷ *We learnt it here*

The second Taylor polynomial hence degenerates to

$$\mathbb{T}_p^2(G) = \tfrac{1}{2} \partial_{\mu\nu} G(0) x^\mu x^\nu \tag{6.6.2}$$

Let us then define local functions $g_{\mu\nu} : U \to \mathbb{R}$

$$g_{\mu\nu}(p) := \tfrac{1}{2} \partial_{\mu\nu} G(p) \tag{6.6.3}$$

In general the 2nd partial derivatives of a scalar does not define a tensor.

> By changing coordinates as $x'^\mu = x'^\mu(x)$, one has
>
> $$\begin{cases} G(x') = G(x) \\ \partial'_\mu G(x') = \bar{J}_\mu^\alpha \partial_\alpha G(x) \\ \partial'_{\mu\nu} G(x') = \bar{J}_\mu^\alpha \bar{J}_\nu^\beta \partial_{\alpha\beta} G(x) + \bar{J}_{\mu\nu}^\alpha \partial_\alpha G(x) \end{cases} \tag{6.6.4}$$

This would lead directly to the non-tensorial transformation rules

$$g'_{\mu\nu} = \bar{J}^\alpha_\mu \bar{J}^\beta_\nu g_{\alpha\beta} + \tfrac{1}{2} \bar{J}^\alpha_{\mu\nu} g_\alpha \tag{6.6.5}$$

which involves the 2nd order partial derivatives, as well as the 1st order partial derivative g_α.

However, in this case, the value of the differential $g_\alpha = \partial_\alpha G(0)$ (as well as the value of function $G(0)$) is zero; as a consequence, the 2nd order partial derivatives define a (symmetric) 2-tensor $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$.

> Here we have a problem with notation. We shall soon show that light-like vectors v obey $g(v, v) = 0$. Eventually, we shall show that all vectors v such that $g(v, v) = 0$ are in fact light-like, so that, finally, we shall know that a vector is light-like iff $g(v, v) = 0$. However, in the meanwhile, we cannot know whether there can be vectors with $g(v, v) = 0$ which are not light-like. Thus we need a temporary name for vectors with $g(v, v) = 0$.
> We could use the standard name of *null* vectors. However, I think this name is misleading. When one says: *consider a non-null vector*, does that mean a vector for which $g(v, v) \neq 0$ or a non-zero vector $v \neq 0$?
> For this reason, we prefer to avoid any reference to null vectors.

Let us call a ^{0}*vector* any v such that $g(v, v) = 0$. Let us also say that a vector v is time-like iff $g(v, v) < 0$ and *space-like* iff $g(v, v) > 0$.

> A similar ambiguity is between particle-like vectors and time-like vectors. Soon we shall show that a vector is particle-like iff it is time-like. Thus eventually we shall only have *time-like* (=particle-like), *light-like* (=^{0}vector), and *space-like* (=ghost-like) vectors.

We know a couple of extra facts about the tensor g. Let us consider a light ray parameterised as $L : s \mapsto \gamma^\mu(s)$ based at p (i.e. $\gamma^\mu(0) = 0$). The composition $G \circ \gamma : \mathbb{R} \to \mathbb{R}$ vanishes identically since L is a light ray through p.

Hence by taking the second derivative at $s = 0$ one has

$$0 = g_{\mu\nu}(p) \dot{\gamma}^\mu(0) \dot{\gamma}^\nu(0) \tag{6.6.6}$$

where $\dot{L} = \dot{\gamma}^\mu(0)\partial_\mu$ is the tangent vector to the light ray L at p, which is hence a ^{0}vector. Hence we know that $g(\dot{L}, \dot{L}) = 0$ along any light-like vector. Thus any light-like vector is a ^{0}vector.

Analogously, we can consider the parameterisation $\sigma : \mathbb{R} \to P : s \mapsto x^\mu(s)$ of the particle P used in the construction of G. If one fixes $x = \sigma(s) \in P$ then by construction $p_1 = p_2 = x$ and $s_1 = s_2 = s$; accordingly, one has $G(\sigma(s)) = -s^2$. Again, by taking the second derivative at $s = 0$ of both hand sides, one has

$$-2 = g_{\mu\nu}(p) \dot{x}^\mu \dot{x}^\nu \tag{6.6.7}$$

Let us denote by $\dot{x} = \dot{x}^\mu(0)\partial_\mu$ the vector tangent to P at p, then we know that $g(\dot{x}, \dot{x}) = -2$.

By summarising, we know a lot about vectors at p; all light-like vectors $\dot{L} \in (TL)_p$ are ^{0}vectors, i.e. $g(\dot{L}, \dot{L}) = 0$ and at least one particle-like vector, the vector $\dot{x}$ tangent to P, is time-like, i.e. $g(\dot{x}, \dot{x}) < 0$. On the other hand, g is a symmetric tensor and can be put in canonical form, as any other quadratic form. The procedure to find a canonical form depends on a metric on M, but that is irrelevant. The relevant fact is that one can find a coordinate system in which the tensor g is in canonical form, i.e. diagonal with all entries on the diagonal which are either zero or ± 1.

All we have to do is analysing, one by one, all possible canonical forms and check which ones of them agree with what we know about g. Before proceeding, let us denote by $\mathbb{Q}_p = \{v \in T_p M : g(v, v) = 0\}$ the set of ^{0}vectors with respect to g. We obviously know that $(TL)_p \subset \mathbb{Q}_p$.

Let us start by considering low dimensions, case by case, before sketching a discussion for the general case.

Dimension 1

In dimension 1 there are three possible signatures, namely $(+)$, $(-)$, and (0).

In cases $(+)$ and $(-)$ the tensor g is non-degenerate and has no ^{0}vectors other than the zero vector. Thus there will be no light ray in those cases.

In case (0) the tensor g is identically zero and there is no time-like vector (as the tangent vector to P should be).

Accordingly, dimension $m = 1$ cannot meet the requirements and it is incompatible with our axioms.

Dimension 2

In dimension 2, there are three possible signatures, namely $(++)$, $(+-)$, $(--)$ for the non-degenerate cases and three, namely $(+0)$, (-0), (00), for the degenerate cases.

The signatures $(++)$, $(--)$ cannot account for ^{0}vectors; the signatures $(+0)$, (00) cannot account for the time-like vector.

In the signature (-0), the ^{0}vectors are the ones along the y-axis. Notice that, in that case, light-like vectors must be the ones along the y-axis as well; if there is a non-zero light-like vector then all its multiples are light-like as well. Unfortunately, the y-axis singles out one point in the projective space $\mathbf{P}_x(M)$ of spacetime directions which hence fails to split into two connected components; in other words, one cannot define inner and outer-cone vectors.

Thus one is left with the signature $(+-)$ only. In this case, ^{0}vectors are the two bisector lines $y = \pm x$, they are again necessarily light-like, and they single out two points in the projective space which split the directions into two connected components, namely inner and outer-cone vectors.

Hence in dimension $m = 2$ there is only one possible signature $(+-)$ and $\mathbb{Q}_p \equiv (TL)_p$.

However, both the inner and outer-cone vectors disconnect into two connected components when the zero vector $v = 0$ is removed. Accordingly, dimension 2 is a sort of borderline case: strictly speaking there is no signature compatible with all the axioms, if the second part of Axiom CS is included.

Dimension 3

In dimension 3 there are four possible non-degenerate signatures, namely $(+++)$, $(++-)$, $(+--)$, $(---)$ and six for the degenerate cases, namely $(++0)$, $(+-0)$, $(--0)$, $(+00)$, (-00), (000).

The signatures $(+++)$, $(---)$ cannot account for ^{0}vectors; the signatures $(++0)$, $(+00)$, (000) cannot account for the time-like vector.

In the signature (-00), the ^{0}vectors are the ones in the form $v = (0, y, z)$, namely the yz-plane. If one has $\mathbb{Q}_p = (TL)_p$ then the light-like vectors do not split into two connected components when the zero vector is removed. If $(TL)_p \subset \mathbb{Q}_p$ properly, then some line in the yz-plane has been removed and particle-like directions form one connected component and not two as required.

In the signature $(--0)$, the ^{0}vectors are the ones in the form $v = (0, 0, z)$; they split correctly in two connected components when the zero vector is removed, but the complement is connected.

In the signature $(+-0)$, the equation for ^{0}vectors is $x^2 - y^2 = 0$ the solutions of which form two planes intersecting along the z-axis. One cannot leave out any line on those plane (otherwise $\mathbb{L}_x$ would not split the projective space into two connected components); hence $\mathbb{Q}_p = (TL)_p$. Then, dropping the zero vector, the two planes do not disconnect into two connected components. Thence also this signature cannot be accepted.

In the signature $(+--)$, the equation for ^{0}vectors is $x^2 - y^2 - z^2 = 0$ being a cone (a cone in the usual sense with a circle as a directrix) with the axis along the x-axis. The vectors "inside the cone" are space-like, while the ones "outside the cone" are the time-like. Unfortunately, the particle-like vectors are in the wrong connected components, i.e. the one which does not disconnect into two connected components when the zero vector is removed. Thence also this signature cannot be accepted.

We are left again with the Lorentzian signature $(+ + -)$. The situation in this case is pretty much the same as for the signature $(+ - -)$, though with the vectors "inside the cone" which are time-like, while the ones "outside the cone" are space-like. Now, of course, the particle-like vectors are inner-cone vectors which do disconnect into two connected components when the zero vector is removed. The signature can be accepted. Again we are forced to accept Lorentzian signature, g is non-degenerate, and $\mathbb{Q}_p \equiv (TL)_p$.

Dimension m

The situation is getting more and more involved as the dimension grows and we need to devise a more general method. Let us suppose we have proved *should be reviewed* that in dimension $m-1$ the only allowed signature is Lorentzian $(-+++\ldots)$ and that $\mathbb{Q}_p \equiv (TL)_p$. A signature in dimension m is obtained adding a 0, $+$ or $-$ to a signature in dimension $m-1$. In dimension m, we already can discard signatures $(+++\ldots)$, $(---\ldots)$; the degenerate signatures have entire planes of ^{0}vectors and they do not disconnect into two connected components when the zero vector is removed.

We are left with all mixed signatures of the type $(+ + \ldots - - \ldots)$ with at least one $+$ and one $-$. The equation for ^{0}vectors is $(x^1)^2 + \ldots + (x^r)^2 - (x^{r+1})^2 - \ldots - (x^m)^2 = 0$. If $r > 2$ then let us define $\rho^2 = (x^{r-1})^2 + (x^r)^2$ and we get the equivalent equation for cones in the form $(x^1)^2 + \ldots + (x^{r-2})^2 + \rho^2 - (x^{r+1})^2 - \ldots - (x^m)^2 = 0$. It is easy to check that one has a continuous path connected two points in $\mathbb{R}^m$ iff one has the continuous path in $\mathbb{R}^{m-1}$ (the essential point is that $\rho = 0$ iff $x^{r-1} = x^r = 0$). Accordingly the only case which can accomodate a suitable $(LM)_p$ in dimension $m-1$ is the Lorentzian case $(+++\ldots+-)$ which is Lorentzian again when a $+$ is added. If $r \le 2$ then define $\rho^2 = (x^{r+1})^2 + (x^{r+2})^2$ and we get the equivalent equation in the form $(x^1)^2 + \ldots + (x^r)^2 - \rho^2 - (x^{r+3})^2 - \ldots - (x^m)^2 = 0$. This has at least two $-$ in dimension $m-1$ and was discarded; accordingly it is discarded in dimension m.

To summarise, in dimension $m > 2$, the only signature allowed is Lorentzian $(++\ldots+-)$. In particular, it is non-degenerate and g is a Lorentzian metric over the spacetime. Moreover, $\mathbb{Q}_p \equiv (TL)_p$; hence there is no difference between ^{0}vectors and light-like vectors. Accordingly, we shall forget about ^{0}vectors which will be called *light-like* and are the same things as light-like vectors. Similarly, we shall forget about particle-like vectors which, from now on, will be called *time-like* and ghost-like vectors which will be called *space-like* vectors.

Conformal metrics

We just saw that one can build a Lorentzian metric g on the spacetime M only relying on a clock and light rays. However, we noticed that the construction depends on the choice of a particle P as a clock, i.e. with a parameterisation over it.

Let us suppose we want now to change the parameterisation we fixed along P by defining a new parameter $s' = \phi(s)$. Instead of defining the function G, we would define a different function G' which is defined to be

$$G'(x) = -\phi(s_1)\phi(s_2) \tag{6.6.8}$$

By taking the second Taylor expansion we have

$$g'_{\mu\nu}(p) = \tfrac{1}{2}\partial_{\mu\nu}G'(p) = (\phi'(0))^2 g_{\mu\nu}(p) \tag{6.6.9}$$

i.e. we define a metric g' conformal to g.

We say that two metrics g and g' are *conformal*, if there exists a positive function Φ such that $g' = \Phi(x)g$.

> Notice that Weyl conformal transformation is defined to be a field transformation, no coordinate transformation is associated to it. Weyl conformal transformations have not to be confused with coordinate dilatations $x'^\mu = \alpha x^\mu$, which in fact induce a transformation on the metric $g' = \alpha^2 g$, though with α constant. Of course, a coordinate dilatation is a coordinate transformation, i.e. a local diffeomorphism, and, as such, it leaves automatically invariant any generally covariant theory.
> Here we are claiming that Weyl conformal transformations leave the physical situation invariant and they correspond only to changing observer conventions. They are however, an extra gauge freedom of observers, beside diffeomorphisms.

Let us stress also that two conformal metrics define the same sheaf of light cones, i.e. the same conformal structure on spacetime. Thence we can equivalently define the conformal structure on spacetime to be an equivalence class of conformal metrics $[g]$.

Notice also that a time-like vector for a metric g is time-like for any conformal metric g'. The same for light-like and space-like vectors.

Analogously, if we change particle P to a nearby P' passing through the same point p, let us denote by $\phi : P \to P'$ a message map from P to P'. Again, one can prove that the effect of changing particle is a change in the representative g in the conformal structure $[g]$.

Accordingly, light rays select a conformal structure on spacetime and, by choosing a clock, one can define a specific representative for it.

The conformal structure on spacetime can also be described by an invariant object.

> Originally, EPS used this representation. I think the discussion is clearer without using it, though we mention it to be complete.

Let us consider a conformal structure $[g]$ and g be one of its representatives. Let us define the tensorial density

$$\mathfrak{g}_{\mu\nu} = g^{-1/m} g_{\mu\nu} \tag{6.6.10}$$

where g denotes the absolute value of the determinant of the metric $g_{\mu\nu}$.

The quantities $\mathfrak{g}_{\mu\nu}$ are a tensorial density of weight $2/m$; in fact, by changing coordinates $x' = x'(x)$, with the usual notation about Jacobians, one $\triangleright$*Standard Jacobians* has $g' = \bar{J}^2 g$ and one has $\mathfrak{g}'_{\mu\nu} = J^{-2/m}\mathfrak{g}_{\rho\sigma}\bar{J}^\rho_\mu \bar{J}^\sigma_\nu$, where J is the determinant of the Jacobian matrix.

The tensorial density $\mathfrak{g}_{\mu\nu}$ is also invariant with respect to Weyl conformal transformations; in fact, if $g' = \Phi \cdot g$ is another conformal metric, then $g' = \Phi^m g$ and

$$\mathfrak{g}'_{\mu\nu} = g'^{-1/m}g'_{\mu\nu} = g^{-1/m}g_{\mu\nu} = \mathfrak{g}_{\mu\nu} \tag{6.6.11}$$

Hence the tensorial density $\mathfrak{g}_{\mu\nu}$ is a characteristic of the conformal class $[g]$, more than of a specific metric g.

7. Geodesics

We shall need to specify an equation for freely falling particles. We know a number of things about this equation. First, there is a particle for
each initial event and any (time-like) direction; so, we expect a second order, ordinary differential equation. Second, we expect that initial conditions uniquely determine the particle worldline; thus we expect the equation to be in normal form so that Cauchy theorem holds for it. Third, the equations should determine a worldline, not a motion; thus the equation must be invariant with respect to reparameterisations. Fourth, the equation should be covariant with respect to coordinate changes (or spacetime diffeomorphisms).

In view of first and second observations, we can postulate an equation in the form

$$\ddot{x}^\mu = f^\mu(s, x, \dot{x}) \tag{6.7.1}$$

and we can expand the right hand side in powers of velocities

$$\ddot{x}^\mu = A^\mu(s, x) + A^\mu_\alpha(s, x)\dot{x}^\alpha + A^\mu_{\alpha\beta}(s, x)\dot{x}^\alpha\dot{x}^\beta + \dots \tag{6.7.2}$$

Let us consider reparameterisations; suppose we have a solution $x^\mu(s)$ and consider a reparameterisation $\gamma^\mu(s') = x^\mu(\phi(s'))$ of it. We wish to impose that such a curve is again a solution of the equation with the new parameter and possibly new coefficients so that the equation (or the family of equations) identifies a whole class of curves and induces an equation on trajectories. Of course, we have

$$\dot{\gamma}^\mu(s') = \dot{\phi}(s')\dot{x}^\mu(\phi(s')) \qquad \ddot{\gamma}^\mu(s') = \dot{\phi}^2(s')\ddot{x}^\mu(\phi(s')) + \ddot{\phi}(s')\dot{x}^\mu(\phi(s')) \tag{6.7.3}$$

and the new equation is

$$\ddot{\gamma}^\mu = A'^\mu(s', x) + A'^\mu_\alpha(s', x)\dot{\gamma}^\alpha + A'^\mu_{\alpha\beta}(s', x)\dot{\gamma}^\alpha\dot{\gamma}^\beta + \dots \tag{6.7.4}$$

Thus γ is a solution iff

$$\dot{\phi}^2\ddot{x}^\mu = A'^\mu(s', x) + \left(\dot{\phi}A'^\mu_\alpha(s', x) - \ddot{\phi}\delta^\mu_\alpha\right)\dot{x}^\alpha + \dot{\phi}^2 A'^\mu_{\alpha\beta}(s', x)\dot{x}^\alpha\dot{x}^\beta + \dots \tag{6.7.5}$$

Notice that, in this equation, $\ddot{x}^\mu$ and $\dot{x}^\alpha$ denote the derivatives of the function $x^\mu(\cdot)$ with respect to its argument, then evaluated at $\phi(s')$.

Equation (6.7.5) can be expanded using (6.7.2) (just evaluated at $\phi(s')$) to obtain

$$\left(\dot{\phi}^2 A^\mu(s', x) - A'^\mu(s', x)\right) + \left(\dot{\phi}^2 A^\mu_\alpha(s', x) - \dot{\phi}A'^\mu_\alpha(s', x) + \ddot{\phi}\delta^\mu_\alpha\right)\dot{x}^\alpha + \dot{\phi}^2\left(A^\mu_{\alpha\beta}(s', x) - A'^\mu_{\alpha\beta}(s', x)\right)\dot{x}^\alpha\dot{x}^\beta +$$
$$+ \dot{\phi}^2\left(A^\mu_{\alpha\beta\gamma}(s', x) - \dot{\phi}A'^\mu_{\alpha\beta\gamma}(s', x)\right)\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma + \dots = 0 \tag{6.7.6}$$

which gives the transformation rules with respect to reparameterisations for the coefficients

$$\begin{cases} A'^\mu = \dot{\phi}^2 A^\mu \\ A'^\mu_\alpha = \dot{\phi}A^\mu_\alpha + \frac{\ddot{\phi}}{\dot{\phi}}\delta^\mu_\alpha \\ A'^\mu_{\alpha\beta} = A^\mu_{\alpha\beta} \\ A'^\mu_{\alpha\beta\gamma} = \frac{1}{\dot{\phi}}A^\mu_{\alpha\beta\gamma} \\ \dots \end{cases} \tag{6.7.7}$$

Item 3 is easy to be interpreted: second order coefficients must be invariant with respect to reparameterisations. Then the coefficient $A_{\alpha\beta}^\mu$ does not depend on s and it is a field on spacetime.

If we had a reason to require that all coefficients vanish in one parameterisation then one would obtain easily ($A'^\mu = 0, A_{\alpha\beta\gamma}'^\mu = 0, \dots$) in any parameterisation; however, the first order coefficient cannot be dismissed so easily since, if it is set to zero in one parameterisation, a generic reparameterisation brings it back into the game (since the second equation is not homogeneous). It is in view of this consideration that one is forced to work with a family of equations, one for each parameterisation, instead simply with spacetime fields.

Once we resign ourself to work with a family of equations, we have a list of coefficient fields $(A^\mu, A_\alpha^\mu, A_{\alpha\beta}^\mu, \dots)$ in a fixed parameterisation. This identifies an equation (6.7.2) which, in turn, identifies a class of motions x^μ as solutions. When one considers a reparameterisation ϕ one has to define a new family of fields $(A'^\mu, A_\alpha'^\mu, A_{\alpha\beta}'^\mu, \dots)$ according to (6.7.7) which, in turn, defines a new equation (6.7.5) such that the motion $x'^\mu = x^\mu \circ \phi$ is automatically solution iff x^μ was originally a solution of (6.7.2).

We still have to implement general covariance, i.e. consider spacetime diffeomorphisms. Let is consider a spacetime diffeomorphism $\Phi : M \to M :$ ▷*Standard Jacobians* $x^\mu \mapsto x'^\mu(x)$ (with usual notation about Jacobians). The diffeomorphism Φ defines a dragging of any curve $x^\mu(s)$ into $x'^\mu(s) := x'^\mu(x(s))$. Velocities and (Lagrangian) accelerations are given by

$$\dot{x}'^\mu = J_\nu^\mu \dot{x}^\nu \qquad \ddot{x}'^\mu = J_\nu^\mu \ddot{x}^\nu + J_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda \tag{6.7.8}$$

If the original motion x^μ was a solution of equation (6.7.2), the dragged curve is again a solution of the same equation with new coefficients, namely

$$\ddot{x}'^\mu = A'^\mu + A_\alpha'^\mu \dot{x}'^\alpha + A_{\alpha\beta}'^\mu \dot{x}'^\alpha \dot{x}'^\beta + \dots \tag{6.7.9}$$

i.e.

$$\ddot{x}^\nu = \bar{J}_\mu^\nu A'^\mu + \bar{J}_\mu^\nu A_\alpha'^\mu J_\rho^\alpha \dot{x}^\rho - \bar{J}_\mu^\nu \left(-A_{\alpha\beta}'^\mu J_\rho^\alpha J_\sigma^\beta + J_{\rho\sigma}^\mu\right) \dot{x}^\rho \dot{x}^\sigma + \dots \tag{6.7.10}$$

Since equation (6.7.2) holds, then one has

$$\left(A'^\mu - J_\nu^\mu A^\nu\right) + J_\rho^\alpha \left(A_\alpha'^\mu - J_\nu^\mu A_\lambda^\nu \bar{J}_\alpha^\lambda\right) \dot{x}^\rho + J_\epsilon^\alpha J_\tau^\beta \left(A_{\alpha\beta}'^\mu - J_\nu^\mu A_{\rho\sigma}^\nu \bar{J}_\alpha^\rho \bar{J}_\beta^\sigma - J_{\rho\sigma}^\mu \bar{J}_\alpha^\rho \bar{J}_\beta^\sigma\right) \dot{x}^\epsilon \dot{x}^\tau + \dots = 0 \tag{6.7.11}$$

which leads to transformation rules for the coefficients with respect to spacetime diffeomorphisms

$$\begin{cases} A'^\mu = J_\nu^\mu A^\nu \\ A_\alpha'^\mu = J_\nu^\mu A_\lambda^\nu \bar{J}_\alpha^\lambda \\ A_{\alpha\beta}'^\mu = J_\nu^\mu \left(A_{\rho\sigma}^\nu \bar{J}_\alpha^\rho \bar{J}_\beta^\sigma - \bar{J}_{\alpha\beta}^\nu\right) \\ A_{\alpha\beta\gamma}'^\mu = J_\nu^\mu A_{\rho\sigma\lambda}^\nu \bar{J}_\alpha^\rho \bar{J}_\beta^\sigma \bar{J}_\gamma^\lambda \\ \dots \end{cases} \tag{6.7.12}$$

where we did an integration by part on the second order term.

The integration by part is

$$-J_{\rho\sigma}^\mu \bar{J}_\alpha^\rho \bar{J}_\beta^\sigma = -\partial_\beta' J_\rho^\mu \bar{J}_\alpha^\rho = \partial_\beta' \left(J_\rho^\mu \bar{J}_\alpha^\rho\right) + J_\rho^\mu \partial_\beta' \bar{J}_\alpha^\rho = \partial_\beta' \left(\delta_\alpha^\mu\right) + J_\rho^\mu \bar{J}_{\alpha\beta}^\rho = J_\rho^\mu \bar{J}_{\alpha\beta}^\rho \tag{6.7.13}$$

Hence the coefficients are all tensor fields except the second order one which is minus a torsionless connection $A^\nu_{\rho\sigma} := -\Gamma^\nu_{\rho\sigma}$. In equation (6.7.2), only the symmetric part (with respect to lower indices) of coefficients is relevant so one can assume the connection $\Gamma^\nu_{\rho\sigma}$ to be torsionless.

> Alternatively, one could assume the connection with torsion, though only the torsionless part enters in the equation (6.7.2). However, at this stage particles are the only primitive objects we are supposed to observe; accordingly, the torsion would not be observable.
>
> Nevertheless, when we shall couple gravity with matter fields, there could be some matter field which couples to torsion. In those models, torsion could be back on stage.

This is the most that can be done by requiring the generic characteristics listed above. One has to list a (torsionless) connection $\Gamma^\nu_{\rho\sigma}$, a vector field A^μ, a $(1,1)$ tensor A^μ_α, a tensor field $A^\mu_{\alpha\beta\gamma}$ of rank $(1,3)$ symmetric in the lower indices, and so on. These fields define an equation

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = A^\mu + A^\mu_\alpha\dot{x}^\alpha + A^\mu_{\alpha\beta\gamma}\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma + \dots \tag{6.7.14}$$

which, by Cauchy theorem, defines a family of motions x^μ as solutions. This equation is not invariant with respect to reparameterisations and thence it does not define trajectories as it should. However, given a list of coefficients $(A^\mu, A^\mu_\alpha, \Gamma^\mu_{\rho\sigma}, A^\mu_{\alpha\beta\gamma}, \dots)$, one can use (6.7.7) to define an *equivalent* list of coefficients $(A'^\mu, A'^\mu_\alpha, \Gamma'^\mu_{\rho\sigma}, A'^\mu_{\alpha\beta\gamma}, \dots)$. This defines equivalence classes of lists of coefficients and then equivalence classes of equations.

We say that a trajectory $\boldsymbol{\gamma} = [\gamma]$ is a solution of a class of equations iff, for any representative motion γ, there exists a representative of the class of equations for which γ is a solution.

We have a lot of freedom in selecting a class of equations which can be used for describing (possibly affected by all sort of forces) particles. This reflects the fact that we want particles to be coupled to gravity *only*, while, of course, we have all kind of equations which correspond to all sort of force fields that can be applied to the particle. We need an extra axiom to select which equation corresponds to pure gravity coupling.

> Here one could do many things, from requiring SR to hold at a point to postulate *free* motion along a curved spacetime. The essential point is that an extra axiom is required and none of them is much simplier than the others, all being essentially fixing a class of equations of the kind described above.
>
> Notice however, that one could define *gravity* as the force which cannot be shielded and which is felt by all particles in the same way. This is a kind of weak form of *equivalence principle*. In this case, let us notice that the coefficients $(A^\mu, A^\mu_{\alpha\beta\gamma}, \dots)$ can be coherently set to zero; they are tensor fields (thus zero has a meaning which is independent of observer, i.e. it is covariant) and, if they are set to zero in a representative of the equation class, they are zero in all representatives. Consequently, we can assume that they represent forces that can be switched off.
>
> The same cannot be said for the coefficients $(A^\mu_\alpha, \Gamma^\mu_{\rho\sigma})$; A^μ_α is a tensor (so one could set it to zero in a covariant way) but, once set to zero in one parameterisation, it is not zero in the other parametersations. For the connection $\Gamma^\mu_{\rho\sigma}$, it has no covariant meaning in putting it to zero since it is not tensorial.

The situation for A^μ_α can be improved a bit. Any $(1,1)$ tensor can be uniquely split as

$$A^\mu_\alpha = \lambda\delta^\mu_\alpha + a^\mu_\alpha \tag{6.7.15}$$

where a^μ_α is a traceless tensor field. By tracing one obtains $A = m\lambda$ which determines λ.

The transformation laws with respect to reparameterisations is thence

$$\lambda'\delta^\mu_\alpha + a'^\mu_\alpha = \dot\phi\lambda\delta^\mu_\alpha + \dot\phi a^\mu_\alpha + \frac{\ddot\phi}{\dot\phi}\delta^\mu_\alpha \tag{6.7.16}$$

This can be split, by tracing, into the conditions

$$
\begin{cases}
\lambda' = \dot\phi\lambda + \dfrac{\ddot\phi}{\dot\phi} \\[2mm]
a'^\mu_\alpha = \dot\phi a^\mu_\alpha
\end{cases}
\tag{6.7.17}
$$

Now we can set a^μ_α to zero coherently as above for the fields $(A^\mu, A^\mu_{\alpha\beta\gamma}, \ldots)$, while the same cannot be done for λ.

Moreover, let us notice that, on a curve $x(s)$, for any $\lambda(s) := \lambda(s, x(s))$ one can fix a reparameterisation ϕ so that $\lambda' = 0$. In fact, one can fix ϕ so that $\ddot\phi = -\lambda\dot\phi^2$, which is obtained by integration from

$$
\frac{1}{\dot\phi} = \int \lambda(s)ds \qquad\qquad \dot\phi = \frac{1}{\int \lambda(s)ds}
\tag{6.7.18}
$$

The equation $\ddot\phi = -\lambda\dot\phi^2$ can be seen as the equation for integral curves for the vector field on $T\mathbb{R}$ (with coordinates (s, v))

$$
\Xi = v\frac{\partial}{\partial s} - \lambda(s)v^2\frac{\partial}{\partial v}
\tag{6.7.19}
$$

If the function λ is sufficiently regular, than the flow exists at least locally.

A further consequence of this fact is that the function λ can be set to any function by fixing the parameterisation.

Geodesics trajectories

We are now ready to define geodesic trajectories which will be used to formulate the *(weak) equivalence principle* in precise mathematical terms. A trajectory $\gamma = [\gamma]$ is a *geodesic trajectory* for the connection $\Gamma^\mu_{\alpha\beta}$ if there exists a function $\lambda(s)$ such that a representative γ is a solution of the equation

$$
\ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = \lambda\dot\gamma^\mu
\tag{6.7.20}
$$

This equation (which is, in fact, a family of equations) will be called the *equation for geodesic trajectories* of the (torsionless) connection $\Gamma^\mu_{\alpha\beta}$.

This equation is not really associated to a single connection $\Gamma^\mu_{\alpha\beta}$. Let us, in fact, consider the new pair $(\Gamma'^\mu_{\alpha\beta}, \lambda')$ given by

$$
\Gamma'^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \delta^\mu_{(\alpha}V_{\beta)} \qquad\qquad \lambda' = \lambda + (V_\mu\dot\gamma^\mu)
\tag{6.7.21}
$$

for any 1-form V. It is easy to show that a curve γ is a solution for the geodesic equation of the pair $(\Gamma^\mu_{\alpha\beta}, \lambda)$ iff it is a solution of the geodesic equation for the new pair $(\Gamma'^\mu_{\alpha\beta}, \lambda')$.

In fact,

$$
\ddot\gamma^\mu + \Gamma'^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = \lambda'\dot\gamma^\mu \quad\Longleftrightarrow\quad \ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta + \dot\gamma^\mu V_\beta\dot\gamma^\beta = \lambda\dot\gamma^\mu + V_\beta\dot\gamma^\beta\dot\gamma^\mu \quad\Longleftrightarrow\quad \ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = \lambda\dot\gamma^\mu
\tag{6.7.22}
$$

Let us say that two connections Γ and Γ' are *projectively equivalent* iff there exists a 1-form V such that $\Gamma'^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \delta^\mu_{(\alpha}V_{\beta)}$; in this case, we have $\Gamma \sim \Gamma'$. The equivalence class $[\Gamma]$ is called a *projective structure* on the spacetime M, while a connection Γ is called an *affine structure* on spacetime.

Now, obviously, an affine structure induces a projective structure, while vice versa is not true in general, though we shall see that a particular compatibility requirement between the projective structure and the conformal structure will enable us to determine a canonical affine structure out of a projective structure.

In the meantime, the equation for the geodesic structure is related to a projective structure on spacetime; changing the representative for the projective structure can in fact be reabsorbed by a suitable reparameterisation.

Let us finally stress that we never required the affine structure Γ or the projective structure $[\Gamma]$ to be somehow related to the conformal structure. In other words, we never required the connection Γ to be the Levi Civita connection of any metric on spacetime, in particular of a representative of the conformal structure. We shall see below that a specific compatibility axiom between the conformal and projective structure will be required. This will allow to partially determine a connection out of the metric structure. The extra freedom of the connection will be parameterised by a 1-form $\boldsymbol{\alpha}$ and standard GR corresponds to the choice $\boldsymbol{\alpha} = 0$, for which the connection is completely determined in terms of the metric structure.

Let us now proceed leaving the discussion about compatibility axioms for due times. Until then, we shall not assume any *a priori* relation between conformal and projective structure.

Geodesic motions

Given a connection $\Gamma^{\mu}_{\alpha\beta}$ (i.e. an affine structure on spacetime) a curve $\gamma^{\mu}(s)$ is a *geodesic motion* for Γ iff it satisfies the following equation

$$\ddot{\gamma}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{\gamma}^{\alpha}\dot{\gamma}^{\beta} = 0 \tag{6.7.23}$$

This equation is already known in mechanics (even if for a specific metric connection $\Gamma^{\mu}_{\alpha\beta} = \{g\}^{\mu}_{\alpha\beta}$) and it describes free motion along configuration space of a system of material points subjected to a number of holonomic constraints. The metric structure g on configuration space is endowed by the kinetic energy of the system. The system motion is *free* in the sense that there are no forces other than constraint forces. The motion of the system is completely determined in terms of the geometry of configuration space, in turn determined by the metric structure coming from kinetic energy bilinear form.

The equation is also known in differential geometry and it determines a special class of curves on manifolds, namely the *geodesics*. If the connection is metric, the geodesics turns out to be the curves of shortest length (or at least they have stationary length).

Of course, here the situation is different since the equation in mechanics is expected to provide a motion as a solution while here in spacetime it is expected to provide a trajectory. Moreover, there is no preferred metric structure on spacetime, just a conformal structure i.e. an equivalence class of metrics, and the geometry of spacetime can be expressed also in terms of a connection (or a projective class of connections).

Let us stress that the equation for geodesic motions is variational if the connection is metric, while it is not for a generic non-metric connection.

Consider the Lagrangian

$$\boldsymbol{L} = \tfrac{1}{2}g_{\mu\nu}u^{\mu}u^{\nu}ds \tag{6.7.24}$$

Its Euler–Lagrange equation are in fact

$$\frac{d}{ds}\left(g_{\mu\alpha}\dot{q}^{\mu}\right) - \tfrac{1}{2}\partial_{\alpha}g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = 0 \tag{6.7.25}$$

By expanding the total derivative, one gets

$$g_{\mu\alpha}\ddot{q}^{\mu} + \partial_{\nu}g_{\mu\alpha}\dot{q}^{\mu}\dot{q}^{\nu} - \tfrac{1}{2}\partial_{\alpha}g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = 0 \tag{6.7.26}$$

which, after few index manipulations and collecting, becomes

$$g_{\mu\alpha}\ddot{q}^{\mu} + \tfrac{1}{2}\left(\partial_{\nu}g_{\mu\alpha} + \partial_{\mu}g_{\nu\alpha} - \partial_{\alpha}g_{\mu\nu}\right)\dot{q}^{\mu}\dot{q}^{\nu} = 0 \tag{6.7.27}$$

Then let us multiply the equation by $g^{\lambda\alpha}$ to get

$$\ddot{q}^{\lambda} + \{g\}^{\lambda}_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = 0 \tag{6.7.28}$$

Of course, the Lagrangian (6.7.24) is not invariant with respect to reparameterisations. In fact, when a reparameterisation is performed then one has

$$\boldsymbol{L}' = \tfrac{1}{2}g_{\mu\nu}u'^{\mu}u'^{\nu}ds' = \tfrac{1}{2}g_{\mu\nu}\dot{\phi}^2 u^{\mu}u^{\nu}ds' = \tfrac{1}{2}g_{\mu\nu}\dot{\phi}u^{\mu}u^{\nu}ds = \dot{\phi}\boldsymbol{L} \tag{6.7.29}$$

If the Lagrangian were invariant, then reparameterisation would be symmetries and the equations would be invariant as well, while we know that the equations for geodesic motions are not.

When the connection is metric, also the equation for the geodesic trajectories is variational

Consider the Lagrangian

$$\boldsymbol{L} = \sqrt{|g_{\mu\nu}u^{\mu}u^{\nu}|}\,ds \tag{6.7.30}$$

This is in fact invariant under (orientation preserving) reparameterisations since one has

$$\boldsymbol{L}' = \sqrt{|g_{\mu\nu}u'^{\mu}u'^{\nu}|}\,ds' = \sqrt{|g_{\mu\nu}\dot{\phi}^2 u^{\mu}u^{\nu}|}\,ds' = \dot{\phi}\sqrt{|g_{\mu\nu}u^{\mu}u^{\nu}|}\,ds' = \sqrt{|g_{\mu\nu}u^{\mu}u^{\nu}|}\,d\sigma = \boldsymbol{L} \tag{6.7.31}$$

The Euler–Lagrange equations are then

$$\frac{d}{ds}\left(\frac{g_{\mu\alpha}\dot{q}^{\mu}}{\sqrt{|g_{\rho\sigma}u^{\rho}u^{\sigma}|}}\right) - \frac{\partial_{\alpha}g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}}{2\sqrt{|g_{\rho\sigma}u^{\rho}u^{\sigma}|}} = 0 \quad \Rightarrow \quad g_{\mu\alpha}\dot{q}^{\mu}\frac{d}{ds}\left(\frac{1}{|u|}\right) + \frac{d}{ds}\left(g_{\mu\alpha}\dot{q}^{\mu}\right)\frac{1}{|u|} - \tfrac{1}{2}\partial_{\alpha}g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}\frac{1}{|u|} = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \dot{q}^{\lambda}\frac{d}{ds}\left(\frac{1}{|u|}\right)|u| + \ddot{q}^{\lambda} + \{g\}^{\lambda}_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = 0 \quad \Rightarrow \quad \ddot{q}^{\lambda} + \{g\}^{\lambda}_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = \dot{q}^{\lambda}\frac{d}{ds}\left(\ln|u|\right) \tag{6.7.32}$$

where we set $|u| = \sqrt{|g_{\rho\sigma}u^{\rho}u^{\sigma}|}$. That is the equation for geodesic trajectories with $\lambda = \frac{d}{ds}\left(\ln|u|\right)$. When one changes parameterisation the coefficient λ changes as

$$\lambda' = \frac{d}{ds'}\left(\ln|u'|\right) = \frac{d}{ds'}\left(\ln(\dot{\phi}|u|)\right) = \frac{d}{ds'}\left(\ln\dot{\phi} + \ln|u|\right) = \frac{\ddot{\phi}}{\dot{\phi}} + \frac{d}{ds'}\left(\ln|u|\right) = \frac{\ddot{\phi}}{\dot{\phi}} + \dot{\phi}\frac{d}{ds}\left(\ln|u|\right) = \frac{\ddot{\phi}}{\dot{\phi}} + \dot{\phi}\lambda \tag{6.7.33}$$

which agrees with the required transformation rules with respect to reparameterisations; compare to (6.7.17).

The equations for a generic connection are not variational. However, let us notice that *particles* and *light rays* which are here considered as primitive objects, in fact are not fundamental but they arise from fields, matter fields which obey their field equations (i.e. PDE) from which geodesic equations are obtained as an approximation. The relation between matter field equations and geodesics will be discussed elsewhere (Link). For now, it is sufficient and necessary to notice that matter field equations are more fundamental and they should be required to be variational.

Let us notice that given a geodesic motion γ for the connection Γ, then any reparameterisation $\gamma' = \gamma \circ \phi$ is another representative of the same trajectory $\boldsymbol{\gamma} = [\gamma] = [\gamma']$ which is a geodesic trajectory for the projective structure $[\Gamma]$.

On the other hand, given a geodesic trajectory γ for the projective structure $[\Gamma]$ then any representative γ of the trajectory is a geodesic motion for some representative $\Pi \in [\Gamma]$ and for any representative connection $\Pi \in [\Gamma]$ there exists a reparameterisation $\gamma' = \gamma \circ \phi$ which is a geodesic motion for Π. In fact, one could define *geodesic trajectories* aa those trajectories which have a representative which is a geodesic motion. Although this approach is often useful for computation, we prefer to regard geodesic trajectories as more fundamental than geodesic motions.

Geodesic vector fields

There is a third way to consider the issue of geodesics that is often used in differential geometry and it will have a certain relevance for GR where it can be used to represent acts of motion of dust (i.e. a family of particles which moves in spacetime, each along its geodesic worldline—hence without collisions or mutual interactions in view of Cauchy uniqueness theorem).

Let us consider a spacetime vector field X. This is called a *geodesic field* with respect of a connection Γ iff its integral curves are geodesic motions of Γ. A curve $\gamma^\mu(s)$ is an integral curve of the vector field $X = X^\mu(x)\partial_\mu$ iff $\dot{\gamma}^\mu = X^\mu \circ \gamma$. By differentiation, one has $\ddot{\gamma}^\mu = \partial_\alpha X^\mu \dot{\gamma}^\alpha$; moreover, if γ is a geodesic motion for Γ, we also have $\ddot{\gamma}^\mu = -\Gamma^\mu_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta$. Thence the vector field X is geodesics iff

$$\partial_\alpha X^\mu \dot{\gamma}^\alpha + \Gamma^\mu_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0 \qquad\qquad X^\alpha\left(\partial_\alpha X^\mu + \Gamma^\mu_{\lambda\alpha}X^\lambda\right) = 0 \qquad\qquad X^\alpha\nabla_\alpha X^\mu = 0 \qquad\qquad (6.7.34)$$

where ∇_α denotes the covariant derivative with respect to the connection Γ.

Hence, we can define a geodesic field to be a vector field such that $X^\alpha\nabla_\alpha X^\mu = 0$. If one finds a hypersurface S which is transverse to the vector field X, one can consider a point $p \in S$, the field provides a spacetime vector $X(p)$ and there is a unique geodesics which is also an integral curve of X. The field X describes a smooth family of geodesics which fills in a region of spacetime. These geodesics cannot intersect each other in view of Cauchy uniqueness theorem and they form a line foliation of the region of spacetime where they are defined.

8. Axioms: part II

It is now time to finish providing axioms for our spacetime geometry.

Particles move under the influence of the gravitational field, only. In its weakest form, the equivalence principle claims that all (pointlike) bodies fall along a particle worldline; in particular, their motion is determined *uniquely* by their initial position and spatial velocity (i.e. spacetime direction). This makes a new description of gravitational field possible: if any body falls the same way, the cause of their motion (to be precise, of their spatial acceleration or spacetime curvature) can be interpreted as a property of spacetime, not as a relation with particles. The gravitational field *is* a deformation of geometry of spacetime and then any body reacts to the geometry of spacetime.

> The trick is not new to physics. Free holonomic systems in mechanics move along geodesic motions which are precisely a manifestation of the geometry of configuration space which is encoded into the kinetic energy form of the system. In that case, the geometry of configuration space completely determines the free motions.

Here we are proposing about the same thing at the level of spacetime with the only difference that, in principle, one has a different holonomic system for any possible geometry on configuration space. A gravitational field determines (or better it is represented by) one geometry of spacetime instead and then any body moves along, feeling the same geometry. Another difference is dictated by the fact that we are now on spacetime and, for example, only trajectories are endowed with a physical meaning and we are thence forced to resort to geodesic trajectories. We are instead more liberal on what has to be understood as the *geometry* of spacetime. We not allow only metric information, but we are forced to describe geometry in terms of conformal (i.e. less than a metric) and projective structure (i.e. less than a connection).

Axiom EP (Equivalence principle):

Particles move along geodesic trajectories of a projective structure [Π].

 This is enough to define a projective structure of spacetime: free falling particles identify it. Until now, conformal and projective structures are completely unrelated. For example, there is nothing preventing a geodesics from starting as a time-like curve and ending to be space-like along its worldline. This would correspond to a particle which under the interaction with gravity ends up to travel faster than light, something against which we have quite strong evidences. In fact, we had never been able to see a particle, free falling or not, travelling faster than light.

Axiom CC (Compatibility condition):

For any event $x \in M$ there exists a neighbourhood U such that $p \in U$ $(p \neq x)$ is on a particle P through x iff p is inside the wavefront ν_x in the spacetime.

 Particles determine a projective structure Π on spacetime. On the other hand, at an event x for any $[g]$-time-like direction v there exists one and only one particle through x with direction v. Such a particle is a $[\Pi]$-geodesic trajectory. And by Cauchy theorem, there is one and only one $[\Pi]$-geodesic trajectory through x with direction v. Accordingly, particles coincide with $[\Pi]$-geodesic trajectories which are $[g]$-time-like at some point.

 Moreover, for the compatibility Axiom, a particle cannot become $[g]$-space-like at any point. If it did, at some point e it would be light-like. At that point it would contradict Axiom CC since the points along the particles would not be inside the light cone ν_e.

Wavefronts in M

 We are now ready to discuss more in detail the structure of wavefronts $\nu_p \subset M$ in spacetime. Let us start by proving that the hypersurface ν_p is regular everywhere in a neighbourhood of p, except at p itself. Let us consider a light ray L through p and a point $q \in L$ in a suitable neighbourhood of p. One can choose a parameterisation $x^\mu(s)$ of the light ray (such that $x^\mu(0) = p$). The gradient $g_\alpha(q)$ can be expressed as a line integral along L depending on the metric:

$$g_\alpha(q) = g_\alpha(q) - g_\alpha(p) = \int_p^q \frac{d}{ds}\left(\partial_\alpha G(x(s))\right) ds = \int_p^q g_{\alpha\beta}(x(s))\dot{x}^\beta(s)ds \tag{6.8.1}$$

where we used the fact that $g_\alpha(p) = 0$, which was proven above.

The covariant velocity $\dot{x} = \dot{x}^\mu(s)\partial_\mu$ along the light ray is a non-zero vector, and since the metric is non-degenerate then the integrand covector $g_{\alpha\beta}(x(s))\dot{x}^\beta(s)$ is non-zero. The integral is thence non-zero, if q is near enough to p. Letting the initial direction and different components vary, since there parameters run in a compact set, the radius guaranteed before the integral vanishes has a minimum; in other words, one can find a neighbourhood U in which the integral never vanishes when $q \neq p$. Hence the gradient $g_\alpha(q)$ never vanishes for $q \in U - \{p\}$; then, by the implicit function theorem, the hypersurface ν_p is regular at any $q \in U$ except p itself.

Next, we wish to show that the hypersurface ν_p is light-like and that, at any $x \in \nu_p$ with $x \neq p$, one has only one light-like direction in $T_x\nu_p$. Let us fix $x \in \nu_p$ with $x \neq p$. Since $x \in \nu_p$, then there exists a light ray L connecting p and x and any point on L is also on ν_p. Hence we have a light ray L which lies on ν_p and passes through x. Let v be tangent to L. Since L lies on ν_p then $v \in T_x\nu_p$. Since v is tangent to a light ray, it is light-like. Then we have a light-like vector in $T_x\nu_p$. Hence ν_p cannot be space-like at x.

Let us notice that any point x on the light ray L is also on the light cone ν_x of x. In other words, L is in the intersection of ν_p and ν_x.

If ν_p were time-like at x then there would exist a time-like vector $\xi \in T_x\nu_p$. In this case, the wavefront ν_x cuts into the wavefront ν_p and one can find a point $q \in \nu_x$ which is inside ν_p. Then there is a particle P passing through q and p. If q is closed enough to x (i.e. into a U-type neighbourhood U_x) then there are two light rays echoing x on P (one, say L_1, through q, the other, say L_2, hits P at $r \in P$). In other words the point x is connected to the particle P by three light rays L, L_1, L_2 hitting P at p, q, r, respectively. Since we begun by choosing x in an arbitrarily small neighbourhood of p this contradicts Axiom AE. Hence ν_p cannot be time-like at x. Since it is not space-like or time-like, ν_p must be light-like at x.

The wavefront ν_x is tangent to the cone ν_p and the intersection is L. Accordingly, the direction of L, which is spanned by $v \in T_x\nu_p$, is the only light-like direction in $T_x\nu_p$. Then one can define a light-like vector field X on $\nu_p^+ - \{p\}$ in a neighbourhood of p.

Of course, the light ray L is everywhere tangent to X. We could say L is an integral trajectory of X: if you wish there exists a parameterisation of L which is an integral curve of X. On the other hand, take a point x and the light-like vector $X(x)$: for Cauchy theorem (applied to geodesic motion equations) there is a unique g-geodesics σ from those initial conditions. Since X is light-like, then σ is light-like, it lies on $\nu_p^+ - \{p\}$. However, we showed that at any point on $\nu_p^+ - \{p\}$ there is only one light-like direction. Then the trajectory $[\sigma]$ necessarily coincides with L. Hence, finally we have that L is a $[g]$-geodesic trajectory (since it can be reparameterised to be a g-geodesic motion). On the other hand, if one has a light-like $[g]$-geodesic trajectory, there should be a light ray with the same initial conditions, which, by the argument above, actually coincides with the geodesic. In other words, a trajectory is a light ray iff it is light-like $[g]$-geodesics. Notice that, unlike for particles, we did not need a specific axiom to impose the equation for light rays. Such an equation is basically forced by the way a family of compact sets (namely the wavefronts) intersect.

9. Affine structure of spacetime

On a spacetime with a conformal structure $[g]$ and a projective structure $[\Pi]$, let us consider a $[\Pi]$-geodesic trajectory $\boldsymbol{\gamma} = [\gamma]$, with γ obeying

$$\ddot{\gamma}^\mu + \Pi^\mu_{\rho\sigma}\dot{\gamma}^\rho\dot{\gamma}^\sigma = \lambda\dot{\gamma}^\mu \tag{6.9.1}$$

for some function $\lambda(s)$.

Let us then consider the quantity

$$n(s) := \frac{d}{ds}\left(g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu\right) = \partial_\lambda g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu\dot{\gamma}^\lambda + 2g_{\mu\nu}\ddot{\gamma}^\mu\dot{\gamma}^\nu = 2g_{\nu\mu}\left(\{g\}^\mu_{\sigma\rho} - \Pi^\mu_{\rho\sigma}\right)\dot{\gamma}^\rho\dot{\gamma}^\sigma\dot{\gamma}^\nu + 2\lambda g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \tag{6.9.2}$$

If $\boldsymbol{\gamma}$ is $[g]$-light-like then it must be $n(s) := 0$. In fact, if it were $n(0) \neq 0$ then $g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu$ (which we know is zero at $s = 0$) would change sign around $s = 0$. Hence we would have a $[\Pi]$-geodesic which is somewhere time-like and somewhere space-like, which we argued would contradict the Axiom CC. The argument can be repeated at any point at which the geodesic is light-like.

> We sooner or later will have to exclude the case of a Π-geodesics which is time-like (or space-like) almost everywhere but a discrete set of points where it is light-like. This would correspond, e.g., to a particle which at some point for a moment tops the speed of light just to slow down to a time-like covariant velocity.
> In the end, when we will have proven that for the compatible connections light-like $[\Pi]$-geodesics are the same as the light-like $[g]$-geodesics the matter will be over. In the meanwhile we have to be careful about circular reasoning. Or excluding the case before. Or add an axiom to exclude the case (though it would be difficult to regard such an axiom as observationally obvious —let us stress the particle becomes light-like *for a moment*!).

Then, along light-like vectors $\dot{\gamma}^\mu$, we have

$$\left(\{g\}^\mu_{\sigma\rho} - \Pi^\mu_{\rho\sigma}\right)\dot{\gamma}^\rho\dot{\gamma}^\sigma\dot{\gamma}^\nu = \left(\{g\}^\mu_{\sigma\rho} - \Pi^\mu_{\rho\sigma} + \tfrac{1}{2}\alpha^\mu_{\cdot}g_{\rho\sigma}\right)\dot{\gamma}^\rho\dot{\gamma}^\sigma\dot{\gamma}^\nu = 0 \tag{6.9.3}$$

for any 1-form $\boldsymbol{\alpha} = \alpha_\mu\,dx^\mu$. The factor $\frac{1}{2}$ is chosen for later convenience and, of course, it could be absorbed in the definition of $\boldsymbol{\alpha}$. Then we fix the projective structure to be

$$\Pi^\mu_{\rho\sigma} = \{g\}^\mu_{\sigma\rho} + \tfrac{1}{2}g^{\mu\epsilon}g_{\rho\sigma}\alpha_\epsilon \tag{6.9.4}$$

> This is not *exactly* what EPS did. They did not use the connection $\Pi^\mu_{\rho\sigma}$ to represent the projective structure, but its traceless version

$$\hat{\Pi}^\mu_{\rho\sigma} := \Pi^\mu_{\rho\sigma} - \tfrac{2}{m+1}\delta^\mu_{(\rho}\Pi^\epsilon_{\sigma)\epsilon} \tag{6.9.5}$$

> This is in fact traceless, i.e.

$$\hat{\Pi}^\mu_{\rho\mu} := \Pi^\mu_{\rho\mu} - \tfrac{1}{m+1}\Pi^\epsilon_{\rho\epsilon} - \tfrac{1}{m+1}m\Pi^\epsilon_{\rho\epsilon} = \Pi^\mu_{\rho\mu} - \Pi^\epsilon_{\rho\epsilon} = 0 \tag{6.9.6}$$

> If we compute $\hat{\Pi}$ for our connection $\Pi^\mu_{\rho\sigma}$ in dimension $m = 4$ we obtain

$$\hat{\Pi}^\mu_{\rho\sigma} := \{g\}^\mu_{\sigma\rho} + \tfrac{1}{2}\left(g^{\mu\epsilon}g_{\rho\sigma} - \tfrac{2}{5}\delta^\mu_{(\rho}\delta^\epsilon_{\sigma)}\right)\alpha_\epsilon - \tfrac{1}{5}\delta^\mu_{(\rho}\partial_{\sigma)}\ln g \tag{6.9.7}$$

> which agrees (modulo the choice of a representative for the projective structure) with EPS (see[18] eq. (25)).

Now it is important to notice that we identified a projective structure $[\Pi]$ compatible to the conformal structure, not an affine structure, i.e. a connection. And there is not much we can do to single out a connection in the class $[\Pi]$ which is better than the others, i.e. *canonical*, on the physical stance. We already showed that all connections in the class $[\Pi]$ produce the same geodesic trajectories and until geodesics is all we can observe there is no way to select a representative which has better observational properties than the others. However, this does not mean that one cannot single out a canonical representative on the mathematical ground. This choice, even if not based on physics, can make our life a lot easier; the only restriction is that the selection of the canonical representative should always be possible.

A connection Γ is said to be a *Weyl connection* for the conformal structure $[g]$ iff there exists a 1-form $\boldsymbol{\alpha} = \alpha_\mu\, dx^\mu$ such that, for any representative $g \in [g]$, one has

$$\overset{\Gamma}{\nabla}_\mu g_{\alpha\beta} = \alpha_\mu g_{\alpha\beta} \tag{6.9.8}$$

If one considers a different representative $g' = \Phi^2 g$, then one has

$$\overset{\Gamma}{\nabla}_\mu g'_{\alpha\beta} = \overset{\Gamma}{\nabla}_\mu \left(\Phi^2 g_{\alpha\beta} \right) = \overset{\Gamma}{\nabla}_\mu \Phi^2 g_{\alpha\beta} + \Phi^2 \overset{\Gamma}{\nabla}_\mu g_{\alpha\beta} = \left(\partial_\mu \Phi^2 + \Phi^2 \alpha_\mu \right) g_{\alpha\beta} = \left(\partial_\mu \left(\ln \Phi^2 \right) + \alpha_\mu \right) g'_{\alpha\beta} \tag{6.9.9}$$

Hence if Γ is a Weyl connection for g (with respect to the 1-form $\boldsymbol{\alpha}$), then it is also a Weyl connection for g'; all one should do is choosing the 1-form $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + d \left(\ln \Phi^2 \right)$. Accordingly, one can say that the connection Γ is a Weyl connection for a conformal structure $[g]$.

Let us consider a manifold M equipped with a conformal structure $[g]$ and a Weyl connection Γ for $[g]$; the triple $(M, [g], \Gamma)$ is called a *Weyl geometry*.

Let us consider a conformal structure $[g]$ and its compatible projective structure $[\Pi^\mu_{\rho\sigma} = \{g\}^\mu_{\sigma\rho} + \frac{1}{2} g^{\mu\epsilon} g_{\rho\sigma} \alpha_\epsilon]$. Any connection $\Gamma \in [\Pi]$ is in the form

$$\Gamma^\mu_{\rho\sigma} = \{g\}^\mu_{\sigma\rho} + \frac{1}{2} g^{\mu\epsilon} g_{\rho\sigma} \alpha_\epsilon - \delta^\mu_{(\rho} \delta^\epsilon_{\sigma)} V_\epsilon \tag{6.9.10}$$

for some 1-form $\mathbf{V} = V_\mu\, dx^\mu$.

Then we can choose the 1-form $\mathbf{V}$ so that the connection Γ is a Weyl connection for the conformal structure $[g]$. In fact

$$\begin{aligned}
\overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} &= \overset{g}{\nabla}_\lambda g_{\mu\nu} - \tfrac{1}{2} g^{\rho\epsilon} g_{\mu\lambda} g_{\rho\nu} \alpha_\epsilon + \tfrac{1}{2} \delta^\rho_\lambda g_{\rho\nu} V_\mu + \tfrac{1}{2} \delta^\rho_\mu g_{\rho\nu} V_\lambda - \tfrac{1}{2} g^{\rho\epsilon} g_{\nu\lambda} g_{\rho\mu} \alpha_\epsilon + \tfrac{1}{2} \delta^\rho_\lambda g_{\rho\mu} V_\nu + \tfrac{1}{2} \delta^\rho_\nu g_{\rho\mu} V_\lambda = \\
&= \tfrac{1}{2} g_{\mu\lambda} \left(V_\nu - \alpha_\nu \right) + \tfrac{1}{2} g_{\lambda\nu} \left(V_\mu - \alpha_\mu \right) + V_\lambda g_{\mu\nu}
\end{aligned} \tag{6.9.11}$$

There is always a canonical choice for the representative, $V_\mu = \alpha_\mu$ for which

$$\overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} = \alpha_\lambda g_{\mu\nu} \tag{6.9.12}$$

Then for any conformal structure $[g]$ and projective structure $[\Pi]$ which obey the Axioms above (in particular, they are compatible) then there is always a Weyl connection

$$\Gamma^\mu_{\rho\sigma} = \{g\}^\mu_{\sigma\rho} + \frac{1}{2} \left(g^{\mu\epsilon} g_{\rho\sigma} - 2\delta^\mu_{(\rho} \delta^\epsilon_{\sigma)} \right) \alpha_\epsilon \tag{6.9.13}$$

In other words, the connections compatible with $[g]$ are parameterised by the extra freedom in choosing a 1-form $\boldsymbol{\alpha}$. A compatible connection Γ depends on the conformal structure only, not on the representative.

If we select a different parameterisation $g' = \Phi^2 g$, first of all we have

$$
\begin{aligned}
\{g'\}^\alpha_{\beta\mu} &= \tfrac{1}{2} g'^{\alpha\lambda} \left(-\partial_\lambda g'_{\beta\mu} + \partial_\beta g'_{\mu\lambda} + \partial_\mu g'_{\lambda\beta} \right) = \{g\}^\alpha_{\beta\mu} + \tfrac{1}{2\varphi} g^{\alpha\lambda} \left(-\partial_\lambda \Phi^2 g_{\beta\mu} + \partial_\beta \Phi^2 g_{\mu\lambda} + \partial_\mu \Phi^2 g_{\lambda\beta} \right) = \{g\}^\alpha_{\beta\mu} + \tfrac{1}{2} \left(-\partial_\lambda \ln \Phi^2 g^{\alpha\lambda} g_{\beta\mu} + \partial_\beta \ln \Phi^2 \delta^\alpha_\mu + \partial_\mu \ln \Phi^2 \delta^\alpha_\beta \right) = \\
&= \{g\}^\alpha_{\beta\mu} - \tfrac{1}{2} \left(g^{\alpha\lambda} g_{\beta\mu} - 2\delta^\lambda_{(\beta} \delta^\alpha_{\mu)} \right) \partial_\lambda \ln \Phi^2
\end{aligned}
\tag{6.9.14}
$$

Then, if a connection Γ is compatible with g, then we have

$$
\Gamma^\mu_{\rho\sigma} = \{g\}^\mu_{\sigma\rho} + \tfrac{1}{2} \left(g^{\mu\epsilon} g_{\rho\sigma} - 2\delta^\mu_{(\rho} \delta^\epsilon_{\sigma)} \right) \alpha_\epsilon = \{g'\}^\mu_{\sigma\rho} + \tfrac{1}{2} \left(g^{\alpha\lambda} g_{\beta\mu} - 2\delta^\lambda_{(\beta} \delta^\alpha_{\mu)} \right) \partial_\lambda \ln \Phi^2 + \tfrac{1}{2} \left(g^{\mu\epsilon} g_{\rho\sigma} - 2\delta^\mu_{(\rho} \delta^\epsilon_{\sigma)} \right) \alpha_\epsilon = \{g'\}^\mu_{\sigma\rho} + \tfrac{1}{2} \left(g^{\mu\epsilon} g_{\rho\sigma} - 2\delta^\mu_{(\rho} \delta^\epsilon_{\sigma)} \right) \left(\alpha_\epsilon + \partial_\lambda \ln \Phi^2 \right)
\tag{6.9.15}
$$

and it is compatible, just with respect to the 1-form $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + d \left(\ln \Phi^2 \right)$.

The $[g]$-time-like Γ-geodesics motions identify particle worldlines. The $[g]$-light-like Γ-geodesics motions coincide with g-light-like g-geodesics and with light rays.

A Γ-geodesics motion reads as

$$
\ddot\gamma^\mu + \Gamma^\mu_{\rho\sigma} \dot\gamma^\rho \dot\gamma^\sigma = \ddot\gamma^\mu + \{g\}^\mu_{\sigma\rho} \dot\gamma^\rho \dot\gamma^\sigma + \tfrac{1}{2} \left(g^{\mu\epsilon} g_{\rho\sigma} - 2\delta^\mu_\rho \delta^\epsilon_\sigma \right) \alpha_\epsilon \dot\gamma^\rho \dot\gamma^\sigma = \ddot\gamma^\mu + \{g\}^\mu_{\sigma\rho} \dot\gamma^\rho \dot\gamma^\sigma + \tfrac{1}{2} \alpha^\mu (\dot\gamma_\sigma \dot\gamma^\sigma) - (\alpha_\epsilon \dot\gamma^\epsilon) \dot\gamma^\mu = 0
\tag{6.9.16}
$$

which, along light-like vectors ($\dot\gamma_\sigma \dot\gamma^\sigma = 0$), corresponds to a g-geodesics (modulo a suitable reparameterisation to cancel the term $\lambda\dot\gamma^\mu$ with $\lambda = (\alpha_\epsilon \dot\gamma^\epsilon)$).

Let us consider a manifold M with a conformal structure $[g]$ and a compatible projective structure $[\Pi]$; the triple $(M, [g], [\Pi])$ is called an *EPS geometry*. We just proved that an EPS geometry defines a *unique* Weyl geometry.

There is also a relation between Γ and $\boldsymbol{\alpha}$. In fact if we trace the compatibility relation

$$
\Gamma^\mu_{\rho\mu} = \{g\}^\mu_{\sigma\mu} + \tfrac{1}{2} \left(\delta^\epsilon_\sigma - (m+1)\delta^\epsilon_\sigma \right) \alpha_\epsilon = \tfrac{1}{2} \partial_\sigma \ln g - \tfrac{m}{2} \alpha_\sigma \qquad \alpha_\rho = \tfrac{1}{m} \partial_\rho \ln g - \tfrac{2}{m} \Gamma_\rho
\tag{6.9.17}
$$

which is useful to compare with the original EPS. That is also why we decided not to use invariant representation of a projective structure. That leads eventually to this form for $\boldsymbol{\alpha}$, i.e. as the sum of two non-tensorial quantities.

Let us finally show that two different Weyl structures can be observed to be different. Let us consider two Weyl structures $(M, [g], \Gamma)$ and $(M, [g], \Gamma')$ with the same $[g]$-time-like geodesic trajectories. A curve $\gamma^\mu(s)$ is a Γ-geodesics iff $\ddot\gamma^\mu + \Gamma^\mu_{\rho\sigma} \dot\gamma^\rho \dot\gamma^\sigma = 0$. Let us now suppose that any $[g]$-time-like Γ-geodesic is also a Γ'-geodesic, i.e. $\ddot\gamma^\mu + \Gamma'^\mu_{\rho\sigma} \dot\gamma^\rho \dot\gamma^\sigma = 0$.

Let us define $K^\mu_{\rho\sigma} = \Gamma'^\mu_{\rho\sigma} - \Gamma^\mu_{\rho\sigma}$ which is tensor since it is the difference between two connections. By subtracting the two geodesic equations we obtain

$$
K^\mu_{\rho\sigma} \dot\gamma^\rho \dot\gamma^\sigma = 0
\tag{6.9.18}
$$

which hold for any time-like vector $\dot\gamma$. Since one could find a basis of $T_x M$ made only of time-like vector, then necessarily $K^\mu_{\rho\sigma} = 0$.

Time-like vectors fill in the light cone, i.e. an m-dimensional set. One can start considering a time-like vector v_1 which generate a 1 dimensional line. Since $m > 1$ there are time-like vectors which are not generated by v_1. Hence one can choose a time-like vector v_2 which is independent of v_1. The two vectors (v_1, v_2) span a dimension 2 plane. And one can go on to choose m time-like independent vectors.

Consequently, necessarily the two connections coincide $\Gamma'^\mu_{\rho\sigma} = \Gamma^\mu_{\rho\sigma}$.

Hence, for two different Weyl structures, the sheaf of time-like geodesics is different. They can share the same light cones, so that light rays are in common, though free falling particles are necessarily different. Thus, in principle, one can single out the Weyl structure by observing free falling particles.

10. Riemannian geometries

Given an EPS geometry $(M, [g], [\Pi])$, this defines a canonical Weyl geometry $(M, [g], \tilde{\Gamma})$.

We say that a Weyl geometry is a *Riemannian geometry* if there exists a representative $g \in [g]$ such that $\tilde{\Gamma} = \{g\}$. In the special case of a Riemannian geometry, everything is generated by a Lorentzian metric g; thus we denote a Riemannian geometry simply by (M, g).

> Originally, EPS tried to add an extra axiom to force the Weyl geometry to be Riemannian. EPS discussed such an axiom and they raised doubts about this axiom which is certainly less clearly based on observations than the others. Also in the later literature, this extra axiom is considered less convincing than the rest of the theory. The extra axiom is anyway mathematically relevant so we present it here. However, when below will refer to EPS framework we mean EPS without the Riemannian axiom. Accordingly, an EPS geometry endows a Weyl geometry which, in general, can be non-Riemannian.

For a generic Weyl geometry $(M, [g], \tilde{\Gamma})$, let us consider the curvature $\tilde{R}^\mu{}_{\nu\alpha\beta}(\tilde{\Gamma})$ of the connection $\tilde{\Gamma}$ and lower the index with a representative of the conformal class $g_{\mu\nu}$, i.e. consider

$$\tilde{R}_{\mu\nu\alpha\beta}(\tilde{\Gamma}) := g_{\mu\epsilon}\tilde{R}^\epsilon{}_{\nu\alpha\beta}(\tilde{\Gamma}) = g_{\epsilon(\mu}\tilde{R}^\epsilon{}_{\nu)\alpha\beta} + g_{\epsilon[\mu}\tilde{R}^\epsilon{}_{\nu]\alpha\beta} \tag{6.10.1}$$

Let us set $S_{\mu\nu\alpha\beta} := g_{\epsilon(\mu}\tilde{R}^\epsilon{}_{\nu)\alpha\beta}$ and $A_{\mu\nu\alpha\beta} := g_{\epsilon[\mu}\tilde{R}^\epsilon{}_{\nu]\alpha\beta}$.

> Since we have two connections, we can define their tensor difference

$$K_{\alpha\beta\mu} := g_{\alpha\epsilon}\left(\tilde{\Gamma}^\epsilon{}_{\beta\mu} - \{g\}^\epsilon{}_{\beta\mu}\right) = \tfrac{1}{2}\left(g_{\beta\mu}\alpha_\alpha - g_{\alpha\beta}\alpha_\mu - g_{\alpha\mu}\alpha_\beta\right) \tag{6.10.2}$$

> where we used the compatibility relation (6.9.13). The symmetric part is

$$
\begin{aligned}
S_{\mu\nu\alpha\beta} =& g_{\epsilon(\mu}\tilde{R}^\epsilon{}_{\nu)\alpha\beta} = g_{\epsilon(\mu}\left(R^\epsilon{}_{\nu)\alpha\beta}(g) + \overset{g}{\nabla}_\alpha K^\epsilon{}_{\nu)\beta} - \overset{g}{\nabla}_\beta K^\epsilon{}_{\nu)\alpha} + K^\epsilon{}_{\lambda\alpha}K^\lambda{}_{\nu)\beta} - K^\epsilon{}_{\lambda\beta}K^\lambda{}_{\nu)\alpha}\right) = \overset{g}{\nabla}_\alpha K_{(\mu\nu)\beta} - \overset{g}{\nabla}_\beta K_{(\mu\nu)\alpha} + K_{(\mu\underline{\lambda}\alpha}K^\lambda{}_{\nu)\beta} - K_{(\mu\underline{\lambda}\beta}K^\lambda{}_{\nu)\alpha} = \\
=& \tfrac{1}{2}\left(g_{\beta(\nu}\overset{g}{\nabla}_{\underline{\alpha}}\alpha_{\mu)} - g_{\mu\nu}\overset{g}{\nabla}_\alpha\alpha_\beta - g_{\beta(\mu}\overset{g}{\nabla}_{\underline{\alpha}}\alpha_{\nu)}\right) - \tfrac{1}{2}\left(g_{\alpha(\nu}\overset{g}{\nabla}_{\underline{\beta}}\alpha_{\mu)} - g_{\mu\nu}\overset{g}{\nabla}_\beta\alpha_\alpha - g_{\alpha(\mu}\overset{g}{\nabla}_{\underline{\beta}}\alpha_{\nu)}\right) + K_{(\mu\underline{\lambda}\alpha}K^\lambda{}_{\nu)\beta} - K_{(\mu\underline{\lambda}\beta}K^\lambda{}_{\nu)\alpha} = \\
=& -g_{\mu\nu}\overset{*}{\nabla}_{[\alpha}\alpha_{\beta]} + \tfrac{1}{4}\left(-g_{\alpha(\mu}g_{\nu)\beta}\alpha_\lambda\alpha^\lambda + g_{\mu\nu}\alpha_\alpha\alpha_\beta - g_{\beta\alpha}\alpha_{(\mu}\alpha_{\nu)} + g_{\beta(\mu}\alpha_{\nu)}\alpha_\alpha + g_{\alpha(\mu}\alpha_{\nu)}\alpha_\beta + g_{\beta(\mu}g_{\nu)\alpha}\alpha_\lambda\alpha^\lambda - g_{\mu\nu}\alpha_\beta\alpha_\alpha + g_{\beta\alpha}\alpha_{(\mu}\alpha_{\nu)} - g_{\alpha(\mu}\alpha_{\nu)}\alpha_\beta - g_{\beta(\mu}\alpha_{\nu)}\alpha_\alpha\right) = \\
=& -\tfrac{1}{2}g_{\mu\nu}F_{\alpha\beta}
\end{aligned} \tag{6.10.3}
$$

> where we set $F_{\alpha\beta} := 2\overset{*}{\nabla}_{[\alpha}\alpha_{\beta]}$ and we expanded

$$
\begin{aligned}
K_{(\mu\underline{\lambda}\alpha}K^\lambda{}_{\nu)\beta} =& \tfrac{1}{4}\left(g_{\beta(\nu}\alpha_{\mu)}\alpha_\alpha - g_{\beta(\nu}\alpha_{\mu)}\alpha_\alpha - g_{\alpha(\mu}g_{\nu)\beta}\alpha_\lambda\alpha^\lambda - g_{\alpha(\nu}\alpha_{\mu)}\alpha_\beta + g_{\mu\nu}\alpha_\alpha\alpha_\beta + g_{\alpha(\mu}\alpha_{\nu)}\alpha_\beta - g_{\beta\alpha}\alpha_{(\mu}\alpha_{\nu)} + g_{\beta(\mu}\alpha_{\nu)}\alpha_\alpha + g_{\alpha(\mu}\alpha_{\nu)}\alpha_\beta\right) = \\
=& \tfrac{1}{4}\left(-g_{\alpha(\mu}g_{\nu)\beta}\alpha_\lambda\alpha^\lambda + g_{\mu\nu}\alpha_\alpha\alpha_\beta - g_{\beta\alpha}\alpha_{(\mu}\alpha_{\nu)} + g_{\beta(\mu}\alpha_{\nu)}\alpha_\alpha + g_{\alpha(\mu}\alpha_{\nu)}\alpha_\beta\right)
\end{aligned} \tag{6.10.4}
$$

Thus any Riemann tensor can be uniquely expanded as

$$\tilde{R}^{\mu}{}_{\nu\alpha\beta} = A^{\mu}{}_{\nu\alpha\beta} - \tfrac{1}{2}\delta^{\mu}_{\nu}F_{\alpha\beta} \tag{6.10.5}$$

Let us stress that the tensor $A^{\mu}{}_{\nu\alpha\beta}$, in general, is not a Riemann tensor (except, of course, if $F = 0$).

One can show that parallel transport along any curve preserves the length of a vector iff $F = 0$. In that case $A^{\mu}{}_{\nu\alpha\beta}$ is a Riemann tensor and it coincides with $\tilde{R}^{\mu}{}_{\nu\alpha\beta}$.

Since $\mathbf{F} = d\boldsymbol{\alpha} = 0$, then (at least locally) $\boldsymbol{\alpha} = d\omega$ for some potential 0-form ω. Accordingly, one has

$$K^{\alpha}_{\beta\mu} = \tfrac{1}{2}\left(g_{\beta\mu}g^{\alpha\epsilon} - \delta^{\alpha}_{\beta}\delta^{\epsilon}_{\mu} - \delta^{\alpha}_{\mu}\delta^{\epsilon}_{\beta}\right)\alpha_{\epsilon} = \tfrac{1}{2}\left(g_{\beta\mu}g^{\alpha\epsilon} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{\mu)}\right)\overset{*}{\nabla}_{\epsilon}\omega \tag{6.10.6}$$

and consequently

$$\Gamma^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2}\left(g_{\beta\mu}g^{\alpha\epsilon} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{\mu)}\right)\overset{*}{\nabla}_{\epsilon}\ln e^{-\omega} = \{\tilde{g}\}^{\alpha}_{\beta\mu} \tag{6.10.7}$$

where we used equation (6.9.14) and we set $\tilde{g}_{\mu\nu} = e^{-\omega}g_{\mu\nu}$.

Hence the connection is the Levi Civita connection of a conformal metric $\tilde{g}$.

In this case only, one can define a metric $\tilde{g} \in [g]$ such that $\tilde{R}^{\mu}{}_{\nu\alpha\beta}$ and the connection $\tilde{\Gamma}$ is the Levi Civita connection of the metric $\tilde{g}$. As it happens in standard GR, there is one metric $\tilde{g}$ which does everything: it describes free fall and distances in spacetime.

We say that a Weyl structure $(M, [g], \tilde{\Gamma})$ is a *Riemannian structure*, if $\mathbf{F} = 0$ and we denote it by $(M, \tilde{g})$.

As a matter of fact, the EPS analysis shows that spacetime geometry is naturally described by a Weyl geometry and if one wants to recover standard GR one should add a further axiom:

Axiom R (Riemannian Condition):

The curvature $\tilde{R}^{\mu}{}_{\nu\alpha\beta}$ is such that $\tilde{R}_{(\mu\nu)\alpha\beta} = 0$.

The Axiom is less strongly based on observations than the others, in fact, also in EPS the authors questioned if it is really well established.

That is pretty much one can do with a Weyl geometry $(M, [g], \tilde{\Gamma})$. However, we shall find useful to fix a representative of the conformal structure as well to obtain $(M, g, \tilde{\Gamma})$, which is called a *Weyl frame*, sometimes, by an (ugly) abuse of language, simply a *frame*.

Let us stress that Weyl frames are obtained by Weyl geometry and one still has the EPS compatibility condition, partially fixing the relation between the connection and the metric. Still one has that there exists a 1-form $\boldsymbol{\alpha} = \alpha_{\epsilon}\,dx^{\epsilon}$ such that

$$\tilde{\Gamma}^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2}\left(g_{\beta\mu}g^{\alpha\epsilon} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{\mu)}\right)\alpha_{\epsilon} \tag{6.10.8}$$

Let us also remark that, in the case of a metric-affine theory, one selects a preferred representative g for the conformal structure, namely the one used to formulate dynamics, unless the Lagrangian is corformally invariant. In other words, in a metric-affine theory, one uses a Weyl frame $(M, g, \tilde{\Gamma})$ as a fundamental geometric structure. The connection $\tilde{\Gamma}$ determines the free fall, the metric g compares in the specification of the dynamics and it is associated to an operative definition of distances.

Weyl frames are a bit more elastic than Weyl geometries with respect to fixing a metric structure. In a Weyl geometry, if the connection is metric it is necessarily metric with respect to a representative of the conformal structure. Thus it is the connection which fixes the metric structure (up to at least a constant factor) and the geometry is Riemannian.

On the contrary, when dealing with a Weyl frame, one has *already* fixed a representative g for the conformal structure, so, if the connection is metric there are two possibilities:

    i) the connection selects a different metric $\tilde{g}$ which is necessarily conformal to the metric g which has been fixed as a conformal representative;

    ii) or it selects $\tilde{g} = g$.

In the first case, the Weyl frame is called an *integrable Weyl frame*, while in the second case it is called a *purely metric Weyl frame*.

Also, assuming the extra Axiom R holds true for a Weyl geometry $(M, [g], \tilde{\Gamma})$, that implies that both the conformal and the projective structure can be expressed in terms of a Lorentzian metric $\tilde{g}$, in the case of a Weyl frame $(M, g, \tilde{\Gamma})$ there is no reason to assume that such a Lorentzian metric is in fact g. Of course, one can decide to formulate the dynamics in term of the conformal representative $\tilde{g}$. Doing that, everything is formulated in terms of a single Riemannian geometry given by $\tilde{g}$ as in purely metric formalism. However, since the Lagrangian is not covariant with respect to Weyl conformal transformations, by doing so the form of the Lagrangian written with respect to $\tilde{g}$ is different from what it was in terms of g. Quite typically, if matter was minimally coupled in terms of g, it is not in terms of $\tilde{g}$. We shall consider detailed examples in the next Chapter.

In other words, also in these Riemannian dynamics, one does select a preferred representative of the conformal class to be used. Accordingly, we could also say that dynamics selects a preferred Weyl frame.

This situation is typical of extended theories of gravitation. However, in general one could argue that, since dynamics is potentially able to select a conformal frame, why one should worry about an Axiom to fix it at a kinematical level?

For this reason, our attitude will be not to require any extra Axiom and leave the dynamics free to determine the projective structure in terms of the metric structure dynamically.

11. Special relativity

As an example, let us consider the whole EPS construction for Minkowski spacetime. This also proves that the axioms we gave are coherent providing an explicit model for them. Moreover, they define an important spacetime solution of standard GR, that is also needed for the limit of the theory known as Special Relativity (SR).

Let us consider the manifold $M = \mathbb{R}^m$. To identify points in M, let us use canonical Cartesian coordinates $x^\mu = (x^0, x^i)$ in $\mathbb{R}^m$.

> To be precise we are not allowed to call them *coordinates*, yet. We are here considering $\mathbb{R}^m$ as a smooth manifold and coordinates are only given by charts in the smooth atlas defined on the manifold. The axioms given above do in fact define such an atlas as all charts smoothly compatible with parallax coordinates. At this stage, we do not know if Cartesian coordinates will turn out to be compatible with parallax coordinates in Minkowski space. In fact, we should treat x^μ as labels to identify points in $\mathbb{R}^m$. However, in the end we shall show that Cartesian coordinates do in fact belong to the atlas defined on Minkowski space so we call them coordinates, by an abuse of language. Let us also fix $n := m - 1$.

Before considering the general case, let us specify to dimension 2 which is probably easier to follow.

Minkowski spacetime in dimension *2*

In dimension 2, we have $M = \mathbb{R}^2$ with coordinates (t, x). At any event $p = (t_p, x_p)$, we have exactly two light rays $R_p^\pm$ parameterised as

$$\begin{cases} t = t_p + s \\ x = x_p \pm \mathrm{c}s \end{cases} \tag{6.11.1}$$

and (freely falling) particles are parameterised as

$$\begin{cases} t = t_p + s \\ x = x_p + ws \end{cases} \qquad (-\mathrm{c} < w < \mathrm{c}) \tag{6.11.2}$$

At any event $e = (t_e, x_e)$, we select a clock

$$\begin{cases} t = t_e + s \\ x = x_e + ws \end{cases} \qquad (-\mathrm{c} < w < \mathrm{c}) \tag{6.11.3}$$

and consider an event $p = (t_p, x_p) \neq e$.

The two light rays $R_p^\pm$ intersect the clock at

$$\begin{cases} x - x_p = \pm \mathrm{c}(t - t_p) \\ x - x_e = w(t - t_e) \end{cases} \Rightarrow \begin{cases} \mathrm{c}t_\pm = \pm \dfrac{(x_e - x_p) - (wt_e \mp \mathrm{c}t_p)}{1 \mp \beta} \\ x_\pm = \dfrac{\pm wx_e \mp wx_p + w\mathrm{c}t_p \mp w^2 t_e - w\mathrm{c}t_e \pm w^2 t_e + \mathrm{c}x_e \mp wx_e}{\mathrm{c} \mp w} = \dfrac{(x_e \mp \beta x_p) - \beta\mathrm{c}(t_e - t_p)}{1 \mp \beta} \end{cases} \tag{6.11.4}$$

where we set $\beta := \frac{w}{\mathrm{c}}$. Thus we have two intersection points

$$p_+ = \left(\frac{(x_e - x_p) - (\beta\mathrm{c}t_e - \mathrm{c}t_p)}{\mathrm{c}(1 - \beta)}, \frac{(x_e - \beta x_p) - \beta\mathrm{c}(t_e - t_p)}{1 - \beta} \right) \qquad p_- = \left(-\frac{(x_e - x_p) - (\beta\mathrm{c}t_e + \mathrm{c}t_p)}{\mathrm{c}(1 + \beta)}, \frac{(x_e + \beta x_p) - \beta\mathrm{c}(t_e - t_p)}{1 + \beta} \right) \tag{6.11.5}$$

which correspond to the parameters

$$s_+ = \frac{(x_e - x_p) - \mathrm{c}(t_e - t_p)}{\mathrm{c}(1 - \beta)} \qquad s_- = -\frac{(x_e - x_p) + \mathrm{c}(t_e - t_p)}{\mathrm{c}(1 + \beta)} \tag{6.11.6}$$

respectively.

Hence we have the clock readings $s_\pm$ as a function of the unknown event (t_p, x_p) and of the clock parameters (t_e, x_e, β).

If we consider two clocks with parameters (t_e, x_e, β_e) and (t_f, x_f, β_f), and an event (t_p, x_p) in the stripe between the clocks, then one has that the two clocks readings in the future of p are

$$s_1 = \frac{(x_e - x_p) - \mathrm{c}(t_e - t_p)}{\mathrm{c}(1 - \beta_e)} \qquad s_2 = -\frac{(x_f - x_p) + \mathrm{c}(t_f - t_p)}{\mathrm{c}(1 + \beta_f)} \tag{6.11.7}$$

which define a map $r : \mathbb{R}^2 \to \mathbb{R}^2 : (t_p, x_p) \mapsto (s_1, s_2)$, which can be inverted as a map $r^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 : (s_1, s_2) \mapsto (t_p, x_p)$ as

$$\begin{cases} ct_p = \frac{1}{2}\left(c\left(1 + \beta_f\right)s_2 + c\left(1 - \beta_e\right)s_1 + (x_f - x_e) + c(t_f + t_e)\right) \\ x_p = \frac{1}{2}\left(c\left(1 + \beta_f\right)s_2 - c\left(1 - \beta_e\right)s_1 + (x_e + x_f) - c(t_e + t_f)\right) \end{cases} \tag{6.11.8}$$

Since the map r is a local diffeomorphism and (s_1, s_2) are coordinates due to Axiom DST, then (t_p, x_p) are coordinates as well.

The function $G(p)$ is defined as

$$G(p) = \frac{\left(x_e - x_p\right)^2 - c^2(t_e - t_p)^2}{c^2\left(1 - \beta^2\right)} \tag{6.11.9}$$

The second derivatives of this function define the metric tensor

$$g_{\mu\nu}(e)\,dx^\mu \otimes\,dx^\nu = \frac{1}{2}\partial_{\mu\nu}G(e)\,dx^\mu \otimes\,dx^\nu = \frac{dx^2 - c^2\,dt^2}{c^2\left(1 - \beta^2\right)} = \frac{1}{c^2\left(1 - \beta^2\right)}\eta \tag{6.11.10}$$

where we set $\eta := -c^2\,dt^2 +\,dx^2$ for the Minkowski metric.

As we showed, an observer equipped with clocks can define a Cartesian coordinate system and a metric in these coordinates (which is conformal to the Minkowski metric η). One can instruct observers to use (uniform) clocks at rest (i.e. β) so that they define Minkowski metric.

If now one considers two observers freely falling at different speed, one will define Cartesian coordinates (t, x) and define Minkowski metric $\eta = -c^2\,dt^2 +\,dx^2$ and the other will define different coordinates (t', x') and Minkowski metric $\eta = -c^2\,dt'^2 +\,dx'^2$. Since they see the same metric, the transformation between the two observers is an isometry, i.e. a Lorentz (or, more generally, Poincaré) transformation.

> Of course, *uniform* clock does not mean much in absolute terms. Here we just mean the parameterisations such that the covariant velocity is (eventually) a vector of constant length.

Minkowski spacetime in general dimension m

Let us fix an origin point $x_0^\mu \in \mathbb{R}^m$ and a direction (a^μ). Of course, the direction is defined up to an overall non-zero factor and a^μ cannot be all zero at the same time. Let also $\phi : \mathbb{R} \to \mathbb{R}$ be a global orientation preserving diffeomorphism such that $\phi(0) = 0$.

A light ray is thence the image of the following curve:

$$\sigma : \mathbb{R} \to M : s \mapsto (x^\mu = a^\mu\phi(s) + x_0^\mu) \qquad \text{with } -(a^0)^2 + (a^1)^2 + \ldots + (a^n)^2 = 0 \tag{6.11.11}$$

Notice that different values of the parameters (a^μ, x_0^μ) may, in fact, represent the same light ray.

Particles are defined as the image of the following curves

$$\gamma : \mathbb{R} \to M : s \mapsto (x^\mu = a^\mu\phi(s) + x_0^\mu) \qquad \text{with } -(a^0)^2 + (a^1)^2 + \ldots + (a^n)^2 < 0 \tag{6.11.12}$$

In both cases, one cannot have $a^0 = 0$; in fact, if $a^0 = 0$ in light rays, then necessarily $a^i = 0$ and, as noticed above, $a^\mu = 0$ is not allowed. For particles, if $a^0 = 0$ there is no choice of a^i to meet the condition on the direction parameters. Let us first prove that for any $e \in M$, for any particle P through e and for any event $x \in M$, then there exists exactly 2 light rays through x hitting P.

The generic particle through $e = e^\mu$ is given by the curve $\gamma : \mathbb{R} \to M : s \mapsto (x^\mu = a^\mu s + e^\mu)$ with the condition on direction $-|a|^2 := -(a^0)^2 + (a^1)^2 + \ldots + (a^n)^2 < 0$. We anticipate the notation of norm (and inner product by the Minkowski metric even though the Minkowski metric is yet to come; we set $|v|^2 = -(v^0)^2 + (v^1)^2 + \ldots + (v^n)^2$ (which, of course, can be negative) and $v \cdot w = -v^0 w^0 + v^1 w^1 + \ldots + v^n w^n$. Let us also introduce $|\vec{v}|^2 = (v^1)^2 + \ldots + (v^n)^2$ for the standard Euclidean spatial norm so that one can write $|v|^2 = -(v^0)^2 + |\vec{v}|^2$. The direction through $x \in M$ and a point $p \in P$ is $c^\mu = (a^\mu s + e^\mu - x^\mu)$ which is light-like iff

$$-(a^0 s + e^0 - x^0)^2 + (a^1 s + e^1 - x^1)^2 + \ldots + (a^n s + e^n - x^n)^2 = 0 \qquad \Longleftrightarrow \qquad |a|^2 s^2 + 2a \cdot (e - x)s + |e - x|^2 = 0 \tag{6.11.13}$$

This is enough for Axiom AE with $V_e \equiv M$; equation (6.11.13) has at most two solutions for all $x \in M$.

To discuss Axiom ECHO, we have to check the discriminant of the equation (6.11.13) in order to ensure exactly two solutions. The discriminant is

$$\Delta = 4 \left[(a \cdot (e - x))^2 - |a|^2 |e - x|^2 \right] \tag{6.11.14}$$

and it can be shown to be strictly positive by the usual SR techniques (as well as by direct computation). In fact one can always do a Lorentz transformation so that $a = (1, 0, 0, 0)$. In this Lorentz frame, the discriminant reads as

$$\Delta = 4 \left[(e^0 - x^0)^2 - (e^0 - x^0)^2 + (e^1 - x^1)^2 + \ldots + (e^n - x^n)^2 \right] = 4 \left[(e^1 - x^1)^2 + \ldots + (e^n - x^n)^2 \right] > 0 \tag{6.11.15}$$

except when $e^i = x^i$, i.e. $x \in P$. Since the discriminant is a (Lorentz) scalar, one has $\Delta > 0$ also in the original frame.

Since the discriminant is positive, one can find two values of the parameter $s = s_1$ and $s = s_2$ such that the point $p_1 = a^\mu s_1 + e^\mu$ (and $p_2 = a^\mu s_2 + e^\mu$) along P is connected to x by a light ray, namely the light ray defined by the curve $\sigma_1 : \mathbb{R} \to M : \tau \mapsto (x^\mu = c_1^\mu \tau + x^\mu)$ where we set $c_1^\mu = (a^\mu s_1 + e^\mu - x^\mu)$ (and $\sigma_2 : \mathbb{R} \to M : \tau \mapsto (x^\mu = c_2^\mu \tau + x^\mu)$ where we set $c_2^\mu = (a^\mu s_2 + e^\mu - x^\mu)$). Both the directions c_1^μ and c_2^μ are light-like, by construction.

This guarantees Axiom ECHO, again with $U_e \equiv M$.

Notice that, from equation (6.11.13), we are able to compute the value of the parameter ($s = s_1$ and $s = s_2$) at the intersections (p_1 and p_2). One does only need to check that in general

$$s_i = \frac{a \cdot (e - x) \pm \sqrt{(a \cdot (e - x))^2 - |a|^2 |e - x|^2}}{|a|^2} \tag{6.11.16}$$

are solutions of equation (6.11.13).

Accordingly, we are ready to define the function

$$G(x) = -\frac{(a \cdot (e - x))^2 - (a \cdot (e - x))^2 - |a|^2 |e - x|^2)}{|a|^4} = -\frac{1}{|a|^2} |e - x|^2 \tag{6.11.17}$$

Notice how a is related to the parameterisation along P; the factor $\Phi = -|a|^{-2}$ is positive and related to the conformal factor originated by different parameterisations along P. The second derivatives (at $x = e$) defines the spacetime metric

$$g_{\mu\nu}(e) := \tfrac{1}{2} \partial_{\mu\nu} G(e) = -\frac{1}{|a|^2} \eta_{\mu\nu} \tag{6.11.18}$$

which coincides with the Minkowski metric (modulo a conformal factor $-|a|^{-2}$). The conformal structure on Minkowski spacetime is thence $[\eta]$.

Let us now consider Axiom DPR. Given two particles P and Q defined by the curves $\gamma_P : \mathbb{R} \to M : s \mapsto a^\mu s + x_0^\mu$ and $\gamma_Q : \mathbb{R} \to M : s \mapsto b^\mu s + y_0^\mu$ we have to define the message map. When the point $x_0 \in P$ sends a message to Q, it hits a point Q. There is no loss of generality assuming that such a point is y_0. In this way the spacetime vector $x_0 - y_0$ is light-like.

Given a generic point $p = x_0 + \delta x_0 = a\delta s + x_0 \in P$ and a generic point $q = y_0 + \delta y_0 = b\delta s' + y_0 \in Q$, then the points (p, q) are linked by a light ray iff the separation $|p - q|^2$ is light-like, i.e. iff $|as - bs' + x_0 - y_0|^2 = 0$. This leads, at first order, to

$$2(a\delta s - b\delta s') \cdot (x_0 - y_0) = 0 \tag{6.11.19}$$

This equation is fantastic! Imagine you have two identical clocks.
Two identical clocks are, e.g., two particles as P and Q above, parameterised as above with $|a|^2 = |b|^2$. In this way, the parameter flows at the same rate along P and Q; if their worldlines are parallel, one obtains $\delta s' = \delta s$. In general, equation (6.11.19) reads as

$$\delta s' = \frac{a \cdot (x_0 - y_0)}{b \cdot (x_0 - y_0)} \delta s \tag{6.11.20}$$

Now let us simplify the situation, assuming that $a = (1, 0, \dots, 0)$ lies along the x^0-axis and $b = \left(\sqrt{1 + |\vec{b}|^2}, \vec{b}\right)$ any other time-like unit vector. Since $x_0 - y_0$ is light-like, it is in the form $x_0 - y_0 = \left(|\vec{d}|, \vec{d}\right)$. In this situation, one gets

$$\delta s' = \frac{-|\vec{d}|}{-\sqrt{1 + |\vec{b}|^2}|\vec{d}| + \vec{b} \cdot \vec{d}} \delta s = \frac{1}{\sqrt{1 + |\vec{b}|^2} - \frac{\vec{b} \cdot \vec{d}}{|\vec{d}|}} \delta s \tag{6.11.21}$$

The relative velocity between P and Q is given by $v^i = \frac{b^i}{\sqrt{1 + |\vec{b}|^2}}$ so that

$$|\vec{v}|^2 = \frac{|\vec{b}|^2}{1 + |\vec{b}|^2} \qquad \Rightarrow |\vec{b}|^2 = \frac{|\vec{v}|^2}{1 - |\vec{v}|^2} \qquad \Rightarrow |\vec{b}| = |\vec{v}| \frac{1}{\sqrt{1 - |\vec{v}|^2}} =: \gamma |\vec{v}| \tag{6.11.22}$$

By substitution above, we have $(1 + |\vec{b}|^2 = \frac{1}{1 - |\vec{v}|^2} = \gamma^2$ and $b^i = \gamma v^i)$

$$ds' = \frac{1}{\gamma} \frac{1}{1 - v^i \frac{d_i}{|\vec{d}|}} ds \tag{6.11.23}$$

The quantity $\vec{v} \cdot \frac{\vec{d}}{|\vec{d}|}$ is the relative radial velocity of P and Q. Suppose, for example, that $\vec{v}$ is parallel to the direction $\frac{\vec{d}}{|\vec{d}|}$, namely that $\vec{v} = v \frac{\vec{d}}{|\vec{d}|}$ (and v is now the scalar velocity) then we have

$$\delta s' = \frac{1}{\gamma} \frac{1}{1 - v} \delta s \tag{6.11.24}$$

This formula tells that, if P and Q are two identical clocks, moving at constant velocity, one of them judges the rate of the other clock modified of a quantity depending on the relative radial velocity. One could use such a relation to give an operational definition of relative radial velocity.

One also has to stress that this is not the relativistic time dilatation; In fact, the factor can be greater or smaller that 1 depending on the orientation of $\vec{v}$ with respect to $\vec{d}$. This is related to the relativistic Doppler effect. This is the *apparent* time rate modification of a moving clock and it has also a Newtonian counterpart. The rate modification is in part due to time dilatation, but it is also due to the longer/shorter path a wavefront has to go as time passes by, due to the fact the the points are moving apart/closer. In other words, the Doppler effect refers to the time rate an observer sees, while time dilatation refers to the residual effect an observer infers once the light flying time is considered and purged.

From equation (6.11.19), one easily obtain

$$\frac{\delta s'}{\delta s} = \frac{a \cdot (x_0 - y_0)}{b \cdot (x_0 - y_0)} \tag{6.11.25}$$

For this to be zero, one should have $a \cdot (x_0 - y_0) = 0$ which is impossible since a is time-like and $x_0 - y_0 \neq 0$ is light-like. Hence when the point P moves along P its message along Q do in fact move and the message map $m : P \to Q$ is, in fact, a smooth map (Axiom DPR:a).

We can define also an echo of Q on P as the composition of two messages. Let us start from $x_0 \in P$ send a message to $y_0 \in Q$ and send it back to $x_1 \in P$. The particle P is parameterised based at x_0, i.e. $\gamma_P : \mathbb{R} \to M : s \mapsto a^\mu s + x_0^\mu$ while Q is parameterised based at y_0, i.e. $\gamma_Q : \mathbb{R} \to M : s \mapsto b^\mu s + y_0^\mu$. A point on P is connected to y_0 by a light ray iff

$$|as + x_0 - y_0|^2 = 0 \quad \Rightarrow |a|^2 s^2 + 2a \cdot (x_0 - y_0)s = 0 \tag{6.11.26}$$

which has two solutions, one is $s = 0$ (i.e. x_0), the other is $s_* = -2\frac{a \cdot (x_0 - y_0)}{|a|^2}$. The corresponding point $x_1 = -2a\frac{a \cdot (x_0 - y_0)}{|a|^2} + x_0 \in P$ is the echo of $x_0 \in P$ against Q on P.

If one starts from a point $a \, \delta s + x_0 \in P$, the message of it on Q is $b \, \delta s' + y_0 \in Q$, and the message back on P is $x_1 + a \, \delta s \in P$. Accordingly, the echo map is $\epsilon_Q : P \to P : s \mapsto s + s_*(a, x_0 - y_0)$ which is, of course, smooth and invertible. This proves that Axiom DPR:b holds true.

Axiom DPR:c is trivial since straight lines are smooth lines. For Axiom DPR:d a message from P to Q is with usual notation

$$m : P \to Q : x_0 + a \, \delta s \mapsto y_0 + b \frac{a \cdot (x_0 - y_0)}{b \cdot (x_0 - y_0)} \delta s \tag{6.11.27}$$

The direction of the light ray is

$$y_0 - x_0 + \left(b \frac{a \cdot (x_0 - y_0)}{b \cdot (x_0 - y_0)} - a \right) \delta s \tag{6.11.28}$$

which is, of course, smooth in the initial point (which is parameterised by δs).

The Axiom DST is related to parallax coordinates. In dimension $m = 2$, given two parallel particles P and Q, both at rest, one has $P = (s, x_0)$ and $Q = (s, y_0)$; any event $x = (t, x) \in M$ sends a message to P, which hits at

$$|(t - s, x - x_0)|^2 = 0 \quad \Rightarrow s^2 - 2ts + t^2 - (x - x_0)^2 = 0 \quad \Rightarrow s_\pm = t \pm |x - x_0| \quad \Rightarrow s_P := s_+ = t + |x - x_0| \qquad (6.11.29)$$

Analogously, for Q, it hits at $s_Q := t + |x - y_0|$. In other words, the point (t, x) corresponds, in parallax coordinates, to $(s_P, s_Q) = (t + |x - x_0|, t + |x - y_0|)$.

Since we are in dimension 2, we can draw the coordinate grid, looking for the points which correspond to $s_P = s_*$ or $s_Q = s_*$. If $x_0 < x_1$, the coordinates are good coordinates in the strip $x_0 < x < x_1$; in such an interval, they are smoothly compatible with Cartesian coordinates (and vice versa). Of course, one can form an atlas of parallax coordinates, each defined in a strip and covering the whole M.

If the two particles are not parallel, one can anyway fix a point on each particle and draw the light cone (which, in this case, is only the union of two lines) at each point. Then, moving the points fixed on the particles, the grid is recovered. Again they define good coordinates in the region between the two particles; again coordinates are compatible with Cartesian coordinates and then they are compatibile with the parallax atlas defined above.

In higher dimension, one draws light cones at a point which are hypersurfaces. The grid is hence formed by m hypersurfaces which generically intersect at points and, generically, define good coordinates in a region.

For Axiom CS, it is enough to notice that, in Lorentzian signature, the light cones are really *cones*, which thence disconnect inner (i.e. time-like) directions and outer (i.e. space-like) directions (Axiom CS:a); that the light cone itself disconnects into two nappes once the zero vector is removed (Axiom CS:b); and that each nappe contains a connected component of time-like vectors (Axiom CS:c).

Axiom P is trivial: at any point, there is a particle for any time-like direction.
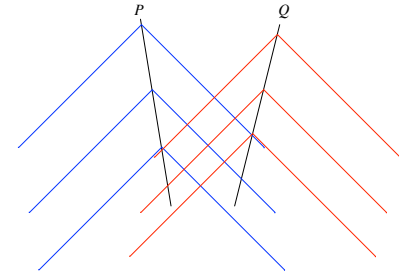
Fig 5.5: The parallax coordinate grid.
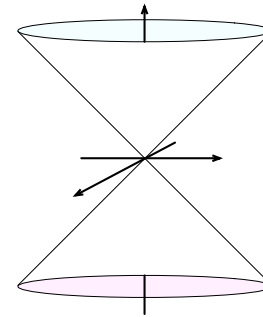
Fig 5.6: The light cone in Minkowski spacetime.

References

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Herbert Pfister ? Markus King, *Inertia and Gravitation. The Fundamental Nature and Structure of Space-Time* there is something like derivation of geodesic equations. ma lo fa nello spazio

spray for geometric definition of acceleration. Also here

Chapter 7. Extended theories of gravitation

So Gandalf, you try to lead them over Caradhras. And if that fails. Where then will
you go? If the mountain defeats you, will you risk the more dangerous road?

(*The Lord of the Rings*, Saruman)

1. Introduction

In Part I, we discussed the inputs from (general or gauge) covariance. That has little or nothing to do with gravity, in particular; it defines how to have a field theory in which we can provide an absolute (i.e. independent of the observers) description of the physical reality.

In Chapter 6, we described EPS framework, which provides a definition and axiomatic approach to gravitational physics, which describes the geometric properties of spacetime so that they are the mathematical counterpart of the physics of gravity. The very bottom content of any *gravitational theory* based on a relativistic framework is *exactly* to describe gravity as a manifestation of a non-trivial geometry of spacetime which is determined by the matter content which lives *on* the spacetime. Wheeler's quotation (see[19]) is still perfectly updated:

Spacetime tells matter how to move; matter tells spacetime how to curve.

Just one has to be somehow more flexible about what it is meant by *to curve*, i.e. by what it is referred as being the *geometry* of spacetime. To some surprise of the Ehlers, Pirani and Schild themselves, their framework ended up to indicate that geometry of spacetime is a Weyl frame, which is somehow more general than the usual metric Lorentzian structure, usually assumed in standard purely metric GR.

Although Weyl frames have been originally introduced as the attempt to unify gravity and electromagnetism (see[20][21]), the same structure appears in EPS framework with a completely different context.

In unified theories, the connection

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = \{g\}^{\mu}_{\sigma\rho} + \tfrac{1}{2}\left(g^{\mu\epsilon}g_{\rho\sigma} - 2\delta^{\mu}_{(\rho}\delta^{\epsilon}_{\sigma)}\right)\alpha_{\epsilon} \tag{7.1.1}$$

contains the information about gravity, which is encoded on g *only*, as in standard GR, together with the information about electromagnetic field, which is encoded into the 1-form $\mathbf{A} = \alpha_{\epsilon}\,dx^{\epsilon}$. This 1-form $\mathbf{A}$ defines the electromagnetic field $\mathbf{F} = d\mathbf{A}$ on spacetime (M, g).

This original unified theory was criticised (and eventually rejected) because of different reasons, one of which was that, if rulers are parallelly transported around a closed loop with respect to the connection $\tilde{\Gamma}$, their length with respect to g is not conserved, as it happens instead for parallel transport with respect to $\{g\}$.

In a Weyl frame $(M, g, \tilde{\Gamma})$, the parallel transport with respect to $\tilde{\Gamma}$ of a vector v does not preserve its g-length.

A vector $v(s)$, defined along a curve γ, is parallelly transported, if it satisfies the equation

$$\dot{v}^\lambda + \tilde{\Gamma}^\lambda_{\alpha\beta} \dot{\gamma}^\alpha v^\beta = 0 \tag{7.1.2}$$

while its g-length is $g(v, v) = g_{\mu\nu} v^\mu v^\nu$. Then one has

$$\begin{aligned}
\frac{d}{ds} g(v,v) &= \partial_\alpha g_{\mu\nu} \dot{\gamma}^\alpha v^\mu v^\nu + 2 g_{\mu\nu} \dot{v}^\mu v^\nu = \left(-\partial_\mu g_{\nu\alpha} + \partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} \right) \dot{\gamma}^\alpha v^\mu v^\nu - 2 g_{\mu\nu} \tilde{\Gamma}^\mu_{\alpha\beta} \dot{\gamma}^\alpha v^\beta v^\nu = \\
&= -2 g_{\mu\lambda} \left(\tilde{\Gamma}^\lambda_{\alpha\nu} - \{g\}^\lambda_{\nu\alpha} \right) \dot{\gamma}^\alpha v^\mu v^\nu = -g_{\mu\lambda} \left(g^{\lambda\epsilon} g_{\alpha\nu} - 2\delta^\lambda_{(\alpha} \delta^\epsilon_{\nu)} \right) \alpha_\epsilon \dot{\gamma}^\alpha v^\mu v^\nu = \\
&= \left(-\delta^\epsilon_\mu g_{\alpha\nu} + g_{\mu\alpha} \delta^\epsilon_\nu + g_{\mu\nu} \delta^\epsilon_\alpha \right) \alpha_\epsilon \dot{\gamma}^\alpha v^\mu v^\nu = \alpha_\alpha \dot{\gamma}^\alpha g(v,v)
\end{aligned} \tag{7.1.3}$$

The g-length of the vector v is conserved by parallel transport along a curve, if the vector is g-light-like (i.e. $g(v, v) = 0$), while in general it is not. The 1-form $\mathbf{A}$ measures exactly to which extent this violation occurs and, if one has $\mathbf{A} = 0$ (as in Riemannian geometries), then the g-length of vectors is preserved by parallel transport along the Levi Civita connection $\tilde{\Gamma} = \{g\}$. In particular, the g-length of vectors is not generally conserved in Weyl frames nor if one restricts to integrable Weyl frames.

Accordingly, let us consider two vectors, v and w, at a point $x \in M$ with the same g-length (i.e. $g(v, v) = g(w, w)$), which are then parallelly transported along two different curves γ_1 and γ_2, both going from x to another event $\tilde{x} \in M$ (so that $\gamma_1 \cdot \gamma_2^{-1}$ is a *closed loop*); let us denote by $\tilde{v}$ and $\tilde{w}$ the vectors, when they get to $\tilde{x}$. Of course, in a general Weyl frame, one has $g(\tilde{v}, \tilde{v}) \neq g(\tilde{w}, \tilde{w})$, since one has to integrate (7.1.3) along two different curves.

Equivalently, if one parallelly transports back to x along a loop a vector v, the parallelly transported vector $\hat{v}$ (which is tangent at x), in general, has a different g-length, i.e. $g(\hat{v}, \hat{v}) \neq g(v, v)$.

If $\mathbf{A}$ is exact, i.e. if the Weyl frame is integrable and one has $\mathbf{A} = d\ln\varphi$, then, by integration along an open path from x to $\tilde{x}$, one gets

$$\ln(g(\tilde{v}, \tilde{v})) - \ln(g(v, v)) = \ln(\varphi(\tilde{x})) - \ln(\varphi(x)) \quad \Rightarrow g(\tilde{v}, \tilde{v}) = \frac{\varphi(\tilde{x})}{\varphi(x)} g(v, v) \tag{7.1.4}$$

Thus, considering two vectors of the same g-length at x, parallelly transported along two different open paths to $\tilde{x}$, we have

$$g(\tilde{v}, \tilde{v}) = \frac{\varphi(\tilde{x})}{\varphi(x)} g(v, v) = \frac{\varphi(\tilde{x})}{\varphi(x)} g(w, w) = g(\tilde{w}, \tilde{w}) \qquad \Rightarrow g(\tilde{v}, \tilde{v}) = g(\tilde{w}, \tilde{w}) \tag{7.1.5}$$

even though in general $g(\tilde{v}, \tilde{v}) \neq g(v, v)$. Equivalently, by parallelly transporting along a closed loop we have

$$g(\hat{v}, \hat{v}) = \frac{\varphi(x)}{\varphi(x)} g(v, v) = g(v, v) \tag{7.1.6}$$

and the g-length at x does not depend on the loop.

And in a unified theory, that was checkmate, since one cannot restrict to the case of exact potentials $\mathbf{A} = d\ln(\varphi)$, because this would imply $\mathbf{F} = 0$, which would be a really poor theory of electromagnetic field!

If one could somehow restrict to $\mathbf{A} = d\ln(\varphi)$, the g-length of vectors would still change when parallelly transported around but, at least, their length would uniquely depend on the position, not on the path to get there.

In fact, in an integrable Weyl frame, one can define a new metric $\tilde{g} = \varphi \cdot g$ and show that $\tilde{\Gamma} = \{\tilde{g}\}$. Thus the g-length of a vector v would not be preserved by $\tilde{\Gamma}$-parallel transport, while the $\tilde{g}$-length would, as it happens in a Riemannian geometry $(M, \tilde{g})$.

> In other words, integrable Weyl frames correspond to relax the comparison between length of vectors at different points, though keeping a notion of equal length at the same point.
>
> Now, this is often sometimes expressed in terms of length of rulers, though, to be precise, here most of the content is in the details: rulers (if they can be defined at all in a relativistic theory) are not tangent vectors. Does the physical length of rulers coincide their geometric length? And the length computed by g or $\tilde{g}$?
>
> As it often happens, shifting a mathematical claim into a physical one does rely on a number of assumptions about the physical meaning of the mathematical objects involved, which, if clear within a well defined model, becomes at least questionable in an undefined context.

In EPS, however, there is no physics directly associated to $\mathbf{F} = d\mathbf{A}$, so that one could also consider restricting to an exact 1-form $\mathbf{A}$. The corresponding Weyl frames are called *integrable*. In integrable Weyl frames, one still has the possibility of comparing lengths at a point, though loosing something about comparing them at a distance.

Of course, in this very same setting, one could also argue that the physical length could be described by $\tilde{g}$ rather than by g. If this were the case, one would define length and parallel transport with one single metric, namely $\tilde{g}$, and the situation would not be very different from standard GR (at least at a kinematical level, while the dynamics will be discussed below). In this case, the integrable Weyl frame is called a *Lorentzian geometry* $(M, \tilde{g})$, since the original metric g has no role any longer.

To summarise, EPS framework gives a precise context to describe geometry of spacetime, more general than standard Lorentzian geometries. Some freedom is left though it contains the standard Lorentzian case as a special case. This description of geometry on spacetimes comes with a definite relation to potentially observable phenomena; the connection $\tilde{\Gamma}$ is, by construction, the connection which describes the free fall of particles.

The metric g is a representative of the conformal structure, which describes light cones and wavefronts (as any other representative of the conformal structure does). However, also the choice of g can be related to some physical quantity; of course, we can decide we want to pursue a description of the physical reality in which distances and clocks play no fundamental role (them being rather conventions of the observer), choosing accordingly a dynamics which is compatible with conformal structures in which Weyl conformal transformations are symmetries.

> In this setting distances would not be gauge invariant and, accordingly, they, together with time lapses, would not play a fundamental role. I still do not know if this option would be compatible with what we know about the physical world around us.
>
> On one hand, we often appear to be fond in measuring the length of tables or the distance of the Moon from the Earth and we strongly believe that the first is shorter than the second. On the other hand, what we call *length* is often (if not always) the ratio of two distances, which as such *is* conformally invariant.

If, instead, we decide that distances mean something, we can find an observable meaning for distances and clocks (from beyond EPS framework, for example, from quantum mechanics). In this case, one can argue that there is a one-to-one correspondence between the choices of conformal representatives $g \in [g]$ and the protocols for distances and time lapses. If the physical length of a table has to be given a meaning, whatever meaning and protocol is used as an operational definition, the physical length cannot agree with the geometric distance defined by g for more than one choice of the conformal representative $g \in [g]$.

Accordingly, we can say that either we describe a world without distances, or we can use distances (however defined) to single out a conformal representative $g \in [g]$, which, by definition, renders physical distances as geometric g-lengths. In the second case, let us stress it once again since we shall use it below, the geometry of spacetime is described by a pair $(g, \tilde{\Gamma})$ in which, by definition, g describes some physical protocol to define distances, and $\tilde{\Gamma}$ describes free fall of particles.

If the Weyl frame $(M, g, \tilde{\Gamma})$ is integrable, it can be described in terms of two (conformal) metrics $(M, g, \tilde{g})$ such that distances are still defined by g and free fall is described by $\tilde{g}$-geodesics which happen to be g-time-like (and, since the two metrics are conformal, also $\tilde{g}$-time-like).

Let us consider a metric g and a connection $\tilde{\Gamma}$ which is EPS compatible with g and metric for another metric $\hat{g}$, hence

$$\tilde{\Gamma}^\mu_{\sigma\rho} = \{\hat{g}\}^\mu_{\sigma\rho} = \{g\}^\mu_{\sigma\rho} + \tfrac{1}{2}\left(g^{\mu\epsilon}g_{\rho\sigma} - 2\delta^\mu_{(\rho}\delta^\epsilon_{\sigma)}\right)\alpha_\epsilon = \{g\}^\mu_{\sigma\rho} + K^\mu_{\sigma\rho} \tag{7.1.7}$$

The Riemann tensors are

$$\hat{R}^\mu{}_{\sigma\nu\rho} = R^\mu{}_{\sigma\nu\rho} + \nabla_\nu K^\mu_{\sigma\rho} - \nabla_\rho K^\mu_{\sigma\nu} + K^\mu_{\lambda\nu}K^\lambda_{\sigma\rho} - K^\mu_{\lambda\rho}K^\lambda_{\sigma\nu} \tag{7.1.8}$$

and their first trace must vanish since they are Riemann tensors of a metric, i.e.

$$\hat{R}^\mu{}_{\mu\nu\rho} = R^\mu{}_{\mu\nu\rho} + \nabla_\nu K^\mu_{\mu\rho} - \nabla_\rho K^\mu_{\mu\nu} + {\color{red}K^\mu_{\lambda\nu}K^\lambda_{\mu\rho} - K^\mu_{\lambda\rho}K^\lambda_{\mu\nu}} \quad \Rightarrow \nabla_\nu K_\rho - \nabla_\rho K_\nu = 0 \tag{7.1.9}$$

where we set $K_\rho := K^\mu_{\mu\rho} = \tfrac{1}{2}\left(g^{\mu\epsilon}g_{\rho\mu} - 2\delta^\mu_{(\rho}\delta^\epsilon_{\mu)}\right)\alpha_\epsilon = -\tfrac{m}{2}\alpha_\rho$. The condition above is hence

$$\nabla_\nu \alpha_\rho - \nabla_\rho \alpha_\nu = d_\nu \alpha_\rho - d_\rho \alpha_\nu = 0 \quad \Longleftrightarrow \quad d\mathbf{A} = 0 \tag{7.1.10}$$

Then the 1-form $\mathbf{A}$ is closed, hence locally exact. Let us denote by ω a local potential so that one has $\alpha_\epsilon = d_\epsilon \omega$ and

$$\tilde{\Gamma}^\mu_{\rho\sigma} = \{\hat{g}\}^\mu_{\sigma\rho} = \{g\}^\mu_{\sigma\rho} + \tfrac{1}{2}\left(g^{\mu\epsilon}g_{\rho\sigma} - 2\delta^\mu_{(\rho}\delta^\epsilon_{\sigma)}\right)d_\epsilon\omega = \{e^{-\omega}g\}^\mu_{\sigma\rho} \tag{7.1.11}$$

Hence there exists a metric $\tilde{g} = e^{-\omega}g$, conformal to g and such that $\tilde{\Gamma} = \{\tilde{g}\}$.

Let us remark that one cannot prove that $\hat{g}$ is conformal to g. For a counterexample, just consider $M = \mathbb{R}^m$ and two constant metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ which are not conformal to each other. Then one has $\{g\} = \{\hat{g}\} = 0$ and $\tilde{\Gamma} = \{\hat{g}\}$ is manifestly metric and EPS compatible with g (just set $\alpha_\epsilon = 0$), while $\hat{g}$ is not conformal to g. Thus the claim:

    if $\tilde{\Gamma} = \{\tilde{g}\}$ is EPS compatible to g then $\tilde{g} \in [g]$

is false, while the claim:

    if $\tilde{\Gamma}$ is metric and EPS compatible to g then there exists a metric $\tilde{g} \in [g]$ such that $\tilde{\Gamma} = \{\tilde{g}\}$

is true.

Being g and $\tilde{g}$ conformal, it means that one has $\tilde{g} = \varphi g$ for some (positive) scalar field φ, called the *conformal factor*, which, in fact, describes the mismatch between distances and free fall.

In standard GR, one *assumes* that such a mismatch is not physically there, i.e. $\varphi \equiv 1$ everywhere, which is a legitimate assumption, still an assumption that we check at human scales (up to, say, the solar system scale) and then we extrapolate it to any scale from Planck scales to cosmological scales (i.e. from $10^{-35}\, m$ to $10^{26}\, m$, i.e. along about 60 orders of magnitude). (That means to believe!)

Finally, if these two metrics g and $\tilde{g}$ coincide, i.e. the conformal factor is trivial, we have a Lorentzian geometry, which is the assumption in standard GR.

To be explicit, let us stress that there is no logical argument which can show that the standard assumption is physically wrong. I am not trying to imply that since the assumption is unmotivated, then it is illegitimate or physically wrong; only observations can decide on that. What I will rather argue is that, by building up a framework in which the assumption is not taken, makes it possible to better test the issue and decide it based on observations.

That is particularly true in gravitational theories in which there is a considerably tight link between the mathematical model and the observational data which is not always easy to disentangle. This is related to the fact that a gravitational theory is *also* a relativistic theory in which (and only in which) one can discuss and introduce physical observables which are independent of the observers. As a consequence, it is particularly tricky to design experiments to measure physical quantities and, at the same time, to keep track of how they depend on the assumptions done in the model.

To be honest, we try to be as conservative as we can and we appreciate EPS approach because it constrains modifications to standard GR more than because it allows them. By the way, it is precisely for this reason that we chose to call our theories of gravitation, *extended*, instead of what it is more common in the literature, i.e. *modified*. To be precise we are not *modifying* standard GR. We are instead providing a wider framework which *contains* standard GR as a specific (degenerate) particular case.

The reason to do that is that standard GR comes from many successes and, at least, it deserves to be treated in a conservative way.

Of course, the different possibilities left open by EPS approach are equivalent on a kinematical level and, in order to discuss them, one needs to introduce dynamics. However, we want to preserve the kinematical meaning of the objects which describe the geometry of spacetime, being them $(g, \tilde{\Gamma})$, $(g, \tilde{g})$, (g, φ), or whatever representation one chooses, instead of simply considering a generic dynamics for a metric and a connection. Let us first argue that if we did that, we would overcount for the freedom we have in choosing dynamics by considering every single dynamics (which, remember, according to our discussion of general covariance, is an *intrinsic* variational principle) infinitely many times by considering it in all possible field coordinates one could choose.

This is not only useless (since we set up geometric methods to discuss intrinsic objects exactly to show that different local representations are irrelevant) but doing it we would also lose track of the relation that objects have with observables, which is a free token we had from EPS.

EPS compatibility

If EPS framework points toward a Palatini formalism based on fields $(g, \tilde{\Gamma})$ with specific physical meaning (g being associated to the physical definition of atomic time, $\tilde{\Gamma}$ being associated to free fall of particles), we need first to remark that this is not purely mathematics.

First of all, we have to remember that, as far as we know, Palatini theories are not, generally, dynamically equivalent to their purely metric counter parts, as it happens for standard GR. Thus, choosing a Palatini framework rather than a purely metric formalism, does not come without consequences, also classical consequences.

Discussing quantum gravity is particularly slippery, since, at the very least, whatever can be said depends on the particular framework one chooses for quantum gravity. Alternatively, one can use general concepts from quantum physics, which probably apply to quantum gravity as well, though the discussion is left somehow vague. That said, let me present some of these arguments anyway.

Whatever quantum gravity will eventually be, it seems that quantum fields (thus including the geometric fields on spacetime) can deviate from their classical counterparts. I know that this is not clearly stated, but *if* the classical limit has a sense, *if* we learn something from the path integral storytelling, classical fields emerge as a mean value of the quantum fields which oscillate around the value prescribed by the corresponding classical field theory. These oscillations may be produced by the mathematical framework or be real.

Anyway, one should expect the classical kinematics to constrain which quantum oscillations are expected. Thus a purely metric formalism, not only forces the connection and the free fall to be a Levi Civita free fall, but the quantum oscillations of the connection, whatever they might be, are determined by the quantum oscillations of the metric field, which is the only fundamental field. On the other hand, in Palatini formalism, we allow the metric field and the connection to have independent quantum oscillations which agree on average, only.

The EPS framework contains a constraint for the connection given by (7.1.1). Before discussing dynamics, one should discuss how to regard this constraint. This constraint, as the whole EPS framework, is expressed for the classical fields which describe the geometry on spacetime. It is hence written for solutions of field equations.

Accordingly, one should take a position about where EPS compatibility comes in, from a variational viewpoint. It is too relevant to regard it as a coincidence of the particular solution which is realised in the real physical spacetime. One does need to have a general basis which guarantees that *all* solutions obey this constraint.

One has two possibilities: either field equations guarantee EPS compatibility on the space of solutions (and hence on solutions only) or it is imposed at kinematical level (and accordingly, all configurations obey to it, not only solutions).

> Of course, if all configurations are EPS compatible, i.e. if EPS compatibility is imposed at kinematical level, then EPS compatibility has to be regarded as exactly preserved by quantum oscillations. On the other hand, if EPS compatibility is enforced by field equations, then quantum oscillation can violate it and it emerges only on average.

For imposing EPS compatibility at kinematical level, one should consider as fundamental fields (g, α) (or, somehow equivalently, (g, φ)). Once field equations find a solution (g, α), then the free fall connection is *defined*, as a derived object, by (7.1.1). Now, even not considering how strange this mechanism would appear, let us notice that, in this case, one is expecting a constraint on quantum oscillations (forcing them to preserve EPS compatibility) which comes from EPS formalism which is a purely classical argument. Not that this needs to be impossible or manifestly wrong, just it sounds strange, also because none of the EPS axioms are expected to hold in a quantum regime.

Instead, the Palatini bet is on $(g, \tilde{\Gamma})$. Once one is clear that all these alternatives should be explored independently, then let me declare what is the option I choose to consider and that we shall call the *extended theories of gravitation*.

2. Extended theories of gravitation

Let us define an *extended theory of gravitation* (ETG) as a field theory for a metric g, a (torsionless) connection $\tilde{\Gamma}$, and some set of matter fields ϕ in which the dynamics, through field equations, implies that, along solutions, the connection $\tilde{\Gamma}$ is EPS-compatible with the metric g. If field equations also imply that the connection $\tilde{\Gamma}$ is metric with respect to $\tilde{g}$ which turns out to be necessarily conformal to g, then the extended theory of gravitation is said to be an *integrable ETG* (iETG).

> Of course, standard GR, in its Palatini formulation, is an integrable extended theory of gravitation.
>
> Depending on matter couplings these theories can be ETG or not. If the matter Lagrangian does not depend on the connection (Palatini formalism) and matter just couples to the metric field (possibly also through its Christoffel symbols), then field equations do imply that the connection is the Levi Civita connection of the metric and one has an iETG.
>
> On the other hand, if matter couples to the connection (metric-affine formalism), one can violate EPS compatibility and produce examples of field theories which are not iETG.

In an iETG the connection is the Levi Civita connection of a metric $\tilde{g}$, then $\tilde{g}$ needs to be conformal to the original metric g. Accordingly, one has $\tilde{g} = \varphi g$ for some (positive) scalar field $\varphi = e^{-\omega}$ which is called the *conformal factor*.

Standard GR as an iETG

In standard Palatini GR, one defines a dynamics by the Hilbert Lagrangian in Palatini formulation

$$\boldsymbol{L} = \tfrac{\sqrt{g}}{2\kappa}\mathcal{R}\, d\sigma + \boldsymbol{L}_m(g, \phi) \tag{7.2.1}$$

where we set $\mathcal{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}(\tilde{\Gamma})$ and ϕ are matter fields which couple with the metric (and not with the connection).

By variation of the Lagrangian, one obtains field equations

$$\begin{cases} \tilde{R}_{(\mu\nu)} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa T_{\mu\nu} \\ \tilde{\nabla}_\mu(\sqrt{g}g^{\alpha\beta}) = 0 \end{cases} \tag{7.2.2}$$

together with some matter field equation $\mathbb{E} = 0$.

The second field equation implies that $\tilde{\Gamma} = \{g\}$ and the first field equation then becomes Einstein equation for the metric g. Accordingly, the metric g defines the free fall, the distances, the coupling with matter. In other words, standard GR is a iETG in which there is no conformal factor (i.e. a single metric g) and distances are defined by the same metric which determines the free fall.

Palatini $f(\mathcal{R})$-theories

Of course, standard GR is not the only iETG. There is a whole class of dynamics which define iETG.

Let us consider a connected paracompact spacetime manifold M of dimension $\dim(M) = m > 2$ and which allows global Lorentzian metrics. Let us consider a metric $g_{\mu\nu}$ and a torsionless connection $\tilde{\Gamma}^\alpha_{\beta\mu}$ as fundamental fields. We shall hereafter consider a Lagrangian in the form

$$\boldsymbol{L} = \left[\tfrac{\sqrt{g}}{2\kappa}f(\mathcal{R}) + L_m(g, \phi)\right] d\sigma \tag{7.2.3}$$

where ϕ is a set of matter fields and f is a generic (regular, to be defined precisely) function.

To be precise by a *regular* function we assume $f : \mathbb{R} \to \mathbb{R}$ to be defined at least in some open set $U \subset \mathbb{R}$, to be differentiable almost everywhere in U, such that

$$F(\mathcal{R}) := f'(\mathcal{R})\mathcal{R} - \tfrac{m}{2}f(\mathcal{R}) \tag{7.2.4}$$

has a discrete set of zeros, and that those zeros are simple so that one can locally solve the equations $F(\mathcal{R}) = T$ for $\mathcal{R}$.

Notice that matter couples directly to the metric only, not to the connection. When this is the case, we call this theory a *Palatini $f(\mathcal{R})$-theory*.

If matter is allowed to couple to the connection, the theory is called a *metric-affine $f(\mathcal{R})$-theory*. We shall consider some metric-affine $f(\mathcal{R})$-theories below, while here we focus on Palatini $f(\mathcal{R})$-theory.

By variation of the Lagrangian, one obtains field equations

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} \\ \tilde{\nabla}_\mu(\sqrt{g}f'(\mathcal{R})g^{\alpha\beta}) = 0 \end{cases} \tag{7.2.5}$$

together with some matter field equation $\mathbb{E} = 0$.

In order to solve the second equation, one can introduce a new conformal metric

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \qquad \varphi := (f'(\mathcal{R}))^{\frac{2}{m-2}} \tag{7.2.6}$$

so that one has $\sqrt{\tilde{g}} = \varphi^{\frac{m}{2}}\sqrt{g}$ and $\varphi\tilde{g}^{\mu\nu} = g^{\mu\nu}$ and, consequently

$$\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta} = \varphi^{\frac{m-2}{2}}\sqrt{g}g^{\mu\nu} = \sqrt{g}f'(\mathcal{R})g^{\alpha\beta} \tag{7.2.7}$$

Hence, the second field equation can be recast and solved as

$$\tilde{\nabla}_\mu(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}) = 0 \qquad \Rightarrow \tilde{\Gamma}^\alpha_{\beta\mu} = \{\tilde{g}\}^\alpha_{\beta\mu} \tag{7.2.8}$$

Then the connection can be replaced in the first field equation to obtain an equation for the metric (either g or $\tilde{g}$), the conformal factor φ, and matter fields. By tracing it by $g^{\mu\nu}$, one obtains the condition

$$f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) = \kappa T \tag{7.2.9}$$

where we set $T := g^{\mu\nu}T_{\mu\nu}$. This condition is called the *master equation* and it must be identically satisfied along solutions.

The master equation is an algebraic (i.e. not differential) equation for $\mathcal{R}$ and T and, if $f(\mathcal{R})$ is regular, then it can be solved for $\mathcal{R}$ to obtain $\mathcal{R} = \mathcal{R}(T)$. Accordingly, the curvature $\mathcal{R}$ (as well as all the functions of it, such as $f(\mathcal{R})$, $f'(\mathcal{R})$, φ) can be either considered as functions of the gravitational field or functions of the matter fields through T.

There is also another function one can obtain from $f(\mathcal{R})$ and which will turn out to be important; it is obtained by solving the definition of the conformal factor for the curvature, i.e.

$$\varphi := (f'(\mathcal{R}))^{\frac{2}{m-2}} \qquad \Rightarrow \mathcal{R} = r(\varphi) \tag{7.2.10}$$

In view of this definition, the curvature (as well as any function of it) can be considered as a function of the conformal factor.

The function $r(\varphi)$ and the function $\mathcal{R}(T)$ defined above are somehow connected together. In fact, in view of the master equation, one can express the trace T as a function of the conformal factor

$$T = \kappa^{-1}\left(f'(r(\varphi))r(\varphi) - \frac{m}{2}f(r\varphi)\right) =: T(\varphi) \tag{7.2.11}$$

so that one has

$$r(\varphi) = \mathcal{R}(T(\varphi)) \tag{7.2.12}$$

On the other hand, one has

$$\varphi(T) := (f'(\mathcal{R}(T)))^{\frac{2}{m-2}} \qquad \Rightarrow \mathcal{R}(T) = r(\varphi(T)) \tag{7.2.13}$$

Since the two functions are somehow equivalent, one could use just one of them.

The real story is that one has three quantities $\mathcal{R}$, T, φ which, in view of master equation, are functionally dependent and one can use the master equation to eliminate one of them and write everything in terms of the other two quantities. However, which two quantities are to be used depends on the problem.

No one will miss the analogy with equation of state for gasses.

Let us set $\tilde{R}_{\mu\nu} := \tilde{R}_{\mu\nu}(\tilde{g}) := \tilde{R}_{\mu\nu}(\{\tilde{g}\})$ for the Ricci tensor of the conformal metric $\tilde{g}$. Then the first field equation can be recast as

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \kappa \left[\varphi^{-\frac{m-2}{2}} \left(T_{\mu\nu} + \tfrac{1}{2\kappa}f(\mathcal{R})g_{\mu\nu} \right) - \tfrac{1}{2\kappa}\mathcal{R}g_{\mu\nu} \right] =: \kappa \tilde{T}_{\mu\nu} \tag{7.2.14}$$

Accordingly, the conformal metric $\tilde{g}$ obeys standard Einstein equations, though with a modified effective source tensor

$$\tilde{T}_{\mu\nu} := \frac{1}{f'(\mathcal{R})} \left(T_{\mu\nu} - \tfrac{1}{2\kappa} \left(f'(\mathcal{R})\mathcal{R} - f(\mathcal{R}) \right) g_{\mu\nu} \right) \tag{7.2.15}$$

Roughly speaking, we shall consider $T_{\mu\nu}$ as the energy momentum tensor of *ordinary matter*, *real* matter. Let us then consider the tensor

$$D_{\mu\nu} := \tilde{T}_{\mu\nu} - T_{\mu\nu} \tag{7.2.16}$$

to be the energy-momentum tensor of *dark sources*.

If we set $f(\mathcal{R}) = \mathcal{R}$, then the conformal factor is identically $\varphi = 1$, so that $\tilde{g} = g$ and one has only one metric, and $\tilde{T}_{\mu\nu} = T_{\mu\nu}$ and one has no dark sources. In other words, standard GR is a particular case of Palatini $f(\mathcal{R})$-theories, for the choice $f(\mathcal{R}) = \mathcal{R}$.

Since we know how Ricci tensor and Ricci scalar transform under conformal transformations (see (2.7.151), (2.7.152), respectively), we can transform it into Einstein equations for the metric g:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \kappa\tilde{T}_{\mu\nu} + \tfrac{m-2}{2\varphi} \left[\nabla_{\mu\nu}\varphi - \tfrac{3}{2\varphi}\nabla_{\mu}\varphi\nabla_{\nu}\varphi - \left(\Box\varphi + \tfrac{m-7}{4\varphi}\nabla_{\rho}\varphi\nabla^{\rho}\varphi \right) g_{\mu\nu} \right] =: \kappa\hat{T}_{\mu\nu} \tag{7.2.17}$$

Hence also g obeys some standard Einstein equations, with a modified tensor, namely $\hat{T}_{\mu\nu}$.

Traditionally, one of the arguments used in discussing whether g or $\tilde{g}$ is the *"physical' metric"* is to argue that $\tilde{g}$ obeys field equations which are in the form of Einstein equations. For that reason, selecting $\tilde{g}$ as a physical metric is sometimes called the *Einstein frame*, meaning the Einstein Weyl frame. On the other hand, using g as a physical metric is sometimes called the *Jordan frame*.

Of course, as we see, the argument is flat wrong. Both the equations (as well as any other equation depending on Ricci tensor) can be recast in the form of Einstein equations. That is not the difference, which has to be looked for somewhere else.

The message here is that how much a theory resembles standard GR is something to be decided at the level of action, not at the level of field equations.

Let us stress that one can write $\mathcal{R}(\varphi)$ so that all extra contributions to energy momentum tensors, both to $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$, are from interactions with matter and conformal factor (or self-interactions of the conformal factor itself). Actually, to be precise, all extra contributions vanish for any constant conformal factor. Accordingly, we could argue that the interacting field is the variation of the conformal factor, namely $\delta\varphi$, and all contributions to dark sources are from the variation of the conformal factor.

Beside Einstein and Jordan frame, we shall also introduce the Helmholtz and the Brans–Dicke (Weyl) frames in due time. Before coping with different Weyl frames in general, we need first to rise and address some issues to set the whole theory on firmer ground.

Conservation of energy-momentum tensors

We have introduced three energy-momentum tensors in any Palatini $f(\mathcal{R})$-theory, namely $T_{\mu\nu}$, $\tilde{T}_{\mu\nu}$, and $\hat{T}_{\mu\nu}$. The last two energy-momentum tensors, $\tilde{T}_{\mu\nu}$ and $T^*_{\mu\nu}$, appear in Einstein-like equations. Both the l.h.s. $\tilde{G}_{\mu\nu} := \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}$ and $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, respectively, are very standard, metric, Einstein tensors, for $\tilde{g}$ and g, respectively, for which Bianchi identity holds true. Then both must be conserved, precisely

$$
\begin{aligned}
\tilde{\nabla}_\mu \tilde{G}^{\mu\nu} = 0 &\qquad \Rightarrow \quad \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \\
\nabla_\mu G^{\mu\nu} = 0 &\qquad \Rightarrow \quad \nabla_\mu \hat{T}^{\mu\nu} = 0
\end{aligned}
\tag{7.2.18}
$$

where the tilde refers to covariant derivative with respect to $\tilde{g}$ and the indices are raised by $\tilde{g}$ in the top line, while on the bottom line the indices are raised by the metric g.

The first energy-momentum tensor $T_{\mu\nu}$ is the ordinary stress tensor of matter Lagrangian, i.e. its variation with respect to the metric g. It does not obey an Einstein-like equation. It obeys field equations in the form (7.2.5), instead. Since the l.h.s. of this equation is not an Einstein tensor, Bianchi identities do not hold true. Accordingly, the same evidence of $T_{\mu\nu}$ being conserved does not hold.

Of course, this does not mean that $T_{\mu\nu}$ is not conserved, with respect to g or $\tilde{g}$, by some other reason. If one wants to interpret $T_{\mu\nu}$ as the energy momentum tensor of visible matter one needs some sort of conservation, meaning that if dark sources are not real fundamental fields, then visible matter cannot exchange energy with them. At most, one can argue that the interaction of matter with dark sources is some sort of gravitational interaction, which actually at solar system scales (as well as when standard GR holds well) is negligible also with respect to ordinary gravitational interaction. However, here the point is that gravity itself is negligible at fundamental level. That is exactly one of the reasons why we have problems with quantum gravity: there is no experimental regime in which we can test quantum gravity and GR at the same time.

Anyway, we can show that $T_{\mu\nu}$ is, in fact, conserved, namely that $\nabla_\mu T^{\mu\nu} = 0$ (with indices raised by g) as a consequence of covariance of the matter Lagrangian.

Let us start from a trivial observation that, even when the total Lagrangian is assumed to split into a purely gravitational part and a matter Lagrangian (which contains matter as well its interaction with gravity), both parts are *separately* generally covariant. In this case, it can be easily and quite generally shown that generalised Bianchi identities for the matter Lagrangian, which hold true off shell because of covariance, when evaluated on shell, produce conservation of energy-momentum tensor of matter, just because the partial work current does not vanish on shell alone, though the total work current, computed together with the gravitational contribution, does.

Let us start by considering a theory for a matter field which is a vector field on spacetime. We shall later generalise to an arbitrary first order geometric object. Accordingly, we can consider the Lagrangian

$$
\boldsymbol{L}_{tot} = \boldsymbol{L}_g(g, \tilde{\Gamma}) + \boldsymbol{L}_m(g, v)
\tag{7.2.19}
$$

Both the Lagrangian $\boldsymbol{L}_g$ and $\boldsymbol{L}_m$ are separately covariant, so that they both obey their covariance identity, both define their Noether currents and their work currents, as well as their superpotentials and reduced currents. Field equations are associated to the total work current, while the two partial work currents do not vanish

separately, due to interaction between gravity and matter. Thus, while the two Noether currents do obey Noether theorem, just the total Noether current is conserved on shell, while the two partial Noether currents separately are not.

The matter Lagrangian has the following local form

$$\boldsymbol{L}_m(g,v) = L_m(g^{\mu\nu}, v^\mu, v^\mu_\lambda)\, d\sigma \tag{7.2.20}$$

where $v^\mu_\lambda = d_\lambda v^\mu + \{g\}^\mu_{\sigma\lambda} v^\sigma$ denotes the covariant derivative of v^μ with respect to g.

By using the standard notation for momenta, the variation of the matter Lagrangian is

$$\delta L_m = p_{\alpha\beta}\delta g^{\alpha\beta} + p_\mu \delta v^\mu + p^\lambda_\mu \delta v^\mu_\lambda = -\tfrac{\sqrt{g}}{2} T_{\alpha\beta}\delta g^{\alpha\beta} + \mathbb{E}_\mu \delta v^\mu + \nabla_\lambda \left(p^\lambda_\mu \delta v^\mu - B^{\;\lambda}_{\alpha\beta}\delta g^{\alpha\beta} \right) \tag{7.2.21}$$

where we set $\mathbb{E}_\mu = p_\mu - \nabla_\lambda p^\lambda_\mu$ for field equations for the matter field, $T_{\alpha\beta} := -\tfrac{2}{\sqrt{g}}\left(p_{\alpha\beta} + \nabla_\lambda B^{\;\lambda}_{\alpha\beta} \right)$ for the matter energy momentum tensor, and $B^{\sigma\lambda\epsilon} := \tfrac{1}{2}p^\epsilon_\mu v^\lambda g^{\mu\sigma} + \tfrac{1}{2}p^\sigma_\mu v^\epsilon g^{\mu\lambda} - \tfrac{1}{2}p^\lambda_\mu v^\sigma g^{\mu\epsilon}$. Thus, on shell, one has $\mathbb{E}_\mu = 0$ but $T_{\mu\nu}$ does not vanish on its own.

Since the matter Lagrangian is covariant, one has the work current and the Noether current

$$\begin{cases} \mathcal{W} = \left[\tfrac{\sqrt{g}}{2} T_{\alpha\beta}\pounds_\xi g^{\alpha\beta} - \mathbb{E}_\mu \pounds_\xi v^\mu \right] d\sigma = [W_\epsilon \xi^\epsilon + W^\rho_\epsilon \nabla_\rho \xi^\epsilon]\, d\sigma \\ \mathcal{E} = \left[p^\lambda_\mu \pounds_\xi v^\mu - B^{\;\lambda}_{\alpha\beta}\pounds_\xi g^{\alpha\beta} - \xi^\lambda L_m \right] d\sigma_\lambda = [E^\lambda_\epsilon \xi^\epsilon + E^{\lambda\rho}_\epsilon \nabla_\rho \xi^\epsilon]\, d\sigma_\lambda \end{cases} \tag{7.2.22}$$

where we used the Lie derivatives

$$\pounds_\xi g^{\alpha\beta} = -2\nabla^{(\alpha}_{\;\cdot} \xi^{\beta)} \qquad\qquad \pounds_\xi v^\mu = \xi^\lambda \nabla_\lambda v^\mu - \nabla_\lambda \xi^\mu v^\lambda \tag{7.2.23}$$

In particular, we have

$$W_\epsilon = -\mathbb{E}_\mu \nabla_\epsilon v^\mu \qquad\qquad W^\rho_\epsilon = -\sqrt{g}T_{\alpha\epsilon}g^{\alpha\rho} + \mathbb{E}_\epsilon v^\rho \tag{7.2.24}$$

from which we see directly that W_ϵ does vanish on shell while W^ρ_ϵ does not. Then we have generalised Bianchi identity for the matter Lagrangian

$$W_\epsilon - \nabla_\rho W^\rho_\epsilon = 0 \qquad\qquad \Rightarrow -\mathbb{E}_\mu \nabla_\epsilon v^\mu - \nabla_\rho \left(-\sqrt{g}T_{\alpha\epsilon}g^{\alpha\rho} + \mathbb{E}_\epsilon v^\rho \right) = 0 \tag{7.2.25}$$

which on shell become

$$\nabla_\rho T^{\rho\epsilon} = 0 \tag{7.2.26}$$

More generally, if the matter fields are first order geometric objects (thus including, e.g., all tensor densities), then one still has

$$\pounds_\xi v^i = \xi^\lambda \nabla_\lambda v^i + Z^{i\rho}_\epsilon(v)\nabla_\rho \xi^\epsilon \tag{7.2.27}$$

so that, for any first order Lagrangian, one has the work current in the form

$$\mathcal{W} = [W_\epsilon \xi^\epsilon + W^\rho_\epsilon \nabla_\rho \xi^\epsilon]\, d\sigma \tag{7.2.28}$$

with $W_\epsilon = 0$ on shell and $W^\rho_\epsilon = -\sqrt{g}T_{\alpha\epsilon}g^{\alpha\rho}$ on shell. Accordingly, the partial Bianchi identities evaluated on shell then still imply conservation of energy-momentum tensor.

The fact that all three energy-momentum tensors are conserved, either with respect to g or $\tilde{g}$, is a potential threat to the theory. In fact, they come with relations among each other; if you fix $T_{\mu\nu}$ and $f(\mathcal{R})$ then $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$ are functionally determined. Thus if one assumes that $T_{\mu\nu}$ is conserved,

then one can *compute* $\tilde{\nabla}_\mu \tilde{T}^{\mu\nu}$ and $\nabla_\mu \hat{T}^{\mu\nu}$, and those might result to be conserved or not. In other words, conservation of $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$ is consequently either true or false, but not our choice. And, of course, for a generic function of $T_{\mu\nu}$ and $f(\mathcal{R})$ the expectation is that it is *not* conserved.

However, $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$ have a specific form and we can show that they are conserved iff $T_{\mu\nu}$ is.

In an $f(\mathcal{R})$-theory, the effective stress tensor $\tilde{T}_{\mu\nu}$ is defined by equation (7.2.15). Its conservation with respect to the conformal metric $\tilde{g}$ reads as $\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$, where indices have been raised by $\tilde{g}$. By considering that $\varphi^2 \tilde{T}^{\mu\nu} = \varphi^2 \tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\tilde{T}_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}\tilde{T}_{\alpha\beta}$, the relation between controvariant stress tensors is then

$$\varphi^2 f'(\mathcal{R})\tilde{T}^{\mu\nu} = (f'(\mathcal{R}))^{\frac{m+2}{m-2}} \tilde{T}^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{2\kappa}\left(f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})\right)g^{\mu\nu} \tag{7.2.29}$$

Let us stress that the indices of $\tilde{T}^{\mu\nu}$ are here raised by $\tilde{g}$, while the indices of $T^{\mu\nu}$ are raised by g.

Let us introduce the tensor

$$K^\alpha_{\beta\mu} = \{\tilde{g}\}^\alpha_{\beta\mu} - \{g\}^\alpha_{\beta\mu} = -\tfrac{1}{2}\left(g^{\alpha\sigma}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\sigma_{\mu)}\right)\overset{*}{\nabla}_\sigma \ln\varphi \tag{7.2.30}$$

which accounts for the difference between the connections of the metrics $\tilde{g}$ and g. Hereafter, the covariant derivatives with respect to g will be denoted by ∇_ϵ, while the ones with respect to $\tilde{g}$ will be denoted by $\tilde{\nabla}_\epsilon$. The symbol $\overset{*}{\nabla}_\epsilon$, as usual, denotes a covariant derivative when it is independent of any connection (in this case, being applied to the scalar $\ln\varphi$).

One can easily show that

$$K^\mu_{\epsilon\mu}\tilde{T}^{\epsilon\nu} = -\tfrac{1}{2}\left(g^{\mu\sigma}g_{\epsilon\mu} - \delta^\mu_\epsilon\delta^\sigma_\mu - \delta^\mu_\mu\delta^\sigma_\epsilon\right)\tilde{T}^{\epsilon\nu}\overset{*}{\nabla}_\sigma \ln\varphi = \tfrac{m}{2}\tilde{T}^{\epsilon\nu}\overset{*}{\nabla}_\epsilon \ln\varphi = \tfrac{m}{m-2}\tilde{T}^{\epsilon\nu}\left(f'(\mathcal{R})\right)^{-1}\overset{*}{\nabla}_\epsilon f'(\mathcal{R}) \tag{7.2.31}$$

and

$$K^\nu_{\epsilon\mu}\tilde{T}^{\epsilon\mu} = -\tfrac{1}{2}\left(g^{\nu\sigma}g_{\epsilon\mu} - 2\delta^\nu_{(\mu}\delta^\sigma_{\epsilon)}\right)\tilde{T}^{\epsilon\mu}\overset{*}{\nabla}_\sigma \ln\varphi = \tilde{T}^{\nu\sigma}\overset{*}{\nabla}_\sigma \ln\varphi - \tfrac{1}{2}g^{\nu\sigma}g_{\epsilon\mu}\tilde{T}^{\epsilon\mu}\overset{*}{\nabla}_\sigma \ln\varphi = \tfrac{2}{m-2}\left(f'(\mathcal{R})\right)^{-1}\tilde{T}^{\nu\sigma}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) - \tfrac{1}{m-2}\left(f'(\mathcal{R})\right)^{-1}g^{\nu\sigma}g_{\epsilon\mu}\tilde{T}^{\epsilon\mu}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) \tag{7.2.32}$$

as well as

$$\tfrac{m+2}{m-2}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\nabla_\mu f'(\mathcal{R})\tilde{T}^{\mu\nu} + \left(f'(\mathcal{R})\right)^{\frac{m+2}{m-2}}\nabla_\mu \tilde{T}^{\mu\nu} = \nabla_\mu T^{\mu\nu} - \tfrac{1}{2\kappa}\left(\nabla_\mu f'(\mathcal{R})\mathcal{R} + f'(\mathcal{R})\nabla_\mu \mathcal{R} - f'(\mathcal{R})\nabla_\mu \mathcal{R}\right)g^{\mu\nu} \quad \Rightarrow$$
$$\Rightarrow \qquad \left(f'(\mathcal{R})\right)^{\frac{m+2}{m-2}}\nabla_\mu \tilde{T}^{\mu\nu} = \nabla_\mu T^{\mu\nu} - \tfrac{1}{2\kappa}\nabla_\mu f'(\mathcal{R})\mathcal{R}g^{\mu\nu} - \tfrac{m+2}{m-2}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\nabla_\mu f'(\mathcal{R})\tilde{T}^{\mu\nu} \tag{7.2.33}$$

Then one has directly that

$$\varphi^2 f'(\mathcal{R})\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = \left(f'(\mathcal{R})\right)^{\frac{m+2}{m-2}}\left(\nabla_\mu \tilde{T}^{\mu\nu} + K^\mu_{\epsilon\mu}\tilde{T}^{\epsilon\nu} + K^\nu_{\epsilon\mu}\tilde{T}^{\epsilon\mu}\right) = \nabla_\mu T^{\mu\nu} - \tfrac{1}{2\kappa}\mathcal{R}g^{\sigma\nu}\nabla_\sigma f'(\mathcal{R}) - \tfrac{m+2}{m-2}\tilde{T}^{\sigma\nu}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\nabla_\sigma f'(\mathcal{R}) +$$
$$+ \tfrac{m}{m-2}\tilde{T}^{\sigma\nu}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) + \tfrac{2}{m-2}\tilde{T}^{\nu\sigma}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) - \tfrac{1}{m-2}g^{\nu\sigma}g_{\epsilon\mu}\tilde{T}^{\epsilon\mu}\left(f'(\mathcal{R})\right)^{\frac{4}{m-2}}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) =$$
$$= \nabla_\mu T^{\mu\nu} - \tfrac{1}{2\kappa}g^{\sigma\nu}\mathcal{R}\nabla_\sigma f'(\mathcal{R}) - \tfrac{1}{m-2}g^{\nu\sigma}T\left(f'(\mathcal{R})\right)^{-1}\nabla_\sigma f'(\mathcal{R}) + \tfrac{1}{2\kappa}\tfrac{m}{m-2}g^{\nu\sigma}\mathcal{R}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) - \tfrac{1}{2\kappa}\tfrac{m}{m-2}g^{\nu\sigma}\tfrac{f(\mathcal{R})}{f'(\mathcal{R})}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) =$$
$$= \nabla_\mu T^{\mu\nu} + \tfrac{1}{2\kappa}\tfrac{2}{m-2}g^{\nu\sigma}\left(f'(\mathcal{R})\mathcal{R} - \tfrac{m}{2}f(\mathcal{R}) - \kappa T\right)\left(f'(\mathcal{R})\right)^{-1}\overset{*}{\nabla}_\sigma f'(\mathcal{R}) = \nabla_\mu T^{\mu\nu} \tag{7.2.34}$$

where we used the master equation (7.2.9) so that one simply has

$$\left(f'(\mathcal{R})\right)^{\frac{m+2}{m-2}}\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = \nabla_\mu T^{\mu\nu} \tag{7.2.35}$$

Hence $\tilde{T}^{\mu\nu}$ is conserved with respect to $\tilde{g}$ iff $T^{\mu\nu}$ is conserved with respect to g.

After we checked that, one should check directly also that $\nabla_\sigma \hat{T}^{\mu\nu} = 0$ iff $T^{\mu\nu}$ is conserved with respect to g.

We leave it to do, as practice, and we shall obtain the same results in an indirect way. We shall, in fact, prove that $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$ are energy-momentum stress tensors for a covariant matter Lagrangian, then they must be necessarily conserved.

Universality theorems

As we remarked above, Palatini $f(\mathcal{R})$-theories are quite conservative with respect to standard GR. For example, if we consider the vacuum sector of the theory, hence $T_{\mu\nu} = 0$, we get field equations in the form

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = -\left(\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{2\left(f'(\mathcal{R})\right)^{\frac{m}{m-2}}}\right)\tilde{g}_{\mu\nu} \qquad \Rightarrow \qquad f'(\mathcal{R})\mathcal{R} - \tfrac{m}{2}f(\mathcal{R}) = 0 \tag{7.2.36}$$

In vacuum, the master equation constrains the curvature $\mathcal{R}$ to be a zero $\mathcal{R}_0$ of the equation $f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) = 0$, which are generically isolated. Then $\mathcal{R} = \mathcal{R}_0$ as well as any function of it, is constant on spacetime. That corresponds to standard GR with a cosmological constant Λ which is dictated by the choice of $f(\mathcal{R})$, namely

$$\Lambda := \frac{f'(\mathcal{R}_0)\mathcal{R}_0 - f(\mathcal{R}_0)}{2\left(f'(\mathcal{R}_0)\right)^{\frac{m}{m-2}}} = \frac{m-2}{2m}\left(f'(\mathcal{R}_0)\right)^{\frac{-2}{m-2}}\mathcal{R}_0 \tag{7.2.37}$$

This is called the *universality theorem* for Palatini $f(\mathcal{R})$-theories.

For standard GR, the only solution of master equation is $\mathcal{R}_0 = 0$ and no cosmological constant is produced.

If we compare $\tilde{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}$ in vacuum, we get

$$\hat{T}_{\mu\nu} = \tilde{T}_{\mu\nu} + \tfrac{1}{\kappa}\tfrac{m-2}{2\varphi}\left[\nabla_{\mu\nu}\varphi - \nabla_\mu\varphi\nabla_\nu\varphi - \left(\Box\varphi + \tfrac{(m-9)}{4\varphi}\nabla_\rho\varphi\nabla^\rho\varphi\right)g_{\mu\nu}\right] = \tilde{T}_{\mu\nu} \tag{7.2.38}$$

since the conformal factor is expected to be constant in vacuum. These two tensors will be, in general, different in matter, though.

Notice that although we set $T_{\mu\nu} = 0$ we have

$$\kappa\tilde{T}_{\mu\nu} = -\Lambda\tilde{g}_{\mu\nu} \tag{7.2.39}$$

which shows that in general $T_{\mu\nu}$, and $\tilde{T}_{\mu\nu}$ are, in fact, different tensors, also in vacuum.

The universality theorem is an important piece of information. It says that, every time $f(\mathcal{R})$ is chosen so that the corresponding Λ is so small to be negligible at the scale of the solar system, then, since the solar system is modelled as a vacuum solution, Palatini $f(\mathcal{R})$-theory and standard GR are indistinguishable theories.

On the other hand, we have to stress that this equivalence holds only in vacuum. If matter is present, then all contributions from the conformal factor enters. For example, in cosmology, one expects the conformal factor to be a function of time, in galaxies it is a function of r. Accordingly, there one expects a contribution of dark sources to be a function of time in cosmology and a function of the radial distance from the center in galaxies.

Let us finally notice how, although gravity is very well probed at the scales of the solar system and in vacuum solutions, it is considerably less well known in non-vacuum solutions. To be true, non-vacuum solutions are used only in cosmology and in galaxies, which is precisely where we see contributions of dark sources.

3. Metric-affine $f(\mathcal{R})$-theories

Before continuing to analyse the equivalences among different Weyl frames, let us consider, on a spacetime of dimension $m > 2$, a class of theories
slightly more general than Palatini $f(\mathcal{R})$-theories. They are based on a matter Lagrangian which couples matter also to the connection $\tilde{\Gamma}$, instead of g
only, i.e. let us consider Lagrangians in the form

$$\boldsymbol{L} = \left[\frac{\sqrt{g}}{2\kappa} f(\mathcal{R}) + L_m(g, \tilde{\Gamma}, \phi) \right] d\sigma \tag{7.3.1}$$

These kind of field theories are called *metric-affine $f(\mathcal{R})$-theories*.

Let us investigate what consequences the matter-connection interactions have in view of EPS formalism.

> These Lagrangians are sometimes (if not often) dismissed arguing that one does not really know how to solve them since the trick of introducing the conformal metric,
> as we shall see below, does not apply any longer.
> Now, besides the fact that dismissing a theory because we do not know how to solve field equations is never a solid argument, let us show that it is even wrong, since
> we *do know* how to solve them, at least in some cases.

The variation of the Lagrangian (7.3.1) gives an extra term ${}^*\tilde{P}_\alpha^{\beta\mu} \delta \tilde{\Gamma}_{\beta\mu}^\alpha$ form the variation of L_m with respect to $\tilde{\Gamma}_{\beta\mu}^\alpha$.

> Since we defined $\tilde{u}_{\alpha\beta}^\lambda = \tilde{\Gamma}_{\alpha\beta}^\lambda - \delta_{(\alpha}^\lambda \tilde{\Gamma}_{\beta)\epsilon}^\epsilon$ to account for the variation of Ricci tensor, we have to notice that this relation is invertible so that one can map this extra term
> in a term in the form $\tilde{P}_\alpha^{\beta\mu} \delta \tilde{u}_{\beta\mu}^\alpha$, by means of simple algebraic manipulations. In fact, one has

$$\tilde{u}_\beta := \tilde{u}_{\alpha\beta}^\alpha = -\frac{m-1}{2} \tilde{\Gamma}_\beta \quad \Longleftrightarrow \quad \tilde{\Gamma}_\beta = -\frac{2}{m-1} \tilde{u}_\beta \quad \Rightarrow \tilde{\Gamma}_{\alpha\beta}^\lambda = \tilde{u}_{\alpha\beta}^\lambda - \frac{2}{m-1} \delta_{(\alpha}^\lambda \tilde{u}_{\beta)} \quad \Rightarrow {}^*\tilde{P}_\alpha^{\beta\mu} \delta \tilde{\Gamma}_{\beta\mu}^\alpha = {}^*\tilde{P}_\alpha^{\beta\mu} \left(\delta_\rho^\alpha \delta_\beta^\sigma \delta_\mu^\nu - \frac{2}{m-1} \delta_\beta^\alpha \delta_\mu^\nu \delta_\rho^\sigma \right) \delta \tilde{u}_{\sigma\nu}^\rho =: \tilde{P}_\rho^{\sigma\nu} \delta \tilde{u}_{\sigma\nu}^\rho \tag{7.3.2}$$

> where we set $\tilde{\Gamma}_\beta := \tilde{\Gamma}_{\beta\epsilon}^\epsilon$ for the trace of the connection, as well as $\tilde{u}_\beta := \tilde{u}_{\beta\epsilon}^\epsilon$, and $\tilde{P}_\rho^{\sigma\nu} := {}^*\tilde{P}_\rho^{\sigma\nu} - \frac{2}{m-1} {}^*\tilde{P}_\alpha^{\alpha(\nu} \delta_\rho^{\sigma)}$.

The field equation of (7.3.1) associated to $\tilde{u}_{\alpha\beta}^\lambda$, are thence in the form

$$\tilde{\nabla}_\lambda \left(\sqrt{g} f'(\mathcal{R}) g^{\alpha\beta} \right) = \tilde{P}_\lambda^{\alpha\beta} \tag{7.3.3}$$

where $\tilde{P}_\lambda^{\alpha\beta}$ is a suitable tensor density of weight 1 which is a function of the matter fields ϕ, the metric g and possibly the connection $\tilde{\Gamma}$ itself.

One can try to *"solve"* the equation by considering $\tilde{P}_\lambda^{\alpha\beta}$ as an additional parameter. This is similar to what one is used to do in Hamiltonian mechanics
in which one considers momenta as independent of the positions and solves the equations. At least in some cases, one could obtain meaningful results,
e.g. when the particular combination expressing $\tilde{\Gamma}$ as a function of $\tilde{P}$ happens to be independent of the connection despite the $\tilde{P}$s may depend on $\tilde{\Gamma}$.

Let us stress that a similar technique is used in ordinary $f(R)$ theories, in which the connection is *"solved"* as a function of the metric *and* the
conformal factor $\varphi = \left(f'(\mathcal{R}) \right)^{\frac{2}{m-2}}$ which (at least in the beginning, i.e. before using the master equation) is itself a function of the connection.

Existence and uniqueness

First of all, one defines, as usual, a conformal metric $\tilde{g}_{\alpha\beta} = \varphi g_{\alpha\beta}$ so that the equation (7.3.3) can be recast as

$$\tilde{\nabla}_\lambda \left(\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \right) = \sqrt{\tilde{g}} P_\lambda^{\alpha\beta} \tag{7.3.4}$$

where we set $\tilde{P}^{\alpha\beta}_\lambda = \sqrt{\tilde{g}}P^{\alpha\beta}_\lambda$.

Let us then consider the Levi Civita connection $\{\tilde{g}\}^\alpha_{\mu\nu}$ of the conformal metric $\tilde{g}$. Accordingly, the difference between the two connections $\tilde{K}^\alpha_{\mu\nu} := \tilde{\Gamma}^\alpha_{\mu\nu} - \{\tilde{g}\}^\alpha_{\mu\nu}$ is a tensor.

Let us stress that here, as long as $\tilde{K}$ does not vanish, we need to pay extra care to distinguish between the covariant derivative with respect to $\tilde{\Gamma}$, which are denoted by $\tilde{\nabla}$, and the covariant derivative induced by $\tilde{g}$, which are here denoted by $\overset{\tilde{g}}{\nabla}$. This is not the case in Palatini $f(\mathcal{R})$-theories, in which one has $\tilde{\Gamma} = \{\tilde{g}\}$ on shell, and consequently $\tilde{\nabla} = \overset{\tilde{g}}{\nabla}$.

By the way, it is precisely because of that reason that we originally decided to use the tilde to denote conformal objects so that, eventually, these conformal objects would coincide on shell.

Equation (7.3.4) can be thence written as

$$\tilde{\nabla}_\lambda\left(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}\right) = \overset{\tilde{g}}{\nabla}_\lambda\left(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}\right) + \tilde{K}^\alpha_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\nu\beta} + \tilde{K}^\beta_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\alpha\nu} - \tilde{K}^\nu_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta} = \tilde{K}^\alpha_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\nu\beta} + \tilde{K}^\beta_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\alpha\nu} - \tilde{K}^\nu_{\nu\lambda}\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta} = \sqrt{\tilde{g}}P^{\alpha\beta}_\lambda \tag{7.3.5}$$

which in turn can be simplified to

$$\tilde{K}^\alpha_{\nu\lambda}\tilde{g}^{\nu\beta} + \tilde{K}^\beta_{\nu\lambda}\tilde{g}^{\alpha\nu} - \tilde{K}^\nu_{\nu\lambda}\tilde{g}^{\alpha\beta} = P^{\alpha\beta}_\lambda \tag{7.3.6}$$

Notice that this is an algebraic (in fact linear!) equation for the tensor $\tilde{K}$.

The homogenous equation (obtained by setting $P^{\alpha\beta}_\lambda = 0$) has been already known in[18] to have a unique solution (i.e. $\tilde{K} = 0$).

Let us set $\tilde{K}^{\alpha\beta\lambda} := \tilde{K}^\alpha_{\nu\sigma}\tilde{g}^{\nu\beta}\tilde{g}^{\sigma\lambda}$. Then the homogeneous equation from (7.3.6) can be recast in terms of the variable $\tilde{K}^{\alpha\beta\lambda}$ as

$$\tilde{K}^{\alpha\beta\lambda} + \tilde{K}^{\beta\alpha\lambda} = \tilde{K}^{\mu\nu\lambda}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta} \quad \Rightarrow \tilde{K}^{\alpha\beta\lambda}\tilde{g}_{\alpha\beta} + \tilde{K}^{\beta\alpha\lambda}\tilde{g}_{\alpha\beta} = m\tilde{K}^{\mu\nu\lambda}\tilde{g}_{\mu\nu} \quad \Rightarrow \tilde{K}^{\mu\nu\lambda}\tilde{g}_{\mu\nu} = 0 \tag{7.3.7}$$

Then one has

$$\tilde{K}^{\alpha\beta\lambda} + \tilde{K}^{\beta\alpha\lambda} = 0 \tag{7.3.8}$$

Thus we are looking for a tensor $\tilde{K}^{\alpha\beta\lambda}$ which is skew-symmetric in the indices $[\alpha\beta]$ as well as symmetric in the indices $(\beta\lambda)$. And, as we have repeatedly shown, the only such tensor is the zero tensor, in fact

$$\alpha\beta\lambda = -\beta\alpha\lambda = -\beta\lambda\alpha = \lambda\beta\alpha = \lambda\alpha\beta = -\alpha\lambda\beta = -\alpha\beta\lambda \quad \Rightarrow \alpha\beta\lambda = -\alpha\beta\lambda \quad \Rightarrow \alpha\beta\lambda = 0 \tag{7.3.9}$$

where, of course, $\alpha\beta\lambda$ stands for $\tilde{K}^{\alpha\beta\lambda}$. Then one has $\tilde{K}^{\alpha\beta\lambda} = 0$ iff $\tilde{K}^\alpha_{\nu\sigma} = 0$, since the metric $\tilde{g}$ is invertible.

Hence we just have to find a particular solution of (7.3.6); then that solution is unique, in view of Rouché-Capelli theorem for linear systems. As usual, one can rely on a bit of luck, write down a number of tensors built with the metric $\tilde{g}$ and the tensor P and search for a particular solution which is a linear combination of such basic tensors.

Let us then define $P_\mu := P^{\alpha\beta}_\mu\tilde{g}_{\alpha\beta}$ and $P^\alpha := P^{\alpha\lambda}_\lambda$ (not to be confused with $P^\alpha_. = P_\epsilon\tilde{g}^{\epsilon\alpha}$). By noticing that the tensor $\tilde{K}$ is defined to be symmetric in its lower indices, we can try with a linear combination

$$\tilde{k}^\alpha_{\mu\nu} = aP_{(\mu}\delta^\alpha_{\nu)} + bP^{\alpha\sigma}_{(\mu}\tilde{g}_{\nu)\sigma} + cP_\sigma\tilde{g}^{\alpha\sigma}\tilde{g}_{\mu\nu} + d\tilde{g}^{\alpha\lambda}P^{\rho\sigma}_\lambda\tilde{g}_{\rho\mu}\tilde{g}_{\sigma\nu} \tag{7.3.10}$$

By substituting back into (7.3.6), one gets

$$\left(\tfrac{b}{2}+d\right)P_\nu^{\alpha\sigma}\tilde{g}_{\lambda\sigma}\tilde{g}^{\nu\beta}+\left(\tfrac{a}{2}+c\right)P_\nu\delta_\lambda^\beta\tilde{g}^{\alpha\nu}+bP_\lambda^{\alpha\beta}+\left(\tfrac{b}{2}+d\right)P_\nu^{\beta\sigma}\tilde{g}_{\lambda\sigma}\tilde{g}^{\alpha\nu}+\left(\tfrac{a}{2}+c\right)P_\sigma\tilde{g}^{\beta\sigma}\delta_\lambda^\alpha-\left(\tfrac{a}{2}(m-1)+\tfrac{b}{2}+c\right)P_\lambda\tilde{g}^{\alpha\beta}-\left(d+\tfrac{b}{2}\right)P_\nu^{\nu\sigma}\tilde{g}_{\lambda\sigma}\tilde{g}^{\alpha\beta}=P_\lambda^{\alpha\beta} \tag{7.3.11}$$

Thus the ansatz (7.3.10) is a solution of equation (7.3.4), if the coefficients are

$$b=1 \qquad d=-\tfrac{1}{2} \qquad c=\tfrac{1}{2(m-2)} \qquad a=-\tfrac{1}{m-2} \tag{7.3.12}$$

Since we have assumed that spacetime has dimension $m>2$, this is a good solution. As usual, two dimensional spacetimes are degenerate under many viewpoints and, in case, they must be treated separately.

EPS compatible connections

Thus we are able to formally solve the field equation (7.3.3) by saying that, if a connection $\tilde{\Gamma}$ is a solution, it must be in the form

$$\tilde{\Gamma}^\alpha_{\beta\mu}=\{\tilde{g}\}^\alpha_{\beta\mu}-\tfrac{1}{m-2}P_{(\mu}\delta_{\nu)}^\alpha+P_{(\mu}^{\alpha\sigma}\tilde{g}_{\nu)\sigma}+\tfrac{1}{2(m-2)}P_\sigma\tilde{g}^{\alpha\sigma}\tilde{g}_{\mu\nu}-\tfrac{1}{2}\tilde{g}^{\alpha\lambda}P_\lambda^{\rho\sigma}\tilde{g}_{\rho\mu}\tilde{g}_{\sigma\nu} \tag{7.3.13}$$

By *formal solution* we mean that, if the matter Lagrangian depends non-linearly on the connection, then P is a function of $\tilde{\Gamma}$ itself. Accordingly, the condition (7.3.13) is not an expression for $\tilde{\Gamma}$, but an equivalent way to write the original equation, a constraint to which a solution has to satisfy.

However, EPS framework requires that, in any ETG, field equations should imply that the connection is EPS compatible, i.e. there must be a 1-form $\tilde{\alpha}_\epsilon$ such that

$$-\tfrac{1}{(m-2)}P_\mu\delta_\nu^\alpha-\tfrac{1}{(m-2)}P_\nu\delta_\mu^\alpha+P_\mu^{\alpha\sigma}\tilde{g}_{\nu\sigma}+P_\nu^{\alpha\sigma}\tilde{g}_{\mu\sigma}+\tfrac{1}{(m-2)}P_\sigma\tilde{g}^{\alpha\sigma}\tilde{g}_{\mu\nu}-\tilde{g}^{\alpha\lambda}P_\lambda^{\rho\sigma}\tilde{g}_{\rho\mu}\tilde{g}_{\sigma\nu}=\left(\tilde{g}^{\alpha\epsilon}\tilde{g}_{\mu\nu}-\delta_\mu^\alpha\delta_\nu^\epsilon-\delta_\nu^\alpha\delta_\mu^\epsilon\right)\tilde{\alpha}_\epsilon \tag{7.3.14}$$

Hence, we can write P as a function of $\tilde{\alpha}$ to see which P are allowed so that, eventually, they lead to EPS compatible connections. By tracing this relation by δ_α^ν, one has

$$P_\mu=\tfrac{m(m-2)}{2}\tilde{\alpha}_\mu \tag{7.3.15}$$

while by tracing it by $\tilde{g}^{\mu\nu}$

$$P^\alpha=P_\lambda^{\alpha\lambda}=\tfrac{m-2}{2}\tilde{g}^{\alpha\epsilon}\tilde{\alpha}_\epsilon \tag{7.3.16}$$

By substituting back these into equation (7.3.14), one obtains

$$P_\mu^{\alpha\sigma}\tilde{g}_{\nu\sigma}+P_\nu^{\alpha\sigma}\tilde{g}_{\mu\sigma}-\tilde{g}^{\alpha\epsilon}P_\epsilon^{\rho\sigma}\tilde{g}_{\rho\mu}\tilde{g}_{\sigma\nu}=-\frac{m-2}{2}\left(\tilde{g}^{\alpha\epsilon}\tilde{g}_{\mu\nu}-2\delta_{(\nu}^\alpha\delta_{\mu)}^\epsilon\right)\tilde{\alpha}_\epsilon \tag{7.3.17}$$

Let us now multiply by $\tilde{g}^{\nu\lambda}$, which leads to an equivalent equation in the following form

$$P_\mu^{\alpha\lambda}+2\tilde{g}^{\epsilon[\lambda}P_\epsilon^{\alpha]\sigma}\tilde{g}_{\sigma\mu}=(m-2)\left(\tilde{g}^{\epsilon[\lambda}\delta_\mu^{\alpha]}+\tfrac{1}{2}\tilde{g}^{\alpha\lambda}\delta_\mu^\epsilon\right)\tilde{\alpha}_\epsilon \tag{7.3.18}$$

One can now split this in the symmetric and skew parts with respect to the indices $(\alpha\lambda)$ obtaining

$$\begin{cases} P_\mu^{\alpha\lambda} = \frac{m-2}{2}\tilde{g}^{\alpha\lambda}\tilde{\alpha}_\mu \\ \tilde{g}^{\epsilon[\lambda}P_\epsilon^{\alpha]\sigma}\tilde{g}_{\sigma\mu} = \frac{m-2}{2}\tilde{g}^{\epsilon[\lambda}\delta_\mu^{\alpha]}\tilde{\alpha}_\epsilon \end{cases} \tag{7.3.19}$$

These equations are not independent; in fact, by substituting the first into the second one, the second is identically satisfied.

As one can easily check

$$\tilde{g}^{\epsilon[\lambda}P_\epsilon^{\alpha]\sigma}\tilde{g}_{\sigma\mu} = \frac{m-2}{4}\left(\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\sigma} - \tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\sigma}\right)\tilde{g}_{\sigma\mu}\tilde{\alpha}_\epsilon = \frac{m-2}{4}\left(\tilde{g}^{\epsilon\lambda}\delta_\mu^\alpha - \tilde{g}^{\epsilon\alpha}\delta_\mu^\lambda\right)\tilde{\alpha}_\epsilon = \frac{m-2}{2}\tilde{g}^{\epsilon[\lambda}\delta_\mu^{\alpha]}\tilde{\alpha}_\epsilon \tag{7.3.20}$$

Accordingly, if we allow the connection to interact directly with matter fields, still pretending the theory to be a ETG, we need to have

$$P_\mu^{\alpha\lambda} = \frac{m-2}{2}\tilde{g}^{\alpha\lambda}\tilde{\alpha}_\mu \tag{7.3.21}$$

In these cases, one reads P for field equations, does check it is in the required form (otherwise the theory is not an ETG) and then uses (7.3.21) to compute $\tilde{\alpha}_\mu$, which then, in turn, defines the connection $\tilde{\Gamma}$. Let us finally remark that, in Palatini $f(\mathcal{R})$-theories, one has $P = 0$, which is in the required form setting $\tilde{\alpha}_\mu = 0$, and consequently $\tilde{\Gamma} = \{\tilde{g}\}$.

Purely metric as metric-affine $f(\mathcal{R})$-theories

As an example of the technique introduced above, one can consider a field theory, for a matter field A_μ being a covector density of weight -1, assuming the dynamics to be described by the Lagrangian

$$\boldsymbol{L} = \frac{1}{2\kappa}\left(\sqrt{g}f(\mathcal{R}) - gg^{\mu\nu}\tilde{\nabla}_\mu A_\nu\right)d\sigma \tag{7.3.22}$$

Here the *"matter"* Lagrangian

$$\boldsymbol{L}_m = -\frac{1}{2\kappa}gg^{\mu\nu}\tilde{\nabla}_\mu A_\nu\, d\sigma = -\frac{1}{2\kappa}gg^{\mu\nu}\left(d_\mu A_\nu - \tilde{\Gamma}_{\nu\mu}^\epsilon A_\epsilon + \tilde{\Gamma}_{\epsilon\mu}^\epsilon A_\nu\right)d\sigma = -\frac{1}{2\kappa}gg^{\mu\nu}\left(d_\mu A_\nu - \tilde{u}_{\nu\mu}^\epsilon A_\epsilon\right)d\sigma \tag{7.3.23}$$

does depend on the connection $\tilde{\Gamma}$, precisely through the combination $\tilde{u}_{\beta\mu}^\alpha = \tilde{\Gamma}_{\beta\mu}^\alpha - \delta_{(\beta}^\alpha\tilde{\Gamma}_{\mu)}$.

The variation of the matter Lagrangian gives us

$$\delta L_m = -\frac{1}{2}\sqrt{g}\,{}^*T_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{2\kappa}gg^{\mu\nu}A_\alpha\delta\tilde{u}_{\mu\nu}^\alpha + \tilde{\nabla}_\mu\left(\frac{1}{2\kappa}gg^{\mu\nu}\right)\delta A_\nu - \tilde{\nabla}_\mu\left(\frac{1}{2\kappa}gg^{\mu\nu}\delta A_\nu\right) \tag{7.3.24}$$

where we set ${}^*T_{\mu\nu} := \frac{1}{\kappa}\sqrt{g}\left(\tilde{\nabla}_{(\mu}A_{\nu)} - g_{\mu\nu}g^{\alpha\beta}\tilde{\nabla}_\alpha A_\beta\right)$, so that field equations take the form:

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa\,{}^*T_{\mu\nu} \\ \tilde{\nabla}_\alpha\left(\sqrt{g}g^{\mu\nu}f'(\mathcal{R})\right) = gg^{\mu\nu}A_\alpha \\ \tilde{\nabla}_\mu\left(gg^{\mu\nu}\right) = 0 \end{cases} \tag{7.3.25}$$

Following the guideline used in Palatini $f(\mathcal{R})$-theories, let us perform a conformal transformation:

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \qquad \Rightarrow \sqrt{\tilde{g}}\tilde{g}^{\mu\nu} = \sqrt{g}g^{\mu\nu}\varphi^{\frac{m-2}{2}}, \quad \tilde{g}\tilde{g}^{\mu\nu}A_\alpha = gg^{\mu\nu}A_\alpha\varphi^{m-1} \tag{7.3.26}$$

so that, setting $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$, the second field equations can be written as

$$\tilde{\nabla}_\alpha\left(\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\right) = \tilde{g}\tilde{g}^{\mu\nu}A_\alpha\varphi^{1-m} \quad \Rightarrow P_\alpha^{\mu\nu} = \sqrt{\tilde{g}}\tilde{g}^{\mu\nu}A_\alpha\varphi^{1-m} \tag{7.3.27}$$

which, in fact, is in the form required by (7.3.21). Then, solving the relation (7.3.21), one has

$$\tilde{\alpha}_\sigma = \tfrac{2}{m-2}\sqrt{g}A_\sigma\varphi^{-\frac{m-2}{2}} \tag{7.3.28}$$

so that the connection is in the following form

$$\tilde{\Gamma}_{\beta\mu}^\alpha = \{\tilde{g}\}_{\beta\mu}^\alpha + \tfrac{1}{2}\left(\tilde{g}^{\alpha\sigma}\tilde{g}_{\beta\mu} - 2\delta_{(\beta}^\alpha\delta_{\mu)}^\sigma\right)\tilde{\alpha}_\sigma = \{g\}_{\beta\mu}^\alpha + K_{\beta\mu}^\alpha \tag{7.3.29}$$

where we set $K_{\beta\mu}^\alpha := \tfrac{1}{2}\left(g^{\alpha\sigma}g_{\beta\mu} - 2\delta_{(\beta}^\alpha\delta_{\mu)}^\sigma\right)\left(\tilde{\alpha}_\sigma - \overset{*}{\nabla}_\sigma\ln\varphi\right)$. However, to be a solution, it needs to satisfy also the other field equations. In particular, the third field equation reads

$$\tilde{\nabla}_\mu\left(gg^{\mu\nu}\right) = \textcolor{red}{\nabla_\mu\left(gg^{\mu\nu}\right)} + \underline{K_{\epsilon\mu}^\mu gg^{\epsilon\nu}} + K_{\epsilon\mu}^\nu gg^{\mu\epsilon} - \underline{2K_{\epsilon\mu}^\epsilon gg^{\mu\nu}} = g\left(K_{\epsilon\mu}^\nu g^{\mu\epsilon} - K_{\epsilon\mu}^\epsilon g^{\mu\nu}\right) = 0 \tag{7.3.30}$$

i.e.

$$K_{\epsilon\mu}^\nu g^{\mu\epsilon} - K_{\epsilon\mu}^\epsilon g^{\mu\nu} = [(m-2) + (m+\textcolor{red}{1}-1)]g^{\sigma\nu}\left(\tilde{\alpha}_\sigma - \overset{*}{\nabla}_\sigma\ln\varphi\right) = 2(m-1)g^{\nu\sigma}\left(\tilde{\alpha}_\sigma - \overset{*}{\nabla}_\sigma\ln\varphi\right) = 0 \quad \Rightarrow \tilde{\alpha}_\sigma = \overset{*}{\nabla}_\sigma\ln\varphi \tag{7.3.31}$$

which, in turn, implies $K_{\beta\mu}^\alpha = 0$, and going back to the definition of the connection one gets

$$\tilde{\Gamma}_{\beta\mu}^\alpha = \{g\}_{\beta\mu}^\alpha \tag{7.3.32}$$

Thus, unlike in Palatini $f(\mathcal{R})$-theories, the connection coincides with the Levi Civita connection of the original metric g, not of the conformal metric $\tilde{g}$. Still it is a metric connection.

Then, solving equation (7.3.28), one has $\sqrt{g}A_\sigma = -\tfrac{m-2}{2}\varphi^{\frac{m-4}{2}}\overset{*}{\nabla}_\sigma\varphi = \overset{*}{\nabla}_\sigma f'(\mathcal{R})$, and the energy-momentum tensor is

$${}^*T_{\mu\nu} = \tfrac{1}{\kappa}\sqrt{g}\left(\nabla_{(\mu}A_{\nu)} - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha A_\beta\right) = \tfrac{1}{\kappa}\left(\nabla_{(\mu}\overset{*}{\nabla}_{\nu)}f'(\mathcal{R}) - g_{\mu\nu}\Box f'(\mathcal{R})\right) \quad \Rightarrow {}^*T = -\tfrac{1}{\kappa}(m-1)\Box f'(\mathcal{R}) \tag{7.3.33}$$

Thus the first field equation can be recast into the form

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} = \kappa\,{}^*T_{\mu\nu} = \nabla_{(\mu}\overset{*}{\nabla}_{\nu)}f'(\mathcal{R}) - g_{\mu\nu}\Box f'(\mathcal{R}) \tag{7.3.34}$$

where now the curvature tensors $R_{\mu\nu}$ and R refer to the curvatures of the metric g. This equation exactly agrees with the field equation for vacuum purely metric $f(R)$-theories; see (1.10.170). Thus this particular metric-affine $f(\mathcal{R})$-theory is dynamically equivalent to vacuum purely metric $f(R)$-theory.

The dynamical equivalence also extends to matter coupling. If we consider the Lagrangian

$$\boldsymbol{L} = \left(\tfrac{1}{2\kappa} \left(\sqrt{g}f(\mathcal{R}) - gg^{\mu\nu}\tilde{\nabla}_\mu A_\nu \right) + L_m(g_{\mu\nu}, v^i, v^i_\mu) \right) \, d\sigma \tag{7.3.35}$$

then field equation are

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa \left(T_{\mu\nu} + {}^*T_{\mu\nu} \right) \\ \tilde{\nabla}_\alpha \left(\sqrt{g}g^{\mu\nu}f'(\mathcal{R}) \right) = gg^{\mu\nu}A_\alpha \\ \tilde{\nabla}_\mu \left(gg^{\mu\nu} \right) = 0 \\ E_i = 0 \end{cases} \tag{7.3.36}$$

where $E_i = p_i - \nabla_\mu p_i^\mu = 0$ are matter field equations and $T_{\mu\nu}$ is the matter energy momentum tensor as in (1.10.170).

One can repeat exactly the same argument presented above to show that they are equivalent to

$$\begin{cases} f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} + \nabla_{(\mu}\overset{*}{\nabla}_{\nu)}f'(\mathcal{R}) - g_{\mu\nu}\Box f'(\mathcal{R}) \\ E_i = 0 \end{cases} \tag{7.3.37}$$

which in fact extends the dynamical equivalence to the non-vacuum cases.

Accordingly, we have a pretty odd result: each purely metric $f(R)$-theory is dynamically equivalent to the corresponding metric-affine $f(\mathcal{R})$-theory, so that purely metric $f(R)$-theories can be (classically) identified with their corresponding metric-affine $f(\mathcal{R})$-theory.

These metric-affine $f(\mathcal{R})$-theories are in fact *iETG* and, by the way, they also show that there are iETG other than Palatini $f(\mathcal{R})$-theories.

In this metric-affine theory, also free fall is described by $\tilde{\Gamma}$, i.e. by g, as in the metric theory. In other Palatini $f(\mathcal{R})$-theories, the free fall is described by $\tilde{\Gamma}$, i.e., in that case, by $\tilde{g}$, instead.

Metric-affine $f(\mathcal{R})$-theory with a scalar field

Looking back into the example above, one can see that most of the trick is independent of the fact that A is a fundamental field, but it is mainly related to the fact that it is a covector density of weight -1, which fixes the dependence of its covariant derivative on $\tilde{u}^\lambda_{\mu\nu}$ as in (7.3.23). Let us consider a Lorentzian metric g, a torsionless connection $\tilde{\Gamma}$ and a scalar field ϕ as fundamental fields. Then consider the Lagrangian

$$\boldsymbol{L} = \tfrac{1}{2\kappa} \left(\sqrt{g}f(\mathcal{R}) - gg^{\mu\nu}\tilde{\nabla}_\mu \left(\tfrac{1}{\sqrt{g}}\overset{*}{\nabla}_\nu\phi \right) \right) \, d\sigma \tag{7.3.38}$$

By variation of this Lagrangian and usual covariant integration by parts, one obtains

$$\delta L = \tfrac{\sqrt{g}}{2\kappa} \left(f'(\mathcal{R})\tilde{R}_{(\alpha\beta)} - \tfrac{1}{2}f(\mathcal{R})g_{\alpha\beta} - \kappa \, {}^{**}T_{\alpha\beta} \right) \delta g^{\alpha\beta} - \tfrac{1}{\kappa} \left(\tilde{\nabla}_\lambda \left(\sqrt{g}g^{\alpha\beta}f'(\mathcal{R}) \right) + \kappa gg^{\alpha\beta}A_\lambda \right) \delta u^\lambda_{\alpha\beta} + \nabla_\nu \left(\tfrac{1}{\sqrt{g}}\tilde{\nabla}_\mu \left(gg^{\mu\nu} \right) \right) \delta\phi + $$
$$+ \tilde{\nabla}_\lambda \left(\tfrac{\sqrt{g}}{\kappa}g^{\alpha\beta}f'(\mathcal{R})\delta\tilde{u}^\lambda_{\alpha\beta} + \tfrac{\sqrt{g}}{2}g^{\lambda\nu}g_{\alpha\beta}\nabla_\nu\phi\delta g^{\alpha\beta} + \sqrt{g}g^{\lambda\nu}\nabla_\nu\delta\phi - \tfrac{1}{\sqrt{g}}\tilde{\nabla}_\mu \left(gg^{\mu\lambda} \right) \delta\phi \right) \tag{7.3.39}$$

where we set

$$^*T_{\alpha\beta} := \tfrac{1}{\kappa}\sqrt{g}\left(g_{\alpha\beta}g^{\mu\nu}\tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_{(\alpha}A_{\beta)}\right) \qquad ^{**}T_{\alpha\beta} := {}^*T_{\alpha\beta} - \tfrac{1}{2\kappa g}\tilde{\nabla}_\mu\left(gg^{\mu\nu}\right)\overset{*}{\nabla}_\nu\phi g_{\alpha\beta} \tag{7.3.40}$$

for the energy-momentum tensor induced by the coupling. Here we denote by $\tilde{\nabla}_\mu$ the covariant derivative with respect to the connection $\tilde{\Gamma}$, while, as usual, $\overset{*}{\nabla}_\mu$ is used for the special cases in which the covariant derivative turns out to be independent of any connection and reduces to a partial derivative.

For the variation of the interaction Lagrangian we can start from the variation of the Lagrangian (7.3.23), namely

$$\delta L_m = -\tfrac{1}{2}\sqrt{g}\,{}^*T_{\alpha\beta}\delta g^{\alpha\beta} + \tfrac{1}{2\kappa}gg^{\mu\nu}A_\alpha\delta\tilde{u}^\alpha_{\mu\nu} + \tilde{\nabla}_\mu\left(\tfrac{1}{2\kappa}gg^{\mu\nu}\right)\delta A_\nu - \tilde{\nabla}_\mu\left(\tfrac{1}{2\kappa}gg^{\mu\nu}\delta A_\nu\right)$$

and consider that since A_ν is not a fundamental field anymore, one has

$$\delta A_\nu = \tfrac{1}{\sqrt{g}}\left(\overset{*}{\nabla}_\nu\delta\phi + \tfrac{1}{2}\overset{*}{\nabla}_\nu\phi g_{\alpha\beta}\delta g^{\alpha\beta}\right) \tag{7.3.41}$$

Then we need to substitute this back into the variation and integrate by parts once again to obtain

$$\delta L_m = -\tfrac{1}{2}\sqrt{g}\left({}^*T_{\alpha\beta} - \tfrac{1}{2\kappa}\tfrac{1}{g}\tilde{\nabla}_\mu\left(gg^{\mu\nu}\right)\overset{*}{\nabla}_\nu\phi g_{\alpha\beta}\right)\delta g^{\alpha\beta} + \tfrac{1}{2\kappa}gg^{\mu\nu}A_\alpha\delta\tilde{u}^\alpha_{\mu\nu} - \tfrac{1}{2\kappa}\overset{*}{\nabla}_\nu\left(\tfrac{1}{\sqrt{g}}\tilde{\nabla}_\mu\left(gg^{\mu\nu}\right)\right)\delta\phi - \tilde{\nabla}_\mu\left(\tfrac{1}{2\kappa}\sqrt{g}g^{\mu\nu}\left(\overset{*}{\nabla}_\nu\delta\phi + \tfrac{1}{2}\overset{*}{\nabla}_\nu\phi g_{\alpha\beta}\delta g^{\alpha\beta}\right) - \tfrac{1}{2\kappa}\tfrac{1}{\sqrt{g}}\tilde{\nabla}_\nu\left(gg^{\mu\nu}\right)\delta\phi\right) \tag{7.3.42}$$

Accordingly, field equations are

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\alpha\beta)} - \tfrac{1}{2}f(\mathcal{R})g_{\alpha\beta} = \kappa\,{}^{**}T_{\alpha\beta} \\[2mm] \tilde{\nabla}_\lambda\left(\sqrt{g}g^{\alpha\beta}f'(\mathcal{R})\right) = \sqrt{g}\overset{*}{\nabla}_\lambda\phi g^{\alpha\beta} \\[2mm] \overset{*}{\nabla}_\nu\left(\tfrac{1}{\sqrt{g}}\tilde{\nabla}_\mu\left(gg^{\mu\nu}\right)\right) = 0 \end{cases} \tag{7.3.43}$$

By performing the usual conformal transformation on the metric to define the conformal metric $\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$, always with $\varphi^{\frac{m-2}{2}} = f'(\mathcal{R})$, the second field equation becomes

$$\tilde{\nabla}_\lambda\left(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}\right) = \sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}\overset{*}{\nabla}_\lambda\phi\left(f'(\mathcal{R})\right)^{-1} \qquad \Rightarrow \tilde{\alpha}_\mu = \tfrac{2}{m-2}\overset{*}{\nabla}_\mu\phi\left(f'(\mathcal{R})\right)^{-1} \tag{7.3.44}$$

The second equation fixes again the connection to be

$$\tilde{\Gamma}^\alpha_{\beta\mu} := \{\tilde{g}\}^\alpha_{\beta\mu} + \tfrac{1}{m-2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\left(f'(\mathcal{R})\right)^{-1}\nabla_\epsilon\phi = \{g\}^\alpha_{\beta\mu} + \tfrac{1}{2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\left(\tfrac{2}{m-2}\left(f'(\mathcal{R})\right)^{-1}\nabla_\epsilon\phi - \nabla_\epsilon\ln\varphi\right) \tag{7.3.45}$$

Accordingly, we can set

$$\alpha_\epsilon := \tfrac{2}{m-2}\left(f'(\mathcal{R})\right)^{-1}\nabla_\epsilon\phi - \nabla_\epsilon\ln\varphi = \tilde{\alpha}_\epsilon - \nabla_\epsilon\ln\varphi \tag{7.3.46}$$

to measure the difference between the connections $\tilde{\Gamma}$ and $\{g\}$. Notice that α_ϵ is closed iff $\tilde{\alpha}_\epsilon$ is closed (iff $\tilde{\Gamma}$ is metric).

However, the third equation does not force the covector $\tilde{\alpha}_\epsilon = \tfrac{2}{m-2}\left(f'(\mathcal{R})\right)^{-2}\nabla_\epsilon\phi$ to be a closed form; thus the connection $\tilde{\Gamma}$ is not necessarily metric.

Let us expand the third equation. First we have

$$\tilde{\nabla}_\mu \left(gg^{\mu\nu}\right) = \nabla_\mu \left(gg^{\mu\nu}\right) + K^\mu_{\epsilon\mu}gg^{\epsilon\nu} + K^\nu_{\epsilon\mu}gg^{\mu\epsilon} - 2K_\mu gg^{\mu\nu} = g\left(K^\nu_{\epsilon\mu}g^{\mu\epsilon} - K_\mu g^{\mu\nu}\right) = g\left(\delta^\nu_\rho g^{\mu\sigma} - \delta^\sigma_\rho g^{\mu\nu}\right)K^\rho_{\sigma\mu} = $$
$$= \tfrac{1}{2}g\left(\delta^\nu_\rho g^{\mu\sigma} - \delta^\rho_\rho g^{\mu\nu}\right)\left(g^{\rho\epsilon}g_{\sigma\mu} - \delta^\rho_\sigma \delta^\epsilon_\mu - \delta^\rho_\mu \delta^\epsilon_\sigma\right)\alpha_\epsilon = (m-1)gg^{\nu\sigma}\left(\tfrac{2}{m-2}\left(f(\mathcal{R})\right)^{-1}\nabla_\sigma - \nabla_\sigma \ln\varphi\right) \tag{7.3.47}$$

Thus the third equation can be written as $d*\boldsymbol{\beta} = 0$, where we set

$$\boldsymbol{\beta} := (m-1)\left(\tfrac{2}{m-2}\left(f(\mathcal{R})\right)^{-1}\nabla_\sigma \phi - \nabla_\sigma \ln\varphi\right)dx^\sigma \in \Omega^1(M) \tag{7.3.48}$$

The third field equation then has

$$*\boldsymbol{\beta} = \sqrt{g}\,\nabla_\nu \theta^{\mu\nu}\,d\sigma_\mu \qquad (\iff \boldsymbol{\beta} = *d\boldsymbol{\theta}) \tag{7.3.49}$$

as a general solution for any $(m-2)$-form $\boldsymbol{\theta} = \tfrac{1}{2}\sqrt{g}\theta^{\mu\nu}\,d\sigma_{\mu\nu}$. And, in view of this, one has

$$\alpha_\sigma = \tfrac{2}{m-2}\left(f(\mathcal{R})\right)^{-1}\nabla_\sigma - \nabla_\sigma \ln\varphi = \tfrac{1}{m-1}g_{\sigma\mu}\nabla_\nu \theta^{\mu\nu} \qquad \left(\Rightarrow \boldsymbol{\alpha} = \tfrac{1}{m-1}*d\boldsymbol{\theta}\right) \tag{7.3.50}$$

Now, for a general $(m-2)$-form $\boldsymbol{\theta}$, there is no need for $d\boldsymbol{\alpha} = \tfrac{1}{m-1}d*d\boldsymbol{\theta}$ to be zero, i.e. for $\tilde{\Gamma}$ to be metric.

For example, if g is Minkowski metric in dimension 3 with Cartesian coordinates (t, x, y) and $\boldsymbol{\theta} = (x)^2\,d\sigma_{12}$ one can prove that $d\boldsymbol{\alpha} \neq 0$ holds. In fact, one has

$$d\boldsymbol{\theta} = 2x\,dx \wedge d\sigma_{12} = -2x\,d\sigma_2 \qquad \boldsymbol{\alpha} \propto *d\boldsymbol{\theta} = -2x\,dy \qquad d\boldsymbol{\alpha} \propto d*d\boldsymbol{\theta} = -2\,dx \wedge dy \neq 0 \tag{7.3.51}$$

Then the second and third field equations are solved by an EPS connection which is not necessarily metric and, in general, we have a ETG. Since, in general, the connection is not metric, thus this ETG is not integrable. Accordingly, we learn that:

    a) there are ETG which are not integrable;

    b) metric-affine $f(\mathcal{R})$-theories are not necessary integrable;

    c) thus there exist metric-affine $f(\mathcal{R})$-theories which are not equivalent to Palatini $f(\mathcal{R})$-theories.

We still do not know whether there are ETG more general than metric-affine $f(\mathcal{R})$-theories.

Let us finally notice how this result is essentially related to the non-commutativity of the differential with the Hodge duality. Now imagine to search for this result in Minkowski spacetime, in Cartesian coordinates, making no distinction between vectors and covectors. This is a further motivation to use a precise notation which accounts for geometric type of objects.

4. Conformal Weyl frames

The choice of a specific set of fundamental fields in an ETG is called a *conformal Weyl frame.* One should not discuss about the form of field equations simply because that structure in not rigid enough to say anything conclusive, and conformal Weyl frames have to be discussed at the level of action. Similarly, one really needs to use bundle framework to make the action transformations clear.

If the transformation between conformal Weyl frames is algebraic, then that amounts to change coordinates on the configuration bundle. Then as long as the variational principles are global (hence independent of the coordinates) the theories obtained by changing the conformal Weyl frame are trivially dynamically equivalent. However, we shall see also some cases in which the transformation on fields cannot be reduced to a transformation on the configuration bundle.

Of course, if they depend on a finite number of derivative they can be regarded as maps between suitable jet prolongations of the configuration bundles.

In these cases, we shall transform sections of the configuration bundle into sections though the new fields will depend on derivatives as well on the value of the old fields. Dynamical equivalence in these cases is not trivial and needs to be proven on a case by case basis.

Let us start by showing dynamical equivalence between Palatini $f(\mathcal{R})$-theories and a suitable subclass of Brans–Dicke theories at the level of field equations. Then we shall recover the same equivalence at the level of actions and bundles.

Dynamical equivalence with Brans–Dicke theories

We already introduced and studies Brans–Dicke theories; see (1.10.176). Their field equations can be recast in the form (see (1.10.182)):

$$
\begin{cases}
\varphi^\alpha R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{1}{m-2}Tg_{\alpha\beta}\right) + \alpha\varphi^{\alpha-1}\nabla_{\alpha\beta}\varphi + \frac{\alpha}{m-2}\varphi^{\alpha-1}\Box\varphi g_{\alpha\beta} + \left(\alpha(\alpha-1)\varphi^{\alpha-2} + \frac{\omega}{\varphi^\beta}\right)\nabla_\alpha\varphi\nabla_\beta\varphi + \frac{\alpha(\alpha-1)}{m-2}\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} - \frac{1}{m-2}Ug_{\alpha\beta} \\
\left(\alpha\varphi^{\alpha-1}(m-1) + \frac{m-2}{\alpha}\frac{\omega}{\varphi^\beta}\varphi\right)\Box\varphi + \left((m-1)\alpha(\alpha-1)\varphi^{\alpha-2} + \frac{m-2}{2\alpha}\frac{(\alpha-\beta)\omega}{\varphi^\beta}\right)\nabla_\lambda\varphi\nabla^\lambda\varphi + \left(\frac{m-2}{2\alpha}\varphi U' - \frac{m}{2}U\right) = \kappa T \\
E_i = 0
\end{cases}
\tag{7.4.1}
$$

By setting $\beta = 2 - \alpha$ and $\omega = -\frac{\alpha^2(m-1)}{m-2}$ these become

$$
\begin{cases}
\varphi^\alpha R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{1}{m-2}Tg_{\alpha\beta}\right) + \alpha\varphi^{\alpha-1}\nabla_{\alpha\beta}\varphi + \frac{\alpha}{m-2}\varphi^{\alpha-1}\Box\varphi g_{\alpha\beta} + \left(\alpha(\alpha-1) - \frac{\alpha^2(m-1)}{m-2}\right)\varphi^{\alpha-2}\nabla_\alpha\varphi\nabla_\beta\varphi + \\
\qquad + \frac{\alpha(\alpha-1)}{m-2}\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} - \frac{1}{m-2}Ug_{\alpha\beta} \\
\boxed{\left(\alpha(m-1) - \frac{m-2}{\alpha}\frac{\alpha^2(m-1)}{m-2}\right)\varphi^{\alpha-1}\Box\varphi} + \boxed{\left((m-1)\alpha(\alpha-1) - \frac{m-2}{2\alpha}2(\alpha-1)\frac{\alpha^2(m-1)}{m-2}\right)\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi} + \left(\frac{m-2}{2\alpha}\varphi U' - \frac{m}{2}U\right) = \kappa T \\
E_i = 0
\end{cases}
\tag{7.4.2}
$$

i.e.

$$
\begin{cases}
\varphi^\alpha R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{1}{m-2}Tg_{\alpha\beta}\right) + \alpha\varphi^{\alpha-2}\left(\varphi\nabla_{\alpha\beta}\varphi + \frac{2-m-\alpha}{m-2}\nabla_\alpha\varphi\nabla_\beta\varphi\right) + \frac{1}{m-2}\left(\alpha\varphi^{\alpha-1}\Box\varphi + \alpha(\alpha-1)\varphi^{\alpha-2}\nabla_\lambda\varphi\nabla^\lambda\varphi - U\right)g_{\alpha\beta} \\
\frac{m-2}{\alpha}\varphi U' - mU = 2\kappa T \\
E_i = 0
\end{cases}
\tag{7.4.3}
$$

Then we can set $\alpha = \frac{m-2}{2}$ and $U = f(r(\varphi)) - \varphi^\alpha r(\varphi)$ to obtain

$$
\begin{cases}
f'(\mathcal{R})R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \frac{1}{m-2}Tg_{\alpha\beta}\right) + \frac{m-2}{2}\varphi^{\frac{m-2}{2}-1}\nabla_{\alpha\beta}\varphi - \frac{3(m-2)}{4}\varphi^{\frac{m-2}{2}-2}\nabla_\alpha\varphi\nabla_\beta\varphi + \frac{1}{2}\varphi^{\frac{m-2}{2}-1}\Box\varphi g_{\alpha\beta} + \\
\qquad\quad + \frac{m-4}{4}\varphi^{\frac{m-2}{2}-2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\alpha\beta} - \frac{1}{m-2}\left(f(r(\varphi)) - \varphi^\alpha r(\varphi)\right) g_{\alpha\beta} \\
\varphi^\alpha r(\varphi) - \frac{m}{2}f(r(\varphi)) = \kappa T \\
E_i = 0
\end{cases}
\tag{7.4.4}
$$

Thus we can compare these equations, which are equivalent to Brans–Dicke equations for the choices we did, to equations of Palatini $f(\mathcal{R})$-theories, which we already considered in (7.2.17). By tracing them, we obtain

$$
R = -\frac{2\kappa}{m-2}\tilde{T} + \left[(m-1)\frac{1}{\varphi}\Box\varphi + \frac{(m-6)(m-1)}{4\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi\right]
\tag{7.4.5}
$$

so that we can recast the original equations as

$$
R_{\mu\nu} = \kappa\left(\tilde{T}_{\mu\nu} - \frac{1}{m-2}\tilde{T}g_{\mu\nu}\right) + \frac{m-2}{2\varphi}\nabla_{\mu\nu}\varphi + \frac{1}{2\varphi}\Box\varphi g_{\mu\nu} - \frac{3(m-2)}{4\varphi^2}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{(m-4)}{4\varphi^2}\nabla_\rho\varphi\nabla^\rho\varphi g_{\mu\nu}
\tag{7.4.6}
$$

Then we can also expand the energy momentum tensor $\tilde{T}_{\mu\nu}$ as

$$
\kappa f'(\mathcal{R})\left(\tilde{T}_{\alpha\beta} - \frac{1}{m-2}\tilde{T}g_{\alpha\beta}\right) = \kappa\left(T_{\alpha\beta} - \frac{1}{m-2}Tg_{\alpha\beta}\right) + \frac{1}{m-2}\left(f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})\right)g_{\alpha\beta}
\tag{7.4.7}
$$

Accordingly, one can recast the equations for Palatini-$f(\mathcal{R})$ theories in the equivalent form

$$
\begin{aligned}
f'(\mathcal{R})R_{\mu\nu} = {}& \kappa\left(T_{\mu\nu} - \frac{1}{m-2}Tg_{\mu\nu}\right) - \frac{1}{m-2}\left(f(\mathcal{R}) - f'(\mathcal{R})\mathcal{R}\right)g_{\mu\nu} + \\
& + \frac{m-2}{2}\varphi^{\frac{m-2}{2}-1}\nabla_{\mu\nu}\varphi + \frac{1}{2}\varphi^{\frac{m-2}{2}-1}\Box\varphi g_{\mu\nu} - \frac{3(m-2)}{4}\varphi^{\frac{m-2}{2}-2}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{(m-4)}{4}\varphi^{\frac{m-2}{2}-2}\nabla_\rho\varphi\nabla^\rho\varphi g_{\mu\nu}
\end{aligned}
\tag{7.4.8}
$$

which, in fact, agree with Brans–Dicke equations, i.e. the first item in (7.4.4), while the second item there coincides with the master equation in Palatini $f(\mathcal{R})$-theories

$$
f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) = \kappa T
\tag{7.4.9}
$$

Accordingly, if we consider a Palatini $f(\mathcal{R})$-theory, so that the function $f(\mathcal{R})$ is fixed, on a spacetime M of dimension m, then we can compute the conformal factor $\varphi = f'(\mathcal{R})^{\frac{2}{m-2}}$, invert this relation to define the function $\mathcal{R} = r(\varphi)$ and the potential $U(\varphi) = f(r(\varphi)) - \varphi^\alpha r(\varphi)$ as well as the parameters $\alpha = \frac{m-2}{2}$, $\beta = \frac{6-m}{2}$, and $\omega = -\frac{(m-2)(m-1)}{4}$. By using this potential and these parameters, we define a Brans–Dicke theory which we say is *associated* the original Palatini $f(\mathcal{R})$-theory.

On the other hand, if we consider a Brans–Dicke theory with a potential $U(\varphi)$ and parameters $\alpha = \frac{m-2}{2}$, $\beta = \frac{6-m}{2}$, and $\omega = -\frac{(m-2)(m-1)}{4}$ *and* there exists a function $f(\mathcal{R})$ such that the equation $f'(\mathcal{R}) = \varphi$ is inverted to $\mathcal{R} = r(\varphi)$ and the potential can be expressed as $U(\varphi) = f(r(\varphi)) - \varphi^\alpha r(\varphi)$ then we can define a Palatini $f(\mathcal{R})$-theory using that function $f(\mathcal{R})$, which is also called the theory *associated* to the Brans–Dicke theory under consideration.

In this way we establish a one-to-one correspondence between Palatini $f(\mathcal{R})$-theories and a subclass of Brans–Dicke theories and corresponding theories are dynamically equivalent. In fact, let us consider a solution $(g, \tilde{\Gamma}, \phi)$ of a Palatini $f(\mathcal{R})$ theory. we can define a conformal factor $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$ so that (g, φ, ϕ) is a solution of the associated Brans–Dicke theory. On the other hand, if (g, φ, ϕ) is a solution of the Brans–Dicke theory, then we can define

$$\tilde{\Gamma}^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{u)}\right)\overset{*}{\nabla}_{\epsilon}\ln\varphi \tag{7.4.10}$$

so that $(g, \tilde{\Gamma}, \phi)$ is a solution of the associated Palatini $f(\mathcal{R})$-theory.

Thus we define also a one-to-one correspondence between solutions which shows dynamical equivalence. Let us stress that this equivalence is established at the level of field equations and solutions. We shall establish it at the level of actions below.

Before entering into it, however, let us make few comments. First, notice that Brans–Dicke theories, as we introduced them, are purely metric theories, while $f(\mathcal{R})$-theories are Palatini. Accordingly, configuration bundles have different dimensions. Then dynamical equivalence cannot come from a simple isomorphism of configuration bundles. In fact, one can see that the connection $\tilde{\Gamma}$ defined by the correspondence (7.4.10) depends on fields *and* on the first derivatives of φ and g.

Second, usually, in Brans–Dicke theories macroscopic objects fall freely along geodesics of g, while in Palatini $f(\mathcal{R})$-theories they fall along geodesics of $\tilde{\Gamma}$. This shows that free fall is something which is superimposed to the variational principle, as a somehow independent hypothesis. Accordingly, one should be careful to claim generically that Brans–Dicke theories and Palatini $f(\mathcal{R})$-theories are physically equivalent, even pointing out that just *some* Brans–Dicke theories participate to the correspondence.

Third, the two conformal transformations performed in Palatini-$f(\mathcal{R})$ theories and in Brans–Dicke theories are somehow *different*. In the first place, to solve Palatini-$f(\mathcal{R})$ theories, we defined the conformal metric $\tilde{g} = \varphi g$ (leaving the connection unchanged). Then the connection is eliminated by solving the second field equation. Then one goes back to the original metric by the transformation $g = \varphi^{-1}\tilde{g}$. However, at this point the connection $\tilde{\Gamma}$ has been eliminated as an independent field and substituted by $\tilde{\Gamma} = \{\tilde{g}\}$. Accordingly, the transformation $g = \varphi^{-1}\tilde{g}$ does act both on the metric and connection (indeed, one transforms the whole Ricci tensor $\tilde{R}_{\alpha\beta}$). In other words, these two transformations are not the inverse to each other as it seems at first and, in fact, one does not obtain the original theory but Brans–Dicke theory which is a purely metric theory. In the following Section we shall consider the same transformations at the level of Lagrangians to clarify their roles.

Helmholtz conformal frame

For any Palatini $f(\mathcal{R})$-theory, one can define the function $\mathcal{R} = r(\varphi)$ by inverting the equation $\varphi^{\alpha} = f'(\mathcal{R})$, where we set $\alpha = \frac{m-2}{2}$. Then one can consider the Helmholtz Lagrangian

$$\boldsymbol{L}_H = \left[\tfrac{\sqrt{g}}{2\kappa}\left(\varphi^{\alpha}(\mathcal{R} - r(\varphi)) + f(r(\varphi))\right) + L_m(g, \phi)\right]d\sigma = \left[\tfrac{\sqrt{g}}{2\kappa}\left(\varphi^{\alpha}\mathcal{R} + (f\left(r(\varphi)\right) - \varphi^{\alpha}r(\varphi))\right) + L_m(g, \phi)\right]d\sigma \tag{7.4.11}$$

which is considered as a Lagrangian for the independent fields $(g_{\mu\nu}, \tilde{\Gamma}^{\alpha}_{\beta\mu}, \varphi, \phi)$.

Field equations of this Lagrangian are

$$\begin{cases} \varphi^\alpha \tilde{R}_{(\alpha\beta)} - \frac{1}{2} f(r(\varphi)) g_{\alpha\beta} = \kappa T_{\alpha\beta} + \frac{1}{2} \varphi^\alpha \left(\mathcal{R} - r(\varphi) \right) g_{\alpha\beta} \\ \tilde{\nabla}_\lambda \left(\sqrt{g} \varphi^\alpha g^{\mu\nu} \right) = 0 \end{cases} \qquad \begin{cases} \mathcal{R} = r(\varphi) \\ E_i = 0 \end{cases} \tag{7.4.12}$$

For the variation of the Lagrangian we have

$$\begin{aligned} \delta L_H &= \frac{\sqrt{g}}{2\kappa} \left(\alpha\varphi^{\alpha-1} \mathcal{R} + f'\left(r(\varphi)\right) r'(\varphi) - \alpha\varphi^{\alpha-1} r(\varphi) - \varphi^\alpha r'(\varphi) \right) \delta\varphi + \frac{\sqrt{g}}{2\kappa} \left(\varphi^\alpha \tilde{R}_{(\mu\nu)} - \frac{1}{2} \left(\varphi^\alpha \mathcal{R} + f\left(r(\varphi)\right) - \varphi^\alpha r(\varphi) \right) g_{\mu\nu} \right) \delta g^{\mu\nu} + \\ &\quad + \frac{\sqrt{g}}{2\kappa} \varphi^\alpha g^{\beta\mu} \tilde{\nabla}_\lambda \delta \tilde{u}^\lambda_{\beta\mu} - \frac{\sqrt{g}}{2} T_{\mu\nu} \delta g^{\mu\nu} + \left(p_i - \nabla_\mu p_i^\mu \right) \delta\phi^i + \nabla_\mu \left(p_i^\mu \delta\phi^i \right) = \\ &= \frac{\sqrt{g}}{2\kappa} \alpha\varphi^{\alpha-1} \left(\mathcal{R} - r(\varphi) \right) \delta\varphi + \frac{\sqrt{g}}{2\kappa} \left(\varphi^\alpha \tilde{R}_{(\mu\nu)} - \frac{1}{2} f(r(\varphi)) g_{\mu\nu} - \frac{1}{2} \varphi^\alpha \left(\mathcal{R} - r(\varphi) \right) g_{\mu\nu} - \kappa T_{\mu\nu} \right) \delta g^{\mu\nu} - \tilde{\nabla}_\lambda \left(\frac{\sqrt{g}}{2\kappa} \varphi^\alpha g^{\beta\mu} \right) \delta \tilde{u}^\lambda_{\beta\mu} + E_i \delta\phi^i + \\ &\quad + \overset{*}{\nabla}_\mu \left(p_i^\mu \delta\phi^i + \frac{\sqrt{g}}{2\kappa} \varphi^\alpha g^{\beta\mu} \delta \tilde{u}^\lambda_{\beta\mu} \right) \end{aligned} \tag{7.4.13}$$

where we set, as usual, $E_i = p_i - \nabla_\mu p_i^\mu$ for the matter field equations. Then we have field equations in the form

$$\begin{cases} \varphi^\alpha \tilde{R}_{(\alpha\beta)} - \frac{1}{2} f(r(\varphi)) g_{\alpha\beta} = \kappa T_{\alpha\beta} + \frac{1}{2} \varphi^\alpha \left(\mathcal{R} - r(\varphi) \right) g_{\alpha\beta} \\ \tilde{\nabla}_\lambda \left(\sqrt{g} \varphi^\alpha g^{\mu\nu} \right) = 0 \\ \mathcal{R} = r(\varphi) \\ E_i = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} f'(\mathcal{R}) \tilde{R}_{(\alpha\beta)} - \frac{1}{2} f(\mathcal{R}) g_{\alpha\beta} = \kappa T_{\alpha\beta} \\ \tilde{\nabla}_\lambda \left(\sqrt{g} \varphi^\alpha g^{\mu\nu} \right) = 0 \\ f'(\mathcal{R}) = \varphi^\alpha \\ E_i = 0 \end{cases} \tag{7.4.14}$$

while from the trace of the first equation by $g^{\alpha\beta}$ one has $\varphi^\alpha \mathcal{R} - \frac{m}{2} f(\mathcal{R}) = \kappa T$.

These equations are equivalent to field equations of the original Palatini $f(\mathcal{R})$-theory, namely (7.2.5) together with the definition of the conformal factor $\varphi^\alpha = f'(\mathcal{R})$. Accordingly, the correspondence

$$(g_{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i) \quad \longleftrightarrow \quad (g_{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu}, \varphi, \phi^i) \tag{7.4.15}$$

defined by $\varphi^\alpha = f'(\mathcal{R})$ does in fact send solutions into solutions and proves that the two theories are dynamically equivalent.

The Lagrangian $\boldsymbol{L}_H$ is defined on a suitable jet prolongation of the configuration bundle

$$\mathrm{Hel} = \mathrm{Lor}(M) \times_M (M \times \mathbb{R}) \times_M \mathrm{Con}(M) \times_M B \tag{7.4.16}$$

which has coordinates $(x^\mu, g_{\mu\nu}, \varphi, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i)$. Let us refer to this choice of independent fields as the *Helmholtz conformal frame*, also called *twiddle frame*. The original choice $(g, \tilde{\Gamma}, \phi)$ is instead often called the *Jordan conformal frame*.

Jordan conformal frame

The Helmholtz conformal frame is obtained from the Jordan conformal frame by introducing the momentum φ conjugated to $\mathcal{R}$ as an independent variable.

The procedure is similar, in principle if not (yet) in practice, to the Routh transform and Helmholtz technique, as introduced in Section 5.3. It would be worth doing that way explicitly, though we still need some more theory about it.

Starting from the Helmholtz frame, one can obtain the Jordan frame by freezing the conformal factor by setting $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$ (or, equivalently, $r(\varphi) = \mathcal{R}$). Let us remind that, usually, it is not a good practice to use on shell identities (as $r(\varphi) = \mathcal{R}$ is) into the Lagrangian, since sometimes one obtains equivalent Lagrangians, sometimes one does not.

However, one does not need here to derive the Lagrangian in the Jordan frame from a general prescription, since we did check that the reduced Jordan Lagrangian is dynamically equivalent to the Helmholtz Lagrangian; see Subsection 7.4.2.

At bundle level, let us define

$$\mathrm{Jor} = \mathrm{Lor}(M) \times_M \mathrm{Con}(M) \times_M B \tag{7.4.17}$$

Of course, the map $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$ cannot be seen as a bundle map, certainly not as a map $\Phi : \mathrm{Jor} \to \mathrm{Hel}$, since $\mathcal{R}$ depends on derivatives of the connection. However, we can define the maps:

$$\Phi : \mathrm{Hel} \to \mathrm{Jor} : (g, \varphi, \tilde{\Gamma}, \phi) \mapsto (g, \tilde{\Gamma}, \phi)$$

$$\Psi : J(\mathrm{Jor}) \to \mathrm{Hel} : (g, j^1\tilde{\Gamma}, \phi) \mapsto \left(g, \varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}, \tilde{\Gamma}, \phi\right) \tag{7.4.18}$$

where we set $J(\mathrm{Jor}) = \mathrm{Lor}(M) \times_M J^1\mathrm{Con}(M) \times_M B$.

The map Φ simply forgets about the conformal factor (which is not that loss, since the information about it is built in the connection and the metric anyway). The map Ψ takes the information encoded into the connection (and its derivatives) and the metric and it defines the conformal factor.

The situation can be summarised by the diagram here on the side.

Let us remark that we have grown so confident in the notation to overlook some global aspects, taking them for granted. We defined the map locally, in a coordinate system, and we assumed that they define a global bundle map. This is done because we *know* that the quantity $(f'(\mathcal{R}))^{\frac{2}{m-2}}$ is a scalar and it defines a point in $M \times \mathbb{R}$, which was precisely defined to describe scalar fields.
Thus we know that local expressions of the map glue together to define a global map.

Now these maps define maps between sections. First of all, to any section $\sigma \in \mathrm{Sec}(\mathrm{Hel})$ we can associate a section $\Phi_*\sigma \in \mathrm{Sec}(\mathrm{Jor})$ defined as $\Phi_*\sigma := \Phi \circ \sigma$. Similarly, we can define

$$\Psi_* : \mathrm{Sec}(\mathrm{Jor}) \to \mathrm{Sec}(\mathrm{Hel}) : \rho \mapsto \Psi \circ j\rho \tag{7.4.19}$$

where $j\rho$ denotes the prolongation of the section $\rho \in \mathrm{Sec}(\mathrm{Jor})$ to $J(\mathrm{Jor})$.

These two maps between sections are not one-to-one. One cannot obtain any possible conformal factor from a connection and metric. Thus, for example, Ψ_* is not surjective. And, of course, Φ is not injective since all conformal factors are forgotten to the same point in Jor. However, one has the property

$$\Phi_* \circ \Psi_* = \mathbb{I} \tag{7.4.20}$$

(which, in fact, implies that Ψ_* is injective and Φ_* is surjective, as they happen to be) while $\Psi_* \circ \Phi_* \neq \mathbb{I}$. In fact, starting from a section σ of Hel in which the conformal factor is not given by $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$, the section obtained by $\Psi_* \circ \Phi_*(\sigma)$ does have $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$, instead.

However, if one starts from a section $\sigma \in \text{Sec}(\text{Hel})$ which is a solution of field equations, then it needs to have $\varphi = \left(f'(\mathcal{R})\right)^{\frac{2}{m-2}}$ and, in that case, $\Psi_* \circ \Phi_*(\sigma) = \sigma$. In other words, one has that the property

$$\Psi_* \circ \Phi_* = \mathbb{I} \tag{7.4.21}$$

holds true *on shell* (and on shell, only).

Thus the maps Φ_* and Ψ_*, restricted on shell, are in fact, the inverse one of the other.

> To be more geometric, one can go back to the definition of differential equations as sub-manifolds of a suitable jet prolongation. Then if we prolong the maps Ψ and Φ *and* we restrict to the sub-manifolds representing field equations we do obtain invertible maps.
> In a sense, that is precisely what one checks when proving dynamical equivalence at the level of field equations.
> (Take a breath and be sure you get this point. It does not save you any computation but it gives a meaning to it.)

At the level of actions, we have the Helmholtz Lagrangian which is defined on

$$J_k\text{Hel} := J^k\text{Lor}(M) \times_M (M \times \mathbb{R}) \times_M J^1\text{Con}(M) \times_M J^1B \tag{7.4.22}$$

If the matter fields need a connection to define covariant derivatives, then one has $k = 1$, otherwise $k = 0$ is enough.

The original Lagrangian for Palatini $f(\mathcal{R})$-theories, which we could call the *Jordan Lagrangian*, is defined on

$$J_k\text{Jor} := J^k\text{Lor}(M) \times_M J^1\text{Con}(M) \times_M J^1B \tag{7.4.23}$$

Thus, we have the Lagrangians as horizontal forms on these prolongations and, consequently, on all higher prolongations, since any form $\boldsymbol{\omega}$ on J^kC defines forms $(\pi_k^h)^*\boldsymbol{\omega}$ on any prolongation J^hC with $h \geq k$. The prolonged forms $(\pi_k^h)^*\boldsymbol{\omega}$ have the same local expression of $\boldsymbol{\omega}$ just they are thought as forms on J^hC.

In any event, in our current situation, both the maps Ψ and Φ extend as maps between $J_k\text{Hel}$ and $J_k\text{Jor}$.

Let us denote the prolongations as

$$J_k\Phi : J_k\text{Hel} \rightarrow J_k\text{Jor} \qquad\qquad J_k\Psi : J_k\text{Jor} \rightarrow J_k\text{Hel} \tag{7.4.24}$$

> Notice that if the map Ψ depended on the second derivatives of the connection, then we could not prolong it to $J_k\text{Jor}$.
> Again, these two maps are not the inverse one of the other (nor they are invertible) though they induce maps on sections which are invertible on shell and they prolong to maps which are invertible on the sub-manifolds which describe field equations.

Given this situation, we can drag the Lagrangians along the maps and define

$$(J_k\Phi)^* \boldsymbol{L}_J = \boldsymbol{L}'_H \qquad\qquad (J_k\Psi)^* \boldsymbol{L}_H = \boldsymbol{L}'_J \tag{7.4.25}$$

which both are again horizontal forms on $J_k\text{Hel}$ and $J_k\text{Jor}$, respectively. Since both the maps are vertical, for the pull-back one simply has to replace the map expression in the Lagrangian densities. Thus we have

$$
\begin{aligned}
(J_k\Phi)^* \boldsymbol{L}_J &= \boldsymbol{L}'_H = \left[\tfrac{\sqrt{g}}{2\kappa} f(\mathcal{R}) + L_m(g,\phi) \right] \, d\sigma \\
(J_k\Psi)^* \boldsymbol{L}_H &= \left[\tfrac{\sqrt{g}}{2\kappa} \left(f'(\mathcal{R})\mathcal{R} + f(\mathcal{R}) - f'(\mathcal{R})\mathcal{R} \right) + L_m(g,\phi) \right] \, d\sigma \equiv \boldsymbol{L}_J
\end{aligned}
\tag{7.4.26}
$$

Hence the theory in the Jordan frame is defined by the one in the Helmholtz frame.

On the other hand, the theory defined on Hel by the Lagrangian form $\boldsymbol{L}'_H$ is not dynamically equivalent to the original Palatini $f(\mathcal{R})$-theory. In fact, the Lagrangian $\boldsymbol{L}'_H$ does not depend on φ which is then free to get any value on shell. Hence, although any solution of the Palatini $f(\mathcal{R})$-theory is a solution of the theory based on $\boldsymbol{L}'_H$, the latter has many more solutions which are not solutions of $\boldsymbol{L}_H$.

If we try to apply the general results obtained in Chapter 5, the result is inconclusive since $\boldsymbol{L}_H \neq \boldsymbol{L}'_H$, i.e. it is not a map which preserves the Lagrangian *(check this is enough)* dynamics. We know that $\boldsymbol{L}_H$ is an extension of $\boldsymbol{L}_J$ (since $(J_k\Psi)^* \boldsymbol{L}_H = \boldsymbol{L}_J$), though to establish the dynamical equivalence further information is required.

In particular, we could show that, on any solution of $\boldsymbol{L}_H$, one has $\varphi = \left(f'(\mathcal{R}) \right)^{\frac{2}{m-2}}$, i.e. it is contained in the image of the map Ψ. That is of course true, so the theories are equivalent, though establishing it is a bit uncomfortable.

Let us also point out that in Jordan, as in Helmholtz frame, the distances are measured by g and the free fall is described by $\tilde{\Gamma}$. That is the input we have from EPS and the construction of ETG which is based on it.

> Let us also point out that the motion of test particles in a gravitational theory is not obtained by eikonal approximation of matter field, if not as a wishful thinking. On the contrary, it is imposed as an independent axiom which has nothing to do with matter fields.
>
> I mean, of course it would be cool, however, it is not the way it is done in practice. Think about classical tests in solar system.
>
> There one considers a Schwarzschild solution g—which is a vacuum solution—and Mercury is moving along the geodesics of g anyway. What should be the matter field which produce the geodesics equations for g and which represents Mercury's constituents at a fundamental level?
>
> By the way, Schwarzschild solution is a spherically symmetric solutions, thus any field consistently coupled to that should be spherically symmetric as well, while Mercury, is not since it stands at some point of its orbits and not others. Consequently, even if one would think of it as a macroscopic manifestation of a fundamental matter field, back-reaction would need to be neglected. Hence there is no (known) consistent model for the solar system to describe the classical test, without introducing geodesics equation by hand or, alternatively, by introducing background non-dynamical fields, spoiling the theory as a relativistic theory.

Einstein conformal frame

The Einstein conformal frame is a field coordinate system in which the theory seems standard GR. On the bundle Ein = Hel (the same on which Helmholtz frame is defined) one has coordinates $(x^\mu, g_{\mu\nu}, \varphi, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i)$. However, one can use equivalent fields

$$
(x^\mu, \tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}, \varphi, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i)
\tag{7.4.27}
$$

This is simply a field transformation (i.e. it does not depend on fields' derivatives) and, accordingly, it does not change the theory. The Lagrangian in the new coordinates reads as

$$\boldsymbol{L}_E(\tilde{g}_{\mu\nu}, \tilde{\Gamma}^\alpha_\beta, \varphi, \phi) = \left[\sqrt{\tilde{g}} \left(\tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} + \varphi^{-\frac{m}{2}} \left(f\left(r(\varphi)\right) - \varphi^\alpha r(\varphi) \right) \right) + L_m(\varphi^{-1}\tilde{g}, \phi) \right] d\sigma \tag{7.4.28}$$

which is in fact a standard metric-affine GR Lagrangian with an additional matter field φ (which enters at order zero in the dynamics) and the effective matter Lagrangian

$$\tilde{L}_m(\tilde{g}, \varphi, \phi) = L_m(\varphi^{-1}\tilde{g}, \phi) + \sqrt{\tilde{g}}\varphi^{-\frac{m}{2}} \left(f\left(r(\varphi)\right) - \varphi^{\frac{m-2}{2}} r(\varphi) \right) \tag{7.4.29}$$

Since the conformal factor φ enters at zero order, its field equation is algebraic and, through that, it inherits its dynamics from the gravitational field mediating together with $\tilde{g}$ the interaction with matter.

> In Einstein conformal frame, one has fields $(\tilde{g}_{\mu\nu}, \varphi, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i)$. In view of the dynamical equivalence with the Jordan frame, $\tilde{\Gamma}$ still describes free fall, while distances are described by $g = \varphi^{-1}\tilde{g}$. Accordingly, the conformal field exactly measures the discrepancy between the metrics describing free fall and distances.
>
> At the same time, although the dynamics is similar to the one of standard GR for the metric $\tilde{g}$, the theory that we are considering is not standard GR for the metric $\tilde{g}$. If it were, distances would be described by $\tilde{g}$, not by g. That is not an inessential difference: if one allows an atomic clock going around and probing regions at different curvatures (i.e. at different values of values of the conformal factor), in the two theories a drifting of the frequency of the clock is experienced, which makes the two theories observationally different; end of the story.
>
> Let us also remark that depending on the Weyl frame one considers, the conformal factor, as a dynamical field, is included in the gravitational field (in the Jordan frame) or in the matter content (as in the Einstein frame). It is not much of a difference from something we just see to happen in mechanics: the term for centrifugal forces, depending on the reference frame one chooses, sometimes appears as a term in kinetic energy (when the frame is inertial), sometimes it appears as a force (for example, when the reference frame is fixed on a rotating plane). Here and there it is just because the distinction between kinetic energy and forces, as well as gravitational and inertial field, is to some extent fictional and unessential.

In this theory, keeping the EPS interpretation of Palatini $f(\mathcal{R})$-theories as ETG, one has dark sources described by the interactions mediated through φ (together with φ itself), the free fall of test particles is dictated by $\tilde{g}$ while operational definitions for distances and clocks (original encoded by the metric g) are now described by $g = \varphi^{-\frac{m-2}{2}} \tilde{g}$.

In this case the transformation $\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$ can be regarded as an invertible map

$$\Phi : \text{Hel} \to \text{Ein} \qquad \bar{\Phi} : \text{Ein} \to \text{Hel} \tag{7.4.30}$$

which, of course, defines a one-to-one correspondence Φ_* between Sec(Hel) and Sec(Ein) (this time the correspondence being one-to-one also off shell) and can be prolonged to $J_k\Phi : J_k\text{Hel} \to J_k\text{Ein}$. One can check that the dynamics is given by

$$\boldsymbol{L}_E = \left(J_k\bar{\Phi} \right)^* \boldsymbol{L}_H \tag{7.4.31}$$

> To compute the pull-back $\left(J_k\bar{\Phi} \right)^* \boldsymbol{L}_H$, one needs to write everything in terms of the conformal metric $\tilde{g}$ (without touching the connection $\tilde{\Gamma}$ which is an independent field, untouched by the transformation Φ). The one has
>
> $$\left(J_k\bar{\Phi} \right)^* L_H = \left[\frac{\sqrt{\tilde{g}}}{2\kappa}\varphi^{-\frac{m}{2}} \left(\varphi\varphi^\alpha \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} + \left(f\left(r(\varphi)\right) - \varphi^\alpha r(\varphi) \right) \right) + L_m(\varphi^{-1}\tilde{g}, \phi) \right] = \left[\frac{\sqrt{\tilde{g}}}{2\kappa} \left(\tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} + \varphi^{-\frac{m}{2}} \left(f\left(r(\varphi)\right) - \varphi^\alpha r(\varphi) \right) \right) + L_m(\varphi^{-1}\tilde{g}, \phi) \right] \equiv L_E \tag{7.4.32}$$

EPS conformal frame

If the matter Lagrangian L_m is conformally invariant (as, e.g., Maxwell Lagrangian in 4d) then $L^*(\varphi, \tilde{g}, \psi) = L_m(\tilde{g}, \psi)$ and one still has Maxwell Lagrangian in the new frame as well. If not, as it happens for example with a Klein–Gordon field, one can extend the conformal transformation to act on ϕ as

$$\hat{E}_\omega : \text{Ein} \to \text{EPS} : (g, \varphi, \tilde{\Gamma}, \phi) \mapsto (\tilde{g}, \varphi, \tilde{\Gamma}, \tilde{\phi} = \varphi^\omega \phi) \tag{7.4.33}$$

where we set EPS = Ein = Hel and ω is a real exponent to be chosen later, which is a one-to-one bundle map. The frame $(\tilde{g}, \varphi, \tilde{\Gamma}, \tilde{\phi})$ is called the *EPS conformal frame* and the Lagrangian reads as

$$\boldsymbol{L}_{\text{EPS}} = (J_k \hat{E}_\omega)^* \boldsymbol{L}_E = \left[\sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} + \hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) \right] d\sigma \tag{7.4.34}$$

where we set $\hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) = L_m(\varphi^{-1}\tilde{g}, \varphi^{-\omega}\tilde{\phi}) + \sqrt{\tilde{g}} U(\varphi) \varphi^{-\frac{m}{2}}$.

If one sets the matter Lagrangian to represent a Klein–Gordon field in the Jordan frame, in the EPS frame, setting $4\omega := 2 - m$, one has

$$\hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) = -\frac{\sqrt{\tilde{g}}}{2} \left(\tilde{g}^{\mu\nu} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} + \tilde{\mu}^2 \tilde{\phi}^2 \right) + \frac{\sqrt{\tilde{g}}}{2\kappa} U(\varphi) \varphi^{-\frac{m}{2}} + \overset{*}{\nabla}_\mu \left(\frac{\sqrt{\tilde{g}}}{2} \frac{2-m}{4} \tilde{g}^{\mu\nu} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \tilde{\phi}^2 \right) \tag{7.4.35}$$

where one has

$$\tilde{\mu}^2 = \frac{\mu^2}{\varphi} - \frac{m-2}{4\varphi} \Box\varphi - \frac{(m-6)(m-2)}{16\varphi^2} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \tag{7.4.36}$$

Let us first prove an identity from

$$\overset{*}{\nabla}_\mu \left(-\frac{\sqrt{\tilde{g}}}{2} \tilde{g}^{\mu\nu} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \tilde{\phi}^2 \right) = \frac{\sqrt{\tilde{g}}}{2} \left(\left(\frac{m-2}{2} + 2\omega + 1 \right) \varphi^{-\frac{m-2}{2} - 2\omega - 2} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - \varphi^{-\frac{m-2}{2} - 2\omega - 1} \tilde{g}^{\mu\nu} \overset{\tilde{g}}{\nabla}_\mu \overset{*}{\nabla}_\nu \varphi \right) \tilde{\phi}^2 - \sqrt{\tilde{g}} \tilde{g}^{\mu\nu} \tilde{\phi} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \nabla_\mu \tilde{\phi} \tag{7.4.37}$$

which will be used for integration by parts in the form

$$\omega \sqrt{\tilde{g}} \tilde{g}^{\mu\nu} \tilde{\phi} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \nabla_\mu \tilde{\phi} = \omega \frac{\sqrt{\tilde{g}}}{2} \left(\frac{m-2+4\omega+2}{2} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - \left(\varphi \Box\varphi + \frac{m-2}{2} \tilde{g}^{\alpha\epsilon} \nabla_\epsilon \varphi \nabla_\alpha \varphi \right) \right) \varphi^{-\frac{4\omega+2+m}{2}} \tilde{\phi}^2 + \overset{*}{\nabla}_\mu \left(\omega \frac{\sqrt{\tilde{g}}}{2} \tilde{g}^{\mu\nu} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \tilde{\phi}^2 \right) =$$
$$= \omega \frac{\sqrt{\tilde{g}}}{2} \left(-\varphi \Box\varphi + (2\omega+1) \tilde{g}^{\alpha\epsilon} \nabla_\epsilon \varphi \nabla_\alpha \varphi \right) \varphi^{-\frac{4\omega+2+m}{2}} \tilde{\phi}^2 + \overset{*}{\nabla}_\mu \left(\omega \frac{\sqrt{\tilde{g}}}{2} \tilde{g}^{\mu\nu} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \tilde{\phi}^2 \right) \tag{7.4.38}$$

Then the matter Lagrangian in EPS framework is

$$\hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) = -\frac{\sqrt{\tilde{g}}}{2} \varphi^{-\frac{m}{2}} \left(\varphi \tilde{g}^{\mu\nu} \nabla_\mu \left(\varphi^{-\omega}\tilde{\phi} \right) \nabla_\nu \left(\varphi^{-\omega}\tilde{\phi} \right) + \mu^2 \varphi^{-2\omega} \tilde{\phi}^2 \right) + \frac{\sqrt{\tilde{g}}}{2\kappa} U(\varphi) \varphi^{-\frac{m}{2}} =$$
$$= -\frac{\sqrt{\tilde{g}}}{2} \left(\varphi^{-\frac{m-2}{2}} \tilde{g}^{\mu\nu} \left(\varphi^{-2\omega} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} - 2\omega \varphi^{-2\omega-1} \tilde{\phi} \nabla_\mu \tilde{\phi} \nabla_\nu \varphi + \omega^2 \varphi^{-2\omega-2} \nabla_\mu \varphi \nabla_\nu \varphi \tilde{\phi}^2 \right) + \mu^2 \varphi^{-\frac{m}{2} - 2\omega} \tilde{\phi}^2 \right) + \frac{\sqrt{\tilde{g}}}{2\kappa} U(\varphi) \varphi^{-\frac{m}{2}} =$$
$$= -\frac{\sqrt{\tilde{g}}}{2} \left(\varphi^{-\frac{4\omega+m-2}{2}} \tilde{g}^{\mu\nu} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} + \left(\omega^2 \varphi^{-\frac{4\omega+m+2}{2}} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi + \mu^2 \varphi^{-\frac{m}{2} - 2\omega} \right) \tilde{\phi}^2 \right) + \frac{\sqrt{\tilde{g}}}{2\kappa} U(\varphi) \varphi^{-\frac{m}{2}} + \omega \sqrt{\tilde{g}} \tilde{g}^{\mu\nu} \tilde{\phi} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \nabla_\mu \tilde{\phi} =$$
$$= -\frac{\sqrt{\tilde{g}}}{2} \left(\varphi^{-\frac{4\omega+m-2}{2}} \tilde{g}^{\mu\nu} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} + \left(\omega^2 \varphi^{-\frac{4\omega+m+2}{2}} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi + \mu^2 \varphi^{-\frac{m}{2} - 2\omega} \right) \tilde{\phi}^2 \right) + \frac{\sqrt{\tilde{g}}}{2\kappa} U(\varphi) \varphi^{-\frac{m}{2}} +$$
$$- \omega \frac{\sqrt{\tilde{g}}}{2} \left(\varphi \Box\varphi - (2\omega+1) \tilde{g}^{\alpha\epsilon} \nabla_\epsilon \varphi \nabla_\alpha \varphi \right) \varphi^{-\frac{4\omega+2+m}{2}} \tilde{\phi}^2 + \overset{*}{\nabla}_\mu \left(\omega \frac{\sqrt{\tilde{g}}}{2} \tilde{g}^{\mu\nu} \varphi^{-\frac{m-2}{2} - 2\omega - 1} \nabla_\nu \varphi \tilde{\phi}^2 \right) \tag{7.4.39}$$

Since we want to fix ω so that one gets the correct kinetic energy for the field $\tilde{\phi}$, let us set $4\omega := 2 - m$. Thus we get

$$\hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) = -\frac{\sqrt{\tilde{g}}}{2}\left(\tilde{g}^{\mu\nu}\nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi} + \left(\frac{(m-2)^2}{16\varphi^2}\tilde{g}^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + \mu^2\varphi^{-1} - \frac{m-2}{4\varphi}\Box\varphi - \frac{(m-2)(m-4)}{8\varphi^2}\tilde{g}^{\alpha\epsilon}\nabla_\epsilon\varphi\nabla_\alpha\varphi\right)\tilde{\phi}^2\right) + \frac{\sqrt{\tilde{g}}}{2\kappa}U(\varphi)\varphi^{-\frac{m}{2}} +$$

$$+ \overset{*}{\nabla}_\mu\left(\frac{\sqrt{\tilde{g}}}{2}\frac{2-m}{4}\tilde{g}^{\mu\nu}\varphi^{-\frac{m-2}{2}-2\omega-1}\nabla_\nu\varphi\tilde{\phi}^2\right) = \tag{7.4.40}$$

$$= -\frac{\sqrt{\tilde{g}}}{2}\left(\tilde{g}^{\mu\nu}\nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi} + \left(\mu^2\varphi^{-1} - \frac{m-2}{4\varphi}\Box\varphi - \frac{(m-6)(m-2)}{16\varphi^2}\tilde{g}^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right)\tilde{\phi}^2\right) + \frac{\sqrt{\tilde{g}}}{2\kappa}U(\varphi)\varphi^{-\frac{m}{2}} + \overset{*}{\nabla}_\mu\left(\frac{\sqrt{\tilde{g}}}{2}\frac{2-m}{4}\tilde{g}^{\mu\nu}\varphi^{-\frac{m-2}{2}-2\omega-1}\nabla_\nu\tilde{\phi}^2\right)$$

This is not a free Klein–Gordon field (since the mass is not constant, it has a coupling with the conformal factor). However, for a conformal factor which is almost constant, namely $\varphi = \varphi_0 + \delta\varphi$, the new field can be considered as a (minimally coupled) Klein–Gordon field (with a weak interaction with the variation of the conformal factor) and a mass of $\tilde{\mu}^2 = \varphi_0^{-1}\mu^2$. Accordingly, the conformal transformation maps Klein–Gordon fields into (different) Klein–Gordon fields.

$$\hat{L}_m(\varphi, \tilde{g}, \tilde{\phi}) = -\frac{\sqrt{\tilde{g}}}{2}\left(\tilde{g}^{\mu\nu}\nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi} + \frac{\mu^2}{\varphi_0}\tilde{\phi}^2\right) + \frac{\sqrt{\tilde{g}}}{2\kappa}\tilde{U}(\delta\varphi, \tilde{g}, \tilde{\phi}) + \overset{*}{\nabla}_\mu\left(\frac{\sqrt{\tilde{g}}}{2}\frac{2-m}{4}\tilde{g}^{\mu\nu}\varphi^{-\frac{m-2}{2}-2\omega-1}\nabla_\nu\delta\varphi\tilde{\phi}^2\right) \tag{7.4.41}$$

where we defined the potential

$$\tilde{U}(\delta\varphi, \tilde{g}, \tilde{\phi}) = U(\varphi_0 + \delta\varphi)(\varphi_0 + \delta\varphi)^{-\frac{m}{2}} + \kappa\left(\frac{\mu^2}{\varphi_0}\frac{\delta\varphi}{\varphi_0 + \delta\varphi} + \frac{m-2}{4(\varphi_0+\delta\varphi)}\Box\delta\varphi + \frac{(m-6)(m-2)}{16(\varphi_0\delta\varphi)^2}\tilde{g}^{\mu\nu}\nabla_\mu\delta\varphi\nabla_\nu\delta\varphi\right)\tilde{\phi}^2 \tag{7.4.42}$$

Let us also remark that the pure divergence term is there to indicate that the change of conformal frame also affects conservation laws.

Brans–Dicke conformal frame

To go from Helmholtz to the Einstein conformal frame we kept the connection fixed and we redefined the metric. Instead, aiming to write everything in terms of $\tilde{g}$ one can again start from the Helmholtz Lagrangian and do the transformation

$$\Gamma^\alpha_{\beta\mu} = \tilde{\Gamma}^\alpha_{\beta\mu} + \frac{1}{2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\nabla_\epsilon\ln\varphi \tag{7.4.43}$$

this time choosing independent fields as $(g_{\mu\nu}, \Gamma^\alpha_{\beta\mu}, \varphi, \phi^i)$. The transformation has been chosen so that it transforms the independent connection $\tilde{\Gamma}$ as it would change under conformal transformation of the metric if it were the Levi Civita connection of the corresponding metric. We shall see below that then field equations in fact will fix $\Gamma^\alpha_{\beta\mu}$ to be the Levi Civita connection of the metric g.

This transformation needs to be treated more carefully than the one to Einstein frame, since the new fields do not depend on old fields only but also on their derivatives (on the first derivatives of φ, in particular). Let us define the bundle BD = Hel with fibered coordinates $(x^\mu, g_{\mu\nu}, \varphi, \tilde{\Gamma}^\alpha_{\beta\mu}, \phi^i)$ for the Helmholtz conformal frame and $(x^\mu, g_{\mu\nu}, \varphi, \Gamma^\alpha_{\beta\mu}, \phi^i)$ for the Brans–Dicke frame. Let us also define the bundle $J(\text{Hel}) = \text{Lor}(M) \times_M J^1(M \times \mathbb{R}) \times_M \text{Con}(M) \times_M B$ so that transformation (7.4.43) induces a map

$$\Phi : J(\text{Hel}) \to \text{BD} : (g, j^1\varphi, \tilde{\Gamma}, \phi) \mapsto (g, \varphi, \Gamma, \phi) \tag{7.4.44}$$

Let us remark that this map is not one-to-one.

The original Helmholtz Lagrangian is a horizontal form on J_kHel. Unfortunately, there is no jet projection from J(Hel) to J_kHel.

That is precisely where the lack in total order of jet prolongations enters.

There is no morphism connecting J(Hel) to J_kHel. In fact, the bundle J(Hel) contains a factor $J^1(M \times \mathbb{R})$ while J_kHel contains only $(M \times \mathbb{R})$, so that J(Hel) is not higher than J_kHel. On the other hand, the bundle J_kHel contains a factor $J^1\mathrm{Con}(M)$ while J(Hel) contains only $\mathrm{Con}(M)$, so that J_kHel is not higher than J(Hel).

Thus we have to look a bundle over both J(Hel) and J_kHel to discuss dynamics. Let us define

$$J_2\mathrm{Hel} \equiv J\mathrm{BD} = J^1\mathrm{Lor}(M) \times_M J^2(M \times \mathbb{R}) \times_M J^1\mathrm{Con}(M) \times_M J^1B$$

which is both higher than J(Hel), J_kHel, as well as BD.

Then we can prolong the map Φ to a map $J\Phi : J_2\mathrm{Hel} \to J_k\mathrm{BD} : (j^1g, j^2\varphi, j^1\tilde{\Gamma}, j^1\phi) \mapsto (j^1g, j^2\varphi, j^1\Gamma, j^1\phi)$.

Let us remark that the map $J\Phi$ defined in this way is invertible.

In this situation, one has a one-to-one correspondence between sections of Sec(Hel) and sections of Sec(BD) given by

$$\Phi_* : \mathrm{Sec(Hel)} \to \mathrm{Sec(BD)} : (g(x), \varphi(x), \tilde{\Gamma}(x), \phi(x)) \mapsto (g(x), \varphi(x), \Gamma(x), \phi(x)) \tag{7.4.45}$$

As long as the dynamics is concerned, since $J\Phi$ is invertible, the Helmholtz Lagrangian L_H can be pulled back on the bundle $J\mathrm{BD}$ along the map $p \circ (J\Phi)^{-1} : J\mathrm{BD} \to J_k\mathrm{Hel}$ to obtain the Lagrangian

$$\boldsymbol{L}_{BD} = \left((J\Phi)^{-1}\right)^* \circ p^* \boldsymbol{L}_H = L_{BD}(j^1g, j^2\varphi, j^1\Gamma, j^1\phi)\, d\sigma = L_H(j^1g, \varphi, j^1\tilde{\Gamma}(\Gamma, g, j^1\varphi), j^1\phi)\, d\sigma \tag{7.4.46}$$

Since we know the variation of L_H to be in the form

$$\delta L_H = E_{\alpha\beta}\delta g^{\alpha\beta} + E\delta\varphi + \tilde{E}^{\mu\nu}_\lambda \delta\tilde{\Gamma}^\lambda_{\mu\nu} + E_i\delta\phi^i + (\mathrm{Div}) \tag{7.4.47}$$

where Div denotes pure divergence terms.

Then the variation of the Lagrangian L_{BD} is

$$
\begin{aligned}
\delta L_{BD} =& E_{\alpha\beta}\delta g^{\alpha\beta} + E\delta\varphi + \tilde{E}^{\mu\nu}_\lambda \delta\tilde{\Gamma}^\lambda_{\mu\nu} + E_i\delta\phi^i + \mathrm{Div} = \\
=& E_{\alpha\beta}\delta g^{\alpha\beta} + E\delta\varphi + \tilde{E}^{\mu\nu}_\lambda \left(\delta\Gamma^\lambda_{\mu\nu} + \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial\varphi}\delta\varphi + \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial\varphi_\sigma}\nabla_\sigma\delta\varphi\right) + E_i\delta\phi^i + \mathrm{Div} = \\
=& \left(E_{\alpha\beta} + \tilde{E}^{\mu\nu}_\lambda \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial g^{\alpha\beta}}\right)\delta g^{\alpha\beta} + \left(E + \tilde{E}^{\mu\nu}_\lambda \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial\varphi} - \nabla_\sigma\left(\tilde{E}^{\mu\nu}_\lambda \frac{\partial\tilde{\Gamma}^\lambda_{\mu\nu}}{\partial\varphi_\sigma}\right)\right)\delta\varphi + \tilde{E}^{\mu\nu}_\lambda\delta\Gamma^\lambda_{\mu\nu} + E_i\delta\phi^i + \mathrm{Div}
\end{aligned}
\tag{7.4.48}
$$
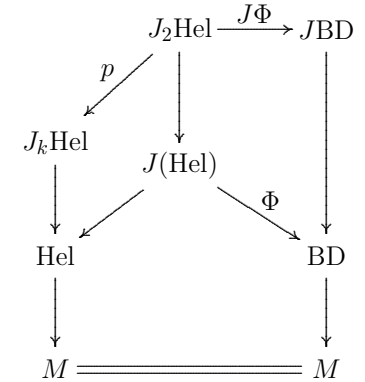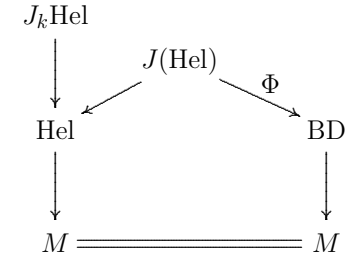
Accordingly, field equations for the Lagrangian L_{BD} are

$$\begin{cases} E_{\alpha\beta} + \tilde{E}^{\mu\nu}_\lambda \dfrac{\partial \tilde{\Gamma}^\lambda_{\mu\nu}}{\partial g^{\alpha\beta}} = 0 \\[2mm] E + \tilde{E}^{\mu\nu}_\lambda \dfrac{\partial \tilde{\Gamma}^\lambda_{\mu\nu}}{\partial \varphi} - \nabla_\sigma \left(\tilde{E}^{\mu\nu}_\lambda \dfrac{\partial \tilde{\Gamma}^\lambda_{\mu\nu}}{\partial \varphi_\sigma} \right) = 0 \\[2mm] \tilde{E}^{\mu\nu}_\lambda = 0 \\[1mm] E_i = 0 \end{cases} \qquad (7.4.49)$$

which, though different from the field equation in the Helmholtz frame, are equivalent to the ones of them, and hence equivalent to the original Palatini $f(\mathcal{R})$-theory.

In fact, matter and connection equations are always unchanged just written in the new variables. Then in particular the field equation of the connection keep prescribing that $\tilde{\Gamma} = \{\tilde{g}\}$, i.e., equivalently, that $\Gamma = \{g\}$. Once this equation is satisfied, then the additional terms in the field equations of the metric g and conformal factor φ vanishes and also these equations determine the same conformal factor and metric as the Helmholtz Lagrangian. In conclusion, also the Lagrangian L_{BD} is dynamically equivalent to the Helmholtz Lagrangian and hence to Brans–Dicke and the original Palatini $f(\mathcal{R})$-theory.

Let us also stress that we showed that L_{BD} is an equivalent formulation of the metric Brans–Dicke theory in a metric-affine formulation. Moreover, it is to be noticed that the metric-affine formulation of the metric Brans–Dicke theory, is not at all obtained by simply promoting the Levi Civita connection to be an independent field. That is a trick that happens to work almost only in standard Palatini GR.

That is the only thing which emerges clearly from the quite intricate analysis. Except that, one has to proceed on a case-by-case stance and also in that case, while it is quite easy to show that two theories are dynamically equivalent, it is almost impossible, with what we have, to prove that they are not.

In BD conformal frame, we have fields $(g_{\mu\nu}, \Gamma^\alpha_{\beta\mu}, \varphi, \phi^i)$, where g represents distances, while Γ does *not* represent free fall,

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} - \tfrac{1}{2}\left(g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)} \right) \nabla_\epsilon \ln\varphi \qquad (7.4.50)$$

does, instead. Here the conformal factor still parameterises, in a different way, the mismatch between the metrics representing distances and free fall, and it shows up as a deviation of particles from their expected orbits in a Brans–Dicke theory, were free fall is *assumed* to be represented by g.

The theory here obtained has the same dynamics as a Brans–Dicke theory, though it is not physically equivalent since particles move differently in the two theories. In particular, one cannot use the fact that Brans-Dicke theories are ruled out by Mercury classical test to rule out the corresponding Palatini $f(\mathcal{R})$-theory as well. The two theories are two different theories, in which Mercury is predicted to move differently.

Summary about conformal frames

The transformations among conformal frames can hence be summarised as:

Helmholtz frame
$(g, \tilde{\Gamma}, \varphi, \phi)$

$\Gamma(g, \tilde{\Gamma}, \varphi)$

$f'(\mathcal{R}) = \varphi$

$\tilde{g} = \varphi g$

Brans–Dicke frame
$(g, \Gamma, \varphi, \phi)$

Jordan frame
$(g, \tilde{\Gamma}, \phi)$

Einstein frame
$(\tilde{g}, \tilde{\Gamma}, \varphi, \phi)$

$\tilde{\phi} = \varphi^\omega \phi$

EPS frame
$(\tilde{g}, \tilde{\Gamma}, \varphi, \tilde{\phi})$

$\Gamma = \{g\}$

Purely metric Brans–Dicke frame
(g, φ, ϕ)
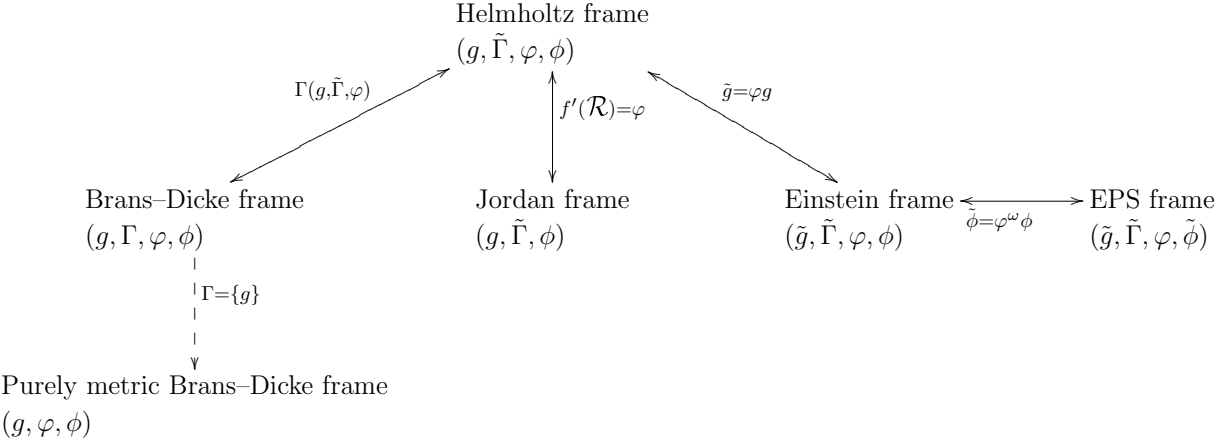
References

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Chapter 8. Palatini $f(\mathcal{R})$-cosmology

> *From the dawn of time we came, moving silently down through the centuries.*
> *Living many secret lives, struggling to reach the time of the Gathering, when the few*
> *who remain will battle to the last. No one has ever known we were among you..*
>
> (Juan Sánchez Villa–Lobos Ramírez, *Highlander*)

1. Introduction

Cosmology can be defined, from a mathematical viewpoint, as the theory of spatially homogeneous and isotropic solutions of a gravitational theory.
Of course, this just shifts the issue to defining what are *spatially homogeneous and isotropic* spacetimes.

Traditionally, one defines them as the ones in which the metric takes a specific form, namely the so-called *Friedmann–Lemaître–Robertson–Walker (FLRW) form*, i.e. in dimension $m = 4$, in coordinates (t, r, θ, ϕ)

$$g = -\mathrm{c}^2 \, dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2(\theta) \, d\phi^2 \right) \right) \tag{8.1.1}$$

where $a(t)$ is called the *scale factor* and k is a real constant which is called the *spatial curvature*.

> Of course, one has $[r] = L$, $[\theta] = [\phi] = 1$, $[t] = T$, and, consequently, $[k] = L^{-2}$ and $[a] = 1$.
>
> Let us stress that one should be cautious in endowing the coordinates with the meaning of spatial-spherical coordinates (or spacetime-cylindrical coordinates). For example, only in a flat space, r is both the length of the radius of the sphere and the number to obtain the area of the sphere as $A = 4\pi r^2$. In a general manifold, whatever *sphere* means, these two numbers are different. In a Riemannian manifold (M, g), where g is in FLRW form, the area of the surface $S = \{r = r_0, t = t_0\}$ is $A = 4\pi a^2(t_0) r_0^2$. Thus $\rho = a(t)r$ is the so-called *area radius* (i.e. the number for which the area is given by $A = 4\pi\rho^2$) and, in general, it has nothing to do with the distance of the surface from the center.
>
> The general attitude is to avoid any reference to possible geometric meanings of coordinates, unless they can be proven to hold true. In a FLRW, the geometry of the spheres $r = r_0$ and $t = t_0$ is that of metrical spheres (with the induced metric ${}^*g = a^2(t_0) r_0^2 \left(d\theta^2 + \sin^2(\theta) \, d\phi^2 \right)$) which have a scale modulated in time by the scale factor $a(t)$. If the scale factor is increasing the spheres are inflating and becoming greater.

This definition of what it means to be *spatially homogeneous and isotropic* is probably unacceptable, certainly a bit awkward. The metric takes a FLRW form in specific coordinate systems, though the FLRW form is not invariant with respect to changes of coordinates. Accordingly, one should first select the good coordinates in which the metric has the expected form. However, the good coordinates are characterised in terms of the metric

so that, practically, it would be hard to prove that a specific metric is spatially homogeneous and isotropic. One should take an arbitrary coordinate system, consider the metric in that coordinate system, and then prove that there are new coordinates in which the metric is in FLRW form.

For that reason, we prefer to give a definition of *spatially homogeneous and isotropic* (as well as, for future convenience, of *spherically symmetric* and *spherically symmetric and stationary*) which is intrinsic and then prove as a theorem that, if a metric obeys this intrinsic condition, then one can define a coordinate system in which the metric is in a special form (in FLRW form, in this case). This is a small detour, with not too many practical applications, though I think is the way one should do it and the way in which one disentangles coordinates from geometric properties and in which the relation with general covariance is best and most clearly expressed.

It is also a template of how one should proceed in similar situations, for example to define spatially axisymmetric solutions. However, we have to remark that all these specially symmetric subspaces of solutions are theoretically quite unessential. They altogether are a zero-measure subset of metrics and there is no reason why nature should privilege these geometries instead the generic ones. On the other hand, all the known exact solutions have some additional isometries and actually they are *catalogued* in terms of their additional isometries. Thus, at the very least, classifying them in terms of their isometries is important for historical reasons, since all exact solutions we know are of this type.

The reason of this strange situation is in the very difficult form of Einstein equations (which are a system of non-linear PDE). We are not clever enough to solve them analytically in general, so either we resort to numerical methods, or we add some symmetry request. The role of symmetry request in this context is also pretty clear: for a metric which has a special isometry group, we are often able to prove there exist special coordinate systems in which it takes a special form, depending on few functions of few coordinates. For example, for spatially homogeneous and isotropic spacetimes we can prove there are coordinates in which the metric is in FLRW form and, in this special form, it depends on the scale factor only, which is a function of time only. In view of that, Einstein equations become a system of ODE, which we are better able to deal with.

Of course, one could as well say that cosmology is a trick to untruthfully represent a general physical situation, which is a general field theoretic situation, as if it were well described by mechanics (or, more generally, by a dynamical system) in order to be able to solve it. Which is also a fair description of the situation, still it is all we are able to do, before going to numerical methods.

> Let us remark that this simplification is not new and it is quite similar to what one does in classical thermodynamics of equilibrium. In a truthful description at some mesoscale of a gas, the density and pressure, as well as the temperature and the entropy of a gas are fields, depending on time and on the position. However, if one restricts to situation in which transformations are slow enough to be able to reach at least a spatial homogeneity (or the equilibrium) then they become constant in space and functions of time only (or constant).

Restricting Einstein equation to spatially homogeneous and isotropic geometries is also called the *cosmological principle*. One can argue that there are good evidences that our universe is approximately spatially homogeneous and isotropic, so it does (approximately) obey cosmological principle.

Anyway, once cosmological principle has been required, there is a good deal of structures one can define on these spacetimes and Einstein equations are considerably simplified. Sources of the gravitational field also are strongly constrained: as a consequence of cosmological principle (not of Einstein equations) the Einstein tensor is equal to the energy-momentum stress tensor of a perfect fluid (with pressure and density which depend on time only) and hence Einstein equations necessarily take the form of ordinary differential equations, precisely two ODE, called the *Friedmann equations*.

This is quite a strange situation: Friedmann equations follow from a kinematic request (the cosmological principle) and they have nothing to do with dynamics. They are true in any cosmology, regardless what gravitational dynamics or matter content is chosen. The Friedmann equations are kinematics, not dynamics.

That made clear, in Friedmann equations a density function and a pressure function appear. While kinematically one already knows that that *some set of Friedmann equations* holds, for some pressure and density functions, dynamics actually really determines that a specific set of Friedmann equations holds true, for a specific pair of pressure and density functions. Different theories prescribe different pairs of pressure and density functions to write down Friedmann equations.

From another viewpoint, Friedmann equations are two equations for the scale factor $a(t)$, for the density $\rho(t)$ and pressure $p(t)$. On the other hand, if one considers a non-vacuum dynamics, Friedmann equations are equivalent to Einstein equations, while matter equations are not represented anywhere by them.

One needs to characterise the matter content of the model. This is usually done by setting some *equation of state* (EoS) for matter. Once EoS has been provided, one has a representation of the cosmological model in terms of a mechanical Lagrangian, the so-called *point-like Lagrangian*. That is a Lagrangian for the scale factor (and its first order derivative) and it completely characterises the evolution of the scale factor, i.e. the evolution of the universe on its own.

About EoS, usually, a simple form is chosen, often in view of physical assumptions which appear to be sound. This is fine, though sometimes it is difficult to extrapolate the model to extreme conditions of very early or very late universe. Getting use to more general EoS would improve our ability to extrapolate models to cover extreme situations in different conditions and extend our application scope.

The material of this Chapter is organised as follows: In Section 2, we shall deal with geometries with special isometry groups, restricting to dimension $m = 4$. We shall first define *axisymmetric* and *spherically symmetric* geometries, as well as stationary geometries. Finally, we shall define spatially homogeneous and isotropic geometries and show that, for them, there exists a coordinate system in which the metric takes the form of a FLRW metric.

In Section 3 we shall discuss Friedmann equations, point-like Lagrangians, EoS. We shall also show how to solve Friedmann equations in some simple situation and define mixtures of perfect fluid to discuss how canonical is the expansion of a generic perfect fluid along a mixture.

In Section 4, we shall discuss cosmologies steaming out of extended theories of gravitation, which are called *extended cosmologies* (ETC). In ETC, the spacetime geometry is represented in terms of Weyl frames, so we try to trace which Weyl frames account for the observables measured in astrophysics and cosmology.

In Section 5, we shall consider a specific Palatini $f(\mathcal{R})$-theory and the cosmology it produces, in order to give an example of how one should proceed in the wider context of Weyl frames and geometries. At this stage, that does not want to be a serious proposal for a physically sound cosmology. We are instead presenting what we are able to say about the theoretical models and how good we are to connect them to physical observations.

2. Geometries with isometries

For any metric g on a spacetime M, one can define the group of isometries and the corresponding Lie algebra $\mathcal{K}(M, g) \subset \mathfrak{X}(M)$, which is a finite dimensional subalgebra of all vector fields on M, called the *Killing algebra*. The dimension of the algebra $\mathcal{K}(M, g)$ refers to how many Killing vectors

one needs to express all others as a real-coefficients linear combination. We know that such a dimension cannot exceed $\frac{m}{2}(m+1)$, which is obtained as a limit on *maximally symmetric* manifolds.

If we consider a point $x \in M$, each Killing vector $X \in \mathcal{K}(M, g)$ defines a vector $X(x) \in T_x M$; the set of the vectors

$$K_x = \{v \in T_x M : \exists X \in \mathcal{K}(M, g) : v = X(x)\} \subset T_x M \tag{8.2.1}$$

is, in fact, a subspace, which has not much to do with the dimension of the algebra. Killing vectors can be independent as vector fields though define dependent vectors at a point. The dimension of the subspace K_x is called the *rank* of the Killing algebra at the point $x \in M$.

The dimension of the Killing algebra considers real-coefficient linear combinations of Killing vector fields, i.e. it regards $\mathcal{K}(M, g)$ as a real vector space. On the contrary, the rank of the Killing algebra refers to linear combinations of Killing vectors with function coefficients, i.e. to the dimension as an $\mathcal{F}(M)$-module. Of course, the rank of the Killing algebra cannot exceed the dimension m of the manifold.

If $\Phi : M \to M$ is an isometry, then $\Phi_* X$ is a Killing vector iff X is. Thus, as a consequence, $T_x \Phi(K_x) = K_{\Phi(x)}$ and the rank of the Killing algebra is constant on the orbits of the isometry group. Accordingly, the Killing algebra defines a distribution $K_x \subset T_x M$, called the *Killing distribution* on (M, g), and its rank is constant on the orbits of the isometry group. Being the Killing distribution produced by a Lie algebra, it is involutive by construction, hence, by Frobenius theorem, it is integrable and it defines a foliation of M in sub-manifolds, the tangent spaces of which are spanned by Killing vectors. This foliation is somehow characteristic of the Killing algebra and it is called the *Killing foliation*.

Let us now focus on a point $x \in S$ on a leaf of the Killing foliation. In the tangent space $T_x M$, we have the subspace which defines the distribution K_x, say of rank k, as well as its orthogonal complement H_x, hence of dimension $m - k$, so that $T_x M = K_x \oplus H_x$.

For any vector $v \in H_x$ one has a unique geodesic motion $\gamma_{(x,v)} : \mathbb{R} \to M$ such that $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$. These geodesic curves can intersect or focus after a while, though there is a neighbourhood U_x of x in which they do not intersect. In the neighbourhood U_x, they establish a one-to-one correspondence $\gamma : U_x \to V \subset S \times \mathbb{R}^{m-k}$; any point $y \in U_x$ is associated to the initial conditions (x, v) for the geodesic motion $\gamma_{(x,v)}$ such that $\gamma_{(x,v)}(1) = y$.

One can relatively easily show that the geodesic trajectories locally foliates a neighbourhood of x. Then, if one considers a point y on a specific geodesic motion $\gamma_{(x,v)}$, then one, of course, has $y = \gamma_{(x,v)}(\hat{s})$ for some value $s = \hat{s}$ of the parameter s along the geodesic motion.

Then we can show that, if we rescale the initial condition $w = \alpha v$ (by a constant α), one has another geodesic motion $\gamma_{(x,w)}$ defined as

$$\gamma_{(x,w)}(s) = \gamma_{(x,v)}(\alpha s) \tag{8.2.2}$$

In fact, since we have

$$\dot{\gamma}_{(x,w)}(s) = \alpha \dot{\gamma}_{(x,v)}(\alpha s) \qquad \ddot{\gamma}_{(x,w)}(s) = \alpha^2 \ddot{\gamma}_{(x,v)}(\alpha s) \tag{8.2.3}$$

then the curve $\gamma_{(x,w)}$ is a geodesic motion

$$\ddot{\gamma}^\mu_{(x,w)} + \{g\}^\mu_{\rho\sigma} \dot{\gamma}^\rho_{(x,w)} \dot{\gamma}^\sigma_{(x,w)} = \alpha^2 \left(\ddot{\gamma}^\mu_{(x,v)} + \{g\}^\mu_{\rho\sigma} \dot{\gamma}^\rho_{(x,v)} \dot{\gamma}^\sigma_{(x,v)} \right) = 0 \qquad \begin{cases} \gamma_{(x,w)}(0) = \gamma_{(x,v)}(0) = x \\ \dot{\gamma}_{(x,w)}(0) = \alpha \dot{\gamma}_{(x,v)}(0) = \alpha v = w \end{cases} \tag{8.2.4}$$

That fully accounts for the notation $\gamma_{(x,w)}$ as the geodesic motion from initial conditions (x, w) and we can notice that, choosing $\alpha = \hat{s}$, then one has $\gamma_{(x,w)}(s) = \gamma_{(x,v)}(\hat{s} s)$, which, in fact, specialises to $\gamma_{(x,w)}(1) = \gamma_{(x,v)}(\hat{s}) = y$, by setting $s = 1$.

In other words, if k^A are coordinates on the leaf S around the point $x \in S$, then (k^A, v^i) are coordinates on U_x. The block of the k coordinates k^A are coordinates on the leaf of the Killing foliation, while the block v^i of $m - k$ coordinates parameterises on which leaf one is.

These are called *adapted coordinates* to the Killing foliation and they locally define transverse sub-manifolds

$$T_x = \{y \in M : y = \gamma_{(x,v)}(s) \text{ for some } v \text{ and } s\} \tag{8.2.5}$$

Of course, there is no guarantee that these transverse manifolds extend globally, nor that, letting x go along S, they define a global foliation. However, they do define a local foliation around x.

Since the neighbourhood U_x is foliated in geodesics, one can define a vector $X = \dot{\gamma}_{(x,v)}(s)$ at the point $\gamma_{(x,v)}(s)$. That defines a (local) vector field which is everywhere tangent to a geodesic motion, hence it obeys

$$X^\mu \nabla_\mu X^\alpha = 0 \tag{8.2.6}$$

i.e. it is a geodesic field. Notice that along S, X is not only transverse, but also orthogonal, by construction.

As we shall see, the Killing algebra allows us to define special coordinate systems, adapted to the Killing foliation, and, in these coordinates, the metric tensor has a specially simple form which is characteristic of the Killing algebra as well.

Let us hence prove the following

Lemma: *if X is a geodesic field, γ an integral curve of X, and K a Killing vector, then the inner product $g(X, K)$ is constant along γ.*

Proof: *One can compute directly*

$$\frac{d}{ds}\left(g(X,K)|_\gamma\right) = \dot{\gamma}^\alpha \left(\partial_\alpha X^\mu g_{\mu\nu} K^\nu + X^\mu \partial_\alpha g_{\mu\nu} K^\nu + X^\mu g_{\mu\nu} \partial_\alpha K^\nu\right) = \dot{\gamma}^\alpha \left(\nabla_\alpha X^\mu g_{\mu\nu} K^\nu + \boxed{X^\mu \nabla_\alpha g_{\mu\nu} K^\nu} + X^\mu g_{\mu\nu} \nabla_\alpha K^\nu\right) = X^\alpha \nabla_\alpha X^\mu K_\mu + X^\alpha X^\mu \nabla_{(\alpha} K_{\mu)} = 0 \tag{8.2.7}$$

∎

Thus $g(X, K)$ is constant and, being it zero on S, it vanishes anywhere X is well defined. In other words, the geodesic field X defined above is orthogonal to the Killing distribution (and hence to the leaves S of the Killing foliation) anywhere in a neighbourhood of x. That means that the metric g, when written in coordinates (k^A, v^i), which are adapted to the Killing foliation, is necessarily in the block-form

$$\|g_{\mu\nu}\| = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{8.2.8}$$

Here the block A is $k \times k$, while the block B is $(m - k) \times (m - k)$.

All coefficients are functions of all coordinates, however, we still have to impose Killing equations for the Killing vectors which span the Killing foliation, which further constrain the dependence of the coefficients A and B on the adapted coordinates. This form could be further simplified by changing coordinates v^i. We shall prove it in the specific examples below.

This construction can be repeated for any Killing subalgebra as well. In particular, when one has a Killing algebra which has subalgebras, each of them defines its own Killing foliation. Accordingly, one has a number of foliations, nested one into the other, and, preserving all of them, one can give a rather strict constraint on special coordinates and, consequently, on the form of the metric.

That is quite beautiful, though on a generic Riemannian manifold (M, g) one has no isometry other than the identity and the corresponding Killing distribution is of rank 0. However, we shall see how to proceed when this is not the case and one has a non-trivial Killing algebra.

Stationary metrics

Let us start with a very simple example. A Lorentzian spacetime (M, g) is called a *stationary* spacetime if its Killing algebra contains a 1 dimensional subalgebra spanned by a (never-vanishing) time-like Killing vector ξ.

In this case, the Killing distribution is of rank 1 and it defines a foliation in sub-manifolds S_k of dimension 1. By the flow-box theorem, one can select adapted coordinates (t, k^i) in which $\xi = \partial_t$, so that the leaves of the Killing foliation are the lines parameterised by t, namely $S_k = \{(t, k)\}$.

These adapted coordinates are not uniquely determined. If (t, k) are adapted, then also (t', k') are, provided that

$$\begin{cases} t' = t'(t, k) \\ k' = k'(k) \end{cases} \tag{8.2.9}$$

Let us remark that this has not much to do with an ADM fibration $t : M \to \mathbb{R}$, in which the fibres are isochronous hypersurfaces and changes of ADM observers are

$$\begin{cases} t' = t'(t) \\ k' = k'(t, k) \end{cases} \tag{8.2.10}$$

The two structures are somehow complementary. In this case, one can define equivalent events on the same leaf and define a quotient $S \simeq M/\mathbb{R}$. If the quotient set S is a manifold, which in general is not guaranteed, then one has a fibration $p : M \to S$ which has the 1 dimensional leaves as fibers.

Of course, one can have $M \simeq \mathbb{R} \times S$ and the two structures can coexist.

The metric in adapted coordinates is in the form

$$g = -A(t, k)\, dt^2 + g_{ij}(t, k)\, dk^i \otimes dk^j \tag{8.2.11}$$

Since $\xi = \partial_t$ needs to be a Killing vector, the coefficients are independent of the time coordinate.

The condition for ξ to be a Killing vector reads as

$$0 = g^{\alpha\mu} \nabla_\mu \xi^\beta + g^{\beta\mu} \nabla_\mu \xi^\alpha \tag{8.2.12}$$

One can specialise $(\alpha\beta) = (00)$

$$0 = g^{00}\Gamma^0_{00} + g^{00}\Gamma^0_{00} = g^{00}g^{0\epsilon}\left(-\partial_\epsilon g_{00} + 2\partial_0 g_{\epsilon 0}\right) = A^{-2}\partial_0 A \qquad \Rightarrow \partial_0 A = 0 \tag{8.2.13}$$

or $(\alpha\beta) = (0i)$

$$0 = g^{0\mu}\nabla_\mu \xi^i + g^{i\mu}\nabla_\mu \xi^0 = g^{00}\Gamma^i_{00} + g^{ij}\Gamma^0_{0j} = \tfrac{1}{2}g^{00}g^{ij}(-\partial_j g_{00} + 2\partial_0 g_{0j} - \partial_0 g_{0j} + \partial_0 g_{0j} + \partial_j g_{00} = g^{00}g^{ij}\partial_0 g_{0j}) = 0 \tag{8.2.14}$$

which is identically satisfied, or $(\alpha\beta) = (ij)$

$$0 = g^{ik}\Gamma^j_{0k} + g^{jk}\Gamma^i_{0k} = \tfrac{1}{2}g^{ik}g^{jl}(-\partial_l g_{0k} + \partial_0 g_{kl} + \partial_k g_{0l}) + \tfrac{1}{2}g^{jk}g^{il}(-\partial_l g_{0k} + \partial_0 g_{kl} + \partial_k g_{0l}) = g^{ik}g^{jl}\partial_0 g_{kl} \qquad \Rightarrow \partial_0 g_{kl} = 0 \tag{8.2.15}$$

Thus we have the metric in the form

$$g = -A(k)\, dt^2 + g_{ij}(k)\, dk^i \otimes dk^j \tag{8.2.16}$$

A spacetime is *static* if it is stationary and, moreover, ξ is surface forming.

Being *surface forming* means that the orthogonal distribution to ξ

$$\Delta = \{v \in T_x M : g(v,\xi) = 0\} \tag{8.2.17}$$

is involutive (and hence integrable). For Δ to be involutive, one needs that any two vector fields X, Y orthogonal to ξ happen to have a commutator $[X, Y]$ which is still orthogonal to ξ, i.e.

$$g(\xi, [X,Y]) = \xi^\alpha g_{\alpha\beta}(X^\lambda \partial_\lambda Y^\beta - Y^\lambda \partial_\lambda X^\beta) = -A(X^\lambda \partial_\lambda Y^0 - Y^\lambda \partial_\lambda X^0) = -X^\lambda \partial_\lambda(AY^0) + Y^\lambda \partial_\lambda(AX^0) + (X^\lambda Y^0 - Y^\lambda X^0)\partial_\lambda A = 0 \tag{8.2.18}$$

that is $(X^i Y^0 - Y^i X^0)\partial_i A = 0$, which, since needs to be true for any X, Y orthogonal to ξ, implies $\partial_i A = 0$.

Thus the spacetime is static iff the metric is in the form

$$g = -c^2\, dt^2 + g_{ij}(k)\, dk^i \otimes\ dk^j \tag{8.2.19}$$

Then a vector tangent to the surface $t = t_0$ is in the form $v = v^i \partial_i$ and one has

$$g(\xi, v) = g_{0i} v^i = 0 \tag{8.2.20}$$

i.e. ξ is orthogonal to the surface $t = t_0$. That is another possible definition of a static spacetime: a static spacetime is stationary and the time-like Killing vector is orthogonal to the surfaces at $t = t_0$.

Stationary axisymmetric metrics

A spacetime (M, g) is *stationary axisymmetric* if it allows two non-vanishing Killing vectors ξ and ζ, time-like and space-like, respectively, in involution (so that each of them spans a subalgebra). The orbits are curves, the orbits of ξ being topological lines and those of ζ are topological circles.

Since they are in involution, there exit coordinates in which they both are in flow-box form, i.e. $\xi = \partial_t$ and $\zeta = \partial_\phi$. The metric takes the form

$$g = -A\, dt^2 + C\,(\, dt \otimes\ d\phi +\ d\phi \otimes\ dt) + B\, d\phi^2 + g_{ij}\, dk^i \otimes\ dk^j \tag{8.2.21}$$

By requiring that ζ and ξ are Killing vectors, the coefficients do not depend on (t, ϕ). Notice how this is a particular stationary metric, with a further isometry which specialises the spatial $(m-1)$-metric.

In dimension $m = 4$, the coordinates k^i spans a 2 dimensional space; any two dimensional metric, in particular $g_{ij}(k)\, dk^i \otimes\ dk^j$, is conformally flat, so one can change coordinates k to obtain new coordinates (ρ, z) in which the metric further simplifies to be

$$g = -A(\rho, z)\, dt^2 + C(\rho, z)\,(\, dt \otimes\ d\phi +\ d\phi \otimes\ dt) + B(\rho, z)\, d\phi^2 + D(\rho, z)\left(\, d\rho^2 +\ dz^2\right) \tag{8.2.22}$$

If one wants to consider *static axisymmetric* spacetimes, for a vector v orthogonal to the surface $t = t_0$, we have that

$$g(\xi, v) = g_{03} v^3 + g_{0i} v^i = C v^3 = 0 \tag{8.2.23}$$

That should be true for any such a v, which implies $C = 0$. Thus the most general static axisymmetric metric is

$$g = -A(\rho, z)\, dt^2 + B(\rho, z)\, d\phi^2 + D(\rho, z)\left(d\rho^2 + dz^2 \right) \tag{8.2.24}$$

Spherically symmetric metrics

A spacetime (M, g) of dimension $m = 4$, is *spherically symmetric* if the Killing algebra contains a space-like copy of the algebra $\mathfrak{so}(3)$. One has three space-like Killing vectors L_i such that

$$[L_i, L_j] = -\epsilon^k{}_{ij} L_k \tag{8.2.25}$$

and at each point they span a rank 2 distribution.

Since the Killing vectors do not commute, one cannot put them in flow-box form simultaneously. The best one can do is choosing a coordinate system (u^1, u^2, v, ϕ), adapted to the Killing foliation so that (v, ϕ) are coordinates on the leaves, in which one of them is in flow-box form, e.g. $L_3 = \partial_\phi$.

Since also L_1 and L_2 are tangent to the leaves, they are in the form

$$L_1 = \alpha_1 \partial_v + \beta_1 \partial_\phi \qquad\qquad L_2 = \alpha_2 \partial_v + \beta_2 \partial_\phi \tag{8.2.26}$$

Since they need to satisfy the commutation relation, one can show that there is a new coordinate system (θ, ϕ) on the leaves, such that L_1 and L_2 can be recast in a canonical form

$$L_1 = -\sin(\phi)\partial_\theta - \cos(\phi)\frac{\cos(\theta)}{\sin(\theta)}\partial_\phi \qquad\qquad L_2 = \cos(\phi)\partial_\theta - \sin(\phi)\frac{\cos(\theta)}{\sin(\theta)}\partial_\phi \tag{8.2.27}$$

which, in fact, implement the commutation relations.

... insert details and link to Maple

Since we followed the general guidelines to define adapted coordinates (u^1, u^2, θ, ϕ) and, by requiring that L_i are Killing vectors, then the metric is generally in the form

$$g = \hat{g}_{ij}(u)\, du^i \otimes du^j + a^2(u)\left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{8.2.28}$$

Then we can still change coordinates (u^1, u^2) to simplify this form. Let us set new coordinates (u, r) defined by $u = u^1$ and $r^2 = a^2(u)$.

$$g = A\, du^2 + 2B\left(du \otimes dr + dr \otimes du \right) + C\, dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{8.2.29}$$

And then define $t = t(u, r)$ so that for some function $\alpha(u, r)$

$$\frac{\partial t}{\partial u} = \sqrt{-\frac{A}{\alpha}} \qquad\qquad \frac{\partial t}{\partial r} = -\frac{B}{\sqrt{-\alpha A}} \tag{8.2.30}$$

and, for later convenience, let us set $\beta := C - \frac{B^2}{A}$.

One can choose α so that one has

$$\frac{\partial}{\partial r}\left(\sqrt{-\frac{A}{\alpha}}\right) = \frac{\partial}{\partial u}\left(-\frac{B}{\sqrt{-\alpha A}}\right) \tag{8.2.31}$$

and, consequently, the function $t(u, r)$ does exist.

Then in the new coordinates (t, r, θ, ϕ) the metric reads as

$$g = -\alpha(t, r)\, dt^2 + \beta(t, r)\, dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right) \tag{8.2.32}$$

Thus we proved that, for any spherically symmetric spacetime, there exists a coordinate system in which the metric is in the special form (8.2.32).

A *stationary, spherically symmetric spacetime* is when the Killing algebra contains $\mathbb{R} \times \mathfrak{so}(3)$, i.e. one has Killing vectors (ξ, L_i) with commutation relations

$$[L_i, L_j] = -\epsilon^k{}_{ij} L_k \qquad\qquad [\xi, L_i] = 0 \tag{8.2.33}$$

As a consequence

$$g = -\alpha(r)\, dt^2 + \beta(r)\, dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right) \tag{8.2.34}$$

Spatially homogenous and isotropic metrics

We are now ready to discuss cosmological principle for spacetimes of dimension $m = 4$.

We say that a spacetime obeys *cosmological principle* or it is *spatially homogeneous and isotropic* if its Killing algebra contains an algebra $C_h(4)$, i.e. if there exists 6 space-like Killing vectors (L_i, P_i) which obeys the following commutation relations

$$[L_i, L_j] = -\epsilon^k{}_{ij} L_k \qquad\qquad [L_i, P_j] = -\epsilon^k{}_{ij} P_k \qquad\qquad [P_i, P_j] = -h\epsilon^k{}_{ij} L_k \tag{8.2.35}$$

and they define a distribution of rank 3 (while L_i alone define a distribution of rank 2).

The algebra $C_h(4)$ corresponds to $C_0(4) = \mathbb{R}^3 \times \mathfrak{so}(3)$ when $h = 0$, to $C_+(4) = \mathfrak{so}(4)$ when $h > 0$, and to $C_-(4) = \mathfrak{so}(3, 1)$ when $h < 0$.

Since a spatially homogeneous and isotropic spacetime is, in particular, spherically symmetric, there exists coordinates (t', ρ, θ, ϕ) in which the metric is

$$g = -\alpha(t', \rho)\, dt'^2 + \beta(t', \rho)\, d\rho^2 + \rho^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right) \tag{8.2.36}$$

and the generators L_i are

$$L_1 = -\sin(\phi)\partial_\theta - \cos(\phi)\frac{\cos(\theta)}{\sin(\theta)}\partial_\phi \qquad\qquad L_2 = \cos(\phi)\partial_\theta - \sin(\phi)\frac{\cos(\theta)}{\sin(\theta)}\partial_\phi \qquad\qquad L_3 = \partial_\phi \tag{8.2.37}$$

Then one can assume P_i as generic vector fields with no component along t and impose commutation relations, obtaining

$$P_1 = -\frac{4 + h\rho^2}{4} \sin(\theta)\cos(\phi)\partial_\rho - \frac{4 - h\rho^2}{4\rho}\left(\cos(\theta)\cos(\phi)\partial_\theta - \frac{\sin(\phi)}{\sin(\theta)}\partial_\theta\right)$$

$$P_2 = -\frac{4 + h\rho^2}{4} \sin(\theta)\sin(\phi)\partial_\rho - \frac{4 - h\rho^2}{4\rho}\left(\cos(\theta)\sin(\phi)\partial_\theta + \frac{\cos(\phi)}{\sin(\theta)}\partial_\theta\right) \qquad (8.2.38)$$

$$P_3 = -\frac{4 + h\rho^2}{4} \cos(\theta)\partial_\rho + \frac{4 - h\rho^2}{4\rho}\sin(\theta)\partial_\theta$$

Then imposing that P_i are Killing vectors, one gets the metric in the form

$$g = -f^2(t')\,dt'^2 + \frac{a^2(t')}{(4 + h\rho^2)^2}\left(d\rho^2 + \rho^2\,d\theta^2 + \rho^2\sin^2(\theta)\,d\phi^2\right) \qquad (8.2.39)$$

Finally, we can redefine the time coordinate so that $\mathrm{c}\,dt = f(t')\,dt'$ as well as

$$r^2 = \frac{\rho^2}{(4 + h\rho^2)^2} \qquad \Rightarrow \quad dr^2 = \frac{(4 - h\rho^2)^2}{(4 + h\rho^2)^4}\,d\rho^2, \qquad 1 - 16hr^2 = \frac{(4 + h\rho^2)^2 - 16h\rho^2}{(4 + h\rho^2)^2} = \frac{(4 - h\rho^2)^2}{(4 + h\rho^2)^2} \qquad (8.2.40)$$

Hence in the new coordinates (t, r, θ, ϕ) the metric takes the form

$$g = -\mathrm{c}^2\,dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \qquad (8.2.41)$$

which is in FLRW form by setting $k = 16h$ and $a(t) = a(t'(t))$. Hence we proved that, for any spatially homogeneous and isotropic spacetime, there exists a coordinate system, called *comoving coordinates*, in which the metric takes the form of a FLRW metric.

Special structures on FLRW spacetimes

Now let us consider a spacetime which implements the cosmological principle. Then one can find a coordinate system (t, r, θ, ϕ) of comoving coordinates in which the metric is in FLRW form, i.e. is locally expressed as

$$g = -\mathrm{c}^2\,dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \qquad (8.2.42)$$

for a scale factor $a(t)$ and a spatial curvature $k \in \mathbb{R}$.

Let us consider the metric ${}^*g$ induced by g on the Killing leaves $S_s = (t = s)$. The (strictly) Riemannian manifold $(S_s, {}^*g)$ is called the *space* (at time $t = s$) and its Ricci scalar is

$${}^*R = \frac{6k}{a^2(s)} \qquad (8.2.43)$$

Accordingly, the number k is called the *spatial curvature.*

At this point we can further normalise the notation in two different, mutually exclusive, ways.

The first way, used when $k \neq 0$, is to consider $k = K\alpha^2$ with $K = \pm 1$ (so that $\alpha^2 = |k|$ and let us assume $\alpha > 0$) and to try to use K instead of k. One can recast $kr^2 = K\alpha^2 r^2 = K\rho^2$, where we set $\rho := \alpha r$. Accordingly, one can recast the metric as

$$g = -\mathrm{c}^2\,dt^2 + \frac{a^2(t)}{\alpha^2}\left(\frac{d\rho^2}{1-K\rho^2} + \rho^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) = -\mathrm{c}^2\,dt^2 + A^2(t)\left(\frac{d\rho^2}{1-K\rho^2} + \rho^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{8.2.44}$$

where we set $A(t) := a(t)\alpha^{-1}$. That essentially corresponds to use the length scale implicitly defined by the spatial curvature k (which, in fact, is $[k] = L^{-2}$) as a unit for measuring all lengths, including r and the scale factor. As a consequence, ρ, which is measured in multiples of the unit scale α becomes adimensional $[\rho] = 1$, while the scale factor $[A] = L$ is now a length. Of course, if $k = 0$ there is no length scale induced by the spatial curvature and one cannot normalise in this way.

The second possible normalisation that one can do is to choose a time $t = t_0$ (which is called *today* and at which initial conditions will be set) and normalise the scale factor to be $a_0 = a(t_0) = 1$.

$$g = -\mathrm{c}^2\,dt^2 + \frac{a^2(t)}{a_0^2}\left(\frac{a_0^2\,dr^2}{1-kr^2} + a_0^2 r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{8.2.45}$$

and define $\bar{r} = a_0 r$, as well as $a_0(t) := a(t)a_0^{-1}$ and $k_0 := ka_0^{-2}$, so that the metric takes the form

$$g = -\mathrm{c}^2\,dt^2 + a_0^2(t)\left(\frac{d\bar{r}^2}{1-k_0\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{8.2.46}$$

Of course, one has $a_0^2(t_0) = a(t_0)a_0^{-1} = 1$. That is still adimensional ($[a_0] = 1$), so that this normalisation is incompatible with the one induced by the spatial curvature (in which the scale factor is a length). We shall use this second normalisation. Hence, for us, the scale factor $a_0(t)$ will be adimensional, $\bar{r}$ a length and today the scale factor is $a_0(t_0) = 1$.

Accordingly, let us consider a spacetime obeying cosmological principle, use normalisation of the scale factor, and fix coordinates so that the metric takes the form

$$g = -\mathrm{c}^2\,dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \qquad (a(t_0) = 1) \tag{8.2.47}$$

This metric comes with a Killing distribution of rank 3 (spanned by 6 Killing vectors L_i and P_i) which defines a preferred foliation of isochronous hypersurfaces $S_s = (t = s)$. One can define an equivalence relation (two events $x, y \in M$ are equivalent iff they belong to the same leaf) which defines an ADM fibration $t : M \to \mathbb{R}$. In other words, spatially homogenous and isotropic spacetimes come with a preferred ADM foliation fixed on them.

As we discussed, fixing an ADM foliation is an observer convention, it should not be something endowed with an absolute physical meaning. Since cosmology comes with a fixed ADM foliation, one should ask whether cosmological principle does not spoil general covariance.

For, let us argue that, as we shall see, we are imposing cosmological principle in order to being able to solve Einstein equations. That is a trick, a specific situation in which we are able to solve field equations. There is nothing fundamental with it. On the other hand, it is not surprising that once one fixes a particular solution then, in *that* solution, one could have preferred observers. For example, if we are considering a star, or any point mass, the observers at rest with respect to the central mass are naturally privileged.

That does not spoil general covariance. General covariance states that there is no *a priori* preferred observer, i.e. observers which are selected *before* selecting a particular solution. Here, the preferred ADM observers are selected *a posteriori*, they are specific of spatially homogenous and isotropic spacetimes. Nobody ever meant that once a solution is fixed there cannot be coordinate systems in which the solution is simpler than in other coordinates.

Since the Killing distribution is space-like and of rank $(m-1)=3$ and we have a metric g on spacetime, we can define the orthogonal distribution, which is of rank 1 and it is time-like. That orthogonal distribution is generated by $\xi = \partial_t$ (which, in fact, is orthogonal to the spatial leaves S_s because of the special FLRW form of the metric). The vector field ξ is not a Killing vector (the metric is not stationary, at all), though it generates a distribution which, being of rank 1, is involutive (hence integrable). It generates a foliation in (time-like) trajectories, which are hence worldlines and are locally represented by motions as $\gamma_k : \mathbb{R} \to M : s \mapsto (s, k)$, which are parameterised by points on a leaf, i.e. $k \in S_0$.

Such curves γ_k are not only worldlines; they are automatically g-geodesics.

We have $\dot{\gamma}_k^\mu(s) = \delta_0^\mu$ and $\ddot{\gamma}_k^\mu(s) = 0$. Thus the geodesic equation, in comoving coordinates, reads as

$$\ddot{\gamma}_k^\mu + \{g\}_{\alpha\beta}^\mu \dot{\gamma}_k^\alpha \dot{\gamma}_k^\beta = \{g\}_{00}^\mu = \tfrac{1}{2} g^{\mu\nu} \left(-\partial_\nu g_{00} + 2\partial_0 g_{0\nu}\right) = 0 \tag{8.2.48}$$

Hence, if g describes the free fall in our model, then γ_s are also freely falling particles and one has a preferred parameterisation set on them (the one defined by the coordinate t which makes the metric in FLRW form). They are called *universal rest clocks* or *comoving clocks*.

To summarise, spatially homogeneous and isotropic spacetimes comes with a canonical notion of synchronisation, i.e. a global universal time. As such, they are similar to Newton-Galilei spacetime, with a global time. However, it is even stricter, since the universal rest clocks establish a global notion of *being at rest*, which (again *a posteriori*) breaks also Galilei covariance!

The vector field ξ is not Killing, though one can define $X = a(t)\partial_t$ which is a conformal Killing vector.

Being a conformal Killing vector means that, instead of having $\mathcal{L}_\xi g_{\mu\nu} = 0$ (as it happens for Killing vectors), one has $\mathcal{L}_X g_{\mu\nu} \propto g_{\mu\nu}$.

For X, one has $X_\mu = g_{\mu\nu} X^\nu = a g_{\mu 0}$ (i.e. $X_0 = -\mathrm{c}^2 a$ and $X_i = 0$) so that

$$\mathcal{L}_X g_{\mu\nu} = 2\nabla_{(\mu} X_{\nu)} = 2\partial_{(\mu} X_{\nu)} + 2\mathrm{c}^2 a\{g\}_{\mu\nu}^0 = 2\partial_{(\mu} X_{\nu)} - \mathrm{c}^2 a(-\partial_0 g_{\mu\nu} + \partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu}) \tag{8.2.49}$$

$(\mu\nu) = (00)$

$$\mathcal{L}_X g_{\mu\nu} = -2\mathrm{c}^2 \dot{a} = 2\dot{a} g_{00} \tag{8.2.50}$$

$(\mu\nu) = (0i)$

$$\mathcal{L}_X g_{\mu\nu} = 2\partial_{(0} X_{i)} - \mathrm{c}^2 a(-\partial_0 g_{0i} + \partial_0 g_{i0} + \partial_i g_{00}) = 0 \tag{8.2.51}$$

$(\mu\nu) = (ij)$

$$\mathcal{L}_X g_{\mu\nu} = 2\partial_{(i} X_{j)} - \mathrm{c}^2 a(-\partial_0 g_{ij} + \partial_i g_{j0} + \partial_j g_{0i}) = 2\dot{a} g_{ij} \tag{8.2.52}$$

Then one has

$$\mathcal{L}_X g_{\mu\nu} = 2\dot{a} g_{\mu\nu} \tag{8.2.53}$$

and X is a conformal Killing vector.

Other useful coodinates

For future reference, it is useful to provide some other coordinates for FLRW metrics as well as their geometric interpretation. The coordinate t is unessential and it will be let unchanged so let us do it on the space manifold.

For $k > 0$ (i.e. $K = 1$), we have to check that the space surface Σ_t is a sphere S^3. To do that we show that we can define a metric on the sphere S^3 which agrees with the metric induced by the spacetime metric g, namely

$$^*g = a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.54}$$

That is not trivial since the metric $^*g$ does not seem the canonical metric on S^3.

On the other hand, the coordinates (r, θ, ϕ) are not the coordinates one would use on a sphere. In fact, we should see if there is a change of coordinates which transforms the local expression of the metric in a form which looks more familiar.

The manifold S^3 is defined as a sub-manifold in $(\mathbb{R}^4, \delta)$, where δ is the canonical, Euclidean flat metric, obeying equation $\delta(x, x) = A^2$. If we choose Cartesian coordinates $x = (x^0, x^1, x^2, x^3)$ in $\mathbb{R}^4$ the equation becomes

$$|x|^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = A^2 \tag{8.2.55}$$

We can use cylindrical $(x^0, \rho, \theta, \phi)$ or spherical $(a, \hat{\chi}, \theta, \phi)$ coordinates on $\mathbb{R}^4$ to provide a parameterisation of S^3, namely

$$\begin{cases} x^0 = \pm A\sqrt{1 - \rho^2} \\ x^1 = A\rho \sin(\theta)\cos(\phi) \\ x^2 = A\rho \sin(\theta)\sin(\phi) \\ x^3 = A\rho \cos(\theta) \end{cases} \qquad \begin{cases} x^0 = A\cos(\hat{\chi}) \\ x^1 = A\sin(\hat{\chi})\sin(\theta)\cos(\phi) \\ x^2 = A\sin(\hat{\chi})\sin(\theta)\sin(\phi) \\ x^3 = A\sin(\hat{\chi})\cos(\theta) \end{cases} \tag{8.2.56}$$

and we transform one into the other by $\rho = \sin(\hat{\chi})$. In the first case (ρ, θ, ϕ) are parameters on the surface, in the second case the parameterisation is in terms of $(\hat{\chi}, \theta, \phi)$.

The induced metric in the first case is

$$^*g = A^2 \left(\frac{d\rho^2}{1 - K\rho^2} + \rho^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.57}$$

By comparing with (8.2.44), we see that transforming $\rho = \sqrt{k}r$ and setting $a := \sqrt{k}A$ leads to the canonical FLRW form parameterised by coordinates (r, θ, ϕ). That proves that the surface Σ is a sphere S^3, with a radius a expanding on time.

By making the transformation $\rho = \sin(\hat{\chi})$ we obtain the second canonical form, associated to spherical coordinates in $\mathbb{R}^4$, namely

$$^*g = A^2 \left(d\hat{\chi}^2 + \sin^2(\hat{\chi}) \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.58}$$

By setting $A^2 k^2 = a^2$ and transforming $\hat{\chi} = \sqrt{k}\,\chi$, we obtain

$$^*g = a^2 \left(d\chi^2 + S^2(\chi; k) \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.59}$$

where we set $S(\chi; k) = k^{-\frac{1}{2}} \sin(\sqrt{k}\,\chi)$ (with $k > 0$).

That confirms that the space manifold is a sphere S^3, this time parameterised by usual angles (χ, θ, ϕ). It also shows how ρ is the area radius, i.e. the radius a sphere S^2 should have to account for its area, while the distance from the center is $d = A\hat{\chi} = a\chi$.

Fig 5.7: different radii in FLRW geometry.

We can repeat the computation for $k < 0$, i.e. $K = -1$. In this case we have to check that the space surface Σ_t is a hypersphere H^3.

The manifold H^3 is defined as a sub-manifold in $(\mathbb{R}^4, \eta)$, where η is the canonical, Minkowskian flat metric, obeying equation $\eta(x, x) = -A^2$. If we choose Cartesian coordinates $x = (x^0, x^1, x^2, x^3)$ in $\mathbb{R}^4$ the equation becomes

$$-|x|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = A^2 \tag{8.2.60}$$

We can use cylindrical $(x^0, \rho, \theta, \phi)$ or spherical $(a, \hat{\chi}, \theta, \phi)$ coordinates on $\mathbb{R}^4$ to provide a parameterisation of S^3, namely

$$\begin{cases} x^0 = A\sqrt{1 + \rho^2} \\ x^1 = A\rho \sin(\theta) \cos(\phi) \\ x^2 = A\rho \sin(\theta) \sin(\phi) \\ x^3 = A\rho \cos(\theta) \end{cases} \qquad \begin{cases} x^0 = A\cosh(\hat{\chi}) \\ x^1 = A\sinh(\hat{\chi}) \sin(\theta) \cos(\phi) \\ x^2 = A\sinh(\hat{\chi}) \sin(\theta) \sin(\phi) \\ x^3 = A\sinh(\hat{\chi}) \cos(\theta) \end{cases} \tag{8.2.61}$$

and we transform one into the other by $\rho = \sinh(\hat{\chi})$. In the first case (ρ, θ, ϕ) are parameters on the surface, in the second case the parameterisation is in terms of $(\hat{\chi}, \theta, \phi)$.

The induced metric in the first case is

$$^*g = A^2 \left(\frac{d\rho^2}{1 - K\rho^2} + \rho^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.62}$$

By comparing with (8.2.44), we see that transforming $\rho = \sqrt{|k|}\,r$ and setting $a := \sqrt{|k|}A$ leads to the canonical FLRW form parameterised by coordinates (r, θ, ϕ). That proves that the surface Σ is a sphere H^3, with a scale factor a expanding on time.

By making the transformation $\rho = \sinh(\hat{\chi})$ we obtain the second canonical form, associated to spherical coordinates in $\mathbb{R}^4$, namely

$$^*g = A^2 \left(d\hat{\chi}^2 + \sinh^2(\hat{\chi}) \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.63}$$

By setting $A^2 k^2 = a^2$ and transforming $\hat{\chi} = \sqrt{|k|}\,\chi$, we obtain

$$^*g = a^2 \left(d\chi^2 + S^2(\chi; k) \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.64}$$

where we set $S(\chi; k) = |k|^{-\frac{1}{2}} \sin(\sqrt{|k|}\,\chi)$ (with $k < 0$).

Finally, for $k = 0$, we obtain a 3-plane.

The space manifold Σ is defined as a sub-manifold in $(\mathbb{R}^4, \delta)$, where δ is the canonical, Euclidean flat metric. If we choose Cartesian coordinates $x = (x^0, x^1, x^2, x^3)$ in $\mathbb{R}^4$ the equation is

$$x^0 = A \tag{8.2.65}$$

We can use cylindrical $(x^0, \rho, \theta, \phi)$ coordinates on $\mathbb{R}^4$ to provide a parameterisation of Σ, namely

$$\begin{cases} x^0 = A \\ x^1 = A\rho \sin(\theta) \cos(\phi) \\ x^2 = A\rho \sin(\theta) \sin(\phi) \\ x^3 = A\rho \cos(\theta) \end{cases} \tag{8.2.66}$$

The induced metric is

$$^*g = A^2 \left(\frac{d\rho^2}{1 - K\rho^2} + \rho^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.67}$$

for $K = 0$. By comparing with (8.2.44), we see that transforming $\rho = r$ and setting $a := A$ leads to the canonical FLRW form parameterised by coordinates (r, θ, ϕ). That proves that the surface Σ is a 3-plane, with a scale factor a expanding on time. We can also recast this metric in the form

$$^*g = a^2 \left(d\chi^2 + S^2(\chi; 0) \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \right) \tag{8.2.68}$$

where we set $\chi = r$ and $S(\chi; 0) = r$.

3. Friedmann equations

Let us forget, for a while, dynamics and consider a metric in FLRW form. We want to show that the cosmological principle already constraints, at a kinematical level, such metrics (essentially showing that there exists a class of field equations, called *Friedmann equations*, which they necessarily obey to) and giving guidelines for solving these equations. In the next Section we shall discuss the meaning of Friedmann equations in different Weyl conformal frames.

Let us now consider the Einstein tensor $G_{\mu\nu}$ for a metric in FLRW form.

By direct computation, we can compute that

$$G_{\mu\nu}\, dx^\mu \otimes\, dx^\nu = \tfrac{\kappa}{\mathrm{c}}\left((\rho+p)u_\mu u_\nu + p g_{\mu\nu}\right)\, dx^\mu \otimes\, dx^\nu =: \kappa T_{\mu\nu}\, dx^\mu \otimes\, dx^\nu \tag{8.3.1}$$

where we set

$$\kappa\rho(t) := 3\frac{\dot a^2 + \mathrm{c}^2 k}{\mathrm{c}a^2} \qquad \kappa p(t) := -\frac{2a\ddot a + \dot a^2 + \mathrm{c}^2 k}{\mathrm{c}a^2} \qquad \boldsymbol{u} := u_\mu\, dx^\mu = \mathrm{c}\, dt \qquad T_{\mu\nu} := \mathrm{c}^{-1}\left((\rho+p)u_\mu u_\nu + p g_{\mu\nu}\right) \tag{8.3.2}$$

A symmetric tensor $T_{\mu\nu}$ is called a *perfect fluid stress tensor*, if it is in the form

$$T_{\mu\nu} = \mathrm{c}^{-1}\left((\rho+p)u_\mu u_\nu + p g_{\mu\nu}\right) \tag{8.3.3}$$

for some function ρ, called the *density*, a function p, called the *pressure*, and some (future directed) unit time-like vector u^μ, called the *comoving velocity of the fluid*.

Let us check the dimensions of everything, just to be sure. In Cartesian coordinates $[x^\mu] = L$ and $[g_{\mu\nu}] = 1$, so that $[u^\mu] = 1$ and $[G_{\mu\nu}] = L^{-2}$. Thus $[T_{\mu\nu}] = ML^{-2}T^{-1}$ $[G] = M^{-1}L^3T^{-2}$
and $[p] = [\rho] = ML^{-1}T^{-2}$. Of course, the dimensions of ρ and p do not depend on coordinates, since they are scalars. $[\kappa] = M^{-1}T$
Notice that p is a pressure, namely a force on an area, i.e. $MLT^{-2}L^{-2} = ML^{-1}T^{-2}$. Let us also notice that an energy density has dimension $ML^{-1}T^{-2} = [p]$ as well. Thus ρ is an *energy density*.
Sometimes, in the literature, the quantity $\hat\rho = \rho\mathrm{c}^{-2}$ is used instead of ρ. One has $[\hat\rho] = ML^{-3}$ which is the *mass density*.
Check that the dimension of equations (8.3.2) are compatible.

Accordingly, the Einstein tensor of a spatially homogeneous and isotropic metric g, just in view of cosmological principle, is necessarily a perfect fluid stress tensor and the metric g obeys Einstein equations for a suitable perfect fluid stress tensor.

Let us remark that, given the functions $\rho(t)$ and $p(t)$, the metric is determined. If one also has a dynamics, field equations also determine the metric. Accordingly, field equations are bound to be equivalent to Einstein equations with a perfect fluid stress tensor (together with a suitable EoS to replace matter equations).
The dynamics simply selects the form of the pressure and density functions which, in fact, corresponds to field equations.

Einstein tensor is a symmetric tensor, so it can be put in canonical form with respect to the metric $g_{\mu\nu}$. In particular, it has a basis of eigenvectors, in particular, it allows one eigenvector which is time-like. Let u be a (future directed) unit time-like eigenvector

$$G_{\mu\nu}u^\nu = -\tfrac{\kappa}{\mathrm{c}}\rho g_{\mu\nu}u^\nu \qquad g(u,u) = -1 \qquad u^0 > 0 \tag{8.3.4}$$

and let $-\kappa\mathrm{c}^{-1}\rho$ be the corresponding eigenvalue; then one has $\kappa\mathrm{c}^{-1}\rho = G_{\mu\nu}u^\mu u^\nu$. Let us also define p as the function such that $\kappa\mathrm{c}^{-1}p = \tfrac{1}{3}G_{\mu\nu}\left(g^{\mu\nu} + u^\mu u^\nu\right)$. That defines the density, the pressure and the comoving velocity of fluid as functions of the Einstein tensors.

If we now consider a dynamics, matter fields produce a *energy-momentum stress tensor* $T_{\mu\nu}$ which, on-shell, is necessarily a perfect fluid tensor, with a matter density $\rho(t)$ and pressure $p(t)$ which obey

$$\frac{\dot a^2}{a^2} + \frac{\mathrm{c}^2 k}{a^2} = \frac{\kappa\mathrm{c}}{3}\rho \qquad\qquad \frac{\ddot a}{a} = -\frac{\kappa\mathrm{c}}{6}\left(\rho + 3p\right) \tag{8.3.5}$$

which are called *Friedmann equations.* $[\kappa\mathbf{c}\rho] = (M^{-1}T)(LT^{-1})(ML^{-1}T^{-2}) = T^{-2}$

Let us stress that the density and pressure are to be understood as the total matter density and pressure. When, in the model, we allow more than one matter field, then each matter species contributes with its own density and pressure and both ρ and p are the sum of all partial densities and pressures, one for each species.

Let us also define the *curvature density* and *pressure*

$$\rho_k = -\frac{3\mathbf{c}k}{\kappa a^2} \qquad\qquad p_k = \frac{\mathbf{c}k}{\kappa a^2} \tag{8.3.6}$$

so that we can put Friedmann equations in canonical form

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa\mathbf{c}}{3}(\rho + \rho_k) = \frac{\kappa\mathbf{c}}{3}\rho_T \qquad\qquad \frac{\ddot{a}}{a} = -\frac{\kappa\mathbf{c}}{6}\left(\rho_T + 3p_T\right) \tag{8.3.7}$$

for the *total density* $\rho_T = \rho + \rho_k$ and the *total pressure* $p_T = p + p_k$.

In other words, one can encode the spatial curvature k as an extra form of effective matter (described by (ρ_k, p_k)) and pretend that the geometry is spatially flat (i.e. $k = 0$). Thus is particularly useful, as one can restrict to Friedmann equations with no spatial curvature, without loss of generality.

Point-like Lagrangian

As we noticed, Friedmann equations are ODE. They can be regarded as a dynamical system which is obtained from a Lagrangian depending on the scale factor and its first derivative.

There is no *a priori* guarantee that, taking the standard Hilbert Lagrangian and substituting into it the cosmological principle, one obtains a Lagrangian $L(a, \dot{a})$, the Euler–Lagrange equation of which has something to do with Friedmann equations, even though Friedmann equations, as a matter of fact, *are* standard Einstein equations written in view of the cosmological principle.

Once again, replacing constraints on configurations into the Lagrangian is not a good practice, since, in general, this operation does not commute with the Euler–Lagrange operator.

However, in this case, if one considers the Ricci scalar of a metric in FLRW form one gets

$$R = \frac{6(a\ddot{a} + \dot{a}^2 + \mathbf{c}^2 k)}{a^2 \mathbf{c}^2} \tag{8.3.8}$$

so that the standard Hilbert Lagrangian on FLRW metric, integrated along a spatial comoving volume V, reads as

$$\boldsymbol{L} = \frac{6}{2\kappa\mathbf{c}}a(a\ddot{a} + \dot{a}^2 + \mathbf{c}^2 k)\, dt \int_V \frac{r^2 \sin(\theta)}{\sqrt{1 - kr^2}}\, dr \wedge d\theta \wedge d\phi = \frac{3V_0}{\kappa\mathbf{c}}(a^2\ddot{a} + a\dot{a}^2 + \mathbf{c}^2 ka)\, dt = \frac{3V_0}{\kappa\mathbf{c}}(-a\dot{a}^2 + \mathbf{c}^2 ka)\, dt + \frac{d}{dt}\left(\frac{3V_0}{\kappa\mathbf{c}}a^2\dot{a}\right)\, dt \tag{8.3.9}$$

where we set $V_0 = \int_V \frac{r^2 \sin(\theta)}{\sqrt{1-kr^2}}\, dr \wedge d\theta \wedge d\phi$ for the comoving volume of the spatial region V ($[V_0] = L^3$), which does not depend on t or a, thus it does not affect the evolution of $a(t)$.

This does not even account for matter, thus it does not give us more than an educated guess to be checked by computing its equation of motion to be compared with Friedmann equations.

For some function $\rho(a)$ and a constant λ to be fixed, let us consider the Lagrangian

$$\boldsymbol{L} = -\frac{3V_0}{\kappa\mathbf{c}}\left(a\dot{a}^2 - \mathbf{c}^2 ka + \lambda\rho(a)a^3\right)\, dt \tag{8.3.10}$$

which is called the *point-like Lagrangian.* $([\lambda] = M^{-1}L.)$

The point-like Lagrangian does not depend explicitly on t, thus its *"energy"*

$$\mathcal{H} = \frac{3V_0}{\kappa \mathbf{c}} \left(-2a\dot{a}^2 + a\dot{a}^2 - k\mathbf{c}^2 a + \lambda\rho(a)a^3 \right) = -\frac{3V_0}{\kappa \mathbf{c}} \left(\frac{\dot{a}^2}{a^2} + \frac{k\mathbf{c}^2}{a^2} - \lambda\rho(a) \right) a^3 \tag{8.3.11}$$

is conserved. Hence, by fixing $\lambda = \frac{\kappa \mathbf{c}}{3}$, the first Friedmann equation is equivalent to requite $\mathcal{H} = 0$.

The equation of motion for the Lagrangian (8.3.10) is

$$2\ddot{a} = \frac{\kappa \mathbf{c}}{3} \left(2\rho + \rho'a \right) a \quad \Longleftrightarrow \quad \frac{\ddot{a}}{a} = \frac{\kappa \mathbf{c}}{6} \left(2\rho + \rho'a \right) \tag{8.3.12}$$

which becomes the second Friedmann equation, provided one has the pressure defined to be

$$-(\rho + 3p) = 2\rho + \rho'a \quad \Longleftrightarrow \quad p = -\left(\rho + \tfrac{1}{3}\rho'a \right) \tag{8.3.13}$$

On the other hand, in Friedmann equations, the density and the pressure do not come as independent functions. In view of Bianchi identities, the Einstein equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ imply the conservation of the energy-momentum stress tensor, i.e. $\nabla_\mu T^{\mu\nu} = 0$, which, in turn, implies

$$0 = \mathbf{c}\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho + p)u^\mu u^\nu + (\rho + p)\nabla_\mu u^\mu u^\nu + (\rho + p)u^\mu \nabla_\mu u^\nu + \nabla_\mu p g^{\mu\nu} \tag{8.3.14}$$

We can first compute the quantity

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \{g\}^\nu_{\epsilon\mu} u^\epsilon = \{g\}^\nu_{0\mu} = \tfrac{1}{2} g^{\nu\epsilon}(-\partial_\epsilon g_{0\mu} + \partial_0 g_{\mu\epsilon} + \partial_\mu g_{\epsilon 0}) \quad \Rightarrow \begin{cases} \nabla_\mu u^\mu = \tfrac{1}{2} g^{ij}\partial_0 g_{ij} = \tfrac{1}{2}\frac{2a\dot{a}}{a^2} g^{ij} g_{ij} = 3\frac{\dot{a}}{a} \\ u^\mu \nabla_\mu u^\nu = \tfrac{1}{2} g^{\nu\epsilon}(-\partial_\epsilon g_{00} + \partial_0 g_{0\epsilon} + \partial_0 g_{\epsilon 0}) = 0 \end{cases} \tag{8.3.15}$$

> Notice that u^μ is automatically a geodesic field, as a direct consequence of cosmological principle, only.

Then the conservation of the energy-momentum stress tensor is

$$(\dot{\rho} + \dot{p})u^\nu + 3(\rho + p)\frac{\dot{a}}{a}u^\nu + \dot{p}g^{0\nu} = 0 \quad \Longleftrightarrow \begin{cases} (\dot{\rho} + \dot{p}) + 3(\rho + p)\frac{\dot{a}}{a} - \dot{p} = 0 \quad \Longleftrightarrow \quad \left(a\rho' + 3(\rho + p) \right) \frac{\dot{a}}{a} = 0 \quad \Longleftrightarrow \quad p = -\left(\rho + \tfrac{1}{3}\rho'a \right) \\ (\dot{\rho} + \dot{p})u^i + 3(\rho + p)\frac{\dot{a}}{a}u^i + \dot{p}g^{0i} = 0 \quad \Longleftrightarrow \quad 0 = 0 \end{cases} \tag{8.3.16}$$

Accordingly, one has a pair of Friedmann equations for any pair of functions $(\rho, p = -\left(\rho + \tfrac{1}{3}\rho'a \right))$, i.e. it is completely determined by the the function $\rho(a)$, only. Hence, Friedmann equations are equivalently described by the point-like Lagrangian. The solutions of Friedmann equations are in one-to-one correspondence with solutions of the point-like Lagrangian with zero energy $\mathcal{H} = 0$.

Since the point-like Lagrangian is a Lagrangian with one degree of freedom (described by the Lagrange coordinate a) and it has a first integral $\mathcal{H}$, solutions are completely characterised by the conservation law. Solving the first Friedmann equation gives $a(t)$, which also solves the second equation.

The conservation reads as

$$\mathcal{H} = 0 \quad \Longleftrightarrow \quad \dot{a}^2 = \tfrac{\kappa c}{3}\rho(a)a^2 - kc^2 =: \Phi(a) \tag{8.3.17}$$

That is a Weierstrass equation and can be solved by separation

$$\pm \int_{a_0}^{a} \frac{da}{\sqrt{\Phi(a)}} = \int_{t_0}^{t} dt \quad \Longleftrightarrow \quad t - t_0 = F(a) - F(a_0) \quad \Longleftrightarrow \quad a(t) = a(t - t_0; a_0) \tag{8.3.18}$$

that, in principle, together with the initial condition $a(t_0) = a_0$, completely determines the evolution of the scale factor (of course, generically, with the exceptions of degenerate cases). Let us stress that, obviously, initial conditions needs to be given at a non-singular time, e.g. today.
The first Friedman equation does imply the second, in fact:

$$0 = \tfrac{d}{dt}\left(\dot{a}^2 - \tfrac{\kappa c}{3}\rho(a)a^2 + kc^2\right) = \left(2\ddot{a} - \kappa c\left(\tfrac{1}{3}\rho' a + \rho - \tfrac{1}{3}\rho\right)a\right)\dot{a} = 2\left(\ddot{a} + \tfrac{\kappa c}{6}(3p + \rho)a\right)\dot{a} \quad \Rightarrow \frac{\ddot{a}}{a} = -\frac{\kappa c}{6}(\rho + 3p) \tag{8.3.19}$$

Accordingly, we can forget about the second Friedmann equation (which is, in fact, used to define the pressure, once the solution is known) and use the first one to determine the scale factor. Solving the first Friedmann equation is quite simple, since it is a Weierstrass equation and, as such, can be integrated by separation. Actually, in this way, one needs to invert the solution for obtaining $a(t)$ or, as we should do, regard the integral of the Weierstrass equation as a parametric representation of the solution $a(t)$ given as $\gamma : a \mapsto (t(a), a)$, just to avoid inversion, which is a costly operation.
From now on, we shall refer to the first Friedmann equation, as to the *Friedmann equation*, since the second one is irrelevant to solve the model.

Let us stress that this is possible, and relatively easy, just because we introduced the density $\rho(a)$ as a function of the scale factor, not as a function of time t. The original density, which along a solution is expected to be a function of time, is $\rho(t) = \rho(a(t))$, computed along a solution $a(t)$. Accordingly, the Friedmann equation determines $a(t)$, which hence determines $\rho(t)$ and $p(t)$ though equation (8.3.13).

One has to get convinced that the density function $\rho(a)$, which allows to write down the point-like Lagrangian, contains more information than the *naive* on-shell function $\rho(t)$. As usual, we need to be precise and explicit about exactly what extra information $\rho(a)$ contains. In Subsection 8.3.3 we shall provide few examples of solutions.

Equations of state

An *equation of state* (EoS) is a relation between the density and the pressure. It allegedly accounts for the type of matter allowed in the model, ideally to replace matter equations.

If one regards Friedmann equations as equations for an arbitrary triple of functions $(a(t), \rho(t), p(t))$ (i.e. forgetting for a moment that they steam from Einstein equations, hence, in view of Bianchi identities, the energy-momentum stress tensor is conserved, hence the pressure must obey (8.3.13) condition) then, obviously, one cannot determine three functions with two equations and an extra equation is needed, namely the EoS.

Ok, you are right! In view of conservation, the independent unknowns are two only. However, exactly because of the conservation law, the independent equations are only one since the second Friedmann equations is a consequence of the first one. Still one cannot determine two functions with one equation.

Such an extra equation is usually provided in the form of an EoS, namely an algebraic relation $S(\rho, p) = 0$ or, more often, simply as $p = p(\rho)$.

The EoS $p = p(\rho)$ is obtained by solving the more general form $S(\rho, p) = 0$ for p. If one wants to keep it simple and general, the EoS can also be represented in parameterised form

$$\sigma : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (\rho(s), p(s)) \tag{8.3.20}$$

chosen so that one identically has $S(\rho(s), p(s)) = 0$.

> Traditionally, the simplification is even more abrupt and often one considers a *barotropic EoS*, in the form $p = w\rho$, for a constant w, called the *state parameter*. While, of course, this is a good approximation of any reasonable function around any point of its graph, the approximation may fail to be accurate, if extrapolated very far from the initial condition.
>
> Of course, cosmologists are not stupid, and they well know that, in extreme regimes, species of matter which have been neglected may become relevant, as well as species which are relevant today may become irrelevant in other regimes.
>
> On the other hand, we shall practice to account for more complicated EoS, which better describe extrapolation to extreme regimes. At the very least, if possible, we would like to avoid solving approximate equations and then assuming that their exact solutions glue together to produce an approximate solution of the exact equation. That is a bit hard to follow though it is what it is usually done. Often, it is also well supported by evidences, though it is usually hard to have some precise mathematical control over it.

So let us start by a simple EoS as $p = w\rho$.

The pressure p is also related to the density by equation (8.3.13), from which we obtain a differential equation for the function $\rho(a)$, i.e.

$$-w\rho = \rho + \tfrac{1}{3}\rho' a \qquad \Longleftrightarrow \qquad \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \qquad \Longleftrightarrow \qquad \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3-3w} \tag{8.3.21}$$

By changing the state parameters and remembering that today the scale factor is normalised to have $a_0 = 1$, one gets different kind of matter, which are essentially defined accordingly:

– for $w = 0$ one has *dust*, which is described by the EoS $p = 0$ and by the density $\rho = \rho_0 a^{-3}$.

– for $w = \tfrac{1}{3}$ one has *radiation*, which is described by the EoS $p = \tfrac{1}{3}\rho$ and by the density $\rho = \rho_0 a^{-4}$.

– for $w = -1$ one has *dark energy*, which is described by the EoS $p = -\rho$ and the density $\rho = \rho_0$ is constant.

Dust matter is also called *ordinary*, or *non-relativistic* matter. It is appropriate whenever the interaction energy is negligible with respect to the rest energy of the system (its mass). For example, it is used to model galaxies in cosmology.

Radiation is also called *ultra-relativistic* matter and it is appropriate to describe any matter in which the rest energy is negligible with respect to the kinetic energy. That is used, for example, also for primordial gasses.

For dark energy, the energy-momentum stress tensor takes the form

$$T_{\mu\nu} = -\mathbf{c}^{-1}\rho_0 g_{\mu\nu} = -\Lambda g_{\mu\nu} \qquad (\Lambda := \mathbf{c}^{-1}\rho_0) \tag{8.3.22}$$

Thus a cosmological constant can be also modelled, as it happens for the spatial curvature, as an additional effective matter species (of dark energy) and one can restrict to consider cosmological models without cosmological constant, without loss of generality.

In many models, one uses more elaborate EoS. One example which is often used in called *polytropic EoS* and it is given by

$$p = K\rho^{\frac{k+1}{k}} \tag{8.3.23}$$

where K and k are real constants characteristic of the type od matter under consideration. The constant k is called the *polytropic index*, while sometimes $\gamma = \frac{k+1}{k}$ is used instead and it is called the *polytropic exponent*.

Often $k \in (\frac{1}{2}, 1)$ is used for modelling neutron stars. A polytropic index of $k = \frac{3}{2}$ is used for the so-called *fully convective cores*, for example for red giants, gaseous planets, or even rocky planets. Main sequence stars, including the Sun, are usually modelled by $k = 3$.

In the limit $k \to \infty$ (i.e. $\gamma = 1$) one can model globular clusters and the polytropic EoS reduces to a barotropic perfect fluid EoS.

Solutions of Friedmann equation

As usual, before even discussing the meaning of it, let us show that we are able to effectively discuss solutions of Friedmann equation. The terminology will be fully justified later on.

For any metric g in FLRW form, let us define the *Hubble parameter* to be

$$H := \frac{\dot{a}}{a} \tag{8.3.24}$$

Let us first consider the Friedmann equation for dust alone and no spatial curvature. Hence we have a density function $\rho(a) = \rho_0 a^{-3}$ and Friedmann equation is

$$\dot{a}^2 = \frac{\kappa c}{3}\rho(a)a^2 = \frac{\kappa c}{3}\rho_0 a^{-1} = \Phi(a) \tag{8.3.25}$$

The evolution of the scale factor is constrained to stay within the allowed region where $\Phi(a) \geq 0$, that is $a > 0$. If we start at $a_0 = 1$, then the scale factor evolves and goes to zero or infinity, depending if it starts with a positive or negative derivative. How much time it takes to get to zero or infinity depends on the integral

$$t - t_0 = \int_1^\infty \frac{da}{\sqrt{\Phi(a)}} = \sqrt{\frac{3}{\kappa c \rho_0}}\int_1^\infty \sqrt{a}\,da \qquad t - t_0 = -\int_1^0 \frac{da}{\sqrt{\Phi(a)}} = \sqrt{\frac{3}{\kappa c \rho_0}}\int_0^1 \sqrt{a}\,da = \frac{2}{\sqrt{3\kappa c \rho_0}} =: \Delta \tag{8.3.26}$$

That means that if the universe is now expanding, it will expand forever, while it has been expanded for a (coordinate time) time Δ, i.e. it started expanding from a scale factor $a = 0$, Δ years ago and expanded to $a_0 = 1$ today.

> We made an abuse of notation, which is quite substantial on a physical stance. That is why we said that, by now, we do not want to enter the meaning of this integration.
>
> The issue is that we said that between the surface at $t = t_1$ and the surface $t = t_0$ the time passes by for $t_1 - t_0$. Now that is a coordinate time and we still have to prove that it has something to do with a physical time, measured by a some clock. We shall later show that a comoving clock will exactly measure a time $t_1 - t_0$ as it goes from the surface at $t = t_1$ to the surface $t = t_0$, which will give a precise physical meaning to which time we mean in it.
>
> Until them, we shall talk about *coordinate time* which *a priori* has nothing to do with physical time. To some extent, the coordinate time is just a parameterisation of worldlines and surfaces of the ADM foliation. It will become a physical time once it will be related to the reading of some clock.

Einstein said: *time is what clocks measure.*

Let us remark that discussing the coordinate time to go from a value of the scale factor to another is equivalent to solve the evolution of the scale factor. Let us fix today as $t_0 = 0$. Then we have

$$t = \pm \int_{a_0=1}^{a} \frac{da}{\sqrt{\Phi(a)}} = \pm \frac{2}{\sqrt{3\kappa c \rho_0}} a^{\frac{3}{2}} \Big|_{1}^{a} \qquad \Rightarrow t(a) = \pm \frac{2}{\sqrt{3\kappa c \rho_0}} \left(a^{\frac{3}{2}} - 1 \right) \tag{8.3.27}$$

That is already a parameterisation of the evolution of the scale factor, namely $\gamma : a \mapsto (t(a), a)$ draws the graph of the function $a : \mathbb{R} \to \mathbb{R} : t \mapsto a(t)$ which is what we are aiming to determine.

In this specific case though, we are able to invert the function

$$a(t) = \left(\frac{\sqrt{3\kappa c \rho_0}}{2} t + 1 \right)^{\frac{2}{3}} \tag{8.3.28}$$

which, in fact, gives $a(0) = 1$ today and $a(t) = 0$ when $t = -\frac{2}{\sqrt{3\kappa c \rho_0}}$, meaning that the age of the universe is $t_U = \frac{2}{\sqrt{3\kappa c \rho_0}}$, since the universe started expanding at $t = -t_U$ at $a = 0$, it expands to $a = 1$ today, and it will keep expanding forever.

We can try to insert numerical values to evaluate the age of our universe, if it was made of only dust.
By assuming ρ_0 as the density of barionic matter today, we get (see Fig. 7.1: *green* line)

$$t_U = 1.413 \cdot 10^{18} \ s = 44.77 \ By \tag{8.3.29}$$

If we assume ρ_0 to be critical, one has (see Fig. 7.1: **black dash** line)

$$t_U = 3.03 \cdot 10^{17} \ s = 9.60 \ By \tag{8.3.30}$$

Let us now consider a universe with dust but with a positive curvature (see Fig. 7.1: **red** line). The density function is $\rho(a) = \rho_0 a^{-3} - \frac{3ck}{\kappa} a^{-2}$.

$$\dot{a}^2 = \frac{\kappa c \rho_0}{3a} - c^2 k = \Phi(a) \tag{8.3.31}$$

where $k = \frac{\kappa}{3c} (\rho_0 - \rho_0^c) > 0$ is fixed so that the total density is critical.

The allowed region is $a \in (0, a_M)$ for a maximal value $a_M = \frac{\kappa \rho_0}{3ck} = \frac{\rho_0}{\rho_0 - \rho_0^c}$. The universe expands to a scale factor a_M in a finite time t_M, then, since the zero is simple, it is reflected and starts contracting to a_0, again in a time T_U, the total life of this universe being $2T_U$.

The evolution of the scale factor is obtained as

$$t = \int_1^a \frac{da}{\sqrt{\Phi(a)}} = \int_1^a \frac{\sqrt{3a} \, da}{\sqrt{\kappa c (\rho_0 - a(\rho_0 - \rho_0^c))}} \tag{8.3.32}$$
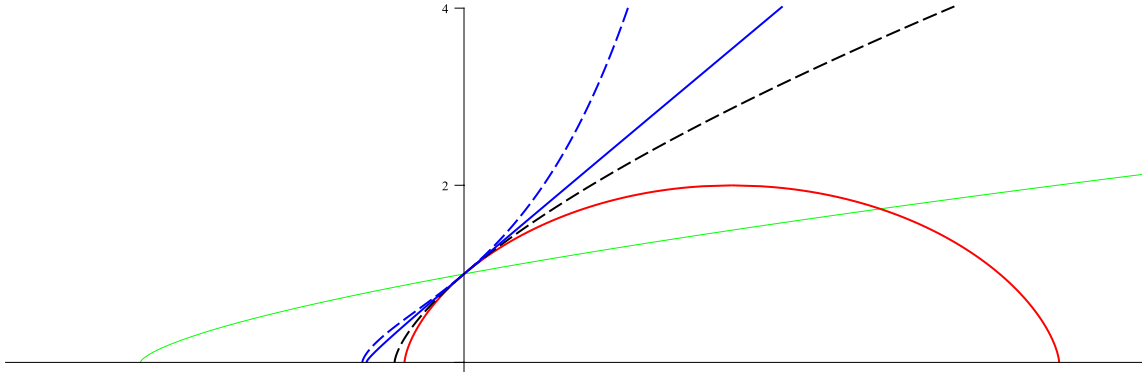
Fig. 7.1: Solutions of Friedmann equation.
*Spatially flat solution: dust $\rho_0 = \rho_0^c$, $k = 0$ (**black dash**)*
*Spatially closed solution: dust, $\rho_0 = 2\rho_0^c$, $k > 0$ is fixed to fill up to the critical density (**red**)*
*Spatially open solution: dust, ρ_0 is taken to be the observed visible barionic density, $k < 0$ is fixed to fill up to the critical density (**solid blue**)*
*Spatially flat solution: dust, ρ_0 is taken to be the observed visible barionic density, $k = 0$ (**green**). Notice different derivative $\dot{a}(t_0)$ today $t_0 = 0$.*
*ΛCDM model: visible barionic matter 0.046, cold dark matter 0.24, dark energy 0.714, spatially flat $k = 0$ (**blue dash**)*

The age of the universe today is

$$\Delta = -t(0) \sim 2.59 \cdot 10^{17} \, s \sim 8.22 \, By \tag{8.3.33}$$

The total age T_U is computed as

$$2T_U = 2\left(\Delta + t(a_M)\right) \sim 2.34 \, 10^{18} \, s \sim 74.06 \, By \tag{8.3.34}$$

Let us now consider a universe with dust but with a negative curvature. With a negative spatial curvature, the density function is again $\rho(a) = \rho_0 a^{-3} - \frac{3ck}{\kappa} a^{-2}$ and the Weierstrass function is still $\Phi(a) = \frac{\kappa c \rho_0}{3a} - c^2 k$. However, since now $k < 0$, the allowed region extends to infinity and the evolution is given by

$$t = \int_1^a \frac{da}{\sqrt{\Phi(a)}} = \int_1^a \frac{\sqrt{3a} \, da}{\sqrt{\kappa c \rho_0 + 3ac^2 k}} \tag{8.3.35}$$

We also have another connected component of the allowed region for $a \in (-\infty, a_m)$ with $a_m = \frac{\kappa \rho_0}{3ck}$. Since the sign of the scale factor is not that important, as it appears in the metric as a^2, that corresponds to an universe with can start at $a_0 = -1$, contract reach a minimal radius a_m and then expand again to infinity.
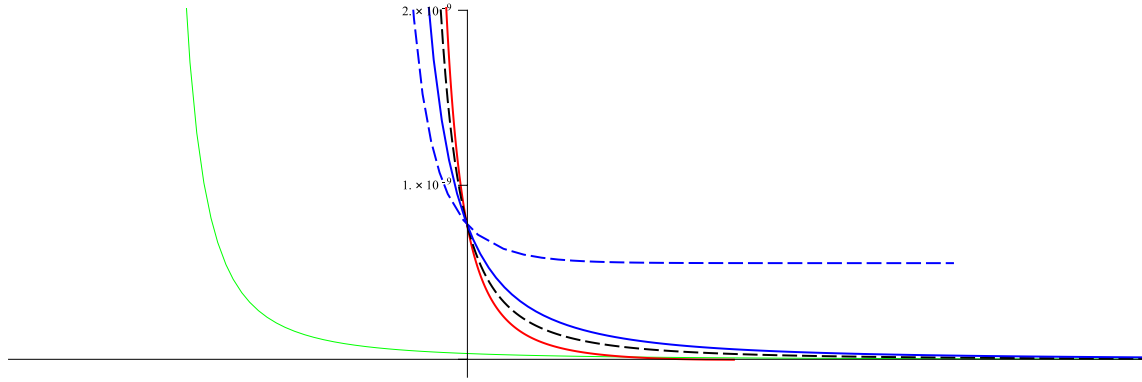
Fig. 7.2a: Total density of Friedmann models as a function of time (same color code as in Fig. 7.1).
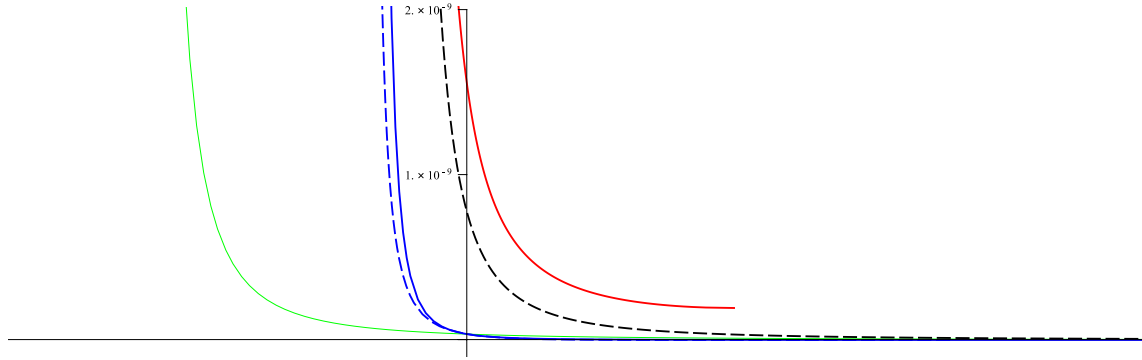
Fig. 7.2b: Visible barionic density of Friedmann models as a function of time (same color code as in Fig. 7.1).

As far as the total density $\rho(t)$ as a function of time is concerned, we have $t(a)$ and $\rho(a)$. This provides a parametric representation of the function $\rho(t)$, where the scale factor a enters as a parameter.

We can then easily draw the function $\rho(t)$ in different models. Let us notice that different models have different characteristic evolution of the total density. If we are able to observe the total density and its evolution is when we can really distinguish the models.

We can also draw each component separately, for example the visible barionic density. In this case, different models do predict different visible densities today, in addition to different evolution.

As we can see in Fig. 7.1, Friedmann equations with dust only cannot catch precisely what we know. The most accurate representation of what we know is the ΛCDM model (**dash blue**) that we shall discuss in a while.

Let us now pretend for a moment that we know how to measure the Hubble parameter H_0 as, we shall see, it is certainly the case in standard GR. If we model our universe with dust only, with a density today ρ_0 compatible with the visible barionic matter that we see, we get a open universe (*green*) with ab age of more than $44\ By$ which moreover has a Hubble parameter H_0 that is definitely incompatible with observations.

So let us say that we account for the Hubble parameter H_0 we measure (and what we are really measuring and why is something still to be addressed in general). If we can assume that the density of barionic matter is what we estimate from visible matter (***solid blue***) we obtain an open universe which falls a bit short in universe age but mainly it shows no universe late acceleration, i.e. the scale factor grows in time though it does so remaining concave (the speed of growing decreases in time).

If we assume that there is some barionic matter dust which for some reason is not visible (from interstellar gasses to a spread population of Dyson spheres) enough to reach spatial flatness (***dash black***) the age of the universe falls considerably short (to less than $10\ By$). This, of course, also does not show any late time acceleration in the evolution of the scale factor.

If the barioric matter dust density exceeded the critical value, then one would obtain a spatially closed universe (***red***) with an even shorter age, no late acceleration, as well as a quite different future destiny. Assuming $\rho_0 = 2\rho_0^c$, the time for re-collapsing is of about $74.06\ By$).

Now at this stage, we still do not control exactly what quantities we can observe and how. Here we only wish to show two things: first, there are a lot of solutions of the Friedmann equation with different behaviours depending on the parameters. Second, we are actually able to do the computations and predict the evolution of the scale factor, at least in many simple cases.

Mixtures and abundances

We already defined the Hubble parameter to be essentially a way of accounting for the growth speed of the scale factor, namely $H = \frac{\dot{a}}{a}$. In particular, today, when $a_0 = 1$, we have $H_0^2 = \frac{\kappa \mathrm{c}}{3}\rho_0 - \mathrm{c}^2 k$; for spatially flat solutions one simply $has H_0^2 = \frac{\kappa \mathrm{c}}{3}\rho_0$. Thus

$$\rho_0^c = \rho_0 = \frac{3}{\kappa \mathrm{c}}H_0^2 \tag{8.3.36}$$

is the total density that one needs today for the metric to be spatially flat. That is called the *critical density (today)*.

> If we assume that $\lim_{a \to +\infty} \rho(a) = 0$, i.e. as the universe expands, the scale factor becoming large, the matter content is diluted and approaches zero as the universe becomes older, the Friedmann equation, recast in the form of a Weierstrass equation reads as
>
> $$\dot{a}^2 = \frac{\kappa \mathrm{c}}{3}\rho(a)a^2 - \mathrm{c}^2 k = \Phi(a) \tag{8.3.37}$$
>
> and one has $\lim_{a \to \infty} \Phi(a) = -\mathrm{c}^2 k$, which is negative if $k > 0$. Accordingly, if $k > 0$ the scale factor cannot become big at will, it cannot approach infinity. It will approach a maximal value $a = a_M$ and (if the function Φ has a simple zero in a_M, as it generically happens) will bounce back and it will begin to contract. On the contrary, if $k \leq 0$ the universe can expand forever and its scale factor can grow without being bound.
>
> The critical density today ρ_0^c is the threshold for the density, which discriminates ever-growing universes, from first-expanding-then-contracting ones. This argument does not really stands in general since we do not know much *a priori* about the density function. However, it is a remnant of the model when one was allowing only ordinary matter (usually dust or dust and radiation), no cosmological constant, no dark sources.
>
> Also notice that we are not meaning that the Hubble parameter is something one can observe, less than ever observe by measuring the receding speed of galaxies. That will be done later. By now, all these quantities are theoretical and geometric quantities still to be given a physical meaning.

Once we define the critical density, we can go back to the Friedmann equation, written in the form (8.3.7), and divide both hand sides by the critical density to obtain

$$H^2 = H_0^2 \left(\frac{\rho}{\rho_0^c} + \frac{\rho_k}{\rho_0^c} \right) = H_0^2 \left(\Omega + \Omega_k \right) \tag{8.3.38}$$

where, for any matter species, we defined the *species abundance* $\Omega_i = \frac{\rho_i}{\rho_0^c}$, for example in particular $\Omega = \frac{\rho}{\rho_0^c}$, $\Omega_k = \frac{\rho_k}{\rho_0^c}$, and let us also set $\Omega_T := \frac{\rho_T}{\rho_0^c} = \Omega + \Omega_k$ for the *total abundance*. By a direct evaluation of this equation today, one readily has that the total abundance $\Omega_T(t_0) = 1$ is always 1.

The abundance of a matter species is naturally a function of the scale factor a, since $\rho(a)$ is, and the Friedmann equation can be recast as

$$\dot{a}^2 = H_0^2 \left(\Omega(a) + \Omega_k(a) \right) a^2 \tag{8.3.39}$$

which is still in the form of a Weierstrass equation, this time the form of the Weierstrass $\Phi(a)$ being modulated by the different abundances we decide to allow.

A *mixture* is a family of matter species, labelled by an index i, each represented by a density $\rho_i(a)$ (or, equivalently, by an EoS $p_i = p_i(\rho_i)$). In the mixture, we always account for all matter species allowed in the model *and* the effective ones, namely, possibly, the cosmological constant and the curvature density.

Thus we have a total abundance in the form

$$\Omega_T(a) := \sum_i \frac{\rho_i(a)}{\rho_0^c} \qquad\qquad \Omega_T(1) = 1 \tag{8.3.40}$$

The sequence of all abundances for each matter species, real and effective, namely $\boldsymbol{\Omega} := (\Omega_k, \Omega_1, \ldots, \Omega_n, \Omega_\Lambda)$ is called a *cosmic pie.*

Today (2016), the best fit comic pie (WMAP) is made of $\Omega_b := 0.046$ of ordinary barionic matter, of $\Omega_{DM} := 0.24$ of cold dark matter, $\Omega_\Lambda := 0.714$ of dark energy. Radiation today does not play a role, though of course it may have a role back in early evolution, since it scales differently from dust. The curvature density is compatible with $\Omega_k = 0$.

Let us stress that the composition of the universe depends on when it is considered, since it is an *energetic* budget of many species each scaling differently in time. For example, in early universe (at the decoupling) the cosmic pie was made of 0.12 of ordinary barionic matter, of 0.63 of cold dark matter, 0.15 of radiation, 0.10 of neutrinos. The dark energy density, contrarily to other matter species, does not increase as the universe become denser, so it was initially negligible.

One can consider a theory with the cosmic pie observed by WMAP. It corresponds to a total abundance function as

$$\Omega(a) = \Omega_b a^{-3} + \Omega_{DM} a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \tag{8.3.41}$$

The Friedmann equation reads as

$$\dot{a}^2 = H_0^2 \left(\left(\Omega_b + \Omega_{DM} \right) a^{-1} + \Omega_k + \Omega_\Lambda a^2 \right) \tag{8.3.42}$$

which can be integrated and gives an evolution of the scale factor (see Fig. 7.1 *dash blue* line), total density (see Fig. 7.2.a *dash blue* line), and visible barionic matter as shown in Fig. 7.2.b (*dash blue* line).

Let us remark that once we have a single matter species, then it is quite reasonable to consider a simple EoS for it, conservation of the energy-momentum stress tensor $T_{\mu\nu}$ is equivalent to matter equation. Still one can discover that the EoS is a good assumption only for a limited time interval

around today and that further unknown effects have been neglected today though they might become important at very early or late time, though the model is clear and simple.

When we allow a composite mixture we are kicked out of this relative paradise. Even assuming each matter species is relatively simple as far as the EoS is concerned, they usually corresponds to a multitude of different matter fields, each with its own matter equation, each with its own energy-momentum stress tensor, which is separately conserved—as long as each partial matter Lagrangian is separately covariant—each conservation corresponding to its matter equations. However, if two matter species interact, only their common energy-momentum stress tensor is conserved. Moreover, even though one has a quite precise correspondence between EoS and the density function $\rho_i(a)$ for each species, when different species are allowed, the mixture has not a simple EoS or a simple total density function.

Still one can compute the total density function $\rho_T(a)$ (as well as the corresponding abundance $\Omega_T(a)$) which, in turn, determine the evolution of the scale factor $a(t)$ from the Friedman equation and the initial condition $a(t_0) = 1$.

However, the total density function can be *a priori* complicated at will. Moreover, we shall see in next Section we shall see that a simple EoS in the Jordan frame, induces an EoS in the Einstein frame which is exactly known, it depends on the function $f(\mathcal{R})$ which describes the gravitational sector, and, generally, is complicated at will. Thus, also for these reasons, when working with one species, it is futile to keep stuck to simple EoS (and, correspondingly, to simple density functions).

Besides these amenities, we also have another issue with mixtures. Suppose we observe accurately and precisely the evolution of the scale factor $a(t)$ and $\rho(t)$ so that we are able to regard these observational facts as a parametric description of the density function $\rho(a)$, and hence of the description of the EoS of the mixture. Imagine, however, that we do not have a direct evidence of which species one should expect to appear in the cosmic pie.

One could argue that knowing $\rho(a)$ uniquely defines a mixture of barotropic perfect fluids, as an expansion of the density function as a sum of powers $\rho_i \propto a^{\gamma_i}$ each corresponding to a barotropic perfect fluid. One expects that to work more or less as a Taylor expansions of an analytic function. And that it is wrong.

It is exactly because it works precisely as a Taylor expansion, even if it is not. If we have a function $\rho(a)$ to be expanded as a power series, if the function is analytical, it has a single Taylor expansion

$$\rho(a) = \sum_k c_k a^k \tag{8.3.43}$$

Now we are allowing negative powers and, moreover, real exponents. Taylor expansions are also canonical, in the sense that if we expand to order $k_0 = 5$ and the coefficient $c_3 = 0$ is missing, then it will be missing from any expansion at any order. The coefficients are *independent* of the expansion order, either they are there or they are not.

That has not much to do with the fact that we allow negative or real powers. It has rather to do with the fact that species are characterised by powers a^{γ_i}, i.e. around the value $a = 0$, at which the function is usually divergent—that is why we need negative exponents—while we impose initial conditions at $a_0 = 1$, i.e. today.

Let us consider the function $f(a) = (1 + a)^{-1}$ and we want to expand around the point $a_0 = 1$ in power series of a^n, instead of $(a - 1)^n$ we have at order 2

$$f(a) \simeq c_0 + c_1 a + c_2 a^2 \tag{8.3.44}$$

and one can use the conditions

$$f(1) = c_0 + c_1 + c_2 = \tfrac{1}{2} \qquad f'(1) = c_1 + 2c_2 = -\tfrac{1}{(1+a)^2}|_{a=1} = -\tfrac{1}{4} \qquad f''(1) = 2c_2 = 2\tfrac{1}{(1+a)^3}|_{a=1} = \tfrac{1}{4} \tag{8.3.45}$$

from which one obtains

$$c_2 = \tfrac{1}{8} \qquad c_1 = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2} \qquad c_0 = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{8} = \tfrac{7}{8} \qquad \Rightarrow f(a) \simeq \frac{7 - 4a + a^2}{8} \tag{8.3.46}$$

If we decide to expand to order 3

$$f(a) \simeq c_0 + c_1 a + c_2 a^2 + c_3 a^3 \tag{8.3.47}$$

and one can use the conditions

$$f(1) = c_0 + c_1 + c_2 + c_3 = \tfrac{1}{2} \qquad\qquad f'(1) = c_1 + 2c_2 + 3c_3 = -\tfrac{1}{(1+a)^2}|_{a=1} = -\tfrac{1}{4}$$
$$f''(1) = 2c_2 + 6c_3 = 2\tfrac{1}{(1+a)^3}|_{a=1} = \tfrac{1}{4} \qquad\qquad f'''(1) = 6c_3 = -6\tfrac{1}{(1+a)^4}|_{a=1} = -\tfrac{3}{8} \tag{8.3.48}$$

from which one obtains

$$c_3 = -\tfrac{1}{16} \qquad c_2 = \tfrac{1}{8} + \tfrac{3}{16} = \tfrac{5}{16} \qquad c_1 = -\tfrac{1}{4} - \tfrac{10}{16} + \tfrac{3}{16} = -\tfrac{11}{16} \qquad c_0 = \tfrac{1}{2} + \tfrac{11}{16} - \tfrac{5}{16} + \tfrac{1}{16} = \tfrac{15}{16} \qquad \Rightarrow f(a) \simeq \frac{15 - 11a + 5a^2 - a^3}{16} \tag{8.3.49}$$

So, you see a strange thing here: the expansion is not canonical, if we add a species to the expansion, the abundance of other species are affected. While in ordinary Taylor expansions, terms are not affected by the order or type of powers we decide to add, here they do. Notice that this happens even if we are considering only positive integer powers; thus that is not the issue. The issue is related to the fact that we expand in monomials which are not vanishing at the point around which we are expanding.

We can check it quite easily by expanding a polynomial $p(a) = a(a - 2)$ to order 1 as $p(a) \simeq c_0 + c_1 a$ to have

$$p(1) = c_0 + c_1 = -1 \qquad p'(a) = c_1 = 2a - 2|_{a=1} = 0 \tag{8.3.50}$$

from which one obtains

$$c_1 = 0 \qquad c_0 = -1 \qquad \Rightarrow f(a) \simeq -1 \tag{8.3.51}$$

or to order $k \geq 2$

$$p(a) = a^2 - 2a \tag{8.3.52}$$

At order 1, one has an approximation in terms of a constant, while at order 2 or higher the expansion is exact and one does not even have a constant term!

To show that something similar happens also with our expansions, which are more general than Taylor expansions, let us consider an example which is more in the line of cosmological applications. Let us consider a density function $\rho(a) = ca^{-2}$ and decide we want to see only dust and cosmological constant. Thus we would find:

$$\rho(a) \sim \tfrac{c}{3} + \tfrac{2c}{3a^3} \tag{8.3.53}$$

which approximates the actual effective density around today $a = 1$.
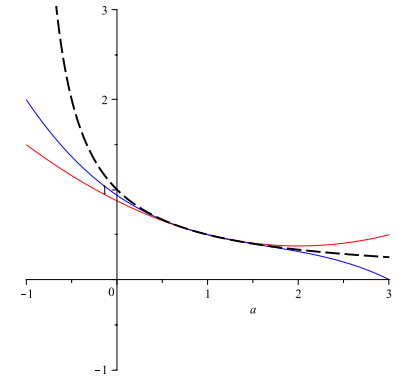
Fig. 7.3: second and third approximations of $f(a)$
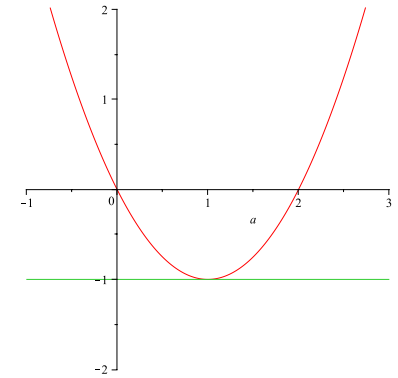
Fig. 7.4: first and second approximations of $p(a)$

This is a pretty strange example: a purely curvature density can be approximated around any given time $t = t_0$ by a spatially flat model with a suitable cosmological constant and dust, the approximation being good around $a_0 = 1$ (i.e. $t = t_0$) though eventually failing as one looks back in the past. One could also decide, in the same situation, that an expansion as $f(a) = c_1 a^{-1/2} + c_2 a^{-4}$ is preferable so to obtain

$$\rho(a) \simeq \tfrac{4c}{7} a^{-1/2} + \tfrac{3c}{7} a^{-4} \tag{8.3.54}$$

One has

$$f(1) = c_1 + c_2 \qquad\qquad f'(1) = \left(-\tfrac{1}{2} c_1 a^{-\frac{3}{2}} - 4 c_2 \tfrac{1}{a^5} \right) |_{a=1} = -\tfrac{1}{2} c_1 - 4 c_2 \tag{8.3.55}$$

and

$$\rho(1) = c \qquad\qquad \rho'(1) = \left(-2 c \tfrac{1}{a^3} \right) |_{a=1} = -2c \tag{8.3.56}$$

Hence the best approximation is obtained for

$$\begin{cases} -\tfrac{1}{2} c_1 - 4 c_2 = -2c \\ c_1 + c_2 = c \end{cases} \qquad \Rightarrow c_1 = \tfrac{4c}{7} \qquad\qquad c_2 = \tfrac{3c}{7} \tag{8.3.57}$$
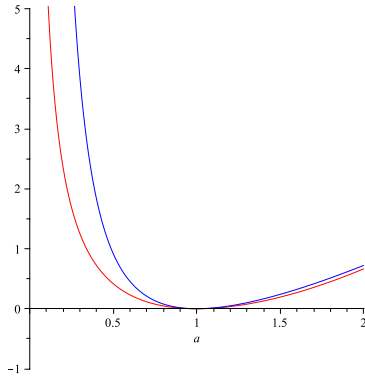
Fig. 7.5: (a) The deviation functions Δ_f (red), Δ_g (blue)   (b) The density function $\rho(a)$ (dash) and the approximations $f(a)$ (red) and $g(a)$ (blue)
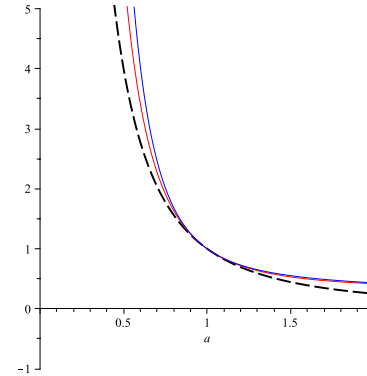
Thus, from this example, we have learnt two things: first, that, as we reasonably have approximate data about $\rho(a)$ which goes back for a while but not forever, the function $\rho(a)$ does not select a canonical expansion in terms of a mixture of perfect fluids. The same situation can be expanded along different species, obtaining different models which reliably reproduced the observed data today and for a while back in the past. Selecting the species we would like to see is something we add (possibly motivated by physical considerations, though external to cosmology itself). This choice is a kind of coloured glasses one decides to wear and it transforms the model, accordingly.

The second thing we learn is that the two approximations are not *exactly equivalent*. They are a good approximation around today $a_0 = 1$, though back in the past they are quantitatively different. If we are able to look back enough in the values of $\rho(a)$ the two approximations and the original density functions should produce different predictions.

In this specific toy example, suppose we have some error (say 1%) in observing $\rho(a)$. Any difference below 0.01 will be covered up by statistical errors.

If we approximate $\rho(a) \simeq f(a) = \frac{c}{3} + \frac{2c}{3a^3}$ ($\rho(a) \simeq g(a) = \frac{4c}{7\sqrt{a}} + \frac{3c}{7a^4}$, respectively) we have a relative deviation of

$$\Delta_f = \frac{f(a) - \rho(a)}{\rho(a)} \qquad\qquad \Delta_g = \frac{g(a) - \rho(a)}{\rho(a)} \tag{8.3.58}$$

We can plot these deviations (Fig. 7.5.a), as well as the approximation (Fig. 7.5.b).

One can define the *look back deviation intervals*, defined as

$$B_f := \{a \in [0,1] : \forall b \in [a,1], \Delta_f(b) < 0.01\} \qquad\qquad B_g := \{a \in [0,1] : \forall b \in [a,1], \Delta_g(b) < 0.01\} \tag{8.3.59}$$

In this case, one has something like

$$B_f = (0.903385, 1] \qquad\qquad B_g = (0.922164, 1] \tag{8.3.60}$$

meaning that, in order to see a deviation between the data and the approximate model, one has to look back at least to $a \sim 0.90$ for the first approximation, to $a \sim 0.92$ for the second.

Of course, the look back deviation interval depends on the available precision and on the approximation. Generally speaking, we could say that the approximation $\rho(a) \simeq f(a)$ is a better approximation, since it extends further back in time.

Though mathematically one could also consider how good is the approximation *within* the look back deviation interval (which in principle could be much better for $g(a)$ than for $f(a)$, at least for most of the values of a), of course, that data is physically unavailable, since it is completely suppressed by observation errors.

The look back deviation intervals will be discussed more in detail below. As a matter of fact, there are a number of surveys scheduled in the future to measure the evolution of the scale factor along the history of the universe. Thus it is important to have an idea of how to build models to describe universe evolution and how to compare different models. Let us remark that, in general, some extra care needs to be devoted to discuss what exactly is the scale factor that these surveys are measuring.

In the next Section we shall start considering how dynamics affects this issue. Until now, our discussion has been completely kinematical. Take a FLRW metric, it is described in terms of a scale factor and it obeys to some sort of Friedmann equations. Coming from a dynamics only constrains the pressure and make the second Friedmann equation irrelevant.

In the next Section we shall discuss in detail the role of the dynamics. A dynamics singles out a specific function $\rho(a)$, which is equivalent to selecting an EoS for matter. In standard GR, one has one single metric to do everything—describing the gravitational field, measuring distances, describing free falling particles, light cones, and wavefronts. It is hence natural in standard GR, assuming that surveys are observing the evolution of the scale factor on that single metric.

One should not write sentences like that!

Even in standard GR, since we have a metric and matter, one could have other metrics in the theory. For example, for any scalar field φ one could define the metric $\tilde{g} = e^\varphi g$. However, one does not even need the scalar field to be fundamental. One always has a scalar curvature to be used to define new metrics $\tilde{g} = e^{f(R)}g$, which is approximately what happens in ETG.

The issue is not whether one has other metrics available. It is rather which of these metrics is in FLRW form and which scale factor is measured in observations. This issue is even more explicit in Palatini $f(\mathcal{R})$-theories, since there one has two metrics metrics g and $\tilde{g}$, both with a physical meaning, and, as we shall see, both in FLRW form.

Vacuum solutions of Friedmann equation

We can consider vacuum solutions of Friedmann equation, i.e. we can consider the case with $\rho = \rho_0$, accounting for a cosmological constant Λ and, possibly, a spatial curvature.

In that special case, the Friedmann equation reads as

$$\dot{a}^2 = -k + \frac{\kappa c^2}{3}\Lambda a^2 \tag{8.3.61}$$

If $\Lambda = 0$ we have essentially two possibilities, depending on the value of k.

If $k = 0$, then the solution is $a(t) = a_0 = 1$ and the FLRW metric becomes

$$\eta = -c^2\,dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right) \tag{8.3.62}$$

One can check that this metric is flat, hence (or since) there exist new coordinates (t, x, y, z)

$$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta) \tag{8.3.63}$$

in which the metric can be recast in the form

$$\eta = -c^2\,dt^2 + dx^2 + dy^2 + dz^2 \tag{8.3.64}$$

which is called the *Minkowski metric*.

If $k = -\omega^2 < 0$, with $\omega > 0$, then the solution is $a(t) = \omega t + 1$ and the FLRW metric becomes

$$g = -c^2\,dt^2 + (\omega t + 1)^2\left(\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{8.3.65}$$

Now we can set $\omega\tau = \omega t + 1$, $\rho = \omega r$ so that metric can be recast as

$$g = -c^2 d\tau^2 + \tau^2\left(\frac{d\rho^2}{1 + \rho^2} + \rho^2\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{8.3.66}$$

One can also redefine $\rho = \sinh(\chi)$ to have the metric in the form

$$g = -c^2 d\tau^2 + \tau^2\left(d\chi^2 + \sinh^2(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \qquad\qquad d\rho^2 = \cosh^2(\chi)\,d\chi^2 = \left(1 - \rho^2\right)\,d\chi^2 \tag{8.3.67}$$

This is called the *Milne universe*.

For $k > 0$, the equation becomes inconsistent and there is no solution.

If we allow a positive cosmological constant $\Lambda > 0$

and $k = 0$
If $\Lambda > 0$ and $k < 0$
If $\Lambda > 0$ and $k > 0$

If we allow a negative cosmological constant $\Lambda < 0$

If $\Lambda < 0$ and $k = 0$
If $\Lambda < 0$ and $k < 0$
If $\Lambda < 0$ and $k > 0$

Tanto per definire AdS e dS

4. Extended cosmologies

Let us consider a Palatini $f(\mathcal{R})$-theory based on the Lagrangian

$$L = \frac{\sqrt{g}}{2\kappa} f(\mathcal{R}) + L_m(g, \phi) \tag{8.4.1}$$

We already discussed in Subsection 7.2.2 field equations induced by this dynamics.

If we go for a cosmological model based on this theory, one should start by restricting to spacetime geometries which are spatially homogeneous and isotropic to impose the cosmological principle.

However, in a Palatini $f(\mathcal{R})$-theory, one has at least two natural metrics, namely the original metric g as well as the conformal metric $\tilde{g}$. Thus, unlike what happens in standard GR, one should decide whether the cosmological principle corresponds to ask that g or $\tilde{g}$ is spatially homogeneous and isotropic. The answer is that, luckily, we do not have to worry about it: if g is spatially homogeneous and isotropic, then also $\tilde{g}$ is.

Let us suppose that g is spatially homogeneous and isotropic. Then there exists a coordinate system (t, r, θ, ϕ) in which g is in FLRW form, i.e.

$$g = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right) \tag{8.4.2}$$

If everything needs to be spatially homogeneous and isotropic, the conformal factor which, in view of the master equation, is a scalar function of metric and matter fields, eventually needs to be a function of time t, only.

Since the metric $\tilde{g}$ is conformal to g and the conformal factor φ depends on t only, then one can redefine

$$\tilde{g} = -c^2 d\tilde{t}^2 + \tilde{a}^2(\tilde{t}) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right) \quad \Rightarrow \begin{cases} \tilde{a}(\tilde{t}) = \sqrt{\varphi(t)} \, a(t) \\ d\tilde{t} = \sqrt{\varphi(t)} \, dt \end{cases} \tag{8.4.3}$$

Here we need to be more precise about how to make the Weyl conformal transformation to define $\tilde{g}$ out of g. We have to remember that g is Lorentzian (so that one cannot have a negative conformal factor which would not preserve the signature). Moreover, we understand that the scale factor $a(t)$, appearing in g when it is in FLRW form, is normalised so that $a(t_0) = a_0 = 1$.

On the other hand, when the conformal factor is defined as $\varphi = f'(\mathcal{R})$ (as it should, since we are lazy and we are discussing cosmology in dimension $m = 4$, only) there is no guarantee that it is positive at least on shell, nor that it preserves the normalisation of the scale factor, so that $\tilde{a}(\tilde{t}_0) = 1$ as well.

We need to go back and check why we introduced the Weyl conformal transformation in the first place. In Palatini $f(\mathcal{R})$-theories, one has a metric g and a connection $\tilde{\Gamma}$. Then, in view of field equations, we showed that there exists a conformal metric $\tilde{g}$ such that $\{\tilde{g}\} = \tilde{\Gamma}$ to partially solve the theory. In other words, as a matter of

fact, the conformal metric $\tilde{g}$ is introduced to produce the connection $\tilde{\Gamma}$, which, contrary to $\tilde{g}$, is endowed with a direct physical meaning of describing the free fall of particles. When we showed that the conformal factor φ does the job, we forgot to mention that any Weyl conformal transformation by a *constant* conformal factor χ_0 does in fact leave the endowed connection unchanged, i.e. one has $\{\chi_0 \tilde{g}\} = \{\tilde{g}\} = \tilde{\Gamma}$. Thus we have a certain freedom in fixing the conformal factor; if $\varphi = f'(\mathcal{R})$ does the trick, also $\tilde{\varphi} = \chi_0 f'(\mathcal{R})$ does. Consequently, we have some freedom to exploit in order to preserve signature and the normalisation of the scale factor.

In practice, on each solution, one has a conformal factor $\varphi(t)$ and one can instead use the conformal factor

$$\tilde{\varphi}(t) = \frac{|\varphi(t)|}{|\varphi(t_0)|} \tag{8.4.4}$$

Now one has $\tilde{\varphi}(t_0) = 1$ and $\tilde{\varphi}(t) \geq 0$. Unfortunately, if $f'(\mathcal{R})$ has zeros, it may happen that, sometimes in spacetime, the metric becomes singular, thus one either excludes that φ can change sign (for example, by assuming that $f(\mathcal{R})$ is monotonic), or one restricts to a region where that does not happen (or one learns what happens physically if the metric goes through this kind of a singularity).

In all cases, in cosmology, one chooses a positive conformal factor, which is unitary today.

Then, if g is spatially homogeneous and isotropic, then also $\tilde{g}$ is. If g is Lorentzian, then also $\tilde{g}$ is. And if $a(t_0) = 1$ is normalised, then also $\tilde{a}(\tilde{t}_0) = 1$ is.

Let us stress that this does not mean that there is a coordinate system in which both metrics are in FLRW form. The original metric g is in FLRW form in a coordinate system, while one needs to change coordinates (and scale factor) to have $\tilde{g}$ in FLRW form.

Then next step is to obtain the relevant energy-momentum stress tensor as a perfect fluid stress tensor. However, we have three energy-momentum tensors in our theory, namely $T_{\mu\nu}$, $\tilde{T}_{\mu\nu}$, and $\hat{T}_{\mu\nu}$, so that one should ask which of them should be a perfect fluid stress tensor. Once again, the good news is that it does not matter, since if $T_{\mu\nu}$ is a perfect fluid stress tensor, then also $\tilde{T}_{\mu\nu}$, and $\hat{T}_{\mu\nu}$ are.

Let us assume that

$$T_{\mu\nu} = c^{-1}\left((\rho + p)u_\mu u_\nu + p g_{\mu\nu}\right) \tag{8.4.5}$$

is a perfect fluid stress tensor.

The fluid velocity u^μ (which is unit with respect to g) can be normalised to have a vector $\tilde{u}^\mu = (\tilde{\varphi})^{-\frac{1}{2}} u^\mu$ which is unit with respect to $\tilde{g}_{\mu\nu} = \tilde{\varphi} g_{\mu\nu}$. Accordingly, one has $\tilde{u}_\mu = (\tilde{\varphi})^{\frac{1}{2}} u_\mu$, where we defined $u_\mu = g_{\mu\nu} u^\nu$ and $\tilde{u}_\mu = \tilde{g}_{\mu\nu} \tilde{u}^\nu$.

Then we have

$$\tilde{T}_{\mu\nu} = \frac{1}{f'(\mathcal{R})}\left(T_{\mu\nu} - \frac{1}{2\kappa}\left(f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})\right) g_{\mu\nu}\right) = \frac{c^{-1}}{f'(\mathcal{R})\tilde{\varphi}}\left((\rho + p)\tilde{u}_\mu \tilde{u}_\nu + \left(p - \frac{c}{2\kappa}\left(f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})\right)\right)\tilde{g}_{\mu\nu}\right) = c^{-1}\left((\tilde{\rho} + \tilde{p})\tilde{u}_\mu \tilde{u}_\nu + \tilde{p}\tilde{g}_{\mu\nu}\right) \tag{8.4.6}$$

which is again a perfect fluid stress tensor, provided that one defines a new density and pressure as

$$\tilde{p} = \frac{p}{f'(\mathcal{R})\tilde{\varphi}} - \frac{c}{2\kappa}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})\tilde{\varphi}} \qquad \tilde{\rho} + \tilde{p} = \frac{\rho + p}{f'(\mathcal{R})\tilde{\varphi}} \qquad \Rightarrow \tilde{\rho} = \frac{\rho}{f'(\mathcal{R})\tilde{\varphi}} + \frac{c}{2\kappa}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})\tilde{\varphi}} \tag{8.4.7}$$

For the tensor $\hat{T}_{\mu\nu}$, instead we first need to notice that, since φ depends on t only, one has

$$\left(\nabla_{00}\varphi - \frac{3}{2\varphi}\nabla_0\varphi\nabla_0\varphi\right)u_\mu u_\nu = \nabla_{\mu\nu}\varphi - \frac{3}{2\varphi}\nabla_\mu\varphi\nabla_\nu\varphi \tag{8.4.8}$$

as one can check by setting $(\mu\nu) = (00)$, $(\mu\nu) = (0i)$, $(\mu\nu) = (ij)$, the first being trivial since $u_0 = 1$, the second because $\{g\}^0_{0i} = 0$, the third because $\nabla_j\varphi = 0$.

$$\hat{T}_{\mu\nu} = c^{-1}\left[\frac{\rho + p}{f'(\mathcal{R})}u_\mu u_\nu + \frac{p}{f'(\mathcal{R})}g_{\mu\nu} - \frac{c}{2\kappa}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})}g_{\mu\nu} + \frac{c}{\varphi}\left(\nabla_{00}\varphi - \frac{3}{2\varphi}\nabla_0\varphi\nabla_0\varphi\right)u_\mu u_\nu - \frac{c}{\varphi}\left(\Box\varphi - \frac{3}{4\varphi}\nabla_\rho\varphi\nabla^\rho\varphi\right)g_{\mu\nu}\right] =$$
$$= c^{-1}\left[\left(\frac{\rho + p}{f'(\mathcal{R})} + \frac{c}{\varphi}\left(\nabla_{00}\varphi - \frac{3}{2\varphi}\nabla_0\varphi\nabla_0\varphi\right)\right)u_\mu u_\nu + \left(\frac{p}{f'(\mathcal{R})} - \frac{c}{2\kappa}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})} - \frac{c}{\varphi}\left(\Box\varphi - \frac{3}{4\varphi}\nabla_\rho\varphi\nabla^\rho\varphi\right)\right)g_{\mu\nu}\right] \tag{8.4.9}$$

which is again a perfect fluid stress tensor, provided that one defines a new density and pressure as

$$\hat{p} = \frac{p}{f'(\mathcal{R})} - \frac{\mathrm{c}}{2\kappa}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})} - \frac{\mathrm{c}}{\varphi}\left(\Box\varphi - \frac{3}{4\varphi}\nabla_\rho\varphi\nabla^\rho\varphi\right) \qquad \hat{\rho} + \hat{p} = \frac{\rho + p}{f'(\mathcal{R})} + \frac{\mathrm{c}}{\varphi}\left(\nabla_{00}\varphi - \frac{3}{2\varphi}\nabla_0\varphi\nabla_0\varphi\right) \tag{8.4.10}$$

Once the cosmological principle has been required, the equation of motion for the theory can be recast as Einstein equations $\tilde{G}_{\mu\nu} = \kappa\tilde{T}_{\mu\nu}$ for the metric $\tilde{g}_{\mu\nu}$ and the energy-momentum stress tensor $\tilde{T}_{\mu\nu}$ (check (7.2.14)). These imply Friedman equations for $(\tilde{a}, \tilde{\rho}, \tilde{p})$. Also g obeys Einstein-like equations (see (7.2.17)) for the energy-momentum stress tensor $\hat{T}_{\mu\nu}$, hence implying Friedman equations for $(a, \hat{\rho}, \hat{p})$.

The last item we need for a cosmological model is the EoS. This time the form of EoS is not preserved in any way from Weyl conformal transformations and, accordingly, the dynamics in different Weyl conformal frames have different EoS. In particular, if we assume an EoS $p = p(\rho)$ for $T_{\mu\nu}$ then one can compute out of (ρ, p) both $(\tilde{\rho}(\rho, p), \tilde{p}(\rho, p))$ and $(\hat{\rho}(\rho, p), \hat{p}(\rho, p))$.

In the original Weyl frame the EoS $p = p(\rho)$ is equivalent to fix $\rho(a)$ and then define $p(a) = p(\rho(a)) = -\left(\rho + \frac{1}{3}\rho' a\right)$. Then one can define

$$\tilde{\rho}(a) = \tilde{\rho}(\rho(a), p(a)) \qquad\qquad \tilde{p}(a) = \tilde{p}(\rho(a), p(a)) \tag{8.4.11}$$

which can be regarded as a parametric form for the EoS for the stress tensor $\tilde{T}_{\mu\nu}$.

Analogously, one can define

$$\hat{\rho}(a) = \hat{\rho}(\rho(a), p(a)) \qquad\qquad \hat{p}(a) = \hat{p}(\rho(a), p(a)) \tag{8.4.12}$$

which can be regarded as a parametric form for the EoS for the stress tensor $\hat{T}_{\mu\nu}$.

In both cases, one can think, at least locally, of eliminating the parameter a and write the EoS in the form $\tilde{p} = \tilde{p}(\tilde{\rho})$ and $\hat{p} = \hat{p}(\hat{\rho})$. However, these usually are not one-to-one, they can be solved in a number of local branches only. Accordingly, it is better to consider EoS in parametric form and learn to live with that. Let us also notice, that any simple EoS for (ρ, p) produces an *exact* EoS for $(\tilde{\rho}, \tilde{p})$ as well as for $(\hat{\rho}, \hat{p})$, which, though complicated, are exact. Further approximating them, especially in view of the non-canonical form of expansions along barotropic perfect fluid mixtures, is definitely not a good idea.

Thus one has to learn to live with complicated EoS and find technical guidelines to solve the Friedmann equation with the less possible effort. Let us stress that by *solving* these cosmological models we mean being able to give a representation of all relevant quantities in each Weyl frame *and* the conformal factor to pass from one frame to the other. After all, in view of the dynamical equivalences proven among different Weyl conformal frames, solving the model in one Weyl frame is equivalent to solve it in each other Weyl frame. The good news is that, since we go for solving in all Weyl conformal frames, we do not need previously to discuss which frames have what physical meaning. We can first learn to solve a model and later discuss about the relation of theoretical quantities with observational quantities.

Let us first analyse some simple, the first non-trivial, examples of Palatini $f(\mathcal{R})$-cosmology. In next Section we shall discuss a more complicated example.

Example: $f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon}{2}\mathcal{R}^2$, $(\epsilon > 0)$

Let us consider the cosmological model from a Palatini $f(\mathcal{R})$-theory endowed by the choice $f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon}{2}\mathcal{R}^2$. This theory should be supported by a description of matter, that eventually will be described by the energy-momentum stress tensor $T_{\mu\nu}$ and hence by (ρ, p) and their EoS.

Notice that $[\epsilon] = L^2$. Notice also that we can define the *Planck area* $A_P := \hbar \mathbf{G}\mathbf{c}^{-3} = 2.612 \cdot 10^{-70}\, m^2$ (with $[A_P] = L^2$). Then we can set

$$\epsilon = \hat{\epsilon} A_P \qquad\qquad [\hat{\epsilon}] = 1 \tag{8.4.13}$$

Since the energy-momentum stress tensor $T_{\mu\nu}$ is a perfect fluid stress tensor, its trace is

$$T = \mathbf{c}^{-1}(-\rho - p + 4p) = \mathbf{c}^{-1}(3p - \rho) \tag{8.4.14}$$

Let us mention that since both (ρ, p) are eventually function of time t only, then the matter trace T is a function of time t only, as well.

For the selected function $f(\mathcal{R})$, the master equation reads as

$$(1 - \epsilon\mathcal{R})\mathcal{R} - 2(\mathcal{R} - \tfrac{\epsilon}{2}\mathcal{R}^2) = -\mathcal{R} = \kappa T = \kappa \mathbf{c}^{-1}(3p - \rho) \qquad \Rightarrow \mathcal{R} = \kappa \mathbf{c}^{-1}(\rho - 3p) \tag{8.4.15}$$

The reason why this model is particularly easy to discuss is exactly that the master equation can be easily solved for the curvature $\mathcal{R}$ to determine it as a function of the matter content. The curvature, as well as all functions of it, in particular the conformal factor $\varphi = f'(\mathcal{R})$, is again a function of time t only.

Part of the motivation for this example is showing that, even if one assumes the EoS for (ρ, p) to be extremely simple, the EoS for $(\tilde{\rho}, \tilde{p})$ is relatively non-trivial—also in this simple case. So, let us assume a simple dust EoS, namely $p = 0$. That corresponds to a density function $\rho(a) = \rho_0 a^{-3}$.

Then we can replace this in the curvature $\mathcal{R} = \kappa \mathbf{c}^{-1}\rho$ and that into the definition of the conformal factor

$$\tilde{\varphi}(\rho) = \frac{1 - \kappa \mathbf{c}^{-1}\epsilon\rho}{1 - \kappa \mathbf{c}^{-1}\epsilon\rho_0} = 1 + \frac{\hat{\epsilon}\,(\rho_0 - \rho)}{\frac{\mathbf{c}^4}{8\pi\hbar\mathbf{G}^2} - \hat{\epsilon}\rho_0} \tag{8.4.16}$$

which shows as the conformal factor will eventually be a function of time t only. Notice that today we have $\tilde{\varphi}(\rho_0) = 1$.

We can assume (safely?) that

$$\epsilon < \frac{1}{\kappa \mathbf{c}^{-1}\rho_0} \simeq 10^{36}\, m^2 \qquad\qquad [\kappa \mathbf{c}^{-1}] = M^{-1}L^{-1}T^2 \qquad ([\kappa \mathbf{c}^{-1}\rho] = 1) \qquad\qquad (\Rightarrow \hat{\epsilon} < 10^{106}) \tag{8.4.17}$$

In that case, one has $\tilde{\varphi}(\rho) > 0$ iff

$$\rho < \frac{1}{\kappa \mathbf{c}^{-1}\epsilon} \tag{8.4.18}$$

which is certainly ok if $\rho < \rho_0$ (i.e. probably in the future), while it may be violated in past, dense, universes. (Of course, *"past universes"* refers to spatial leaves, the *"universe"* being a four dimensional spacetime.)

If at any time during the evolution of the universe we get $\rho \geq (\kappa \mathbf{c}^{-1}\epsilon)^{-1}$, then we have to redefine the conformal factor to be

$$\tilde{\varphi} = \frac{\kappa \mathbf{c}^{-1}\epsilon\rho - 1}{1 - \kappa \mathbf{c}^{-1}\epsilon\rho_0} \tag{8.4.19}$$

which will be still positive for big densities.

Of course, a mess might happen when $\rho \sim (\kappa \mathbf{c}^{-1}\epsilon)^{-1}$, which by now we ignore and start find solutions at small and big densities.

Thus let us first consider the case

$$\epsilon < \frac{1}{\kappa \mathbf{c}^{-1}\rho_0} \qquad \text{and} \qquad \rho < \frac{1}{\kappa \mathbf{c}^{-1}\epsilon} \qquad \left(\Rightarrow a > \left(\rho_0 \kappa \mathbf{c}^{-1}\epsilon \right)^{\frac{1}{3}} =: a_* > 0 \right) \qquad (8.4.20)$$

(which is reasonable today unless $\epsilon \simeq 10^{36} \, m^2$) and see whether, during the evolution of this universe, we manage to get into trouble, anyway.

We can also express a_* in terms of $\hat{\epsilon}$ as

$$a_* = \left(\rho_0 \kappa \mathbf{c}^{-1}\epsilon \right)^{\frac{1}{3}} = \left(\rho_0 \frac{8\pi \hbar \mathbf{G}^2 \hat{\epsilon}}{\mathbf{c}^4} \right)^{\frac{1}{3}} \simeq 6.91 \cdot 10^{-33} \cdot \sqrt[3]{\hat{\epsilon}} \qquad (8.4.21)$$

Again, unless we consider quite a drastic modification of standard GR (with $\hat{\epsilon} \simeq 3 \cdot 10^{96}$) one can assume $a_* \ll 1$. On the other hand, if we wish to go for $\hat{\epsilon} \simeq 3 \cdot 10^{96}$ we should check that it does not contradict observations, classical tests in particular.

Here we shall assume $a_* \ll 1$ and draw graphs for $a_* = 10^{-4}$ and $\rho_0 = 10^{-2}$ and no spatial curvature $k = 0$. That corresponds to $\epsilon = 4.77 \cdot 10^{32} \, m^2$, which is quite irrealistic, though it produced decent graphs. Since we are assuming today density ρ_0 and spatial curvature $k = 0$, the Hubble parameter today H_0 will not match the observations, which anyway is not the purpose of this example.

Under these assumptions, the conformal factor is given by (8.4.16) and, when we need, we can write the conformal factor as a function of a as (see Fig. 7.6)

$$\tilde{\varphi}(a) = \frac{1 - \kappa \mathbf{c}^{-1}\epsilon\rho}{1 - \kappa \mathbf{c}^{-1}\epsilon\rho_0} = \frac{a^3 - a_*^3}{(1 - a_*^3)a^3} \qquad (8.4.22)$$

Fig 7.6: Conformal factor $\tilde{\varphi}(a)$

and it stays positive as long as $a > a_*$ stays satisfied. Of course, this expression for $\tilde{\varphi}(a)$ depends on the EoS we chose.

The scale factor $\tilde{a}$ for the metric $\tilde{g}$ can be obtained as function of a as well, namely

$$\tilde{a}(a) = a\sqrt{\frac{a^3 - a_*^3}{(1 - a_*^3)a^3}} \qquad (8.4.23)$$

and one has $\tilde{a}(1) = 1$, i.e. the normalisation of the scale factor is preserved (see Fig. 7.7). That also implicitly defines $a = a(\tilde{a})$.

We can also compute the effective density and pressure $(\tilde{\rho}, \tilde{p})$ to be

$$\tilde{\rho}(\rho) = \frac{4 - \epsilon \kappa \mathbf{c}^{-1}\rho}{4(1 - \kappa \mathbf{c}^{-1}\epsilon\rho)^2}(1 - \kappa \mathbf{c}^{-1}\epsilon\rho_0)\rho \qquad\qquad \tilde{p}(\rho) = \frac{\epsilon \kappa \mathbf{c}^{-1}\rho}{4(1 - \kappa \mathbf{c}^{-1}\epsilon\rho)^2}(1 - \kappa \mathbf{c}^{-1}\epsilon\rho_0)\rho \qquad (8.4.24)$$
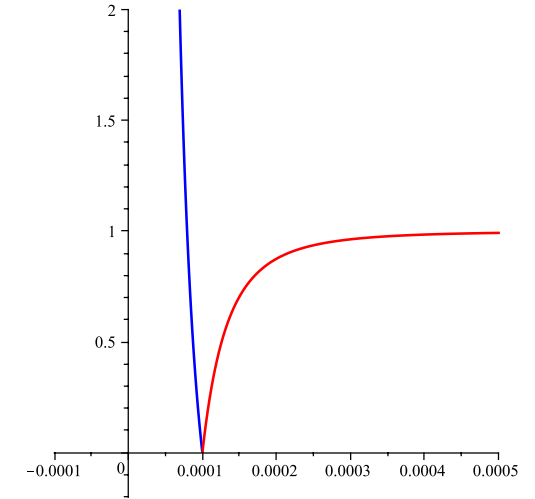
We can start to specify equation (8.4.7), to the EoS $p = 0$

$$\tilde{\rho} = \frac{\rho}{f'(\mathcal{R})\tilde{\varphi}} + \frac{1}{2\kappa\mathbf{c}^{-1}}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})\tilde{\varphi}} \qquad\qquad \tilde{p} = -\frac{1}{2\kappa\mathbf{c}^{-1}}\frac{f'(\mathcal{R})\mathcal{R} - f(\mathcal{R})}{f'(\mathcal{R})\tilde{\varphi}} \qquad (8.4.25)$$

to the function $f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon}{2}\mathcal{R}^2$

$$\tilde{\rho} = \frac{4\kappa\mathbf{c}^{-1}\rho - \epsilon\mathcal{R}^2}{4\kappa\mathbf{c}^{-1}(1 - \epsilon\mathcal{R})\tilde{\varphi}} = \frac{4a^3 - a_*^3}{4(a^3 - a_*^3)\tilde{\varphi}}\frac{\rho_0}{a^3} \qquad\qquad \tilde{p} = \frac{\epsilon\mathcal{R}^2}{4\kappa\mathbf{c}^{-1}(1 - \epsilon\mathcal{R})\tilde{\varphi}} = \frac{a_*^3}{4(a^3 - a_*^3)\tilde{\varphi}}\frac{\rho_0}{a^3} \qquad (8.4.26)$$

and, finally, to the conformal factor (8.4.22)

$$\tilde{\rho} = \frac{4 - \epsilon\kappa\mathbf{c}^{-1}\rho}{4(1 - \kappa\mathbf{c}^{-1}\epsilon\rho)^2}(1 - \kappa\mathbf{c}^{-1}\epsilon\rho_0)\rho \qquad\qquad \tilde{p} = \frac{\epsilon\kappa\mathbf{c}^{-1}\rho^2}{4(1 - \kappa\mathbf{c}^{-1}\epsilon\rho)^2}(1 - \kappa\mathbf{c}^{-1}\epsilon\rho_0) \qquad (8.4.27)$$

Fig 7.7: Scale factor $\tilde{a}(a)$

Notice that, even if the EoS for (ρ, p) is for pure dust, in $(\tilde{\rho}, \tilde{p})$ is not simply dust ($\tilde{p} \neq 0$). Let us also remark that the EoS for $(\tilde{\rho}, \tilde{p})$ is not that of a barotropic perfect fluid ($\tilde{p} \neq w\tilde{\rho}$), with a constant parameter w.

The EoS for $(\tilde{\rho}, \tilde{p})$ is obtained in parametric form by expanding $\rho = \rho_0 a^{-3}$.

$$\tilde{\rho}(a) = \frac{4a^3 - a_*^3}{4(a^3 - a_*^3)^2}(1 - a_*^3)\rho_0 \qquad\qquad \tilde{p}(a) = \frac{a_*^3}{4(a^3 - a_*^3)^2}(1 - a_*^3)\rho_0 \qquad (8.4.28)$$

Hence we can plot the effective EoS.

This EoS is good news. As we go back along the universe history, the scale factor becomes smaller. Today we are on the red branch and going back corresponds to move along the red branch towards infinity, which is reached at $a = a_*$. After that, if the scale factor further reduces, we go on the blue branch (at infinity and we go along it to the finite endpoint which is reached for $a = 0$). Accordingly, as we go through $a = a_*$ we should feel a strong negative pressure, which should prevent the scale factor to further contract and a_* might appear as a minimal scale factor at which the universe bounce back.

This is not a proof, it is just a story telling. One should prove it by analysing Friedmann equation.

We can now write down the Friedmann equation for $\tilde{a}$ as

$$\dot{\tilde{a}}^2 = \frac{\kappa\mathbf{c}}{3}\tilde{\rho}(a)\tilde{a}^2(a) - \mathbf{c}^2 k = \tilde{\Phi}(\tilde{a}(a)) \qquad (8.4.29)$$

Fig 7.8: Effective EoS $\tilde{p}(\tilde{\rho})$

where the dot on $\tilde{a}$ denotes the derivative with respect to $\tilde{t}$. This is quite difficult to obtain analytically (since it requires the function $a(\tilde{a})$ which should be obtained by solving $\tilde{a} = \tilde{a}(a)$ given by the definition (8.4.23) of the transformed scale factor).
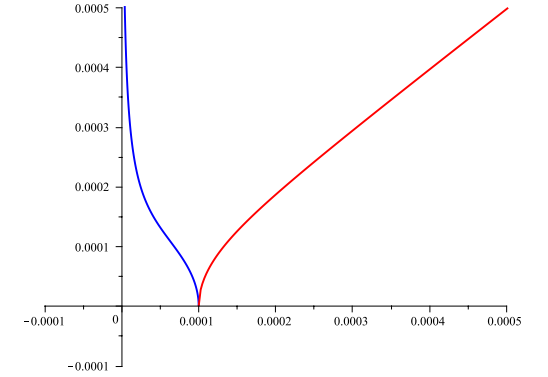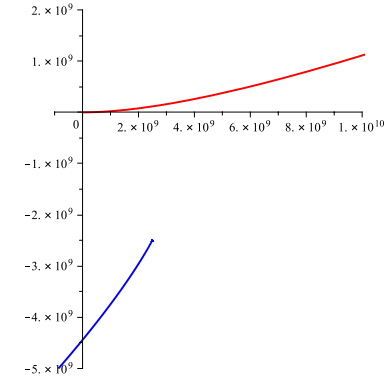
It is, instead, easier to go for writing that equation directly for a as

$$\frac{d\tilde{a}}{d\tilde{t}} = \frac{d\tilde{a}}{da}\frac{da}{dt}\frac{dt}{d\tilde{t}} \qquad \Rightarrow \left(\frac{da}{dt}\right)^2 = \left(\frac{d\tilde{t}}{dt}\right)^2 \left(\frac{d\tilde{a}}{da}\right)^{-2} \tilde{\Phi}(\tilde{a}) =: \Phi(a) \tag{8.4.30}$$

Since we know that $d\tilde{t} = \sqrt{\tilde{\varphi}}dt$ and $\tilde{a}(a)$ is given by (8.4.23), for spatially flat solutions, i.e. with $k = 0$, one has

$$\Phi(a) := \tilde{\varphi}(a)\left(\frac{d\tilde{a}}{da}\right)^{-2}\tilde{\Phi}(\tilde{\rho}(a)) = \frac{\kappa\mathbf{c}}{3}\tilde{\varphi}(a)\left(\frac{d\tilde{a}}{da}\right)^{-2}\tilde{\rho}(a)\tilde{a}^2(a) = \frac{\kappa\mathbf{c}}{3}\frac{(4a^3 - a_*^3)(a^3 - a_*^3)}{(2a^3 + a_*^3)^2 a}\rho_0 \tag{8.4.31}$$

Thus the Friedmann equation can be recast as a (different) Weierstrass equation, namely

$$\dot{a}^2 = \frac{\kappa\mathbf{c}}{3}\frac{(4a^3 - a_*^3)(a^3 - a_*^3)}{(2a^3 + a_*^3)^2 a}\rho_0 = \Phi(a) \tag{8.4.32}$$

Fig 7.9: The Weierstrass function $\Phi(a)$

By inspecting the graph of the Weierstrass function $\Phi(a)$, we see that there are two allowed regions: one as $\left(0, a_M = \frac{a_*}{4^{1/3}}\right]$ and one as $[a_*, +\infty)$.

The first region corresponds to a universe which expands to a maximal scale factor $a = a_M$ and then re-collapses to $a = 0$. The second is an ever-expanding universe which comes from a contracting universe which contracts to the scale $a = a_m$ and then bounces back to $a \to +\infty$.

One can compute the time needed to re-collapse in the first region as

$$T = \int_0^{a_m} \frac{da}{\sqrt{\Phi(a)}} \simeq 8.95 \cdot 10^7 \ s = 2.83 \ y \tag{8.4.33}$$

which is finite, anyway.

If the initial condition is on the red branch, then we have a bouncing universe, which stays on the red branch. The age of the universe today is

$$T_U = \int_{a_*}^1 \frac{da}{\sqrt{\Phi(a)}} \simeq 8.43 \cdot 10^{13} \ s = 2.67 \cdot 10^6 \ y \tag{8.4.34}$$

which is, of course, too short though we remark that we are in fact discussing a model in which $a_* = 10^{-4}$ and $\rho_0 = 10^{-2}$ which are both quite untenable.

Let us focus on the red branch; the time can be obtained as

$$t(a) = t_0 + \int_{a_0=1}^a \frac{da}{\sqrt{\Phi(a)}} \tag{8.4.35}$$

and this time we set today $t_0 = T_U$, so that the bouncing is at $t = 0$.

Since now we have time $t(a)$ we can draw $a(t)$ (see Fig.7.10), then we have $\rho(t) = \rho(a(t))$, $\tilde{\varphi}(t) = \tilde{\varphi}(a(t))$, $\tilde{\rho}(t)$, $\tilde{p}(t)$.
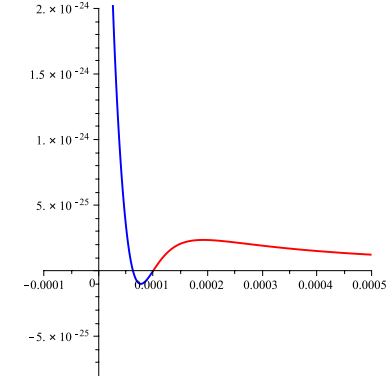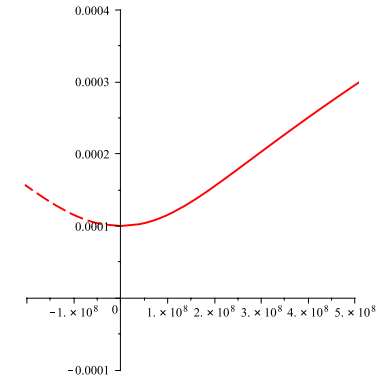
Fig 7.10: The evolution of the scale factor $a(t)$
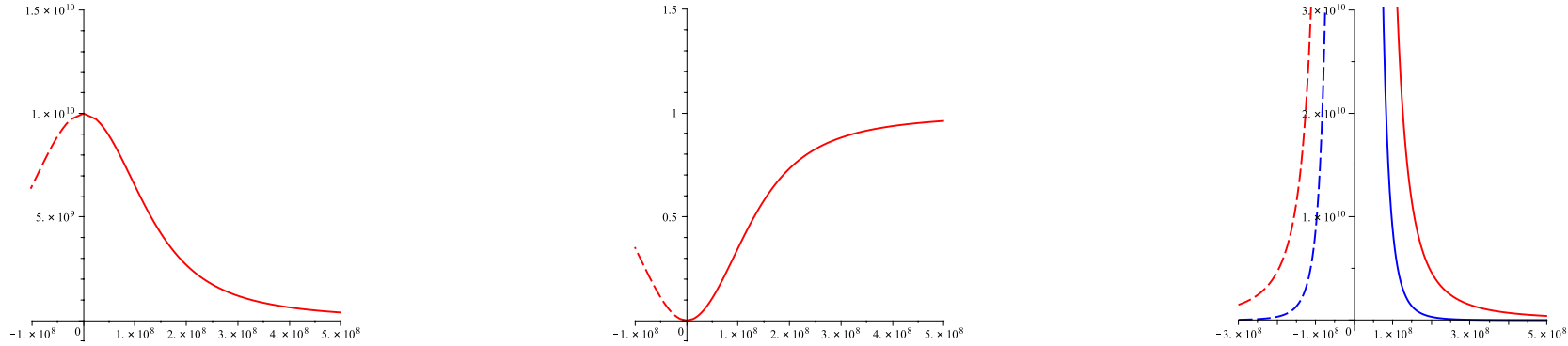
Fig. 7.11: a) The evolution of density $\rho(t)$ b)The evolution of the conformal factor $\tilde{\varphi}(t)$ c) Evolution of effective density and pressure $\tilde{\rho}(t)$, $\tilde{p}(t)$

Then we can compute the conformal time as

$$\tilde{t}(a) = \tilde{t}_0 + \int_{t_0}^t \sqrt{\tilde{\varphi}(a(t))}dt = \tilde{t}_0 + \int_{a_*}^a \sqrt{\tilde{\varphi}(a(t))}\frac{dt}{da}\,da = \int_{a_*}^a \sqrt{\frac{\tilde{\varphi}(a)}{\Phi(a)}}\,da \qquad (8.4.36)$$

where we set $\tilde{t}_0 = 0$ at the bounce. That can be integrated and it allows us to draw the evolution of all quantities in terms of $\tilde{t}$. For example we can draw $\tilde{t}(t)$, $\tilde{a}(\tilde{t})$, as well as $\tilde{\rho}(\tilde{t})$ and $\tilde{p}(\tilde{t})$.
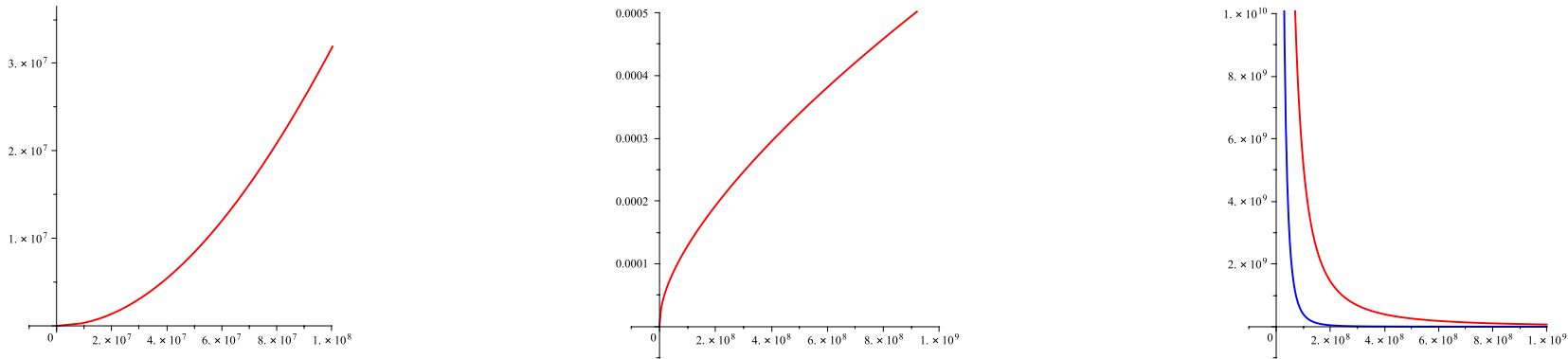
Fig. 7.12: a) The time transformation $\tilde{t}(t)$ b)The evolution of the scale factor $\tilde{a}(\tilde{t})$ c) Evolution of effective density and pressure $\tilde{\rho}(\tilde{t})$, $\tilde{p}(\tilde{t})$

Possibly check that Friedmann equation for a starting from $(\hat{\rho}, \hat{p})$.

As a matter of fact, we can obtain a parametric representation of basically anything in the model. Now parametric representations are not too bad; one can compute exactly a lot of details and draw anything in the model. In return, we obtained that without a single function inversion, so that is the template of solution for less trivial cases.

Example: $f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon}{2}\mathcal{R}^2$, $(\epsilon < 0)$

In the previous Subsection we discussed the model $f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon}{2}\mathcal{R}^2$ for a positive ϵ. Here we want to consider the same model for $\epsilon < 0$. Let us introduce the parameter $\varsigma = -\epsilon > 0$ and repeat the computation above.

Let us consider the cosmological model from a Palatini $f(\mathcal{R})$-theory endowed by the choice $f(\mathcal{R}) = \mathcal{R} + \frac{\varsigma}{2}\mathcal{R}^2$. The master equation still reads as (8.4.15), we still choose dust EoS for (ρ, p), i.e. a density function $\rho(a) = \rho_0 a^{-3}$, and the curvature is still given by $\mathcal{R} = \kappa \mathbf{c}^{-1}\rho$.

Of course, now we have

$$1 + \kappa \mathbf{c}^{-1}\varsigma\rho_0 > 0 \qquad \text{and} \qquad 1 + \kappa \mathbf{c}^{-1}\varsigma\rho > 0 \tag{8.4.37}$$

whenever $\rho > 0 > -\frac{1}{\kappa \mathbf{c}^{-1}\varsigma}$, which is always the case for $a > 0$. The conformal factor is then always given by

$$\tilde{\varphi} = \frac{1 + \kappa \mathbf{c}^{-1}\varsigma\rho}{1 + \kappa \mathbf{c}^{-1}\varsigma\rho_0} = \frac{a^3 + a_\circ^3}{(1 + a_\circ^3)a^3} \tag{8.4.38}$$

where we set $a_\circ^3 := \kappa \mathbf{c}^{-1}\varsigma\rho_0 = -a_*^3$ $([a_\circ] = 1)$ and which is today $\tilde{\varphi}(\rho_0) = 1$.

> Here we shall assume $a_\circ \ll 1$ and draw graphs for $a_\circ = 10^{-4}$ and $\rho_0 = 10^{-2}$ and no spatial curvature $k = 0$. That corresponds again to $\varsigma = 4.77 \cdot 10^{32} \ m^2$.

The scale factor $\tilde{a}$ for the metric $\tilde{g}$ can be obtained as function of a as well, namely

$$\tilde{a}(a) = a\sqrt{\frac{a^3 + a_\circ^3}{(1 + a_\circ^3)a^3}} \tag{8.4.39}$$

and one has $\tilde{a}(1) = 1$, i.e. the normalisation of the scale factor is preserved (see Fig. 7.7). That also implicitly defines $a = a(\tilde{a})$.

We can also compute the effective density and pressure $(\tilde{\rho}, \tilde{p})$ to be

$$\tilde{\rho}(a) = \frac{4a^3 + a_\circ^3}{4(a^3 + a_\circ^3)^2}(1 + a_\circ^3)\rho_0 \qquad\qquad \tilde{p}(a) = -\frac{a_\circ^3}{4(a^3 + a_\circ^3)^2}(1 + a_\circ^3)\rho_0 \tag{8.4.40}$$

> Hence we can plot the effective EoS.
> Analysing equation (8.4.39), it has no vertical asymptotes (it has one in $a = 0$) and the conformal factor is always positive for $a > 0$ (it changes sign at $a = -a_\circ = a_*$ as in the previous case).
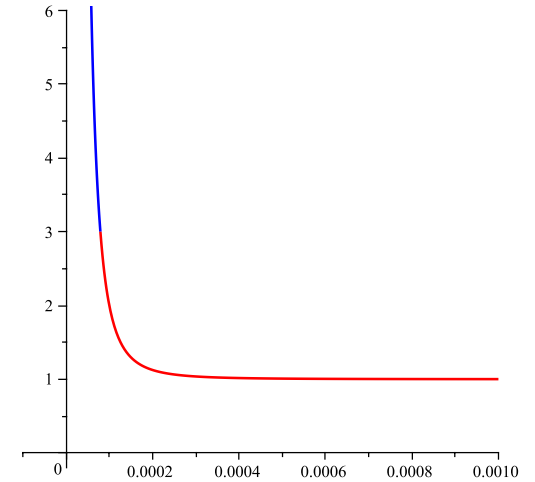
Fig 7.13: Conformal factor $\tilde{\varphi}(a)$

However, we still have a relevant value for the scale factor; at $a = a_m = 2^{-1/3} a_\circ$ the value of $\tilde{a}$ branches, i.e. it has a minimum as a function of a and if one wants to solve for $a = a(\tilde{a})$ that has to be done twice, one in the interval $(0, a_m)$ (**blue**) and one in the interval $(a_m, +\infty)$ (**red**).

All graph in this Subsection will be drawn accordingly. The branching point $a = a_m$ in this case ($\epsilon < 0$) plays pretty much the same role that $a = a_*$ played in the case $\epsilon > 0$; in fact, there $a = a_*$ was also the branching point fo the equation $\tilde{a} = \tilde{(a)}$, though there it was, due to the change of sign in the conformal factor, while here it is simply the minimum of the function.
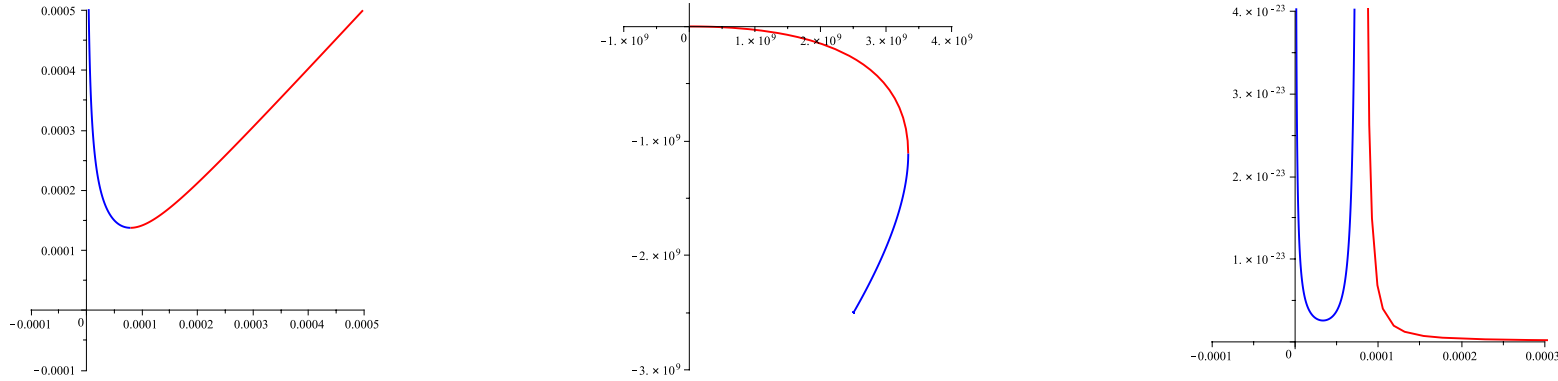
Fig 7.14: a) Scale factor $\tilde{a}(a)$ b) Effective EoS $\tilde{p}(\tilde{\rho})$ c) The Weierstrass function $\Phi(a)$

We can now write down the Friedmann equation for $\tilde{a}$ as

$$\dot{\tilde{a}}^2 = \frac{\kappa \mathbf{c}}{3} \tilde{\rho}(a)\tilde{a}^2(a) - \mathbf{c}^2 k = \tilde{\Phi}(\tilde{a}(a)) \tag{8.4.41}$$

where the dot on $\tilde{a}$ denotes the derivative with respect to $\tilde{t}$. The corresponding Friedmann equation of a is

$$\left(\frac{da}{dt}\right)^2 = \left(\frac{d\tilde{t}}{dt}\right)^2 \left(\frac{d\tilde{a}}{da}\right)^{-2} \tilde{\Phi}(\tilde{a}) =: \Phi(a) \tag{8.4.42}$$

Since we know that $d\tilde{t} = \sqrt{\tilde{\varphi}}dt$ and $\tilde{a}(a)$ is given by (8.4.39), for spatially flat solutions, i.e. with $k = 0$, one has

$$\Phi(a) := \tilde{\varphi}(a)\left(\frac{d\tilde{a}}{da}\right)^{-2} \tilde{\Phi}(\tilde{\rho}(a)) = \frac{\kappa \mathbf{c}}{3}\tilde{\varphi}(a)\left(\frac{d\tilde{a}}{da}\right)^{-2} \tilde{\rho}(a)\tilde{a}^2(a) = \frac{\kappa \mathbf{c}}{3}\frac{(4a^3 + a_\circ^3)(a^3 + a_\circ^3)}{(2a^3 - a_\circ^3)^2 a}\rho_0 \tag{8.4.43}$$

Thus the Friedmann equation can be recast as a (different) Weierstrass equation, namely

$$\dot{a}^2 = \frac{\kappa \mathbf{c}}{3}\frac{(4a^3 + a_\circ^3)(a^3 + a_\circ^3)}{(2a^3 - a_\circ^3)^2 a}\rho_0 = \Phi(a) \tag{8.4.44}$$

which, in fact, agrees with the case $\epsilon > 0$, sending $a_* = -a_\circ$,

By inspecting the graph of the Weierstrass function $\Phi(a)$, we see that there is one allowed region, namely $(0, +\infty)$. The universe is now ever-expanding and one can draw the scale factor $a(t)$, as shown in Fig. 7.15, from

$$t(a) = t_0 + \int_{a_0=1}^{a} \frac{da}{\sqrt{\Phi(a)}} \tag{8.4.45}$$

and this time we set today $t_0 = 0$, when $a = 0$ (which we should not, since it is a singular value).

This corresponds to as universe expanding form $a = 0$, the universe expands to the value $a = a_m$ where the expansion becomes singular ($\dot{a}$ becoming infinite for a moment) and after that it expands more or less as in the standard Big Bang.

Let us stress how the evolution, in this case, is different from the standard one (because of the blue region which is not there in the standard case) and from the case $\epsilon > 0$, in which the universe comes from a bounce, not from a singularity.

Since now we have time $t(a)$ we can draw $a(t)$ (see Fig. 7.15), and then we have a parametric representation of the functions $\rho(t) = \rho(a(t))$, $\tilde{\varphi}(t) = \tilde{\varphi}(a(t))$, as well as of the effective density and pressure $\tilde{\rho}(t)$ (solid line), $\tilde{p}(t)$ (dash line).
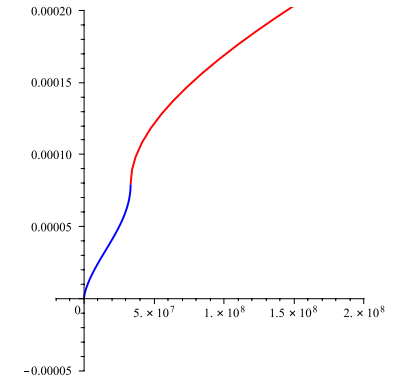
Fig. 7.15: The evolution of the scale factor $a(t)$
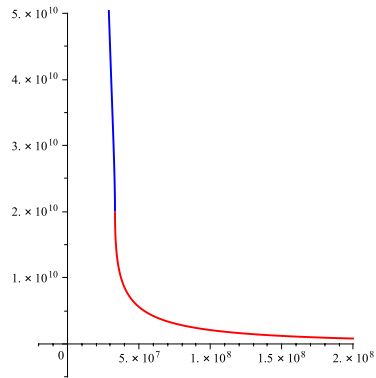
Fig. 7.16: a) The evolution of density $\rho(t)$ *b)The evolution of the conformal factor $\tilde{\varphi}(t)$*   *c) Evolution of effective density and pressure $\tilde{\rho}(t)$, $\tilde{p}(t)$*
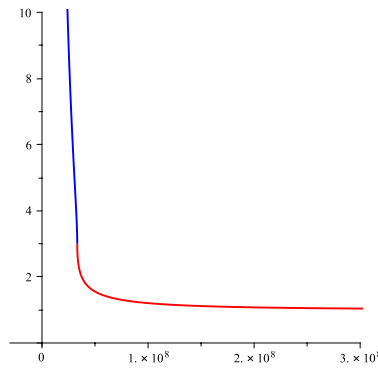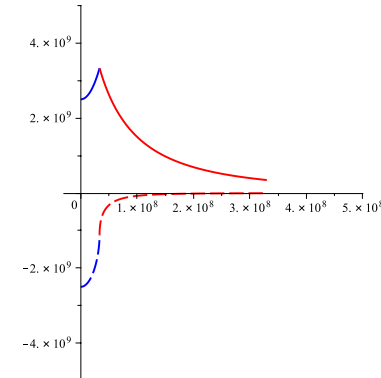
Then we can compute the time coordinate in the Einstein frame as

$$\tilde{t}(a) = \tilde{t}_0 + \int_{t_0}^{t} \sqrt{\tilde{\varphi}(a(t))}dt = \tilde{t}_0 + \int_{a_*}^{a} \sqrt{\tilde{\varphi}(a(t))}\frac{dt}{da}\, da = \int_{a_*}^{a} \sqrt{\frac{\tilde{\varphi}(a)}{\Phi(a)}}\, da \tag{8.4.46}$$

where we set $\tilde{t}_0 = 0$ at the bounce. That can be integrated and it allows us to draw the evolution of all quantities in terms of $\tilde{t}$. For example we can draw $\tilde{t}(t)$, $\tilde{a}(\tilde{t})$, as well as $\tilde{\rho}(\tilde{t})$ and $\tilde{p}(\tilde{t})$.
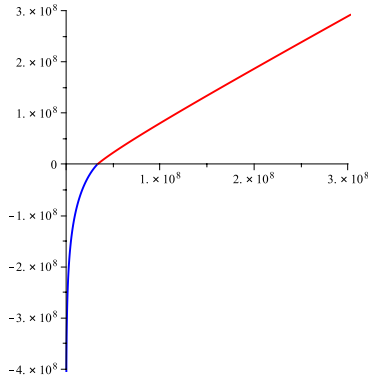
Fig. 7.17: a) The time transformation $\tilde{t}(t)$
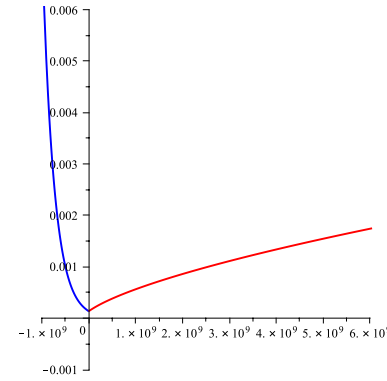
b)The evolution of the scale factor $\tilde{a}(\tilde{t})$

Let us stress that the same system can be described in terms of $a(t)$, or $\tilde{a}(\tilde{t})$. In the first case, the universe has a beginning at $t_0 = 0$ (see Fig. 7.15) while, in the second case, the same universe has no beginning in the time $\tilde{t}$ (see Fig. 7.17.b).
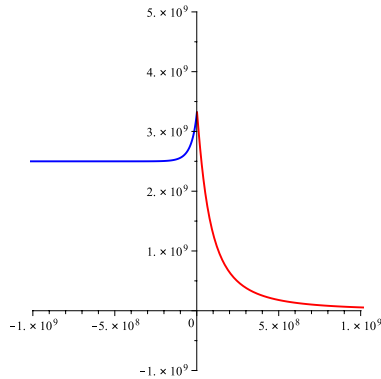
Fig. 7.18: a) Evolution of effective density $\tilde{\rho}(\tilde{t})$
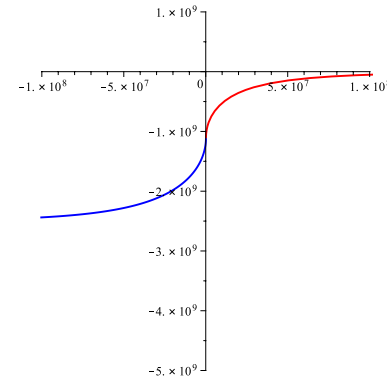
b) Evolution of effective pressure $\tilde{p}(\tilde{t})$

That is similar to what happens in black holes when the same in-falling particle will fall forever for an observer at spatial infinity (measuring time with its clock which is far away from the gravitational source) while it falls in for a finite time for a clock co-moving with the falling particle. There both clocks are identical (i.e. they are both standard with respect the same metric), while here they are in two different Weyl conformal frames, thus we expect one of them to be related to physical clocks, the other being just an unphysical object which provides an equivalent description of the same geometric situation. This will need to be discussed in greater details together with the correspondence between geometric and physical objects.

By now we are satisfied to show that we are actually able to solve the systems, computing the quantities in the different Weyl conformal frames, for whole families of models. Though here the family is pretty simple, we obtain behaviours which are significantly different for standard GR, and may significantly vary with the family parameter.

Example: standard GR

Standard GR cosmology can be recovered from the previous example by setting $\epsilon = 0$. In that case, the master equation becomes

$$\mathcal{R} - 2\mathcal{R} = T \qquad \Rightarrow \mathcal{R} = -T = \rho - 3p = \rho$$

where we used the EoS $p = 0$ for dust. The conformal factor is though $\varphi = f'(\mathcal{R}) \equiv 1$ and cannot be solved for the curvature.

Then in this case $\tilde{g} \equiv g$ (as well as $\tilde{\rho} = \rho$ and $\tilde{p} = p$) and essentially one only has one Weyl frame. The Friedmann equation, for spatially flat solutions $k = 0$, reads as

$$\dot{a}^2 = \frac{\kappa c \rho_0}{3a} = \Phi(a) \qquad \Rightarrow a(t) = \left(\frac{\sqrt{3\kappa c \rho_0}}{2} t + 1 \right)^{\frac{2}{3}} \tag{8.4.47}$$

This shows how the model somehow behaves smoothly with respect to the limit $\epsilon \to 0$ (or, equivalently, $a_* \to 0^+$ or $a_\circ \to 0+$), in the sense that the limit of the conformal factor (8.4.22) gives

$$\lim_{a_* \to 0} \tilde{\varphi}(a) = 1 \tag{8.4.48}$$

the limit of equation (8.4.23) gives

$$\lim_{a_* \to 0} \tilde{a}(a) = a \tag{8.4.49}$$

the limit of the Weierstrass function (8.4.31) gives

$$\lim_{a_* \to 0} \Phi(a) = \frac{\kappa c \rho_0}{3a} \tag{8.4.50}$$

and so on.

Thus we can show how the model is transformed by a transformation of the parameter. For example, for the conformal factor, one starts with positive ϵ and gets a conformal factor as in Fig. 7.7. As the parameter $\epsilon \to 0+$ the conformal factor pointwise approached the conformal factor $\tilde{\varphi} = 1$ for standard GR, though of course the convergence is not uniform. At negative values for ϵ the conformal factor departures from the standard GR one, to become the one shown in Fig. 13. Again, convergence is pointwise away from $a = 0$ and not uniform.

This non-uniform transition accounts for standard GR to be an exceptionally degenerate member of the family of models, and the discontinuities of the global properties of solutions.

The effective EoS is also modified, as shown in Fig. 7.19.a, going from light red, to black, to light blue, as soon as the parameter ranges from $a_* = 10^4$, to $a_* = 0$ for standard GR, to $a_* = -0.5 \cdot 10^{-4}$.

We can trace the variation in the same range in the changes of evolution of the scale factor (see Fig. 7.19.b) as well as of the evolution of the scale factor $a(t)$ (see Fig. 7.19.c).
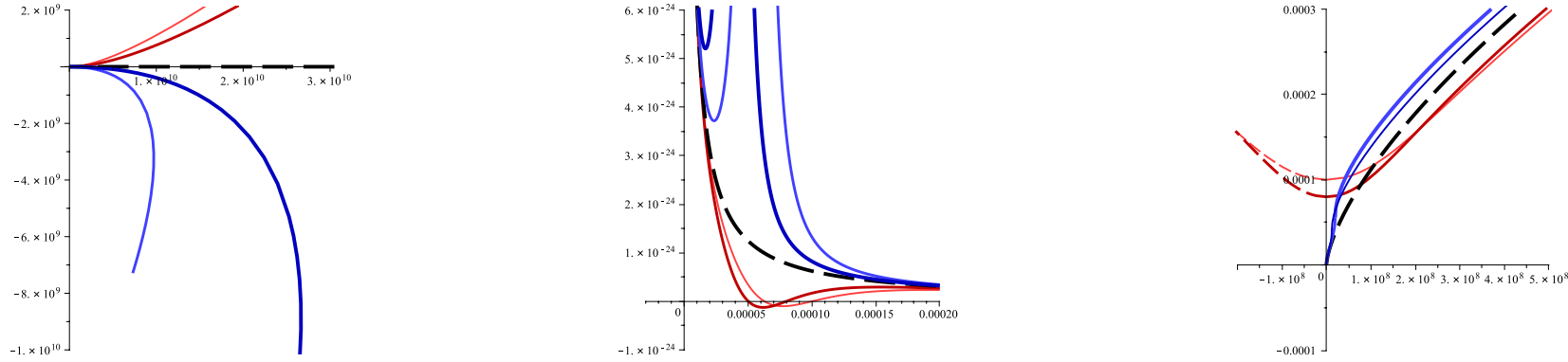
Fig. 7.19: *Model quantities for $\epsilon = 10^{-4}$ (light red), $\epsilon = 8 \cdot 10^{-5}$ (dark red), standard GR ($\epsilon = 0$) (dash black), $\epsilon = -7 \cdot 10^{-5}$ (dark blue), $\epsilon = -5 \cdot 10^{-5}$ (light blue)*

a) EoS for different values of ϵ   *b) The Weierstrass function $\Phi(a)$*   *c) The evolution of the scale factor $a(t)$*

5. A further example: $f(\mathcal{R}) = \mathcal{R} - \frac{\alpha^2}{2}\mathcal{R}^2 - \frac{\beta^2}{3}\mathcal{R}^{-1}$

Now we are ready to consider something more brutal, which is, at least, a good exercise to test our guidelines to solve cosmological models.

We shall focus on the Palatini $f(\mathcal{R})$-theory and its EC, for the function

$$f(\mathcal{R}) = \mathcal{R} - \frac{\alpha^2}{2}\mathcal{R}^2 - \frac{\beta^2}{3}\frac{1}{\mathcal{R}} \tag{8.5.1}$$

We shall here consider $\alpha > 0$, $\beta < 0$, and restrict to $a > 0$. Let us assume dust EoS for visible matter, i.e. $p = 0$. Accordingly, the trace of the energy-momentum tensor is $T = -\mathbf{c}^{-1}\rho$. We shall also restrict to dimension $m = 4$.

We shall show graphs for $\kappa = 1 = \mathbf{c}$, $\alpha = 1/4$, $\beta = -1/8$, $\rho_0 = 1/4$.

The master equation for this theory is

$$(1 - \alpha^2\mathcal{R} + \beta\mathcal{R}^{-2})\mathcal{R} - 2(\mathcal{R} - \frac{\alpha^2}{2}\mathcal{R}^2 - \frac{\beta^2}{3}\mathcal{R}^{-1}) = \underline{\mathcal{R}} - \alpha^2\mathcal{R}^2 + \underline{\frac{\beta^2}{3}\mathcal{R}^{-1}} - \underline{2\mathcal{R}} + \alpha^2\mathcal{R}^2 + \underline{2\frac{\beta^2}{3}\mathcal{R}^{-1}} = \kappa T \qquad \Rightarrow \qquad \mathcal{R} - \beta^2\mathcal{R}^{-1} = \kappa\mathbf{c}^{-1}\rho \tag{8.5.2}$$

which can be solved for the curvature

$$\mathcal{R}_\pm(\rho) = \frac{\kappa\mathbf{c}^{-1}}{2}\left(\rho \pm \sqrt{\rho^2 + 4\beta^2}\right) \tag{8.5.3}$$

Of course, the master equation is not invertible, so that it can be solved only locally, and, because of that, one has two values of the curvature $\mathcal{R} = \mathcal{R}_\pm(\rho)$ for any value of density ρ, depending on the branch one chooses to invert.

These two branches correspond to a positive ($\mathcal{R} \in [\kappa\mathbf{c}^{-1}|\beta|, +\infty)$) or a negative ($\mathcal{R} \in [-\kappa\mathbf{c}^{-1}|\beta|, 0)$) curvature. On each of these two branches, we can compute the quantity $f'(\mathcal{R}) = 1 - \alpha^2\mathcal{R} + \beta\mathcal{R}^{-2}$ as a function of the density ρ and eventually as a function of the scalar factor a:

$$\begin{cases} \varphi_+(\rho) = f'\left(\mathcal{R}_+(\rho)\right) & \left(\mathcal{R} \in [\kappa\mathbf{c}^{-1}|\beta|, +\infty)\right) \\ \varphi_-(\rho) = f'\left(\mathcal{R}_-(\rho)\right) & \left(\mathcal{R} \in [-\kappa\mathbf{c}^{-1}|\beta|, 0)\right) \end{cases} \tag{8.5.4}$$

For later convenience, it is interesting to find the further branching points at which $f'(\mathcal{R}) = 0$.
In the interval $\mathcal{R} \in [\kappa\mathbf{c}^{-1}|\beta|, +\infty)$, there is one zero, at $\mathcal{R} = \mathcal{R}_v$, where

$$\mathcal{R}_v = \frac{\sqrt[3]{\Delta}}{6\alpha^2} + \frac{2}{3\alpha^2 \sqrt[3]{\Delta}} + \frac{1}{3\alpha^2} \tag{8.5.5}$$

and where we set $\Delta := 4\left(2 + 9\alpha^4\beta^2 + 3\alpha^2\beta\sqrt{4 + 9\alpha^4\beta^2}\right)$. That corresponds to values of the density $\rho = \rho_v$ and of the scale factor $a = a_v$, namely

$$\rho_v := \frac{\mathcal{R}_v^2 - \beta^2}{\kappa\mathbf{c}^{-1}\mathcal{R}_v} \qquad a_v := \left(\frac{\kappa\mathbf{c}^{-1}\rho_0\mathcal{R}_v}{\mathcal{R}_v^2 - \beta^2}\right)^{1/3} \tag{8.5.6}$$

respectively.
In the other interval $\mathcal{R} \in [-\kappa\mathbf{c}^{-1}|\beta|, 0)$ there is no zero for $f'(\mathcal{R})$.

Hence, when defining the conformal factor, we have to distinguish three different branches:

B1: the red branch, on which one has $a \in (0, a_v)$, or $\rho \in (\rho_v, +\infty)$, and $\mathcal{R}_+(\rho) \in (\kappa\mathbf{c}^{-1}|\beta|, \mathcal{R}_v)$.
The quantity $f'(\mathcal{R}_+(\rho))$ is negative, so that one defines $\varphi_{(B1)}(\rho) = -f'(\mathcal{R}_+(\rho))$.

B2: the blue branch, on which one has $a \in (a_v, +\infty)$, or $\rho \in (0, \rho_v)$, and $\mathcal{R}_+(\rho) \in (\mathcal{R}_v, +\infty)$.
The quantity $f'(\mathcal{R}_+(\rho))$ is positive, so that one defines $\varphi_{(B2)}(\rho) = f'(\mathcal{R}_+(\rho))$.

B3: the green branch, on which $a \in (0, +\infty)$, or $\rho \in (0, +\infty)$, and $\mathcal{R}_-(\rho) \in (-\kappa\mathbf{c}^{-1}|\beta|, 0)$.
The quantity $f'(\mathcal{R}_-(\rho))$ is positive, so that one defines $\varphi_{(B3)}(\rho) = f'(\mathcal{R}_-(\rho))$.
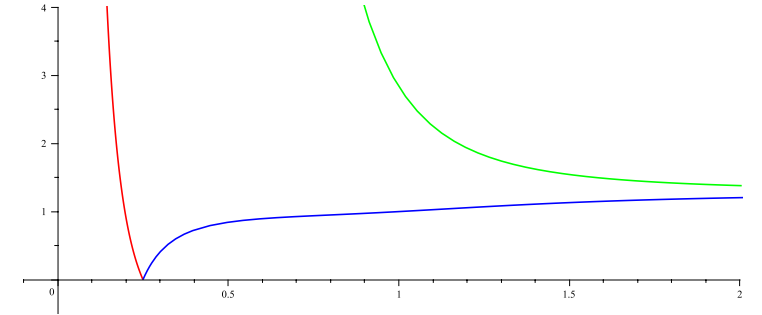
Fig. 7.20: The conformal factor $\tilde{\varphi}(a)$

Let us assume today we have a positive curvature $\mathcal{R}$, i.e. we currently are on the blue or red branch. Then we define the normalised conformal factor (so that $\tilde{\varphi}(\rho_0) = 1$) which is defined as

$$\tilde{\varphi}(\rho) = \begin{cases} -\dfrac{f'\left(\mathcal{R}_+(\rho)\right)}{f'\left(\mathcal{R}_+(\rho_0)\right)} & \rho \in (\rho_v, +\infty) \text{ on red branch } \mathbf{B1}, \mathcal{R} > 0 \\[2ex] \dfrac{f'\left(\mathcal{R}_+(\rho)\right)}{f'\left(\mathcal{R}_+(\rho_0)\right)} & \rho \in (0, \rho_v) \text{ on blue branch } \mathbf{B3}, \mathcal{R} > 0 \\[2ex] \dfrac{f'\left(\mathcal{R}_-(\rho)\right)}{f'\left(\mathcal{R}_+(\rho_0)\right)} & \rho \in (0, +\infty) \text{ on green branch } \mathbf{B3}, \mathcal{R} < 0 \end{cases} \tag{8.5.7}$$

Next step is considering effective density and pressure. We can start from equation (8.4.25) and specialise to the function $f(\mathcal{R})$

$$\tilde{\rho} = \frac{\mathcal{R}}{4\kappa\mathbf{c}^{-1}} \frac{12\kappa\mathbf{c}^{-1}\rho\mathcal{R} - 3\alpha^2\mathcal{R}^3 + 4\beta^2}{(3\mathcal{R}^2 - 3\alpha^2\mathcal{R}^3 + \beta^2)\tilde{\varphi}} \qquad\qquad \tilde{p} = \frac{\mathcal{R}}{4\kappa\mathbf{c}^{-1}} \frac{3\alpha^2\mathcal{R}^3 - 4\beta^2}{(3\mathcal{R}^2 - 3\alpha^2\mathcal{R}^3 + \beta^2)\tilde{\varphi}} \qquad (8.5.8)$$

in which we should substitute $\mathcal{R} = \mathcal{R}_{\pm}(\rho)$, the conformal factor (8.5.7), and the density function $\rho = \rho(a)$, thus obtaining the effective density and pressure as functions of a.

This is a parameterised form for the effective EoS, see Fig. 7.21.
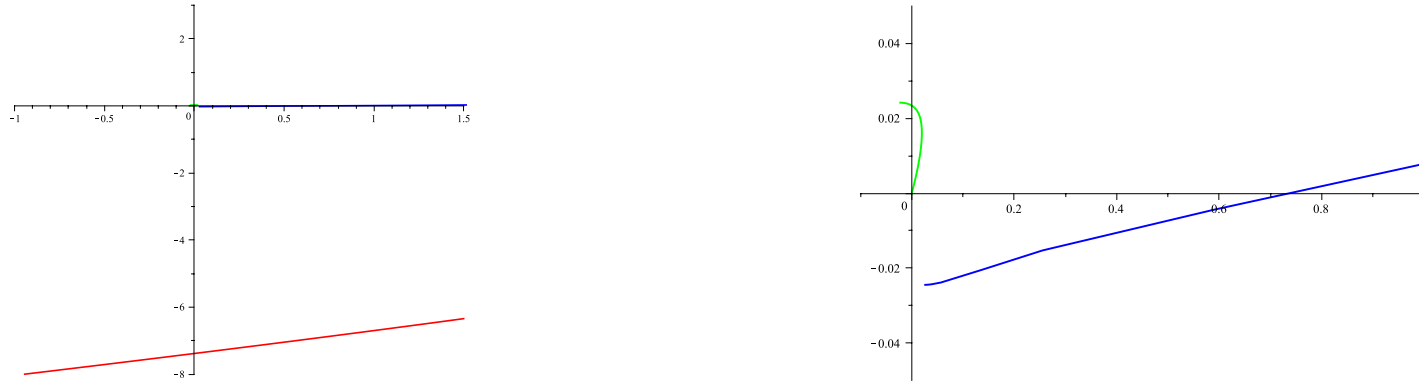
Fig. 7.21: a) Effective EoS   *b) Zoom of EoS at the origin.*

Notice that effective EoS is getting more and more strange. Also notice that as the parameters (α, β) of the theory go to zero, the effective EoS approaches the standard one, i.e. $p = 0$.

Also notice that, as the density becomes small on the blue branch, the pressure eventually becomes negative, which should indicate a late acceleration of the expansion.

As in the previous case, we can compute the Weierstrass function $\Phi(a)$; see Fig. 7.22.a. We already assumed to be on the red/blue branch, and, since today $a_0 = 1$, it means that, for the value of $\kappa\mathbf{c}^{-1}\rho_0 = 1/4$ that we are using for graphs, we are on the blue branch. If our universe were contracting today, it will go on contracting until it reaches $a = a_v$, where the blue branch finish and the Weierstrass function has a (simple) zero, and it would then bounce back. Then it would expand, going through a decelerating phase, and then, when the negative pressure kicks in, it would start a late accelerating phase. Then it expands forever.

In this case, we can also compute the integral for $t(a)$ numerically, see Fig. 7.22.b. However, it is time to realise that we should not. There are many features of the function $a(t)$ that can be inferred using the Friedmann equation instead of going through a solution.

For example, by differentiating the Friedmann equation we obtain

$$2\dot{a}\ddot{a} = \Phi'(a)\dot{a} \qquad \Rightarrow \qquad \ddot{a} = \tfrac{1}{2}\Phi'(a) \qquad (8.5.9)$$

Accordingly, the expansion is accelerated iff the Weirstrass function is growing. Hence by comparing with the graph of the Weierstrass function, Fig. 7.22.a, we see that starting from the minimal scale factor $a = a_v$, the universe at first accelerates, then it goes through a deceleration phase and then, at about $a = 1.3$, it starts accelerating again.
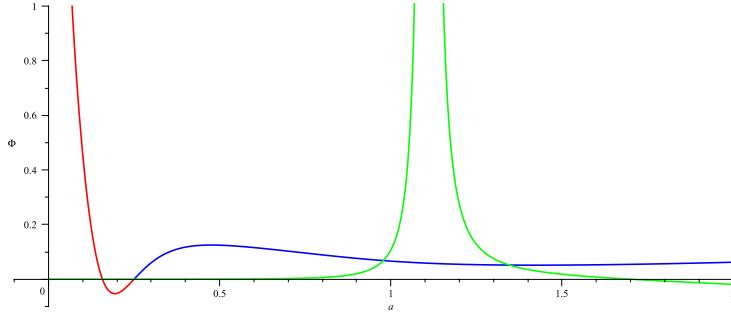
Fig. 7.22: a) Weierstrass function $\Phi(a)$ *b) The evolution of the scale factor $a(t)$*
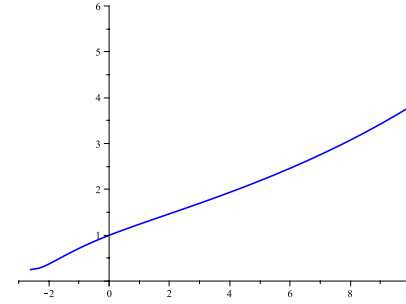
This kind of qualitative analysis is very convenient, since it can be turned into analytic considerations, which are general with respect to the model parameters. For example, if we want to define a *late acceleration*, we can check for the first derivative of $\Phi(a)$ at infinity, i.e. replace $a = u^{-1}$, expand in Maclaurin series around $u = 0$, and replace back $u = a^{-1}$. One obtains

$$\Phi(a) \simeq \frac{|\beta|}{6}a - \frac{\rho_0}{6a^2} + \frac{9(3\alpha^2|\beta| + 2)k\rho_0}{2|\beta|(3\alpha^2|\beta| - 4)a^4} - \frac{(369\alpha^4\beta^2 - 660\alpha^2|\beta| - 100)\rho_0^2}{48|\beta|(3\alpha^2|\beta| - 4)^2 a^5} + \dots \qquad (8.5.10)$$

Fig 7.23: the scale factor $\tilde{a}(a)$.

This is extremely convenient. It shows that the late acceleration remembers the parameter β (only β, not α) and vanishes for standard GR. Before reaching $a = +\infty$, the value of the acceleration has corrections from the other terms which are small though not zero, which in principle remember the parameters of the model (this time both α and β). Thus if the acceleration $\ddot{a}$ will turn out to be observable, that will allow us to *measure* the parameters α and β by measuring corrections to the constant acceleration which is reached to infinity.

> Well, of course, *we* cannot observe the acceleration at $t = +\infty$, since we live today; what we mean is that, *if we live* in the accelerated phase for long enough, *if we are able* to observe the acceleration evolution over a long enough interval, *if we know* we are leaving in such a model family, *if our observations are* precise enough to precisely fit the expansion parameters, then, at least *in principle*, we are able to observe the model parameters.

By using the conformal factor $\tilde{\varphi}$ and the scale factor, we can compute the effective scale factor as a function of a, see Fig. 7.23.

We see that as the scale factor a reaches its minimal value $a = a_v$ at the bounce, the corresponding effective scale factor reaches $\tilde{a}(a_v) = 0$ and the conformal metric $\tilde{g}$ becomes singular. The singularity of $\tilde{g}$ is entirely due to the vanishing of the conformal factor at $a = a_v$. And the dynamics of the conformal factor is entirely dependent on the dynamics of the scale factor a.

This is a somehow strange situation: we have two metrics, which are dynamically equivalent, one becomes singular the other stays regular. That suggests that the dynamics can be solved and continuously prolonged beyond the singularity and, at most, one should check if the free falling of a test particle can be prolonged by continuity as well.

References

add

Part II debrief

If, in Part I, we discussed what is a relativistic theory, in Part II we defined gravitational field as the dynamical geometries on spacetime and we discussed dynamics for them.

Of course, our discussion has been not exhaustive. There are infinitely many alternatives which we did not consider. We, instead, started from Ehlers–Pirani–Schild (EPS) axiomatic approach to gravitational field and took inspiration from it to define Extended Theories of Gravitation (ETG).

ETG are particular theories which implements EPS on shell, in particular with a metric which encodes lengths and the causal structure of spacetime and a connection which describes free falling of particles. Then we specialised to integrable ETG (iETG) which have a connection which is metric on shell.

Finally, we further specialised to Palatini $f(\mathcal{R})$-theories, in which dynamics is proportional to a scalar function $f(\mathcal{R})$ of the curvature $\mathcal{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}(\tilde{\Gamma})$, and $f(\mathcal{R})$-cosmologies which derive from them. We learnt how to solve these models, we discussed dynamical equivalence among them and with different theories.

Palatini $f(\mathcal{R})$-theories are not the most generic ETG, and ETG are not the most general theories which implement EPS. However, they are a wide class which may be of interest. They might eventually produce a sound theory of gravity for applications, though mainly, at least, they provide a more general framework to discuss the relation between geometry and physical observables, as well as a wider setting for (classical and new) tests.

We missed the target to discuss in general ETG. We know for example that there are metric-affine theories which are ETG and other that are not. There is no satisfactory characterisation of ETG out of the family of Palatini $f(\mathcal{R})$-theories.

EPS framework, which is central for us, is itself just an axiomatic setting. One can consider to modify part or all the axioms. For example, few (if not none) of the EPS axioms are expected to hold in a quantum world. Again, at least, what we did is a declaration of good practice: the correct practice to set a fundamental description of reality is to use past knowledge as a qualitative description of what the world is like, to set up axioms which are based and linked with physical experience, so that one knows what it means to drop each of them. For example, if you drop Axiom CS it is quite likely that you will end up to describe a world with no past and future. If you drop Axiom CC sooner or later you'll find particles slower that the speed of light which cannot be further accelerated.

One way of looking at EPS is considering that, rather than allowing theories more general than standard GR, they actually constrain (in fact, quite drastically) modifications to standard GR. If you consider a theory which does not implement EPS, it means that some of EPS axioms are violated and you can have all sort of spurious effects (as violations of the causal structure) which eventually will appear, if your theory is taken seriously.

However, ETG as we defined them are not the most general theories fitting in EPS framework. For example, one could force EPS compatibility at a kinematical level, rather than requiring it at a dynamical level. One can look for a theory in which the fundamental fields are a Lorentzian metric g and a 1-form $\boldsymbol{\alpha}$, and eventually define a compatible connection $\tilde{\Gamma}$ out of them to describe free fall. That would be strange for many reasons (for example, quantum fluctuations would preserve EPS compatibility, or, as far as I can see, there is no real reason to prefer $\boldsymbol{\alpha}$ as a fundamental field over $\tilde{\Gamma}$ which, at least, has a clear, quite direct, physical interpretation of describing free fall of test particles); still it is a viable possibility to explore.

We made a choice, and explored that direction, forgetting other interesting possibilities. As a further example, consider that EPS points to a conformal class rather than to a metric. Accordingly, one could have gone for a dynamics which is covariant with respect to Weyl transformations, which as such, in view of the hole argument, would have a Lorentzian conformal structure as a solution rather than a specific representative of it. Again, that is a mathematically well-defined option and, at least as such, it would be interesting to be explored in details. However, we know in advance that in

such a theory distances and time lapses would not be observable, always in view of the hole argument. If you want to account for a meaning when cosmologists say that the universe is 14BY old, or astronomers say that the Moon is about $3.8 \cdot 10^8\ m$ away from the Earth, that will not be a good setting.

> Since one could say that the distance if the Moon is about 20 times the length of a meridian on the Earth, instead of saying it is about $3.8 \cdot 10^8\ m$ away, honestly it is not clear to me if distances are really needed in physics, thus I cannot exclude conformal invariant theories should be the direction to go.

 Again, we made our choice, we made it clear, and we kept stuck to that. That is part of the good practice to follow in setting up a model of reality. After all, it is amazing to see how far one can go using only what a relativistic theory can offer.

 In this second Part of the book, we discussed dynamics for geometries on spacetime. We did not get fond of a specific interpretation of the objects involved in such models. We somehow kept as a back-thought EPS formalism, so that the metric has been repeatedly said to represent lengths and the connection to describe free fall, though we did not specified how such structures are related to what we measure in astrophysical and cosmological surveys. We shall do that in Part III.

 If what we had said until now is quite independent of the interpretation, what follows in Part III is totally, strictly dependent on it. Again, in dynamics there is nothing that could indicate the meaning of objects, at most one could have information about which objects are potentially observables (excluding the ones which are not determined by the field equations, in view of the hole argument).

 Sooner or later, one needs to start setting a connection with physics by saying how to measure some of them. The EPS framework also provides us with some guidelines. The metric g is there built with clocks and light rays, the connection $\tilde{\Gamma}$ is built with freely falling particles. Hence, test particles (light rays, respectively) are to be identified with g-time like (g-light like) $\tilde{\Gamma}$-geodesics and one should be able to rebuild many information about the metric and the connection in terms of thought experiments with test particles.

> Introducing test particles is a relativistic field theory, is not easy. Physically, being *test* means that one way or the other the back-reaction of test particle on the gravitational field is neglected. On the other hand, relativistic fields are in interaction by the very definition of relativistic theory; no non-dynamical field is allowed, each field in the theory obeys its field equations, which generally depend on all the other fields.
>
> Forcing no back-reaction is the same as dumping some of these interactions. If you do, there is no way in which back-reaction can be forced to exactly vanish. Thus by *neglecting it*, one usually means it to be sufficiently small to be ignored; not that it is not there, just that we ignore it. Also this position is not quite sound: if we add a pointwise particle in Minkowski spacetime, it will become Schwarzschild, no matter how small its mass is. And a Schwarzschild metric is approximately Minkowski only when one is kept few Schwarzschild radiuses away from the mass. If the mass of the particle is small, the region where the Schwarzschild metric is significantly different from Minkowski metric become smaller, though there is always a portion in spacetime in which the curvature grows (in fact to become singular) so that there the spacetime is definitively *not* approximately Minkowski.
>
> Thus by *negligible* we mean that we keep out of the region where the approximation fails. Which sometimes is what one needs, though we have to accept that this is not for anywhere and not forever.
>
> Often, test particle are derived as an eikonal approximation of matter field equations. This is sometimes ok, however one should keep in mind two things: first, the gravitational field is not a linear field, Einstein equations are not linear equations, so we do not much in general about how perturbations affect a solutions. Again, it is fine in a small region of spacetime, but if you want to follow the worldline of a particle for 10BY, one at least should investigate how well the eikonal approximation is maintained at that scale.
>
> Second, eikonal approximation does not account for what *in practice* people call a test particle. It may be that, in general, a particle can be described as propagation of a small perturbation of a reference metric, that one can control that it remains a small perturbation or for how long it does. Even in this case that is *not* what people call a test particle!

Often one does consider test particles on solutions which have a non-trivial group of isometries. That is essentially ubiquitous, since any known exact solution has isometries. When one has a spacetime with isometries (for example, a spatially spherically symmetric solution, as Schwarzschild is) what is a small deformation? Has the deformation to preserve the isometry group or can it break it down? That is equivalent to ask whether the Killing vectors of the reference metric are still Killing vectors of the perturbed metric.

If the answer is yes, then a particle cannot break the isometry group, the perturbed metric is also spatially spherically symmetric, and a particle cannot be at a point along an orbit. Planets do have a position, if they are represented as perturbations they do break rotational invariance. Hence they are not test particles in this sense. Or the answer is no, then test particles can be eikonal approximations of matter fields, though not of the matter fields we assumed in our symmetric model. If the metric on spacetime is spherically symmetric, then matter fields are as well. And, if they are, their eikonal approximation is spherically symmetric as well.

In both cases, when one considers a test particle in a Schwarzschild metric, there is no (known) way to obtain it from the dynamics of the model. At the very least, one should think of a *different* matter field, which has been neglected in the action, which couples with the metric producing small deformations, which are propagated in the full theory (i.e. the one with no symmetry) and remain small deformations of the symmetric metric.

Nothing of this is really under our control. Thus we have to accept that, at least in practice, test particles are something added to the, not something calculated out of, the dynamics.

Since test particles are, in view of EPS, geodetics of the connection $\tilde{\Gamma}$, we can get a lot of geometric information about the geometry by studying worldlines and, potentially, that information is available to be compared with observations, once we assume the Moon, Mercury, a stone, or a galaxy to be test particles.

In principle that is not too straight forwards, maybe it is false. A test particles is something (pointwise) which falls freely in the gravitational field, i.e. it is affected by gravity only.

A rigid body, as a stone could be (if a rigid body made sense), is a collection of particles bound together by electromagnetic and nuclear forces, which, for some reason we cannot completely control in details, move as a single object. Then we should show that there a way to define a point-like position for it which moves along a geodesics, i.e. as if the non-gravitational bounding forces did not exist.

Not that this is manifestly impossible, though, in practice, there is not even agreement on how to define the center of mass of the system. That is why we *assume* the stone to be a test particle, instead of *proving* it is. By the way, in case we need to say it, once we assume it, it works pretty well.

That is quite satisfactory from a classical viewpoint: classical fields, all of them, not only the gravitational field, are not objects to be directly measured. Originally they are there to explain why particles move away from straight motion (and that is true for the gravitational field as well as for the electromagnetic field). That remains appropriate even in a gravitational theory where there is no *straight motion* to deviate from: fields are observed through their effects on particles (and light rays). That is pretty much the end of the interpretation in a relativistic theory until one wants to indirectly observe fields starting from their action on particles.

However, it is not the end word in a dynamical world. Even when we are able to describe the motion of particles in a fixed classical field, we still have to describe the field dynamics. Fields are produced by sources. If we describe the solar system, we have a source, the Sun, which produces a gravitational field, which affects the motion of test particles, e.g. the planets. We identify planets and satellites to be test particles and, by studying their motion, we are able to trace back the gravitational field to be a Schwarzschild metric.

Usually, one *assumes* a Schwarzschild metric and check that its geodesics account for the motion of planets. The EPS framework shows that observing enough test particles the process can be reverted and one can *prove* that the metric is (or measure it to be) Schwarzschild.

We do not know anything, yet, about how it happens that the Sun produces the Schwarzschild metric as its gravitational field. That has nothing to do with test particles, it has to do with Einstein equations, not with geodesic equations.

This is not peculiar at all of the gravitational field. If we consider an electric field, it is observed through its action on charged test particles. That is described by the Coulomb-Lorentz force.

Now, if we consider a charges wire, it generates an electric field which can be measures through its effects on test charges. However, the mechanism by which a charge continuum generates an electric field is not controlled by Coulomb-Lorentz force, it is instead described by a different axiom, which is Biot-Savart law. It is *another* axiom of the theory.

The real essential difference between gravity and electromagnetism is linearity. Being electromagnetism a linear theory, if we assume how a single infinitesimal charge generates a electromagnetic field, then a charged continuum can be modelled as a system of charged point particles and the electromagnetic field can be defined in terms of an integral as the *"sum"* of infinitesimal contributions, one from each charge point particle.

That does not work with gravity which is not linear, so we have to define the gravitational field generated by a source matter field (and that cannot come from, or at least is not trivially the same as, a microscopic description). In a gravitational theory, that is what field equations do: give me a matter field, compute its energy-momentum tensor, then field equations do determine the field out of it. Again, that is also what Maxwell equations do, just we argued that in gravity the connection between sources and test particles has been compromised, partially by the non-linearity of the theory, partially by the lack of general solutions with no symmetry.

In any event, if we wish to apply our gravitational theory to describe something, we have to declare a link between the sources of the gravitational field and the matter field in the model. To describe the solar system we should account for how the Sun can be described as a matter field. In cosmology, we should declare what is the link between galaxies and the density and pressure functions (ρ, p) which define the energy-momentum tensor. In an extended cosmology, we should also declare what galaxies have to do with (ρ, p), $(\tilde{\rho}, \tilde{p})$, or $(\hat{\rho}, \hat{p})$ and that is *another* assumption which has not much to do with the assumption about test particles.

I am not saying that I like it, I am saying that that is what we can do, what we do also in standard GR, where, although one is used to do everything with one metric, we already argued that, in fact, there *are* other metrics available, if we wanted to use them.

So let us declare our choice which has been already embedded into the notation, though it will become really essential in what follows. In an ETG we start from a set of fundamental fields $(g, \tilde{\Gamma}, \phi)$, g being related with lengths, time lapses, and causal structure, the connection $\tilde{\Gamma}$ being related to test particles. The matter fields ϕ are called the *visible matter*. They allegedly account for ordinary matter. For the Sun, they account for the gas of which the star is made, in cosmology it accounts for galaxies which made the universe.

Since gravitational field is described by $\tilde{\Gamma}$, i.e., in the case of Palatini $f(\mathcal{R})$-theories, by the conformal metric $\tilde{g}$, and $\tilde{g}$ obeys Einstein equations with the modified energy-momentum stress tensor $\tilde{T}_{\mu\nu}$, everything related to $\tilde{T}_{\mu\nu}$ (e.g. $\tilde{\rho}$ and $\tilde{p}$, in cosmology) is called *effective matter* and it is meant to include visible, as well dark, sources.

Of course, as the choices we made in studying dynamics, this is not the only possible choice, though they are the choice we decided to study in next Part. In standard GR, it is considered natural to link sources with visible matter, since in standard GR that is all one has since the conformal factor is constant $\tilde{\varphi} \equiv 1$.

Part III

General relativity

Asteroids do not concern me, Admiral! I want that ship, not excuses!

(*The Empire strikes back*), Lord Darth Vader

Life creates the Force, makes it grow. Its energy surrounds us and binds us. Luminous beings are we, not this crude matter. You must feel the Force around you; here, between you, me, the tree, the rock, everywhere, yes. Even between the land and the ship.

(*The Empire strikes back*), Master Yoda

III. Introduction to Part III

In this Part we investigate the physical interpretation and observables in Palatini $f(\mathcal{R})$-theories in particular, as an example of more general ETG.

If we had an absolutistic attitude the issue is easily addressed. If we consider a gravitational theory as a gauge theory with general diffeomorphisms as symmetries, then an observable should be a generally $\mathrm{Diff}(M)$-invariant quantity. Thus a scalar field ψ, in order to be observable, should obey $\pounds_\xi \psi = 0$ for any infinitesimal generator of diffeomorphisms, i.e. for any vector field ξ on spacetime. Then one should have

$$\pounds_\xi \psi = \xi^\mu \partial_\mu \psi = 0 \qquad \Longleftrightarrow \quad \psi \text{ is constant}$$

Any non-constant quantity is not an observable in the stricter gauge sense. Still we have measurements in astronomy, cosmology, everyday life which we would like to account for. Still we say that observations corroborate a theory and rule out another one, thus we need a more general framework to describe what we observe in a gravitational theory.

If what we observe cannot be an absolute quantity (as it happens for gauge invariant quantities in gauge theories) then it must be relative, it must be somehow conventional and observer-dependent.

> That is not too much a surprise. The original *theory of relativity* owes its name to the discovery that some of the quantities which were believed to be fundamental in Newtonian physics (e.g. contemporaneity, time lapses between two events, distances among events, ...) are, in fact, relative to the observer, i.e., in particular, they are not gauge observable.
>
> Einstein acknowledged that the name *theory of relativity* was an unfortunate name for the theory, which claimed exactly that one could (and should) avoid to use these quantities for the fundamental description of physical reality and should focus on quantities which are *absolute*, instead. Although this neat and beautiful start, one has to admit that a gravitational theory seems to melt down during its definition. We used fields (the metric, the connection) which in view of the hole argument have a relative value at a fixed point $x \in M$, since they can be dragged around by spacetime diffeomorphisms. We assumed and used the fact that parameters along a worldline have no direct physical meaning, though they are at the same time assumed to be the readings of clocks. We now discover that our theories have no absolute quantities to account for any of the quantity which are measured in astrophysics, astronomy, or cosmology.
>
> Although the dynamics of the theory is absolutely described (any observer would agree that the Sun generates a gravitational field which is approximately described as a Schwarzschild spacetime) this is done at the price of loosing contact with physical quantities and observables.

The goal of this Part is to sew observables on properly to the geometric gravitational theories.

As a first sketch, the main idea is that, even if there are no gauge invariant quantities in a theory, still there are a lot of absolute claims that can be formulated. That is no much difference from what we implicitly did in studying dynamics after we realised that, by using scalar fields, one cannot account for the physics (for example, accounting for velocities, accelerations, forces, and so on). There we turned to more general fields, to tensor fields and more general geometric objects, even though it is obvious that the components of a tensor field do depend on the observer and as such cannot be observed as absolute physical quantities. The whole original standard GR can be regarded as a school to learn to live with that fact, i.e. to learn how to formulate a meaningful absolute dynamical theory using fields which are relative quantities.

In that case, the answer was already there, provided by the framework of differential geometry which has been developed in the previous 50 years: even if the components of a tensor depend on the basis, even if they are conventional objects, still claiming that two tensor objects are the same (i.e. their difference is zero) is an absolute claim. We can still formulate absolute laws, even if we are using relative fundamental fields.

For observables, something similar applies as well. Even we are quite short of gauge-invariant observables, if we defined conventional quantities, which are not gauge invariant, it may be meaningful, i.e. be absolute, to claim that two such conventional quantities are equal (at a point in spacetime). For example, if we have two scalar fields ψ_1 and ψ_2, they are not gauge invariant every time they are not constant, though claiming that $j_x^1 \psi_1 = j_x^1 \psi_2$ is a gauge invariant statement, thus an absolute claim which is supported by the fact that one has $\pounds_\xi (\psi_1 - \psi_2)|_x = 0$.

Analogously, claiming that two signals arrive at the same event (i.e. at the same space point and the same instant) is something which has an absolute meaning and to which Einstein referred as the *coincidence principle*, which says that only these quantities are absolute.

Also under this viewpoint, gravity is quite different from any other field theory formulated on Minkowski. Even when one has a gauge symmetry, there are a number of quantities which are gauge invariant, thus observables. In a relativistic theory, instead, one needs to learn to live without them, as we learnt to live with fields the values of which are not observable themselves. Let us also stress that, in this viewpoint, relativistic theories are not a special, ugly case of field theories defined on Minkowski spacetime. The absence of observables is not an additional bad feature of gravity. Quite the other way around, since relativistic theories are *more fundamental*—not a particular case of—SR field theories. It is rather the opposite: SR field theories are specific relativistic theories in which we pretend to have solved this issue (by inventing a conventional class of observers, the inertial observers, and pretending that the quantities that they measure, which are still relative, are in fact absolute) and then investigating how to behave in view of some other extra gauge symmetry which there may be there.

Because of this complication, we have an extra work to do, which in fact needs to be done once in any field theory, even in the theories in which we are not used to do it. The idea of defining as observables the comparison of two conventional quantities is so well established that is has a name: it is called *relational approach* and *relational observables*. The implementation is a bit awkward though quite fundamental; instead of saying that an event is observed at a certain time, it is translated to an event is observed at the same time when a certain signal is received from a particular system which is elected as a clock. Both the signal are received in a single event and, just because of it, it is an absolute fact on which any observer would agree.

More generally, connecting mathematical quantities to observation is a long chain of analyses to discuss what depends on which convention and how. For this reason, we shall define observable and tests in relativistic theories and gravitational models in a number of waves, adding conventions one at a time.

In Chapter 9 we shall discuss geodesics in a given geometry. We try to discuss them in an (integrable) Weyl frames so that our analysis will easily apply Palatini $f(\mathcal{R})$-theory. Knowing how to compute geodesic trajectories is important, since fields in general, as well as the gravitational filed in particular, are defined through they action on test particles. The general attitude is that the motion of test particles is observable so that any structure which is defined in term of test particles will be observable as well. Moreover, geodesic analysis is a basis for synchronisations of clocks (discussed in Chapter 10) as well as positioning systems, gravitational lensing, and classical tests.

In Chapter 10 we shall discuss clocks, how one can define uniform clocks (since fundamentally speaking there is no intrinsic absolute definition of what uniform may mean before a standard is chosen). Still we usually make a choice, being it conventional or not, of atomic clocks to be a reference standard for clocks. Once we have a clock and we are able to measure time lapses, we have synchronisations at a distance, as well as a number of definitions of spatial distances. In each of these definitions, we need to discuss how one can reduce extended quantities to a series of measurements at a single event.

In Chapter 10, we discuss models which uses field equations. They are models in which one tests the ability of a theory of reproducing a certain dynamics of observables.

Then we still need to discuss in detail

classical tests

gravitational lensing

relativistic positioning system

astrophysical and cosmological observations

Bariogenesis and structure fonmation

The Novae History of a alien bootstrap theory of

Chapter 9. Geodesics in symmetric geometries

Tra il bufalo e la locomotiva la differenza salta agli occhi:
la locomotiva ha la strada segnata
il bufalo può scartare di lato e cadere.
Questo decise la sorte del bufalo

[The difference between buffalo and locomotive is evident:
The locomotive has a predetermined path,
The buffalo can swerve to one side and fall.
That established the buffalo's fate]

(Francesco De Gregori, *Bufalo Bill*)

1. Introduction

The gravitational field is identified with the geometry of spacetime. Such a geometry can be described in different ways, e.g. by an EPS-geometry $(M, \mathfrak{g}, [\tilde{\Gamma}])$, by a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$ obtained by gauge fixing the projective structure $\tilde{\Gamma} \in [\tilde{\Gamma}]$, by a Weyl frame $(M, g, \tilde{\Gamma})$ obtained by selecting a conformal representative $g \in \mathfrak{g}$. Then in particular this Weyl frame can be integrable, i.e. $\tilde{\Gamma} = \{\tilde{g}\}$ for some (other) conformal representative $\tilde{g} \in \mathfrak{g}$, or even a Riemannien structure (M, g), if $\tilde{g} = g$.

Whatever kinematical choice our gravitational theory does to represent geometrically the gravitational field, the gravitational field, as any other classical field, is not something which can be directly observed.

> Even in Maxwell theory, one cannot observe directly an electric field. The instruments which are used to measure it, in fact, use test particles to measure it. At a fundamental level, one cannot see fields, they are manifest (and entirely defined) by their action on test particles.

Accordingly, the gravitational field manifest itself by its action on test particles, which, as discussed in EPS framework, move along geodesic trajectories (or, one should say *autoparallel trajectories* for a general connection) of the connection $\tilde{\Gamma}$. This is true if $\tilde{\Gamma}$ is a fundamental field, as it happens, e.g., in a non-integrable Weyl frame, or even if it is not a fundamental field and it is defined in terms of other fundamental fields, as in happens in integrable Weyl frames or Riemannian structures.

In general, for a general torsionless connection which is not metric, the geodesic equation is not Lagrangian, though it is, if the connection happens to be metric, as it happens, e.g., in general and on shell, in Palatini $f(\mathcal{R})$-theories.

> The geodesic equation is not a fundamental physical equations as field equations are. It appears in the framework as a trick we envisage to describe the motion of macroscopic bodies. One could argue that having a non-Lagrangian geodesic equation is not a clear criterium to exclude or accept theories, and in fact we agree it is not. However, using Lagrangian formalism is convenient. That means that we restrict to integrable ETG dynamics, e.g. to Palatini $f(\mathcal{R})$-theories. This is convenient also for other reasons, for example to avoid that the length of parallelly transported vectors to the same event depends on the path.
> We are not claiming that these are the only meaningful ETG, though we are here restricting to this case.

Once we know that geodesic equations come from a Lagrangian, which is written in term of metric $\tilde{g}$ for which $\tilde{\Gamma} = \{\tilde{g}\}$, then we get for free many methods from Lagrangian mechanics which allow us to treat geodesics without ever solving the geodesic equation, often even without writing the equation.

We already discussed that what is physically important is the geodesic trajectory, not the geodesic motion, so for us it is quite natural to assume the Lagrangian

$$\boldsymbol{L} = \sqrt{-\tilde{g}_{\mu\nu} u^\mu u^\nu} \, ds \tag{9.1.1}$$

for $\mathfrak{g}$-time-like (or light-like) geodesic trajectories.

Some of the computation can be obtained also by using the Lagrangian

$$\hat{\boldsymbol{L}} = \tfrac{1}{2} \tilde{g}_{\mu\nu} u^\mu u^\nu \, ds \tag{9.1.2}$$

which singles out geodesic motions only. If we fix s to be the arclength parameter, then solving either of the two equations is somehow equivalent, in the sense that any solution of the Lagrangian $\boldsymbol{L}$ is obtained as a reparameterisation of a solution of the Lagrangian $\hat{\boldsymbol{L}}$.

However, the Lagrangian $\boldsymbol{L}$ is for us canonical, since it encodes the invariance with respect to reparameterisation and it does not need to define proper time first.

The Lagrangian $\boldsymbol{L}$ given by (9.1.1) is invariant with respect to reparameterisations, thus s is any arbitrary parameter along the worldline. Not fixing it in the beginning by freezing it to be the proper time is a good thing, since, first, we still do not know what proper time is, and, second, in this way we can treat together particles and light rays, on which proper time, i.e. the arclength affine parameter, is not defined.

Accordingly, we should state our golden rule for geodesics, which is

never solve the geodesic equation, always get information from the corresponding Lagrangian.

Kepler laws

Before going on, let us prove here Kepler laws in a classical Newtonian setting. This is pretty standard as a topic, though it is the template of what we shall do it later on to (fully and exactly) account for relativistic gravity. The point here is to rely on the Lagrangian description and to pretend that we do not know anything and let the math speak to us.

We assume a Lagrangian describing test particles in the form

$$L = \tfrac{m}{2} \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + \tfrac{b}{r} \tag{9.1.3}$$

We do not even assume *a priori* a specific form, or meaning, or even the sign for the model coefficient b.

We define a *bounded state* to be one which cannot reach $r \to \infty$, i.e. a motion confined in a region $r \in (0, r_+]$. We define an *orbital state* as a solution that is confined in a region $r \in [r_-, r_+]$ (which is not obvious, since we do not know what the coordinate r stands for, though the form of the kinetic energy should ring a bell to us).

Then we define a *(bounding) central mass*, (BCM) as something we can orbit around, i.e. something that allows initial conditions which lead to orbital states. Whatever a BCM is, it is encoded in the model coefficient b, thus the issue is to check which values of b correspond to a BCM.

The Lagrangian (9.1.3) has two first integrals

$$
\begin{cases}
E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{b}{r} \\[2mm]
K = mr^2\dot{\phi}
\end{cases}
\iff
\begin{cases}
\dot{r}^2 = \dfrac{2Er^2 + 2br - K^2}{mr^2} = \Phi(r) \\[3mm]
\dot{\phi} = \dfrac{K}{mr^2} \qquad \Rightarrow \left(\dfrac{dr}{d\phi}\right)^2 = \dfrac{mr^2}{K^2}\left(2Er^2 + 2br - K^2\right) = \Psi(r)
\end{cases}
\tag{9.1.4}
$$

both of which are Weierstrass equations, one for $t(r)$, one for the orbit $\phi(r)$. The conservation of K is equivalent to the *second Kepler law*.

These functions do contain all information we need, Kepler laws in particular, it is just a matter of digging them out. Unfortunately, or rather luckily, they depend on one model parameter b, as well as two state parameters (K, E), which are in fact functions of initial conditions. It just takes some time to discuss all possibilities.

Let us start by considering the case $b > 0$.

First of all, one should understand that general initial conditions $(r_0, \dot{r}_0, \phi_0, \dot{\phi}_0)$ do not generate all possible values of (E, K). For, first notice that the value of E is reduced every time we diminish $\dot{r}_0$, reaching a minimal value for $\dot{r}_0 = 0$, without changing the value of K. Moreover, it is invariant with respect to coordinate translations like $\phi' = \phi + \phi_0$, so that we can initially set $\phi = 0$, with no loss in generality. Finally, both the functions $\Phi(r)$ and $\Psi(r)$ depend on K^2, so that the sign of K makes no difference and we can study $K \geq 0$ only.

Let us set $r_0\dot{\phi}_0 = v_0$ and $K = r_0^2\dot{\phi}_0 = r_0 v_0$. Now, we keep the value of K fixed, by moving r_0 around and taking $v_0 = \frac{K}{r_0}$. That of course, changes the value of E as

$$
E = \frac{K^2}{2mr_0^2} - \frac{b}{r_0} \qquad \Rightarrow E' = \frac{-K^2 + mbr_0}{mr_0^3} = 0 \qquad \Rightarrow r_0 = \frac{K^2}{mb}, \quad E_0 = E(r_0) = \frac{K^2 m^2 b^2}{2mK^4} - \frac{mb^2}{K^2} = -\frac{mb^2}{2K^2}
\tag{9.1.5}
$$

and E_0 is the minimum value E can attend for a given value of K. Thus $E < -\frac{mb^2}{2K^2}$ cannot be attained for any initial conditions. These will soon appear again as forbidden regions in parameter space in the discussion of Weierstrass equation.

Most of what follows will be encoded in the ability to represent initial conditions differently. If, in general, initial conditions are $(r_0, \dot{r}_0, \phi_0, \dot{\phi}_0)$, the value of ϕ_0 can always be fixed at will because of the rotational invariance, e.g. it can always be set to $\phi_0 = 0$. Initial conditions can also be moved along the solution and such different initial conditions define the same trajectory in spacetime. Accordingly, initial conditions are defined up to a translation along the solution and there are actually two of them which are relevant.

For example, if one has $\dot{r}_0 = 0$ somewhere along the solution, initial conditions are given by $(r_0, \dot{\phi}_0)$, only. Alternatively, they correspond to a pair (E, K).

In some class of solutions, one can decide to represent initial condition differently. For example, in orbital states, one has two values of r, namely $r_\pm$, at which $\dot{r} = 0$ and, accordingly, one can decide to use $r_\pm$ as initial conditions. This is done by noticing that

$$
\begin{cases}
E = \frac{mv_+^2}{2} - \frac{b}{r_+} = \frac{mv_-^2}{2} - \frac{b}{r_-} \quad \Rightarrow v_-^2 = \frac{2b}{m}\left(\frac{r_+ - r_-}{r_- r_+}\right)\frac{r_+^2}{r_+^2 - r_-^2} = \frac{2b}{m}\left(\frac{1}{r_-}\right)\frac{r_+}{r_+ + r_-} \quad \Rightarrow E = -\frac{b}{r_+ + r_-} \\[3mm]
K = mr_+ v_+ = mr_- v_- \qquad \Rightarrow v_+ = \frac{r_-}{r_+}v_- \qquad \Rightarrow v_+^2 = \frac{2b}{m}\left(\frac{1}{r_+}\right)\frac{r_-}{r_+ + r_-} \qquad \Rightarrow K^2 = 2mb\frac{r_+ r_-}{r_+ + r_-}
\end{cases}
\Rightarrow
\tag{9.1.6}
$$

Thus we can use $r_\pm$ to encode initial conditions and one has

$$
\Phi(r) = -\frac{2b}{mr^2(r_+ + r_-)}(r - r_+)(r - r_-) \qquad\qquad \Psi(r) = -\frac{r^2}{r_+ r_-}(r - r_+)(r - r_-)
\tag{9.1.7}
$$

The function $\Phi(r)$ has the limit $\lim_{r \to +\infty} \Phi(r) = \frac{2E}{m} = -\frac{2b}{m(r_+ + r_-)}$. Thus bounded states are for $E < 0$, while $E \geq 0$ are unbounded.

If $K > 0$, then the limit to $r \to 0^+$ is driven by it to $\lim_{r \to 0^+} \Phi(r) = -\infty$. The maximum of Φ is reached at

$$\Phi'(r) = 2\frac{-br + K^2}{r^3} \qquad \Rightarrow r_* = \frac{K^2}{b} \qquad \Rightarrow \Phi(r_*) = 2E + \frac{2b^2}{K^2} - \frac{K^2 b^2}{K^4} = 2E + \frac{b^2}{K^2} \tag{9.1.8}$$

Accordingly, if $E < -\frac{b^2}{2K^2}$, there is no allowed region since $\Phi(r)$ is always negative. If $-\frac{b^2}{2K^2} < E < 0$, the allowed region is $r \in (r_-, r_+)$ and the function $\Phi(r)$ is in the form

$$\Phi(r) = -\frac{2b}{m} \frac{(r - r_-)(r - r_+)}{r^2(r_+ + r_-)} \tag{9.1.9}$$

If $K = 0$, then the limit to $r \to 0^+$ is driven by b, in this case to $\lim_{r \to 0^+} \Phi(r) = +\infty$ and one has the allowed region $r \in (0, r_+)$.

Hence we can list these cases:

i) $b > 0$, $E < 0$, $K = 0$

The Weierstrass function $\Phi(r)$ is

$$\Phi(r) = -\frac{2b}{m} \frac{r - r_+}{r_+ r} \qquad \left(r_+ := -\frac{b}{E}\right) \tag{9.1.10}$$

The allowed region is $r \in (0, r_+)$. The point $r = r_+$ is a reflection point, the point $r = 0$ is reached in a finite time. Thus the test particle radially falls in to $r = 0$ in a finite time.

ii) $b > 0$, $-\frac{b^2}{2K^2} < E < 0$, $K > 0$

The Weierstrass functions are

$$\Phi(r) = -\frac{2b}{m} \frac{1}{r^2(r_+ + r_-)}(r - r_+)(r - r_-) \qquad \Psi(r) = -\frac{r^2}{r_+ r_-}(r - r_+)(r - r_-) \tag{9.1.11}$$

The allowed region is $r \in (r_-, r_+)$ and both $r = r_\pm$ are reflection points. Going from r_- to r_+ corresponds to incrementing ϕ by

$$\int_{r_-}^{r_+} \frac{dr}{\sqrt{\Psi(r)}} = \pi \tag{9.1.12}$$

Thus the test particles go along closed curves, which, if you want, is part of the *first Kepler law*.

The equation of an ellipse centred in a focus is

$$r = \frac{b^2}{a - c\cos(\phi)} \qquad c^2 = a^2 - b^2 \tag{9.1.13}$$

where a is the semi-major axis and b is the semi-minor axis.

One can express $b^2 = a^2 - c^2$, $2a = r_+ + r_-$, and $2c = r_+ - r_-$ to obtain

$$r = \frac{(r_+ + r_-)^2 - (r_+ - r_-)^2}{4(a - c\cos(\phi))} = \frac{2r_+ r_-}{r_+ + r_- - (r_+ - r_-)\cos(\phi)} \tag{9.1.14}$$

and verify directly that it satisfies the orbit equation $\left(\frac{dr}{d\phi}\right)^2 = \Psi(r)$.

The period of the orbit is

$$T = 2 \int_{r_-}^{r_+} \frac{dr}{\sqrt{\Phi(r)}} = \pi \sqrt{\frac{m}{b}} \left(\frac{r_+ + r_-}{2}\right)^{3/2} \propto \left(\frac{r_+ + r_-}{2}\right)^{3/2} \tag{9.1.15}$$

which is the *third Kepler law*, since $r_+ + r_- = 2a$ is the (double of the) semi-major axis.

The energies $E < -\frac{b^2}{2K^2}$ cannot be attained. For any $b > 0$ there are ortital states, for example all the ones with $-\frac{b^2}{2K^2} < E < 0$ and $K > 0$, hence the solution has a BCM.

If $E \geq 0$, then the limit $\lim_{r \to +\infty} \Phi(r)$ is positive (if $E = 0$, the limit is driven by b and we are assuming $b > 0$), the allowed region extends to infinity, and one has unbounded states. Still there are initial conditions (the one with $E < 0$) which corresponds to orbital states, so they still have a BCM. Having a CBM is something related to the model parameter (b in this case), not to a specific solution represented by specific initial conditions, or by specific values of first integrals.

The reader should practice and show that, in the case $b > 0$ and $E \geq 0$, the orbits are still conics, in particular hyperbolas for $E > 0$ and parabolas for $E = 0$.

The case $b = 0$: for $E \leq 0$, the function

$$\Phi(r) = \frac{2Er^2 - K^2}{mr^2} \tag{9.1.16}$$

is always negative (no allowed region). For $E > 0$, it is positive in $r \in \left(\frac{K}{\sqrt{2E}}, +\infty\right)$. In any case, there is no bounded states. So the case $b = 0$ has no BCM.

Finally, let us consider the cases with $b < 0$.

The function

$$\Phi(r) = \frac{2Er^2 + 2br - K^2}{mr^2} \tag{9.1.17}$$

goes to $\frac{2E}{m}$ for $r \to +\infty$ and, for $r \to 0$, to $-\infty$.

Thus it might have orbital states for $E < 0$, if the function $\Phi(r)$ became positive somewhere. However, the maximum value for it is $r_* = \frac{K}{b} < 0$; there is no allowed regions for $E < 0$, hence no orbital state.

For $E \geq 0$, one has unbounded state since the allowed region extends to $r_+ \to +\infty$

If $K = 0$ then

$$\Phi(r) = \frac{2Er + 2b}{mr^2} r \tag{9.1.18}$$

and again no orbital states.

Of course, we can go through the same discussion, much shorter in this case, based on physical intuition. For $b = 0$, there is no potential, the test particle is free to move on a plane, so no bounded states. For $b < 0$, the potential is repulsive, so no bounded states.

This analysis, however, relies on the assumption that we know what a plane is, how a free test particle is expected to move on it, on concepts as force and potential which have no real counterpart in relativity. Of course, also in relativity, they are *approximately* true in some context, when the Newtonian approximation is reasonable, still that is precisely what one should prove; it is precisely what one proves getting the expected results without using physical intuition.

By that, we do not mean that one should always avoid physical intuition. What we do say is that one should avoid it at least once in the beginning to prove that it is reliable in the considered context.

In this Chapter we shall go for describing geodesic trajectories in different geometries. We will go as far as we can without introducing clocks which will be introduced in the next Chapter.

2. Spherically symmetric solutions

On a static spherically symmetric geometry, the metric will be in the form

$$\tilde{g} = -A(r)\, dt^2 + B(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{9.2.1}$$

as we shown in Section 8.2.3.

In a Palatini $f(\mathcal{R})$-theory, one does not even need to worry if g or $\tilde{g}$ are spherically symmetric. As long as the conformal factor $\varphi(r)$ depends just on r, then $\tilde{g}$ is spherically symmetric iff $g = \varphi^{-1}\tilde{g}$ is.

In fact, we have

$$g = -\varphi^{-1}(r)A(r)\, dt^2 + \varphi^{-1}(r)B(r)\, dr^2 + \varphi^{-1}(r)r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{9.2.2}$$

and then define $\rho^2 = \varphi^{-1}(r)r^2$, hence having

$$d\rho^2 = \frac{\left(2\varphi - r\varphi'\right)^2}{4\varphi^3}\, dr^2 \tag{9.2.3}$$

Thus the metric can be recast as

$$g = -\varphi^{-1}(r)A(r)\, dt^2 + \frac{4B\varphi^2}{(2\varphi - r\varphi')^2}\, d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) = -\tilde{A}(\rho)\, dt^2 + \tilde{B}(\rho)\, d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{9.2.4}$$

where we set $\tilde{A}(\rho) = \varphi^{-1}(r)A(r)$ and $\tilde{B}(\rho) = \frac{4B(r)\varphi^2}{(2\varphi - r\varphi')^2}$. Hence g is spherically symmetric as well, though in the coordinates (t, ρ, θ, ϕ)

The Lagrangian for geodesic trajectories is

$$\boldsymbol{L} = \sqrt{A(r)\left(\frac{dt}{ds}\right)^2 - B(r)\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + \sin^2(\theta)\left(\frac{d\phi}{ds}\right)^2\right)}\, ds \tag{9.2.5}$$

For this Lagrangian, we can use t as a parameter along the trajectories, which recast the Lagrangian as

$$\boldsymbol{L} = \sqrt{A(r) - B(r)\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}\, dt \tag{9.2.6}$$

One can define the momenta

$$
\begin{cases}
p_r = \dfrac{-B\dot{r}}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}} \\[3ex]
p_\theta = \dfrac{-r^2\dot{\theta}}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}} \\[3ex]
p_\phi = \dfrac{-r^2\sin^2(\theta)\dot{\phi}}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}}
\end{cases}
\quad\Rightarrow 1 + \frac{p_r^2}{B} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)} = \frac{A}{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}
\tag{9.2.7}
$$

That corresponds to a Hamiltonian

$$
H = \frac{p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi}}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}} - \frac{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}} = \frac{-A}{\sqrt{A - B\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)}} = -\sqrt{A}\sqrt{1 + \frac{p_r^2}{B} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)}}
\tag{9.2.8}
$$

That corresponds to a Hamiltonian field X_H which can be restricted to the symplectic sub-manifold, $\theta = \pi/2$, $p_\theta = 0$ and it is tangent to this sub-manifold. In fact, one has

$$
\frac{\partial H}{\partial\theta} = \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\frac{\sqrt{A}}{\sqrt{1 + \frac{p_r^2}{B} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)}}}
\qquad
\frac{\partial H}{\partial p_\theta} = -\frac{\sqrt{A}\,p_\theta}{r^2\sqrt{1 + \frac{p_r^2}{B} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)}}}
\tag{9.2.9}
$$

which both vanish on the sub-manifold.

That means that the Hamiltonian flow restricts to the sub-manifold to be the Hamiltonian flow of the reduced Hamiltonian

$$
\hat{H} = -\sqrt{A}\sqrt{1 + \frac{p_r^2}{B} + \frac{p_\phi^2}{r^2}}
\tag{9.2.10}
$$

which corresponds to the reduced Lagrangian on the equatorial plane

$$
\hat{\boldsymbol{L}} = \sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2}\,dt
\tag{9.2.11}
$$

Thus we can consider the Lagrangian on the plane

$$
\hat{\boldsymbol{L}} = \sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}\,dt
\tag{9.2.12}
$$

where now the dots denote the derivatives with respect to t. That Lagrangian does not depend on t or ϕ thus it has two first integrals

$$
\begin{cases}
e = \dfrac{-B\dot{r}^2 - r^2\dot{\phi}^2}{\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2}} - \sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2} = \dfrac{-A}{\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2}} \\[3ex]
k = -\dfrac{r^2\dot{\phi}}{\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2}}
\end{cases}
\quad\Longleftrightarrow\quad
\begin{cases}
\dot{r}^2 = \dfrac{A}{B}\left(1 - \dfrac{A\cdot(r^2 + k^2)}{e^2 r^2}\right) \\[3ex]
\dot{\phi}^2 = \dfrac{A^2 k^2}{e^2 r^4}
\end{cases}
\tag{9.2.13}
$$

One can square the first integrals and solve for velocities to obtain

$$
\begin{cases}
e^2 B \dot{r}^2 + e^2 r^2 \dot{\phi}^2 = e^2 A - A^2 \\
k^2 B \dot{r}^2 + r^2 \left(r^2 + k^2 \right) \dot{\phi}^2 = k^2 A
\end{cases}
\Rightarrow
\begin{cases}
\dot{r}^2 = \dfrac{A}{B} \left(1 - \dfrac{A \cdot (r^2 + k^2)}{e^2 r^2} \right) =: \Phi(r) \\
\dot{\phi}^2 = \dfrac{A^2 k^2}{e^2 r^4}
\end{cases}
\tag{9.2.14}
$$

Then one has a Weierstrass equation depending only on r

$$
\dot{r}^2 = \frac{A}{B} \left(1 - \frac{A \cdot (r^2 + k^2)}{e^2 r^2} \right) = \Phi(r)
\tag{9.2.15}
$$

which is called the *radial equation*, and allows to solve for $r(t)$—actually, for $t(r)$ which gives us a parameterised representation of the inverse function $r(t)$. Once $r(t)$ is known one can obtain $\phi(t)$ from the second equation, or, alternatively, the equation for the *spatial trajectories*

$$
\left(\frac{dr}{d\phi} \right)^2 = \frac{e^2 r^4}{k^2 AB} \left(1 - \frac{A \cdot (r^2 + k^2)}{e^2 r^2} \right) =: \Psi(r)
\tag{9.2.16}
$$

which is again a Weierstrass equation.

> Let us stress that these are exact representations of geodesics, no approximation has been done of the original geodesic equations. We already discussed how Schwarzschild solution approaches Newtonian potential, though restricting to radial motions, and how it gives correction to it; see Subsection 3.5.3. Here we want to be more systematic and extend the claim to more general situations.

 The general discussion as we did for Kepler problem is quite tedious, though we can use those general techniques to go after orbital states, rotational curves, modifications of Kepler laws.

 Before going on, we can express initial conditions differently. Instead of using (E, K) we can consider initial conditions $(r = r_0, \dot{r}_0 = 0, \phi = \phi_0 = 0, \dot{\phi} = r_0 v_0)$, which would be more or less as shooting the test particle at radial *coordinate* r_0, in a direction orthogonal to the radial direction with speed v_0.

 Then, in this case, we have

$$
e^2 = \frac{A_0^2}{A_0 - v_0^2} \qquad\qquad k^2 = \frac{r_0^2 v_0^2}{A_0 - v_0^2}
\tag{9.2.17}
$$

where we set $A_0 := A(r_0)$. Then the Weierstrass functions can be recast in terms of initial conditions (r_0, v_0), the good thing is that we know that at $r = r_0$ one has $\dot{r}_0 = 0$, i.e. we know the function $\Phi(r)$ has a zero in r_0. Then the Weierstrass functions are now

$$
\Phi(r) = \frac{A}{B} \left(1 - \frac{A}{A_0} \left(1 - \frac{(r^2 - r_0^2) v_0^2}{A_0 r^2} \right) \right) \qquad \Psi(r) = \frac{A_0^2 r^4}{r_0^2 v_0^2 AB} \left(1 - \frac{A}{A_0} \left(1 - \frac{(r^2 - r_0^2) v_0^2}{A_0 r^2} \right) \right)
\tag{9.2.18}
$$

> Let us stress that here we did not even restricted to Schwarzschild, yet. We are considering geodesics in a otherwise general static spherically symmetric spacetime. That does not even need to be a solution of vacuum Einstein field equations.

Equatorial rotational curves

We know, by construction, that $\Phi(r_0) = 0$ (and $\Psi(r_0) = 0$). If we want the position to represent a circular motion, we need $r = r_0$ to be an isolated point in the allowed region and this is the case, if $\Phi'(r_0) = 0$ and $\Phi''(r_0) < 0$, so that $r = r_0$ corresponds to a maximum of the function $\Phi(r)$.

In particular the first derivative

$$\Phi'(r_0) = -\frac{A'(r_0)r_0 - 2v_0^2}{r_0 B(r_0)} = 0 \qquad \Longleftrightarrow \quad v_0^2 = \tfrac{1}{2}A'(r_0)r_0 \tag{9.2.19}$$

which, in fact, given the metric, *any* spherically symmetric metric, determines the initial speed v_0 to get a circular motion.

On the other hand, if one observes test particles on the equatorial plane and measures their angular speed v_0 (at aphelion or perihelion), the observed function $v_0(r)$ gives information about the metric coefficient $A(r)$, up to an additive constant.

For the second derivative, setting $v_0^2 = \tfrac{1}{2}A_0'r_0$, one has

$$\begin{aligned}
\Phi''(r_0) &= \frac{\left(-A_0 A_0'' r_0^2 + 4A_0'^2 r_0^2 - 3A_0 A_0' r_0 - 2A_0'^2 r_0^2\right)B_0 + 2\left(A_0 A_0' r_0^2 - A_0 A_0' r_0^2\right)B_0'}{r_0^2 A_0 B_0^2} = \\
&= -\frac{\left(A_0 A_0'' - 2A_0'^2\right)r_0 + 3A_0 A_0'}{r_0 A_0 B_0}
\end{aligned} \tag{9.2.20}$$

for which the sign is quite difficult to address in general. If we consider Schwarzschild solution, i.e. $A_0 = B_0^{-1} = 1 - \frac{\sigma}{r_0}$, $A_0' = \frac{\sigma}{r_0^2}$, and $A_0'' = -\frac{2\sigma}{r_0^3}$

$$\Phi''(r_0) = -\frac{\left(-\frac{r_0-\sigma}{r_0}\frac{2\sigma}{r_0^3} - 2\frac{\sigma^2}{r_0^4}\right)r_0 + 3\frac{r_0-\sigma}{r_0}\frac{\sigma}{r_0^2}}{r_0 A_0 B_0} = -\frac{-2(r_0-\sigma)\frac{\sigma}{r_0^3} - 2\frac{\sigma^2}{r_0^3} + 3(r_0-\sigma)\frac{\sigma}{r_0^3}}{r_0} = -\frac{(r_0 - 3\sigma)\sigma}{r_0^4} \tag{9.2.21}$$

which is, in fact, negative for $r_0 > 3\sigma$.

Fig 8.1: The $\Phi(r)$ function for a Schwarschild with $\sigma = 1$ for $r_0 = 4\sigma$ (red), $r_0 = 3\sigma$ (black), and $r = 2$ (blue)

> That suggests a structure for a cloud of test particles in Schwarzschild which is quite peculiar. If a test particle starts near the center it cannot go along a circular motion, which in fact exists only for $r > 2\sigma$. If the particle starts closer, namely $r < 3\sigma$, there is no velocity which keeps it orbiting along a circle, it falls in asymptotically to $r = \sigma$ or it escapes away (since $\Phi''(r_0) > 0$ and $\Phi(r)$ has a minimum at r_0). The circular motions for $r_0 < 3\sigma$ are unstable. Thus it seems reasonable to expect that any Schwarzschild black hole should appear surrounded by a desert disc which is of (coordinate) radius 3σ.
>
> On the contrary, for $r_0 > 3\sigma$ circular motions are stable and test particles can orbit stably.

Orbital states in Schwarzschild

If we go after orbital states in Schwarzschild, we can consider $A = B^{-1} = 1 - \frac{\sigma}{r}$, $\sigma > 0$, and initial conditions (r_0, v_0) as in the previous Subsection.

> The metric is degenerate at $r = \sigma$, thus we consider the region $(\sigma, +\infty)$. Here, accordingly, we redefine orbital states to be motions in the region $[r_-, r_+]$ with $r_- > \sigma$.

The Weierstrass functions are then

$$\Phi(r) = \frac{(r-\sigma)^2(r-r_0)}{(r_0-\sigma)^2 r^5}p(r) \qquad \Psi(r) = \frac{r(r-r_0)}{r_0^4 v_0^2}p(r) \tag{9.2.22}$$

where we set $p(r) := \left(v_0^2 r_0^2 - (r_0-\sigma)\sigma\right)r^2 + (r_0-\sigma)v_0^2 r_0^2 r - \sigma v_0^2 r_0^3$. The function $\Phi(r)$ has certainly a double zero in $r = \sigma$ (around which it is positive, since $p(\sigma) = -(r_0-\sigma)\sigma^3 < 0$ as well as $(\sigma - r_0) < 0$) and one in $r = r_0$. For $v_0 = 0$, the leading term has a negative coefficient, thus $\lim_{r\to+\infty}\Phi(r) < 0$. For a bounded state, we need to have

$$v_0^2 < \frac{(r_0-\sigma)\sigma}{r_0^2} =: v_{es}^2 \tag{9.2.23}$$

where $v_{es} > 0$ denotes the minimal speed for which the particle can escape to infinity.

For a orbital state, besides having $0 \le v_0 < v_{es}$, we also need $p(r)$ to have 2 more zeros in $(\sigma, +\infty)$.

If $p(r)$ has no real roots, the function $\Phi(r)$ is positive in (σ, r_0) and then negative in $(r_0, +\infty)$. The test particle falls into $r = \sigma$ and no solution is a orbital state.

There are two extra solutions iff

$$\Delta = (r_0-\sigma)^2 v_0^4 r_0^4 + 4\sigma v_0^4 r_0^5 - 4(r_0-\sigma)\sigma^2 v_0^2 r_0^3 = \left((r_0-\sigma)^2 v_0^2 r_0 + 4\sigma v_0^2 r_0^2 - 4(r_0-\sigma)\sigma^2\right)v_0^2 r_0^3$$

$$\Delta = v_0^2 r_0^3 \left((r_0+\sigma)^2 v_0^2 r_0 - 4(r_0-\sigma)\sigma^2\right) \ge 0 \qquad \Rightarrow v_0^2 \ge 4\frac{(r_0-\sigma)\sigma^2}{(r_0+\sigma)^2 r_0} = v_{es}^2 \frac{4r_0\sigma}{(r_0+\sigma)^2} = v_{or}^2 \tag{9.2.24}$$

where $v_{or} > 0$ is the orbital speed, i.e. the minimal speed at which the particle can orbit. Thus, for $0 \le v_0 < v_{or}$, there are no orbital states, all particles fall in.

The particle escapes before orbiting, i.e. $\Phi(r)$ gets two extra solutions, iff $v_{or} > v_{es}$, i.e.

$$\frac{4r_0\sigma}{(r_0+\sigma)^2} > 1 \qquad \Rightarrow 4r_0\sigma > r_0^2 + 2r_0\sigma + \sigma^2 \qquad \Rightarrow (r_0-\sigma)^2 < 0 \tag{9.2.25}$$

i.e. never.

Hence, one always has $0 \le v_0 < v_{or}$ which correspond to particles which necessarily fall in $r = \sigma$, $v_{or} \le v_0 < v_{es}$ which correspond to may orbital states, and $v_0 \ge v_{es}$ which correspond to particles which may escape to infinity.

For $v_0 = v_{or}$, the function $\Phi(r)$ gets a double extra zero at

$$r_* = \frac{(r_0-\sigma)v_0^2 r_0^2}{2\left(v_0^2 r_0^2 - (r_0-\sigma)\sigma\right)} = \frac{2\sigma r_0}{r_0-\sigma} < r_0 \quad \Rightarrow r_0(r_0-3\sigma) > 0 \quad \Rightarrow r_0 > 3\sigma \tag{9.2.26}$$

which, for $\sigma < r_0 < 3\sigma$, is $r_0 < r_*$, for $r_0 = 3\sigma$, is $r_* = r_0$ (which is then a triple zero) while, for $r_0 > 3\sigma$, is $\sigma < r_* < r_0$, in fact $2\sigma < r_* < r_0$.

One can easily check that $r_* = 3\sigma \iff r_0 = 3\sigma$, as well as $r_* > 3\sigma \iff \sigma < r_0 < 3\sigma$ (hence also $2\sigma < r_* < 3\sigma \iff r_0 > 3\sigma$).

For,

$$\frac{2\sigma r_0}{r_0-\sigma} > 2\sigma \qquad \Rightarrow 2\sigma r_0 > 2\sigma(r_0-\sigma) \qquad \Rightarrow 0 > -2\sigma^2 \quad \text{(always)} \tag{9.2.27}$$

and

$$\frac{2\sigma r_0}{r_0 - \sigma} > 3\sigma \qquad \Rightarrow 2\sigma r_0 > 3\sigma(r_0 - \sigma) \qquad \Rightarrow 0 > \sigma(r_0 - 3\sigma) \qquad \Longleftrightarrow r_0 < 3\sigma \tag{9.2.28}$$

Thus we have three cases:

a) $r_0 = 2$ b) $r_0 = 3$ c) $r_0 = 4$
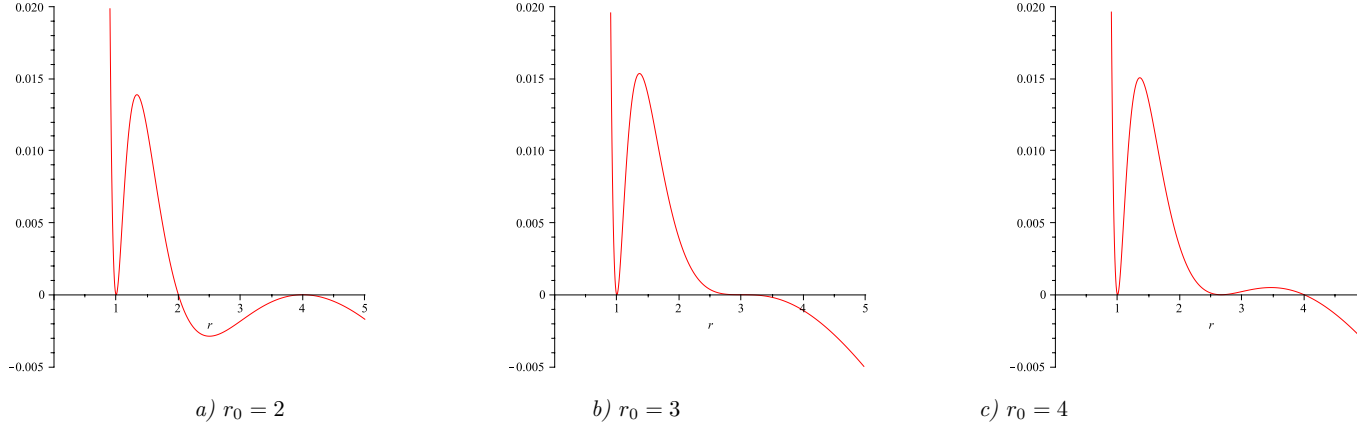
Fig 8.2: The Weierstrass function $\Phi(r)$ for $v_0 = v_{or}$

Accordingly, if $\sigma < r_0 < 3\sigma$ the particle at r_0 falls into, for $r_0 = 3\sigma$ we have an unstable circular orbit which tends to fall in, for $r_0 > 3\sigma$ it orbits in the region (r_*, r_0).

For $v_{or} < v_0 < v_{es}$, one may have orbital states. In these cases, the function $\Phi(r)$ has a double zero (and a minimum) in $r = \sigma$, 1 zero in $r = r_0$, and two more roots in $r = r_-$, and $r = r_+$, where

$$r_\pm = \frac{(r_0 - \sigma)v_0^2 r_0^2 \pm \sqrt{v_0^2 r_0^3 \left((r_0 + \sigma)^2 v_0^2 r_0 - 4(r_0 - \sigma)\sigma^2\right)}}{2\left(v_0^2 r_0^2 - (r_0 - \sigma)\sigma\right)} \tag{9.2.29}$$

One can show (with a quite lengthy though not too difficult computation) that $r_- \in (\sigma, r_0)$ iff $r_0 > 3\sigma$ and $v_{or} < v_0 < v_2$, where we set $v_3^2 = \frac{\sigma}{2r_0}$.

Analogously, $r_- \in (\sigma, r_0)$ in the following cases:

— $2\sigma < r_0 < 3\sigma$ and $v_3 < v_0 < v_{es}$

— $r_0 > 3\sigma$ and $v_{or} < v_0 < v_{es}$

Fig 8.3: Regions in the (r_0, v_0)-plane.

To summarise, in the regions

— $\sigma < r_0 < 2\sigma$ and $v_{or} < v_0 < v_{es}$

— $2\sigma < r_0 < 3\sigma$ and $v_{or} < v_0 < v_3$

there is no zero in (σ, r_0) and the particle falls in.

In the region

— $r_0 > 3\sigma$ and $v_{or} < v_0 < v_3$

there are two zeros, and the particle orbits in the region (r_+, r_0).

In the region

— $r_0 > 2\sigma$ and $v_3 < v_0 < v_{es}$

there is one zero and the particle orbits in the region (r_0, r_+).

For $v_0 = v_{es}$, the limit at infinity of the function $\Phi(r)$ becomes zero and the solutions may become unbounded. One root has escaped to infinity, so the polynomial has only one extra zero at $r_- = \frac{r_0}{r_0 - 1}\sigma$ which, for $r_0 > 2\sigma$, is in (σ, r_0), while, for $\sigma < r_0 < 2\sigma$, it is $r_- > r_0$. Then, in the first case, we get an unbounded solution in the region $(r_0, +\infty)$, while, in the second case, the particle falls in. One can easily check that, in the limit case $r = 2\sigma$ and $v_0 = v_{es}$, the root $r_- = r_0 = 2\sigma$ and one has a double root on r_0 as well, which corresponds to an unstable circular orbit.

Finally, we are left with the region $v_0 > v_{es}$ to be analysed. There, if $r_0 > 2\sigma$, there is one zero in the region (σ, r_0) and the particle escapes to infinity. If $\sigma < r_0 < 2\sigma$, there are two cases depending on the initial speed v_0:

— $\sigma < r_0 < 2\sigma$ and $v_{es} < v_0 < v_3$: there no zero in (σ, r_0) (though one zero in $(r_0, +\infty)$), and the particle falls in.

— $\sigma < r_0 < 2\sigma$ and $v_0 > v_3$, there is one zero in (σ, r_0) and the particle escapes to infinity.

One can see that, even in this relatively simple situation, the discussion is quite complicated. The *"relativistic effects"* have been accounted exactly and they appear to modify, also qualitatively, the Kepler situation.

For example, in Kepler, starting at any position r_0 one has a fall in for small speed v_0; then, after a limit v_{or} is reached, the particle can orbit until, finally, it escapes at infinity for $v_0 > v_{es}$. In Schwarzschild, this is not what happens for $\sigma < r_0 < 2\sigma$, where there is no orbit and one passes directly from falling in to escaping to infinity.

In Kepler, the orbits always start with perihelion closed to the ground $r = \sigma$, it grows until the orbit becomes circular, then the role of perihelion and aphelion is exchanged and the aphelion grows to infinity. After that the solution becomes unbounded and the particle escapes to infinity. This is not what happens in Schwarzschild, where, in the region $2\sigma < r_0 < 3\sigma$ orbits form when they are circular. There is no orbit with a perihelion lower than r_0.

Finally, in the region $r_0 > 3\sigma$, when the first orbital state is formed (i.e. for the minimal speed value $v_0 = v_{or}$), its perihelion is

$$r_- = \frac{2\sigma r_0}{r_0 - \sigma} \tag{9.2.30}$$

Thus, even for very big radius $r_0 \gg 3\sigma$, one cannot get a perihelion smaller that 2σ, unlike what happens in Kepler, where, even considering a finite central planet of radius $r = \sigma$, one can get orbits closed at will to the planet surface, i.e. as closed as one wishes to $r = \sigma$.

However, if one stays away from the BCM, say $r_0 \gg 3\sigma$ *and* $v_0 \gg v_{or}$ (so that, $r_- \gg 3\sigma$ as well, the particle is not expected to go near the BCM nowhere along its motion), then the two situations are, at least qualitatively, comparable. That is the parameter region where we expect to be able to discuss the Newtonian limit of the relativistic theory.

Orbital states in Kepler

Before going on, we can consider the Schwarzschild case as a template and repeat the same computation for Kepler. Instead of focusing on Kepler laws, this time we focus on qualitative study of orbital states.

The function $\Phi(r)$ is

$$\Phi(r) = \frac{2er^2 + 2\beta r - k^2}{r^2} \tag{9.2.31}$$

where we set $E = me$, $K = mk$, and $b = m\beta$. As in Schwarzschild case, we can impose initial conditions $(r = r_0, \dot{r} = 0, \phi = 0, \dot{\phi} = \frac{v_0}{r_0})$ and we have

$$e = \tfrac{1}{2}v_0^2 - \frac{\beta}{r_0} \qquad\qquad k = r_0 v_0 \qquad\qquad \Rightarrow \Phi(r) = \frac{r - r0}{r_0 r^2}\left(\left(v_0^2 r_0 - 2\beta\right)r + r_0^2 v_0^2\right) \tag{9.2.32}$$

which always has two zeros, namely r_0 and $r_* = \frac{r_0^2 v_0^2}{2\beta - v_0^2 r_0}$.

The limit at infinity of $\Phi(r)$

$$\lim_{r \to +\infty} \Phi(r) = \frac{2\beta - v_0^2 r_0}{r_0} > 0 \qquad\qquad \Rightarrow v_0^2 = \frac{2\beta}{r_0} = v_{es}^2 \tag{9.2.33}$$

Fig 8.4: Regions for Kepler in the (r_0, v_0)-plane.

Let us consider a planet of radius σ, so that we say that a particle falls in every time it hits the sphere $r = \sigma < 1$.

If we set $v_1^2 = \frac{\beta}{r_0} = \frac{1}{2}v_{es}^2$ and $v_2^2 = \frac{\sigma}{\sigma+1}v_{es}^2 < v_1^2$, then we have that $r_* \in (\sigma, r_0)$ iff $v_2^2 < v_0^2 < v_1^2$.

Thus for $0 \le v_0 < v_2$ the particle falls in, for $v_1 < v_0 < v_2$ it orbits (and r_0 is at aphelion), for $v_2 < v_0 < v_{es}$ it orbits (and r_0 is at perielion), for $v_0 > v_{es}$, r_* becomes negative and the particle escapes to infinity.

Accordingly, in Kepler, the situation is much simpler than in Schwarzschild: the regions are shown in Fig. 8.x, while the behaviours of the Weierstrass function Φ at $r_0 = 2$ for different speed v_0^2 are shown in Fig. 8.x.

Fig 8.5: The Weierstrass function Φ for different speeds.

Kepler laws in Schwarzschild

Talkin' bout a Revolution (Tracy Chapman)

Let us consider an orbital state in a Schwarzschild spacetime, for example in the region $r \in [r_-, r_+]$. As we did in Kepler, we can use $r_\pm$ as a representation of initial condition and the Weierstrass functions are

$$\Phi(r) = \frac{((\sigma r_+ - r_+ r_- + \sigma r_1)r + \sigma r_+ r_-)(r - r_+)(r - r_-)(r - \sigma)^2}{r^5 (r_+ + r_-)(r_+ - \sigma)(r_- - \sigma)} \qquad \Psi(r) = \frac{r(r - r_+)(r - r_-)((\sigma r_+ - r_+ r_- + \sigma r_-)r + \sigma r_+ r_-)}{r_+^2 r_-^2} \tag{9.2.34}$$

The second Kepler law can be stated as: *the momentum conjugated to the angle ϕ is conserved.*

As such, that is still exactly true in a Schwarzschild spacetime.

For the third Kepler law, we can consider the Weierstrass function $\Phi(r)$, make the transformation $r = a\rho$, $r_- = a(1 - \epsilon)$, $r_+ = a(1 + \epsilon)$, so that the Weierstrass function for ρ is obtained as

$$\left(\frac{d\rho}{dt}\right)^2 = \hat{\Phi}(\rho) = \frac{((2\sigma + a\epsilon^2 - a)\rho + \sigma(1 - \epsilon^2))(\rho - 1 - \epsilon)(\rho - 1 + \epsilon)(a\rho - \sigma)^2}{2a^4\rho^5(a(1 + \epsilon) - \sigma)(a(1 - \epsilon) - \sigma)} \tag{9.2.35}$$

Since we expect Kepler law to hold true away from the BCM, we can expand this function in Taylor series around $a = +\infty$, obtaining

$$\hat{\Phi}(\rho) = \frac{1}{2}\frac{(\rho - 1 + \epsilon)\sigma(1 + \epsilon - \rho)}{\rho^2 a^3} - \frac{3}{2}\frac{(\rho - 1 + \epsilon)\sigma^2(1 + \epsilon - \rho)}{\rho^3 a^4} + O(a^{-5}) \tag{9.2.36}$$

The first term of the series is exactly the Weierstrass function for the classical Kepler problem (with the substitutions $r = a\rho$, $r_- = a(1 - \epsilon)$, $r_+ = a(1 + \epsilon)$ and setting $\sigma = 2\beta$).

The other terms of the series are dominated by the first term and are (small) corrections to the classical Kepler problem to account for GR. Due to corrections, the third Kepler law holds approximately away form the BCM, and it fails as we get near it.

We can play the same game with the first Kepler law and the other Weierstrass function $\Psi(r)$. We make the same substitutions and expand in Taylor series at $a = +\infty$ and obtain

$$\hat{\Psi}(\rho) = \frac{\rho^2(1 + \epsilon - \rho)(\rho - 1 + \epsilon)}{1 - \epsilon^2} + \frac{(1 + \epsilon - \rho)(\rho - 1 + \epsilon)(-2\rho - 1 + \epsilon^2)\rho}{(1 - \epsilon^2)^2 a} + O(a^2) \tag{9.2.37}$$

Again, the first term of the series agrees with the classical Kepler problem, the rest of the series accounts for GR corrections. In this case, the corrections spoils the value of the integral that in Kepler was equal to π. Consequently, the orbit in GR are not closed any longer. The corrections measure by how much the orbits fail to be closed and their integral is the angle by which semi-major axis *precesses*. As we shall see, this effect turns out to be observable for the orbits of Mercury.

Dust rings in Schwarzschild

We can imagine to spray the equatorial plane in a Schwarzschild spacetime with test particles in all possible initial conditions.

> To be true the situation is ill posed since it understands that the initial conditions of test particles are uniformly sampled. The intuitive realisation of that is that a certain volume in phase space has a probability of being hit by a test particle which is proportional to its volume.
> This raises two issues: which metric should I use to compute the volume in phase space (and by that we mean that, first, our metric is on spacetime, not on the phase space, and, second, why should I use the same metric I used in dynamics to define what *uniform* means) and how should I normalise volumes in phase space which is clearly non-compact.
> To be true, there is nothing like a canonical uniform spray of test particles, we just pretend there is.

Then we wait for a while and check what is left.

In $\sigma < r_0 < 2\sigma$, all particles there (or that can reach that region) have fell into or escaped to infinity. That region is thus empty.

In $2\sigma < r_0 < 3\sigma$, some of the particles have fell into, some escaped to infinity. What is left are some particles orbiting, all of which have a part of the orbit farer than $r > 3\sigma$.

In $r_0 > 3\sigma$, some of the particles have fell into, some escaped to infinity. What is left are some particle orbiting. Some of them stay away of the limit $r > 3\sigma$. Some get into the region $2\sigma < r < 3\sigma$ to escape out for a while.

If we have a reason to put some bound on the eccentricity of the orbits (which changes the original notion of *uniform spray*) the number of particles in each region, in particular in $2\sigma < r_0 < 3\sigma$, is affected. For example, if we restrict to circular motions, the region $2\sigma < r_0 < 3\sigma$ is completely deserted.

If this were a model for accretion discs, which is not since particles in accretion discs are interacting and often emitting radiations, we see that accretion discs have a hole at the center which depends on the parameter σ.

Nothing like that in Kepler, were rings around a body have no hole at all in the center.

> Of course, by now, there is no direct observation of a Schwarzschild black hole and its accretion disc. But that is another story.

Light rays in Schwarzschild

According to EPS, light rays can be chased by particles as closed as one wishes. Hence we can regard them as a limit of fast particles. When a particle approaches the speed to light, the quantity $A - B\dot{r}^2 - r^2\dot{\phi}^2 \to 0$ approaches zero. Consequently, $e \to -\infty$ and k diverge as well. Then we cannot use these first integrals for light rays, though we can use their ratio, which in fact stays finite

$$\frac{k}{e} = \frac{r^2\dot{\phi}}{A} \qquad \Rightarrow \epsilon^2 = \frac{k^2}{e^2} = \frac{r^4\dot{\phi}^2}{A^2} \qquad \Rightarrow \dot{\phi}^2 = \frac{A^2\epsilon^2}{r^4} \tag{9.2.38}$$

where we define $\epsilon > 0$ to be positive (rather than defining it as $\frac{k}{e}$), for later convenience.

Moreover, we also have the constraint $A = B\dot{r}^2 + r^2\dot{\phi}^2 = B\dot{r}^2 + \frac{A^2\epsilon^2}{r^2}$ to be satisfied. Hence we have the two Weierstrass equations

$$\dot{r}^2 = \frac{A}{r^2B}\left(r^2 - \epsilon^2 A\right) =: \Phi(r) \qquad \dot{\phi}^2 = \frac{A^2\epsilon^2}{r^4} \qquad \Rightarrow \left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{\epsilon^2 AB}\left(r^2 - \epsilon^2 A\right) =: \Psi(r) \tag{9.2.39}$$

which in fact describe light rays.

> Let us remark how one can obtain these equations from the Weierstrass equations of a particle by taking the limit $\epsilon \to -\infty$ and $k \to \infty$, though keeping the ration $\frac{k}{e} = \pm\epsilon$ constant.
>
> Also notice that equations for light rays depend on one parameter only, namely ϵ, as a consequence of the fact that light speed is fixed. A light ray is uniquely determined by the initial position and spatial direction.

In Schwarzschild, we obtain

$$\Phi(r) = \frac{(r - \sigma)^2}{r^5}\left(r^3 - \epsilon^2 r + \epsilon^2\sigma\right) \qquad \Psi(r) = \epsilon^2 r\left(r^3 - \epsilon^2 r + \epsilon^2\sigma\right) \tag{9.2.40}$$

As for particles, the function $\Phi(r)$ has a double zero (and a minimum) at $r = \sigma$. Other zeros are driven by the polynomial $p(r) = r^3 - \epsilon^2 r + \epsilon^2 \sigma$.

The polynomial $p(r)$ is positive at $r = \sigma$, in fact

$$p(\sigma) = \sigma^3 - \epsilon^2 \sigma + \epsilon^2 \sigma = \sigma^3 > 0 \tag{9.2.41}$$

The limits at infinity of the polynomial $p(r)$ are

$$\lim_{r \to +\infty} p(r) = +\infty \qquad \qquad \lim_{r \to -\infty} p(r) = -\infty \tag{9.2.42}$$

and $p(0) = \sigma > 0$. Accordingly, $p(r)$ always has a negative root r_0. It also has two positive roots $r_\pm$ for small values of ϵ.

In fact, the derivative $p'(r) = 3r^2 - \epsilon^2$ has two zeros, namely $r_*^2 = \frac{\epsilon^2}{3}$, one positive r_*, one negative $-r_*$. Thus $p(r)$ has a minimum at r_* and its minimum value is

$$p(r_*) = \left(r_*^2 - \epsilon^2\right) r_* + \epsilon^2 \sigma = -\frac{2\epsilon - 3\sqrt{3}\sigma}{3\sqrt{3}} \epsilon^2 > 0 \qquad \Rightarrow \epsilon < \frac{3\sqrt{3}}{2}\sigma \tag{9.2.43}$$

Hence for $\epsilon < \frac{3\sqrt{3}}{2}\sigma$, then $p(r)$ has no positive roots (the light ray coming from $r \to \infty$ is capture by the BCM), for $\epsilon = \frac{3\sqrt{3}}{2}\sigma$ a double root is formed at $r_* = \frac{3}{2}\sigma$ (the light ray has an asymptotic circular motion at $r = r_*$), and it has two positive roots $r_\pm$ for $\epsilon > \frac{3\sqrt{3}}{2}\sigma$ (the light ray approaches the BCM, it reaches a minimal radius $r = r_+$, and it is scattered away).

Since the function $\Phi(r)$ depends on one single parameter ϵ, it means that one can recast everything to be a function of r_+ instead of ϵ. At $r = r_+$, the function $\Phi(r)$ has a zero, so one obtains

$$\epsilon^2 = \frac{r_+^3}{r_+ - \sigma} \tag{9.2.44}$$

and the Weierstrass functions can be recast as

$$\Phi(r) = \frac{(r-\sigma)^2}{r^5}(r - r_+)\left(r^2 + r_+ r - \frac{r_+^2 \sigma}{r_+ - \sigma}\right) \qquad \Psi(r) = \frac{(r_+ - \sigma)r}{r_+^3}(r - r_+)\left(r^2 + r_+ r - \frac{r_+^2 \sigma}{(r_+ - \sigma)}\right) \tag{9.2.45}$$

Once we fix r_+, the other roots are frozen to

$$r_- = \frac{-r_+ + \sigma + \sqrt{r_+^2 + 2r_+\sigma - 3\sigma^2}}{2(r_+ - \sigma)} r_+ \qquad r_0 = \frac{-r_+ + \sigma - \sqrt{r_+^2 + 2r_+\sigma - 3\sigma^2}}{2(r_+ - \sigma)} r_+ \tag{9.2.46}$$

Fig 8.6: Light deflection (divided by π).

If we consider $r_+ \in \left(\frac{3}{2}\sigma, +\infty\right)$, then $r_- \in \left(\sigma, \frac{3}{2}\sigma\right)$, $r_0 < -3\sigma$, and r_+ is the greatest positive root. A light ray, approaching the BCM from $r \to +\infty$, gets to the minimal radius $r = r_+$ and then it is scattered back.

If the light ray comes at an angular coordinate ϕ_0, it leaves at an angular coordinate ϕ_1, then

$$\Delta\phi := \phi_1 - \phi_0 = 2\int_{r_+}^{+\infty} \frac{dr}{\sqrt{\Psi(r)}} \tag{9.2.47}$$

The result is shown in Fig. 8.6.

Also in this case, the light rays which approach the BCM nearer than certain radius, in this case $r = \frac{3}{2}\sigma$, are captured and do not escape to infinity or to our eyes. This may mean that we see a Schwarzschild black hole surrounded by a black sphere which is actually of that radius. However, before claiming it, we should first discuss what r has to do with a distance, as well as we should discuss that what we actually see when the light rays reach our eye (or instrument). The result of this process depends on what happens during the scattering, as well as at emission and during whole the travel of the light.

We have to point out that $\Delta\phi$ is a coordinate quantity and sooner or later one should discuss what it has to do with what one measures with a goniometer, possibly at a single point. The same occurs for distances and times and their relations with the corresponding coordinate quantities.

3. Stationary axisymmetric solutions

For a stationary (not necessarily static) axisymmetric spacetime, the metric is

$$g = -\,A(r,\theta)\,dt^2 + C(r,\theta)\,(\,dt \otimes\,d\phi +\,d\phi \otimes\,dt) + B(r,\theta)\,dr^2 + r^2\left(\,d\theta^2 + \sin^2(\theta)\,d\phi^2\right) \tag{9.3.1}$$

The static case is obtained by setting $C = 0$, which however reduces to the previous case.

> We discussed the general form of a stationary, axisymmetric metric to be (8.2.22), which is in pseudo-cylindric coordinates.
> Starting from that form, we can change coordinates $(z = \hat{r}\cos(\theta), \rho = \hat{r}\sin(\theta))$ and recast the metric in the form
>
> $$g = -A\,dt^2 + C\,(\,dt \otimes\,d\phi +\,d\phi \otimes\,dt) + B\,d\phi^2 + D\boldsymbol{d\hat{r}^2} + D\hat{r}^2\,d\theta^2 \tag{9.3.2}$$
>
> Then we can define
> Check if the functions can depend on θ, which however disappera on equatorial plane.

The Lagrangian for it, on the equatorial plane, is

$$L = \sqrt{A\left(\frac{dt}{ds}\right)^2 - 2C\left(\frac{dt}{ds}\right)\left(\frac{d\phi}{ds}\right) - B\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2}\;ds = \sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2 - 2C(r)\dot{\phi}}\;dt \tag{9.3.3}$$

where we set $A(r) := A\left(r, \frac{\pi}{2}\right)$, $B(r) := B\left(r, \frac{\pi}{2}\right)$, and $C(r) := C\left(r, \frac{\pi}{2}\right)$.

Both the coordinates (t, ϕ) are cyclic so that one has two first integrals

$$e = \frac{-A + C\dot{\phi}}{\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2 - 2C\dot{\phi}}} \qquad\qquad k = \frac{-r^2\dot{\phi} - C}{\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2 - 2C\dot{\phi}}} \tag{9.3.4}$$

This can be solved for the velocities

$$\dot{\phi} = \frac{kA - eC}{kC + r^2 e} \qquad \dot{r}^2 = -\frac{\left(r^2 A + C^2\right)\left((r^2 + k^2)(A - e^2) + (ke - C)^2\right)}{B(kC + r^2 e)^2} = \Phi(r; e, k) \tag{9.3.5}$$

For initial conditions $(r(0) = r_0, \dot{r}(0) = 0, \phi(0) = 0, r_0\dot{\phi}(0) = v_0)$, then one has

$$\begin{cases} k = \dfrac{r_0^2 v_0 + r_0 C_0}{v_0 C_0 - r_0 A_0} e = \alpha_0 e \qquad \alpha_0 := \dfrac{r_0 v_0 + C_0}{r_0 A_0 - v_0 C_0} r_0 \\[2ex] e^2 = \dfrac{r_0^2 A_0 + C_0^2}{r_0^2 - \alpha_0(\alpha_0 A_0 - 2C_0)} \end{cases} \tag{9.3.6}$$

We can check that, for $C = 0$, we obtain the results for spherically symmetric cases. In fact, in this case

$$\alpha_0 = \frac{r_0 v_0}{A_0} \tag{9.3.7}$$

and

$$e = -\frac{\sqrt{r_0^2 A_0}}{\sqrt{r_0^2 - \frac{r_0^2 v_0^2}{A_0^2} A_0}} = -\frac{A_0}{\sqrt{A_0 - v_0^2}} \qquad k = -\frac{r_0 v_0}{\sqrt{A_0 - v_0^2}} \tag{9.3.8}$$

Thus we can recast the Weierstrass function in terms of the initial conditions (r_0, v_0) and, on circular motions, one has $\Phi(r_{;0} r_0, v_0) = 0$, $\Phi'(r_0; r_0, v_0) = 0$. One clearly has

$$\Phi(r_0; r_0, v_0) = -\frac{\left(r_0^2 A_0 + C_0^2\right)\left((r_0^2 + k^2)(A_0 - e^2) + (ke - C_0)^2\right)}{B_0(kC_0 + r_0^2 e)^2} = -\frac{\left(r_0^2 A_0 + C_0^2\right)\left((r_0^2 + \alpha_0^2 e^2)(A_0 - e^2) + (\alpha_0 e^2 - C_0)^2\right)}{B_0 e^2(\alpha_0 C_0 + r_0^2)^2} = 0 \tag{9.3.9}$$

since, at initial conditions, $\dot{r}_0^2 = \Phi(r_0; r_0, v_0) = 0$.

Of course, one can double check it directly as

$$\begin{aligned} (r_0^2 + \alpha_0^2 e^2)(A_0 - e^2) + (\alpha_0 e^2 - C_0)^2 &= r_0^2 A_0 - r_0^2 e^2 + \alpha_0^2 e^2 A_0 - \textcolor{red}{\alpha_0^2 e^4} + \textcolor{red}{\alpha_0^2 e^4} + C_0^2 - 2\alpha_0 e^2 C_0 = \\ &= r_0^2 A_0 + C_0^2 - \left(r_0^2 - \alpha_0^2 A_0 + 2\alpha_0 C_0\right)\frac{r_0^2 A_0 + C_0^2}{r_0^2 - \alpha_0(\alpha_0 A_0 - 2C_0)} = 0 \end{aligned} \tag{9.3.10}$$

The other condition $\Phi'(r_0; r_0, v_0) = 0$ reads as

$$\Phi'(r_0; r_0, v_0) = \frac{2v_0^2 + 2C_0' v_0 - A_0' r_0}{B_0 r_0} \tag{9.3.11}$$

Thus the velocity profile is found as a solution of

$$2v_0^2 + 2C_0' v_0 - A_0' r_0 = 0 \qquad \Rightarrow v_0 = \frac{1}{2}\left(-C_0' \pm \sqrt{C_0'^2 + 2A_0' r_0}\right) \tag{9.3.12}$$

Notice how this result is not symmetric: test particles orbit at a different speed depending on whether they are co-rotating or counter-rotating (i.e. depending on the sign of C) and symmetry is recovered for $C = 0$.

By observing test particles along circular orbits in an stationary axisymmetric spacetime, we can determine A and C (up to an additive constant). In order to determine B without using field equations, one needs to observe more test particles.

4. Spatially homogeneous and isotropic solutions

Searching geodesics in spatially homogeneous and isotropic spacetimes is trickier, if not for other reasons, because FLRW metrics depend on time through a function, the scale factor $a(t)$, which is known only after one solves Friedmann equations. The Lagrangian for geodesics depends on time, the generalised energy is not conserved, thus, apparently, no Weierstrass analysis.

However we can exploit the invariance under reparameterisations to do something anyway.

In a spatially homogenous and isotropic spacetime, the metric can be written in FLRW form, and one can change coordinates to define $r = S(\chi; k)$, where we set $\sqrt{|k|}S(\chi; k) = \sinh(\sqrt{|k|}\chi)$ if $k < 0$, $\chi = S(\chi; k)$ if $k = 0$, and $\sqrt{|k|}S(\chi; k) = \sin(\sqrt{|k|}\chi)$ if $k > 0$. In coordinates (t, χ, θ, ϕ), defined as in Subsection 8.2.6, the metric then reads as

$$g = -dt^2 + a^2(t)\left(d\chi^2 + S^2(\chi; k)\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right) \tag{9.4.1}$$

The Lagrangian for radial time-like geodesics reads as

$$\boldsymbol{L} = \sqrt{(t')^2 - a^2(\chi')^2}\,ds \tag{9.4.2}$$

Ok, one should check that the restriction to radial directions is consistent, which is quite well supported by spatial isotropy.

This Lagrangian has a cyclic coordinate χ, thus the conjugate momentum

$$j = \frac{-a^2\chi'}{\sqrt{(t')^2 - a^2(\chi')^2}} \tag{9.4.3}$$

is conserved, as well as it is invariant with respect to reparameterisations.

One can also decide to use χ as a parameter along the worldlines and use the Lagrangian

$$\hat{\boldsymbol{L}} = \sqrt{\left(\frac{dt}{d\chi}\right)^2 - a^2}\,d\chi \tag{9.4.4}$$

instead. This does not depend explicitly on χ so its generalised energy is conserved, i.e.

$$\hat{E} = \frac{\left(\frac{dt}{d\chi}\right)^2}{\sqrt{\left(\frac{dt}{d\chi}\right)^2 - a^2}} - \sqrt{\left(\frac{dt}{d\chi}\right)^2 - a^2} = \frac{a^2}{\sqrt{\left(\frac{dt}{d\chi}\right)^2 - a^2}} = -j \tag{9.4.5}$$

Thus, either way, there is not much of a difference.

In particular, we know there is a class of parameterisations, the affine ones, for which the geodesic trajectory is in fact a geodesic motion. In (a subclass of) such parameterisations, one also has $(t')^2 - a^2(\chi')^2 = 1$.

Then one has

$$
\begin{cases}
\chi' = -\dfrac{j}{a^2} \\[2mm]
(t')^2 = 1 + a^2(\chi')^2 = 1 + \dfrac{j^2}{a^2} \qquad \Rightarrow t' = \dfrac{\sqrt{a^2 + j^2}}{a}
\end{cases}
\qquad \Rightarrow \dot\chi^2 = \frac{j^2}{a^4}\frac{a^2}{a^2 + j^2} = \frac{j^2}{a^2(a^2 + j^2)} = \frac{1}{\Phi(t)}
\tag{9.4.6}
$$

which is, in fact, a Weierstrass equation. By separation, one directly has

$$
\chi(t) - \chi_0 = \pm \int_{t_0}^t \frac{J\,dt}{a\sqrt{a^2 + J^2}}
\tag{9.4.7}
$$

where $J \geq 0$ denotes $|j|$. The sign in front of the integral depends on whether we are considering radial geodesics going to infinity or to the origin.

In particular, for $J = 0$, we obtain $\chi(t) = \chi_0$, which is always (for any scale factor $a(t)$) a geodesic motion; such trajectories are called *comoving particles*.

Unlike comoving particles, the other radial time-like geodesics bring information about the evolution of the scale factor $a(t)$. Accordingly, observing the radial motion of a test particle (for a significant part of the age of the universe) gives us a description of the evolution of the scale factor. That is, of course, beyond our observational ability, however, it provides us with a way of observing scale factor in principle.

We can try to play the same game with a light ray

$$
\begin{cases}
\chi' = -\dfrac{j}{a^2} \\[2mm]
(t')^2 = a^2(\chi')^2 = \dfrac{J^2}{a^2} \qquad \Rightarrow t' = \dfrac{J}{a}
\end{cases}
\qquad \Rightarrow \dot\chi^2 = \frac{J^2}{a^4}\frac{a^2}{J^2} = \frac{1}{a^2} \qquad \Rightarrow \chi(t) - \chi_0 = \pm \int_{t_0}^t \frac{dt}{a}
\tag{9.4.8}
$$

Also in this case, knowing the geodesics of a radial light ray, gives a description of the evolution of the scale factor. We have in fact many candidates to light rays which travelled all along the visible universe (from the light of very far objects, to cosmic background radiation, to cosmological cosmic rays). Unfortunately, we have a very limited control, if any, on their motion between emission and observation.

We have also to stress that, if we consider a projection $p : M \to \Sigma$ on the space manifolds $\Sigma_{t_0} = (t = t_0)$, which are Euclidean manifolds with the induced metric, the projection of a geodesic is not necessarily a geodesic. to be discussed

5. Symplectic methods for geodesics

As we already discussed above, some of the Lagrangians for time-like geodesics are in fact non degenerate and they allow a Hamiltonian formalism. Of course, the general Lagrangian invariant under reparameterisations is not.

> Besides seeing it directly, if it were non-degenerate and allowed a Hamiltonian formulation, geodesic equations in Hamiltonian form would be a first order system in normal form and Cauchy theorem would apply to them. That, by the hole argument, directly contradicts parameterisation invariance.

However, if we decide to use t as a parameter, the corresponding Lagrangian is non-degenerate and it induces a standard Hamiltonian system. As in any standard Hamiltonian system, the evolution is a canonical flow, the generating function of which is a solution of the corresponding Hamilton-Jacobi (HJ) equation. Such a generating function describes geodesics, in the sense that one can write it as a function of the position and the initial position (type I generating function) and that computes the initial and final momenta of the geodesics passing through the two positions.

The computation involved may be difficult, though it formally defines an objects which again contains the whole information about geodesics. In any event, one could guess that, at least on Minkowski, the situation should be easy to be computed explicitly. We shall illustrate its definition and use for Schwarzschild spacetimes, when needed specialising to Minkowski spacetimes.

Hamiltonian formalism for geodesics

Let us consider a 2 dimensional (radial) Schwarzschild spacetime, i.e. with coordinates (t, r) and the metric

$$g = -A(r)\, dt^2 + \frac{dr^2}{A(r)} \tag{9.5.1}$$

which defines the Lagrangian for time-like geodescis

$$\boldsymbol{L} = \sqrt{A(r)\left(\frac{dt}{ds}\right)^2 - \frac{1}{A(r)}\left(\frac{dr}{ds}\right)^2}\, ds = \sqrt{A(r) - \frac{\dot{r}^2}{A(r)}}\, dt \tag{9.5.2}$$

where the dots denote derivatives with respect to t.

That Lagrangianis has one degree of freedom and it reduces to the Minkowski case by choosing $A = 1$.

The momentum associated to the Lagrangian (9.5.2) is

$$p = \frac{\partial L}{\partial \dot{r}} = -\frac{\dot{r}}{\sqrt{A(A^2 - \dot{r}^2)}} \qquad \Longleftrightarrow \qquad \dot{r}^2 = -\frac{pA\sqrt{A}}{\sqrt{1 + Ap^2}} \tag{9.5.3}$$

which corresponds to the Hamiltonian

$$H = -\sqrt{A}\sqrt{1 + Ap^2} \tag{9.5.4}$$

The corresponding (reduced) HJ equation is

$$-\sqrt{A}\sqrt{1 + A\left(W'\right)^2} = E \qquad \Longleftrightarrow \qquad W' = \mp\frac{\sqrt{E^2 - A}}{A} \tag{9.5.5}$$

where prime denotes the derivative with respect to r. The complete integral for HJ is hence

$$S(t, r; E) = -Et \mp \int \frac{\sqrt{E^2 - A}}{A} dr \qquad (9.5.6)$$

The evolution of the system is hence written as

$$p = \frac{\partial S}{\partial r} \qquad\qquad P = -\frac{\partial S}{\partial E} = t \pm \frac{\partial W}{\partial E} \qquad (9.5.7)$$

Of course, that is equivalent to know the general solution of the geodesic equations, since one knows P is a first integral, hence it is constant, hence the corresponding equation can be used to solve for $r(t; E, P)$, hence the first equation gives $p(t)$.

However, this is still not what we were looking for, since the generating function S is a function of E, not of the initial position r_0.

The evolution generator

For our later convenience, we would like to have it expressed as a function of the initial position, i.e.

$$F = S(t, r) - S(t_0, r_0) = -E \cdot (t - t_0) \mp \int_{r_0}^{r} \frac{\sqrt{E^2 - A}}{A} dr \qquad (9.5.8)$$

which will be called the *evolution generator* (once we eliminate E); a similar (different though equivalent) object has been introduced by Synge and called the *world function*.

The evolution generator contains the information for the general solutions of Hamilton equations, i.e. general geodesic trajectories. In fact, one has

$$-\frac{\partial F}{\partial E} = -\frac{\partial S}{\partial E}(t, r) + \frac{\partial S}{\partial E}(t_0, r_0) = P - P_0 = 0 \qquad (9.5.9)$$

which is zero, since the momentum P conjugate to E is conserved, being S a solution of HJ equation.

In principle, one could use this equation to obtain $E(t, r; t_0, r_0)$ and replace it above to obtain the evolution generator $F(t, r; t_0, r_0)$.

Once the evolution generator has been determined, then if we want to determine the geodesic trajectory passing through (t, r) and (t_0, r_0) we can compute

$$\begin{cases} p = \dfrac{\partial F}{\partial r} \\[2mm] p_0 = -\dfrac{\partial F}{\partial r_0} \end{cases} \qquad (9.5.10)$$

where p_0 is the initial momentum to be selected so that the geodesics will eventually pass through (t, r), while p is the momentum when it arrives at (t, r). Of course, one can use the inverse Legendre transform (9.5.3) to obtain the initial and final velocities.

Accordingly, the flow of transformations $\Phi_{t-t_0} : (t_0, r_0) \mapsto (t, r)$ is canonical and it completely describes the geodesic flow.

One can use this method to obtain the geodesics in Minkowski space (setting $A = 1$), this time without resorting to the affine structure but using the manifold structure only.

For Minkowski, we have

$$S(t, r; E) = -Et \mp \int \sqrt{E^2 - 1} dr = -Et \mp \sqrt{E^2 - 1}\, r \tag{9.5.11}$$

and hence

$$F = -E(t - t_0) \mp \sqrt{E^2 - 1}(r - r_0) \qquad \Rightarrow \frac{\partial F}{\partial E} = -(t - t_0) \mp \frac{E}{\sqrt{E^2 - 1}}(r - r_0) = 0 \qquad \Rightarrow E = -\frac{|t - t_0|}{\sqrt{(t - t_0)^2 - (r - r_0)^2}} \tag{9.5.12}$$

Accordingly, for Minkowski, the evolution generator is

$$F(t, r; t_0, r_0) = \frac{(t - t_0)^2}{\sqrt{(t - t_0)^2 - (r - r_0)^2}} - \frac{(r - r_0)^2}{\sqrt{(t - t_0)^2 - (r - r_0)^2}} = \sqrt{(t - t_0)^2 - (r - r_0)^2} \tag{9.5.13}$$

where, for simplicity, we considered $t > t_0$ and $r > r_0$, i.e. we are shooting the particle away from the origin, i.e. the radial speed $\dot{r}$ is positive so we choose the upper sign for W.

Thus, if we want the geodesics which connects (t_0, r_0) to (t, r), we have to choose

$$p_0 = -\frac{\partial F}{\partial r_0} = -\frac{(r - r_0)}{\sqrt{(t - t_0)^2 - (r - r_0)^2}} = -\frac{v}{\sqrt{1 - v^2}} \qquad\qquad p = \frac{\partial F}{\partial r} = \frac{-(r - r_0)}{\sqrt{(t - t_0)^2 - (r - r_0)^2}} = p_0 \tag{9.5.14}$$

where we set $v = \frac{r - r_0}{t - t_0}$.

Thus we see that the momentum (or the radial velocity) is constant, at it coincides with the velocity needed to connect the points with a straight line.

In the Schwarzschild case Maple can analytically solve the integral (9.5.8), but the resulting equation (9.5.9) turns out to be too complicated to be solved for E, so that we need to learn how to go around this issue. For our Schwarzschild–like solution, i.e. for $A = 1 - \frac{\sigma}{r}$, we can introduce an adimensional variable $r = \sigma\rho$ to obtain

$$F(t, r; t_0, r_0) = -E \cdot (t - t_0) \mp \sigma \int_{\rho_0 = \frac{r_0}{\sigma}}^{\rho = \frac{r}{\sigma}} \frac{\sqrt{\rho}\sqrt{(E^2 - 1)\rho + 1}}{\rho - 1} d\rho \tag{9.5.15}$$

The limit to light-like geodesics is obtained by the limit $\dot{r} \to \pm A(r)$, which corresponds to $p \to \mp\infty$, which, in turn, corresponds to the limit $E \to -\infty$.

Thus, for light rays, we are interested to the solutions of (9.5.9) which diverge to $-\infty$. If we are interested in light rays from (t_0, r_0) to (t, r), we want to determine a solution which, for these values, has the corresponding $E(t, r; t_0, r_0)$ which diverges.

Even though the explicit form of $E(t, r; t_0, r_0)$ is hard to find we can make the substitution $E = 1/\epsilon$ in the equation (9.5.9) and then take the limit $\epsilon \to 0$.

In the Schwarzschild case, we directly obtain for (9.5.9)

$$(t - t_0) \mp \left(r - r_0 + 2m \ln\left(\frac{r - 2m}{r_0 - 2m} \right) \right) + O(\epsilon) = 0 \tag{9.5.16}$$

the two signs corresponding to ingoing and outgoing light rays. This allows a divergent solution (i.e. $\epsilon = 0$) iff

$$t - t_0 = \pm\left(r - r_0 + \sigma \ln\left(\frac{r - \sigma}{r_0 - \sigma} \right) \right) \tag{9.5.17}$$

Once we fix the initial condition (t_0, r_0), this provides an implicit definition $t(r)$ of the light-like geodesics trajectories through it, parameterised by r. Thus, in view of separation of variables, HJ method provides us with an exact, analytical, description of light rays as a result of a single integral.

Moreover, before taking the limit to $E \to -\infty$, this is also a good description of test particles. For $-1 < E \leq 0$ one has bounded motions, while for $E \leq -1$ one has unbounded motions.

The bounded motions have a maximal distance they reach before falling in again. This is obtained by conservation of E as the value of $r = r_M$ such that

$$E^2 = A(r_M) \tag{9.5.18}$$

and then one has directly the two branches of the motion as

$$\dot{r}^2 = A^2 \left(1 - \frac{A}{E^2}\right) \qquad \Longleftrightarrow \qquad t - t_M = \mp E \int_{r_M}^{r} \frac{dr}{\sqrt{A}\sqrt{E^2 - A}} \tag{9.5.19}$$

This suggests to use r as a parameter along each branch and in fact it allows us to keep the result analytic and exact.

The unbound motions can be either ingoing or outgoing. In both cases, one can use the parameter r along the whole motion.

If we consider two test particles which exchange light rays one with the other, we can describe the situation exactly; see Fig. 8.7. In this way we are able to find *exactly* the points $(p_0, p_1, p_2, p_3, \dots)$ at which the signals are fired by one particle to the other.

In the example in the Figure, one particle is out going along an unbound trajectory, the other is bounded, it reaches a maximin radius and then falls in again. While they are going around they exchange light rays back and forth, which iare drawn as red lines.
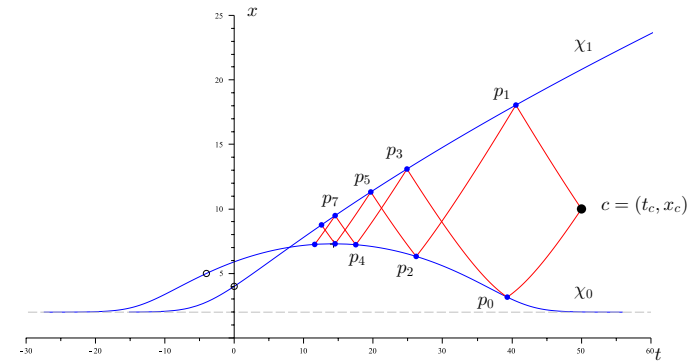
Fig. 8.7: Particles exchanging light rays around Schwarzschild.

6. Special coordinates

In a relativistic theory, coordinates are irrelevant. Using one coordinate system or another is *a priori* irrelevant. On the other hand, exactly because we focus on quantities which are coordinate independent, if we compute such quantities in a specific coordinate system, the result applies to all the other coordinate systems as well.

As we already discussed, on a specific spacetime, there might be *a posteriori* privileged coordinate systems which make our life easier. For example, as discussed in Section 8.2, in spacetimes with Killing symmetries, one can compute *adapted coordinates*, in which the metric takes a specific and simpler form.

However, we can also show that there are *a posteriori* preferred coordinate systems in *any* spacetime. They are still *a posteriori* in the sense that, to define them in details, one needs to know the metric (or the connection—or both, of course) though their general existence is established by non-constructive existence theorems, as, for example, Cauchy theorem.

Moreover, as we shall discuss later on, these coordinate systems are physically important (and at the same time *fundamentally irrelevant*) since they are used as a bridge towards Newtonian setting. They render what it means that, locally in spacetime, one can pretend that the spacetime and its tangent space are the same thing, i.e. one can pretend that (locally) spacetime is a vector space, as it (globally) was in Newton setting. They render what is means that geodesics are straight lines as Galileo claimed. Moreover, they are the proper arena to discuss *equivalence principle*.

> How can something be important for Newtonian physical intuition and, at the same time, fundamentally irrelevant?
> The only answer I can think of is that Newtonian physical intuition itself is fundamentally irrelevant as well.

Normal coordinates

Let us consider a point x in a spacetime $(M, g, \tilde{\Gamma})$ described by a Weyl frame and let us denote by $V = T_x M$ the tangent space to M at x. By an abuse of notation, let us denote by g the inner product induced on V by the metric, so that (V, g) is a (pseudo)-Euclidean vector space.

For any vector $v \in V$, we can define a $\tilde{\Gamma}$-geodesic trajectory $\gamma(s; x, v)$ with initial conditions (x, v). Keeping x fixed and letting v varyimg, we obtain the family $\mathfrak{G}_x$ of all geodesics through x.

In general, this family does all sort of bad things; geodesics can intersect with each other, fail covering some regions of spacetime, different initial conditions can define the same trajectory. However, there is a suitable neighbourhood U_x in which the trajectories foliates U_x.

Even restricting to U_x, two parallel vectors v and αv do still define the same trajectory.

> That is a consequence of reparameterisation invariance. In a nutshell, it amounts to show that if $\gamma : \mathbb{R} \to M$ is a $\tilde{\Gamma}$-geodesic motion for the initial condition (x, v), then $\tilde{\gamma}(s) = \gamma(\alpha s)$ is again a $\tilde{\Gamma}$-geodesic motion, though for the initial condition $(x, \alpha v)$. See Section 8.2.

Accordingly, the geodesics generated by the initial conditions (x, v) and $(x, \alpha v)$ are in fact the same sub-manifold in M. Thus one can use α to adjust which point along the trajectory corresponds to $\tilde{\gamma}(1)$.

If we fix a basis e_μ in V, any vector $v = v^\mu e_\mu \in V$ corresponds to its components v^μ. Thus one has a correspondence between points in U_x and initial conditions (x, v), the correspondence being

$$\exp : V \subset \mathbb{R}^m \to U_x : v \mapsto \gamma(1; x, v) \tag{9.6.1}$$

which, once restricted to a suitable open set $V \subset \mathbb{R}^m$, is one-to-one. That is, in fact, the inverse of a chart on spacetime and it defines coordinates v^μ in U_x which are called *normal coordinates*.

Normal coordinates v^μ at a point x are one possible realisation of the intuitive idea that, in a small enough region, any manifold can be approximately confused with its tangent space, i.e. they are an explanation of why we can approximately think to live on a flat world even though we are actually living on a sphere. They are overloaded with some properties of the curvature, so that the metric, the connection, and the curvature have a special simple form.

In normal coordinates, let us consider a geodesic through x, namely $\gamma(s; x, w)$. Any point y on this geodesic, e.g. $y = \gamma(s_0; x, w)$, can equivalently be written as $y = \tilde{\gamma}(1; x, \alpha w)$ by setting $\alpha = s_0$. Thus, the point y in normal coordinates is written as αw^μ and the geodesic reads as

$$\gamma : \mathbb{R} \to M : s \mapsto v^\mu = sw^\mu \tag{9.6.2}$$

for some fixed w^μ. Hence geodesics in normal coordinates are straight lines.

Let us remark that *"being a straight line"* is a jargon expression and it can have different meanings.

On a manifold, it means to be geodesics with respect to some fixed geometry. That is already ambiguous, unless one specifies the details of the kind of geometry used. For example, in a Weyl frame $(M, g, \tilde{\Gamma})$ one has geodesics of the metric g as well as geodesics of the connection $\tilde{\Gamma}$. It is only in view of EPS that we decided to understand that what matters are geodesics of $\tilde{\Gamma}$. Of course, in this case, saying that γ *"is a geodesic of $\tilde{\Gamma}$"* is as true as obvious.

In a vector (or affine) space, being a straight line means to be linear, to be associated to a subspace. The geodesic γ *is* associated with a linear space in $V \subset \mathbb{R}^m$, though unfortunately, what happens in $\mathbb{R}^m$ is irrelevant, what matters is what happens on the spacetime M. There is no linearity on M, since the map defining the chart does not preserve linearity.

What we have here is that geodesics are *linear in the coordinate space*, which is not a property of the line; it is a property of the coordinate system, instead. To be honest, one should say that geodesics are simple *in normal coordinates*, which just mean that one should ask where the information about geodesics is hidden, the answer being in the definition of normal coordinates. Precisely, what is hard to reconstruct, precisely as hard as solving the geodesic equations, is to find the relation between normal coordinates and some other coordinate system.

Since we know that $\gamma : \mathbb{R} \to M : s \mapsto v^\mu = sw^\mu$ are geodesics, they have to obey geodesic equation. Of course, in normal coordinates, one has $\dot{\gamma} = w^\mu \partial_\mu$ and $\ddot{\gamma} = 0$. Thus one has

$$\tilde{\Gamma}^\mu_{\alpha\beta} w^\alpha w^\beta = 0 \tag{9.6.3}$$

which at x must hold for any w^μ. Accordingly, in normal coordinates, one has $\tilde{\Gamma}^\mu_{\alpha\beta}(x) = 0$.

Of course, the geodesic equations are satisfied for any value of s; however, for $s \neq 0$, the curves with different values of w^μ are not geodesics, and one cannot conclude that for y along a geodesics, namely $y = \gamma(s)$, the connection should be vanishing as well.

Normal coordinates are coordinates on M such that the connection $\tilde{\Gamma}(x) = 0$ at the point $x \in M$.

We can also consider the derivative of the geodesic equations, always at x, from which $\partial_\lambda \tilde{\Gamma}^\mu_{\alpha\beta} w^\lambda w^\alpha w^\beta = 0$ follows and, in view of independence of w, one gets

$$3\partial_{(\lambda} \tilde{\Gamma}^\mu_{\alpha\beta)} = \partial_\lambda \tilde{\Gamma}^\mu_{\alpha\beta} + \partial_\alpha \tilde{\Gamma}^\mu_{\beta\lambda} + \partial_\beta \tilde{\Gamma}^\mu_{\lambda\alpha} = 0 \tag{9.6.4}$$

Another free token from normal coordinates is that covariant derivatives at x reduce to partial derivatives, and, in particular, they commute.

As a consequence of $\tilde{\Gamma}(x) = 0$, the Riemann tensor at x is simply

$$\tilde{R}^\alpha_{\beta\mu\nu} = \partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_\nu \tilde{\Gamma}^\alpha_{\beta\mu} = 2\partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} + \partial_\beta \tilde{\Gamma}^\alpha_{\mu\nu} \qquad \Rightarrow \partial_\nu \tilde{\Gamma}^\alpha_{\beta\mu} = -\left(\partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} + \partial_\beta \tilde{\Gamma}^\alpha_{\mu\nu}\right) = -\tfrac{1}{3}\left(\tilde{R}^\alpha_{\beta\mu\nu} + \tilde{R}^\alpha_{\mu\beta\nu}\right) \tag{9.6.5}$$

which is particularly easy since it involves no quadratic terms.

For example, if one wants to prove first Bianchi identities, we could start by noticing that $\tilde{R}^\alpha_{[\beta\mu\nu]}$ is a tensor, so if it vanishes at a point in one coordinate system, it vanishes there in any coordinates.

Then we can fix a point x and normal coordinates around it. In those coordinates, the Riemann tensor is expressed simply as in equation (9.6.5). Consequently, one has

$$\tilde{R}^{\alpha}_{[\beta\mu\nu]} = \partial_{[\mu}\tilde{\Gamma}^{\alpha}_{\beta\nu]} - \partial_{[\nu}\tilde{\Gamma}^{\alpha}_{\beta\mu]} = 0 \qquad (9.6.6)$$

which proves first Bianchi identities at x in normal coordinates, hence first Bianchi identities at x in any coordinates. Since the point x has been fixed arbitrarily, first Bianchi identities hold true everywhere.

Notice that, once we assume that normal coordinates exist, this is much a shorter proof than the one in general coordinates (compare with Subsection 16.2.6), essentially because we do not need to care about quadratic terms in Riemann tensor expression.

[Notice also that one should learn to prove the identity in general, just using basic tensor calculus.]

If v^{μ} are normal coordinates at x, then also $w^{\mu} = J^{\mu}_{\nu}v^{\nu}$ are normal coordinates at x, where J^{μ}_{ν} is a constant matrix. We can further use this freedom to put the metric g in normal form at x, i.e. to have $g_{\mu\nu}(x) = \eta_{\mu\nu}$. That holds true in any Weyl frame $(M, g, \tilde{\Gamma})$: one can always find a system of normal coordinates at $x \in M$ such that $\tilde{\Gamma}^{\alpha}_{\beta\mu}(x) = 0$ and the metric is in canonical form $g_{\mu\nu}(x) = \eta_{\mu\nu}$. That is *exactly* true at x, while is is approximately true in a small enough neighbourhood U_x. That is the first step to show that the physics in M is locally approximately described as the physics on V, on which the metric is constant and the connection vanishes in the same coordinates in which the metric is in canonical form. That is very similar to the setting one has in SR, so often one says that locally on M, special relativity holds.

It is important to remark that locally means *approximately in a neighbourhood, exactly at the point x*. If one had observations precise enough, then one could see that the metric fails to be exactly in canonical form around x, as well as the connection will fail to be exactly constant. As we shall see below that appears as tidal forces as soon as one considers two particles which are not exactly at x. Moreover, that discussion is in *spacetime* and x is an event, not a point in space. If we consider a particle γ through x, SR is expected to hold exactly at the event x, which it becomes approximate even if one stays exactly on the particle γ though at time before or after x. A better discussion needs us to introduce clocks, distances and better bridge the relativistic setting with the Newtonian one.

Fermi coordinates

Normal coordinates provide a canonical expression for the metric and connection at an event $x \in M$. However, (sometimes) one can do better and provide coordinates in which the metric and connection are in canonical form along a whole worldline of a particle, not at a single event. These coordinates are called *Fermi coordinates* and they exist every time $\tilde{\Gamma} = \{g\}$, i.e. on a Riemannian structure.

Fermi coordinates are a better implementation of the intuitive idea that SR exactly holds true at a spatial point (e.g. at the center of mass of a freely falling lift) instead of holding true at a given time only. That is often considered a milestone for GR interpretation and it is interesting to notice that it holds true only in a Riemannian structure. Most of what will follow in the next Chapters will be to verify whether this assumption is really needed (thus potentially ruling out Palatini $f(\mathcal{R})$-theories) or one can learn to live without it (thus defining a more general framework for relativistic theories).

Let us start by considering a Weyl frame $(M, g, \tilde{\Gamma})$ on a spacetime M of dimension $m = \dim(M)$ with a torsionless connection $\tilde{\Gamma}$.

To define Fermi coordinates around a time-like geodesic γ for the connection $\tilde{\Gamma}$, we start by selecting a complement subspace W_0 transverse to $\dot{\gamma}$ at $\gamma(0)$. Let us fix a basis e_i of W_0 (with $i = 1..(m-1)$), and by parallel transport we can define $e_i(s)$ along the curve γ such that $e_i(0) = e_i$. We can define the subspaces $W_s = \text{Span}(e_i(s))$ which stay transverse to γ for a while, i.e. in an interval $J \subset I$ which contains $0 \in J$. There is no loss of generality to restrict I from the beginning so that $I = J$.

For any vector $w \in W_s$, there is a unique geodesic motion $h(r) = h(r; \gamma(s), w)$ for the initial condition $h(0) = \gamma(s)$ and $\dot{h}(0) = w$. If we start from a vector $w' = \alpha w$ which is a parallel to the initial condition w (i.e. $\alpha \in \mathbb{R}$ is a constant), then we simply end up with a reparameterisation of the same line. In fact, the curve $k(r') := h(\alpha r')$ is again a geodesic motion and it corresponds to initial conditions

$$k(0) = h(0) = \gamma(s) \qquad \dot{k}(0) = \alpha \dot{h}(0) = \alpha w = w' \tag{9.6.7}$$

In other words, one has $h(r; \gamma(s), \alpha w) = h(\alpha r; \gamma(s), w)$. Accordingly, one can fix the initial condition w' so that a point x on this geodesic is obtained for the parameter $r = 1$, i.e. $x = h(1; \gamma(s), w)$.

By dimensional considerations, the family of geodesic motions h fills an open neighbourhood U of $\gamma(0)$ which can be restricted so that none of such geodesics intersect. Accordingly, any point $x \in U$ in the neighbourhood can be (uniquely) written as $x = h(1; \gamma(s), w)$ and, since $w = w^i e_i(s) \in V_s$, it can be parameterised by (s, w^i) which are called *Fermi coordinates*.

In Fermi coordinates, the curve γ reads as $\gamma : s \mapsto (s, 0, \ldots, 0)$ while the geodesics $h(r; \gamma(s), w)$ read as $h(r; \gamma(s), w) : r \mapsto (s, rw^i)$. Choosing $w^i = \delta^i_k$ one obtains coordinate curves associated to Fermi coordinates and the natural basis for vectors are tangent to them.

Hence, in Fermi coordinates, the equation for geodesics and parallel transport must be satisfied, in particular at $\gamma(s)$. The curve $\gamma : s \mapsto (s, 0, \ldots, 0)$ is a geodesic, thus $\tilde{\Gamma}^\alpha_{00}(\gamma(s)) = 0$. The curve $h(r; \gamma(s), e_k(s)) : r \mapsto (s, r\delta^i_k)$ is a geodesic, thus $\tilde{\Gamma}^\alpha_{ij}(\gamma(s)) = 0$. Finally, the vector fields $e_i(s) = \delta^\mu_i \partial_\mu$ are parallelly transported along $\gamma(s)$, i.e. $\tilde{\Gamma}^\mu_{0i}(\gamma(s)) = 0$.

Overall, in Fermi coordinates the coefficients $\tilde{\Gamma}^\mu_{\alpha\beta}$ of the torsionless connection $\tilde{\Gamma}$ vanish along the geodesic γ. Of course, any affine coordinate transformation $x'^\mu = A^\mu_\nu x^\nu + B^\mu$ for constant (A^μ_ν, B^μ) preserves the vanishing of the connection on γ.

We can find a coordinate system x^μ in which the connection vanishes on γ, i.e. $\tilde{\Gamma}^\mu_{\alpha\beta}(\gamma(s)) = 0$. If we single out a point $x = \gamma(0)$ on the time-like geodesic, then we can make a coordinate change $x'^\mu = A^\mu_\nu x^\nu$ to put $g(x)$ in canonical form, i.e. $g'_{\mu\nu}(x) = \eta_{\mu\nu}$.

If we try to repeat the procedure at any point along γ then the Jacobian A^μ_ν would in general depend on $s = x^0$, the transformation would not be linear, and the non-zero Hessian would eventually spoil the vanishing of the connection.

Of course, the fact that the procedure for constructing Fermi coordinates fails, it does not prove that there is no other coordinate system, defined in some other way, in which the conditions for Fermi coordinates hold true γ, anyway. In order to show that Fermi condition is in fact violated, one needs a counterexample.

Let us consider the spacetime $M = \mathbb{R}^m$ with a standard Minkowski metric $\tilde{g} = \eta$ in Cartesian coordinates x^μ and set $\tilde{\Gamma} = \{\tilde{g}\}$ be its Levi Civita connection. Since we are in Cartesian coordinates, one has $\tilde{g} = \eta$ and $\tilde{\Gamma} = 0$ everywhere, in particular on the time-like geodesic $\gamma : \mathbb{R} \to M : s \mapsto (s, 0, \ldots, 0)$.

Then we can consider a conformal metric $g = \bar{\varphi}\tilde{g}$ and we can ask if there exists a coordinate system x'^μ, say in a neighbourhood of the origin, in which $g(s) = \eta$ and $\tilde{\Gamma}(s) = 0$ on γ. Assuming analytical transition functions, we can expand

$$x'^\mu = x'^\mu(x) = A^\mu + A^\mu_\alpha x^\alpha + \tfrac{1}{2} A^\mu_{\alpha\beta} x^\alpha x^\beta + \ldots \tag{9.6.8}$$

with constant coefficients.

Let us first determine the coordinates in which $\tilde{\Gamma} = 0$ on γ is preserved. The expression of γ in the new coordinates is

$$\gamma'^\mu(s) = x'^\mu(\gamma(s)) = A^\mu + A^\mu_0 s + \tfrac{1}{2} A^\mu_{00} s^2 + \ldots \tag{9.6.9}$$

The Jacobian is

$$J_\alpha^\mu = A_\alpha^\mu + A_{\alpha\beta}^\mu x^\beta + \tfrac{1}{2} A_{\alpha\beta\gamma}^\mu x^\beta x^\gamma + \dots \tag{9.6.10}$$

which, on γ, reduces to

$$J_\alpha^\mu(s) = A_\alpha^\mu + A_{\alpha0}^\mu s + \tfrac{1}{2} A_{\alpha00}^\mu s^2 + \dots \tag{9.6.11}$$

Since the Jacobian is invertible its inverse on γ is denoted by $\bar J_\alpha^\mu(s)$.
The second order Jacobian is

$$J_{\alpha\beta}^\mu = A_{\alpha\beta}^\mu + A_{\alpha\beta\gamma}^\mu x^\gamma + \tfrac{1}{2} A_{\alpha\beta\gamma\delta}^\mu x^\gamma x^\delta + \dots \tag{9.6.12}$$

which, on γ, reduces to

$$J_{\alpha\beta}^\mu(s) = A_{\alpha\beta}^\mu + A_{\alpha\beta0}^\mu s + \tfrac{1}{2} A_{\alpha\beta00}^\mu s^2 + \dots \tag{9.6.13}$$

Then we are ready to write the connection in the new coordinates x'^μ on the time-like geodesic γ'; one has

$$\tilde\Gamma_{\beta\mu}'^\alpha(s) = J_\rho^\alpha \left(\tilde\Gamma_{\sigma\nu}^\rho(s) \bar J_\beta^\sigma \bar J_\mu^\nu + \bar J_{\beta\mu}^\rho \right) = -J_{\rho\sigma}^\alpha \bar J_\beta^\rho \bar J_\mu^\sigma \tag{9.6.14}$$

Since the anti Jacobians J_μ^σ are invertible, then $\tilde\Gamma_{\beta\mu}'^\alpha(s) = 0$ iff $J_{\rho\sigma}^\alpha(s)$, i.e. iff

$$A_{\alpha\beta}^\mu = 0 \qquad A_{\alpha\beta0}^\mu = 0 \qquad A_{\alpha\beta00}^\mu = 0 \qquad \dots \tag{9.6.15}$$

Thus the most general (analytical) coordinate system in which $\tilde\Gamma(s) = 0$ on γ is obtained by

$$x'^\mu = A^\mu + A_\alpha^\mu x^\alpha + \tfrac{1}{3!} A_{ijk}^\mu x^i x^j x^k + \tfrac{1}{4!} A_{\alpha ijk}^\mu x^\alpha x^i x^j x^k + \dots = A^\mu + A_\alpha^\mu x^\alpha + f_{ijk}^\mu(x) x^i x^j x^k \tag{9.6.16}$$

For these coordinate systems, one has

$$J_\alpha^\mu(s) = A_\alpha^\mu \tag{9.6.17}$$

which is constant on γ.
Accordingly, if one has a conformal factor φ which is not constant along γ, then the metric g cannot be set to η on γ leaving the connection $\tilde\Gamma(s) = 0$ on γ.

Finally, let us consider a Riemannian manifold (M, g), as in the previous case though setting $\tilde\Gamma = \{g\}$. In this case, parallel transport preserves inner products.

Hence, if we fix W_0 to be orthogonal to $\dot\gamma$ and if we fix e_i to be an orthonormal basis in W_0, then $e_i(s)$ are orthonormal bases as well. Moreover, all $e_i(s)$ remain orthogonal to $\dot\gamma(s)$, since also $\dot\gamma(s)$ is paralelly transported along γ itself. Hence, being $(\dot\gamma(s), e_i(s))$ orthonormal basis (as one can obtain by a reparameterisation of γ) then one has $g(\gamma(s)) = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$.

In a general Weyl context however, the parallel transport does not preserve orthonormal vectors, so that one cannot enforce canonical form of the metric along the whole time-like geodesic γ.

Accordingly, Fermi coordinates always exist in a Riemannian structure, while they do not exist in general in a Weyl frame. When they do exist, we have a coordinate systems which approximately reproduces SR in a neighbourhood, and it exactly matches it on the worldline of a time-like particles.

7. Equivalence principles

Let us here consider three different mathematical formulations of Equivalence Principle (EP). These three formulations are not equivalent and we shall proceed from the weakest to the strongest one.

The weakest formulation we consider is the following:

>*free falling is universal.*

This states that the motion of any test particle freely falling in a gravitational field is described by a unique equation independent of the specific characteristic of the particle.

This is the key to allow geometrization of gravity: just because the free fall is independent of the particle, it can be regarded as a property of spacetime, which affects anything in the same way.

The weak EP is encoded into EPS framework as well as in any other formulation of relativistic theory of the gravitational field. Either by an axiom or as a theorem, in any gravitational theory, test particles moves along time-like geodesics of some connection, a single connection for all test particles.

In standard purely metric GR, they move along time-like geodesics of the Levi Civita connection of the metric g which describes the gravitational field.

In Palatini standard GR, the gravitational field is described by a metric and an independent connection $(g, \tilde{\Gamma})$ and test particles move along geodesics of $\tilde{\Gamma}$ which are time-like with respect to g. Then the dynamics chosen (namely the Palatini-Hilbert-Einstein action) induces field equations for the metric and the connection, the equation for the connection implies (in vacuum and with matter, if matter does not couple to $\tilde{\Gamma}$) that $\tilde{\Gamma} = \{g\}$ and a test particle moves along g-time-like $\tilde{\Gamma}$-geodesics which are also g-geodesics, *on-shell*.

If we choose a different dynamics, e.g. some Palatini $f(\mathcal{R})$-theory, then the field equation for the connection does not imply any longer that $\tilde{\Gamma} = \{g\}$. However, test particles still move along the geodesics of $\tilde{\Gamma}$ (which, in general, is different from $\{g\}$).

In more general theories, one has to declare how test particles fall, since the action principle is unable to determine that.

If the free fall has to be deterministic, covariant, invariant with respect to reparameterisations and the gravito-inertial field obeys weak EP then it must be described by some connection $\tilde{\Gamma}$. Thus, in any theory describing gravitational field, one must be able to built a connection to describe free fall, either $\tilde{\Gamma}$ is a field or a function of the fields.

Hence we can state the weak EP as:

>**Weak equivalence principle***: there is a single projective structure which describes free falling for all test particles.*

The second formulation we consider is somehow stronger:

>*at any event in spacetime, there exist local inertial observers for which, at that event, there is no gravitational field and SR is valid.*

If there exists a coordinate system in which the metric (that defines distances) and the connection (that defines free fall) at the event $x \in M$ have local expression

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \qquad \qquad \Gamma^{\alpha}_{\beta\mu}(x) = 0 \tag{9.7.1}$$

it means that there is a small enough neighbourhood U of x in which the metric components are nearly constant and connection components are negligible. Thus free falling particles follow worldlines which are almost *"straight lines"* and, as long as one stays in U, SR approximately holds true.

Thus we can translate this as the following mathematical principle.

> **Normal equivalence principle**: *at any event x in spacetime, there exist coordinate systems in which $g_{\mu\nu}(x) = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\beta\mu}(x) = 0$.*

That is also called the *EP at a point* (or *pointwise EP*). Of course, if x^{μ} is one of such coordinate systems, one can do an affine coordinate transformation

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu} \qquad\qquad (\Lambda, b) \in \mathrm{SO}(\eta) \rtimes \mathbb{R}^4 \tag{9.7.2}$$

and, in the new coordinate system x'^{μ}, one still has $g_{\mu\nu}(x) = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\beta\mu}(x) = 0$. Accordingly, the coordinate systems singled out in this way could be called *(pointwise or normal) inertial observers at x*.

In the case of the lift, that means that there are freely falling observers for which the gravitational field is exactly balanced by the inertial field at an event x in spacetime, i.e. at a center of the lift *and at a given time.* After and away from it, the cancellation is, in general, only approximated and it manifests as tidal forces which, in principle, can be observed.

The normal EP is a theorem of differential geometry and it holds true for any pair $(g, \tilde{\Gamma})$ (regardless field equations). In fact, normal EP can be rigorously stated by observing that for any metric g and any (torsionless) connection $\tilde{\Gamma}$ there exists a normal coordinate system x^{μ} around an event $x \in M$ in which the metric tensor is constant and the connection vanishes at $x \in M$. One has

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \qquad \tilde{\Gamma}^{\alpha}_{\beta\mu}(x) = 0 \tag{9.7.3}$$

Of course, this holds true, in particular when $\tilde{\Gamma} = \{g\}$, as it happens off shell in purely metric standard GR, as well as on shell in its Palatini formulation. But it is also generally true in Palatini $f(\mathcal{R})$-theory, where the connection $\tilde{\Gamma}$ is off-shell independent of the metric g and on shell it is the Levi Civita connection of another metric $\tilde{g}$ conformal to the original metric g.

The third formulation is somehow stronger:

> *there exist local inertial observers for which, along a particle worldline, there is no gravitational field and SR is valid.*

Accordingly, one can state it as

> **Fermi equivalence principle**: *given the worldline γ of a free falling particle, there exist coordinate systems in which $g_{\mu\nu}(\gamma(s)) = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\beta\mu}(\gamma(s)) = 0$.*

That is also called the EP *along a worldline.* It means, comparing with normal EP, that there are freely falling observers for which the gravitational field is exactly balanced by the inertial field at a spatial point x, i.e. at a center of the lift, and at *any time.* The cancellation is, in general, only approximated away from that spatial point, while it is exact at the center during a whole time interval.

As far as the Fermi EP is concerned, that holds true (off shell) in standard purely metric GR, as well as on shell in Palatini standard GR.

In Palatini $f(\mathcal{R})$-theories, one has on shell that the configuration is parameterised by $(g, \tilde{\Gamma})$ with

$$\tilde{\Gamma}^{\alpha}_{\beta\mu} = \{\tilde{g}\}^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2} \left(g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^{\alpha}_{(\beta} \delta^{\epsilon}_{\mu)} \right) \nabla_{\epsilon} \ln \varphi \tag{9.7.4}$$

Hence, for the Fermi EP, one should choose a time-like geodesic γ of $\tilde{\Gamma}$ and find a coordinate system around it in which $g_{\mu\nu}(\gamma(s)) = \eta_{\mu\nu}$ and $\tilde{\Gamma}^\alpha_{\beta\mu}(\gamma(s)) = 0$. One can see that there exist Fermi coordinates for which $\tilde{\Gamma}^\alpha_{\beta\mu}(\gamma(s)) = 0$, though since parallel transport along γ does not preserve inner product with respect to g (neither off shell nor on shell, because of the relation (9.7.4)) one cannot enforce $g_{\mu\nu}(\gamma(s)) = \eta_{\mu\nu}$ along γ.

The situation is similar in any general ETG theory in which the relation (9.7.4) holds on-shell as a consequence of EPS framework.

Thus, in Palatini $f(\mathcal{R})$-theories, one cannot satisfy Fermi EP in general. However, if $T = 0$, as it happens in vacuum or with electromagnetic field only, and for a generic function $f(\mathcal{R})$ then the master equation has discrete solutions and hence constrains the curvature to be constant $\mathcal{R} = \mathcal{R}_0$. As a consequence one has a constant conformal factor $\varphi_0 = f'(\mathcal{R}_0)$ and, with a constant conformal factor, one has $\{\tilde{g}\} = \{g\}$. Since one can always find Fermi coordinates for $(g, \{g\})$, then one can always find Fermi coordinates for $(g, \{\tilde{g}\})$ and the Fermi EP holds true in Palatini $f(\mathcal{R})$-theories on shell and in T-vacuum. Thus one could hope to see quantum violations of Fermi EP which, however, holds true classically, i.e. on average. Of course, we are not able to see quantum gravity effects at all, so that is only a possibility in principle.

References

add

Chapter 10. Standard clocks

> *Sometimes the only choices you have are the bad ones. But you still have to choose.*
>
> *(The Doctor)*

1. Introduction

Originally, we defined a clock as any parameterised worldline in spacetime. ▷ *Clocks*

> We have to remark that, according to the definition given in Subsection 6.3.2, a *clock* is *any parameterised worldline*.
> While we require the trajectory to be time-like with respect to the causal structure $[g]$ (as a clock is a *material* device that, as such, cannot move faster than light), we did not require the clock to be free falling—since, in fact, there are clocks on Earth's surface—i.e. we did not require the trajectory to be a $\tilde{\Gamma}$-geodesics—nor we required any notion of being, in any sense, *uniform*. How could we? In fact, we could not define a meaning for being *uniform*, not before establishing a geometry on spacetime, either on a mathematical stance or on a physical stance, since something can be *uniform* only with respect to something else.

In standard GR, i.e. on a Lorentzian manifold (M, g), a free falling clock, parameterised by arc-length, is called a *proper clock* and the time measured by it is called its *proper time*. A clock which is subjected to a non-gravitational force deviates from its free falling worldline; it is called a *proper clock* anyway, if it is parameterised by arc-length. At any moment, if it is compared with a free falling proper clock passing at the same event, with the same spatial velocity, *instantaneously* the two clocks have the same rate. So the notion of *proper time* is extended to any worldline.

> Ok, there is a number to details to fix, for example: what does it mean for two clocks to have the same *instantaneous rate*?
> Another issue is that all the standard discussion about clocks is carried over with the back-thought that we have devices which can be replicated, of which we can produce a fleet of them, that they maintain their rate reliably. I personally do not even know what it means, though I understand the suggestion of having atomic clocks, which can be sent around and compared when they eventually meet again. They appear to have the same rate when they are at rest with each other and at the same point, when they move away they have different rates though these differences are well described by SR. They have different rates when compared at different locations, even if they are at rest, though these differences are well described by different gravitational fields in the two locations. Of course, the comparison of clocks at different events is complicated, it relies on synchronisation protocols, though these phenomena have been well within our experience for a number of decades now. However, mathematically speaking, it is hard to imagine an absolute definition of what one should understand by taking a free falling clock and comparing it with a different, though identical, non-freely falling clock at a distance. Of course, this is exactly what *proper clocks* does in the standard GR. We would like to point out only that this is a *definition* within a specific gravitational theory.

We have to remark that, if *the same clock*—if that had a precise mathematical meaning, though it may have a physical meaning—were not freely falling, it can be regarded by other observers (for example, depending on the different gravitational fields they, both, are experiencing) to get slower or faster, while, of course, any clock is *uniform* with respect to itself.

2. Definition of standard clocks

Let us start, by telling a short story.

We have defined different kind of geometries on spacetimes, from EPS geometries, to Weyl geometries, to Weyl frames and Riemannian structures. These structures are different in generality, some of them being special cases of others. Eventually, we should care most of Weyl frames, since we adopt them as fundamental fields in ETG. However, clocks come before the dynamics, so let us keep a general attitude and consider clocks in the most general cases as well.

When we defined a *clock* as a time-like curve in M, i.e. a C^∞ map $\gamma : I \to M : s \mapsto \gamma^\mu(s)$, for a real interval $I \subset \mathbb{R}$, such that its tangent vector $\dot\gamma(s)$ is time-like for all $s \in I$, that is pretty general, since time-like curves are well defined in the general EPS framework, they are well defined every time a conformal structure $\mathfrak{g}$ is given on M. As a particular case, they are well defined if a Lorentzian metric $g \in \mathfrak{g}$ is available.

Since here we aim to discuss which quantities depend on which structures, let us keep stuck for a while to a more verbose notation. In a Weyl geometry $(M, \mathfrak{g}, \tilde\Gamma)$, a $\mathfrak{g}$-*clock* is a parameterised $\mathfrak{g}$-time-like curve $\gamma : I \to M$. That definition specialises nicely to stricter structures, in which one selects a representative $g \in \mathfrak{g}$. In that case, we simply call it a *g-clock*, in fact meaning a *[g]-clock*.

Given a $\mathfrak{g}$-clock $\gamma : I \to M$, for any event $p \in \gamma(I) \subset M$ there is a unique value of the parameter s such that $p = \gamma(s)$; such a value of the parameter s is called the *time-reading*, or simply the *time*, of the clock γ at the event p. The *rate* of a clock γ at time s (or, equivalently, at the event $\gamma(s)$) is the number $\omega > 0$ such that $g(\dot\gamma(s), \dot\gamma(s)) = -\omega^2$.

> Let us stress that, given a clock γ, its *time* is defined on the worldline $\gamma(I) \subset M$, only. There is still no time out of it, not yet. In order to assign a *time* in an open set of spacetime, one needs a *synchronisation protocol*, which we are not fixing, yet.
>
> Once a synchronisation protocol will be fixed, one is able to determine isochronous surfaces of contemporary events, at least locally on spacetime. Until then, time is defined only along clocks, and each clock defines its own time, different clocks defining times which have *a priori* nothing to do one with the others.
>
> This is a consequence of, and it expresses, the fact that in a relativistic world, at a fundamental level, there is no time. At most each particle goes around carrying its own time which applies to itself and only to itself. There is no global time (or even local time) even though we often define coordinates or restrict to situations in which it seems there is one, in honour to Newtonian legacy. However, one should also remark that *coordinate time* is quite different from an absolute time, first of all because coordinates are *completely, in principle*, conventional.

If a Lorentzian metric $g \in \mathfrak{g}$ is available on spacetime, we can define a *g-proper clock* as a clock $\gamma : I \to M$ such that $g(\dot\gamma, \dot\gamma) = -\omega^2$, with $\omega \in \mathbb{R}$ constant, i.e. independent of s. Of course, there is no canonical choice on a Weyl geometry (nor in an EPS geometry), in which one has no special representative for the conformal structure. However, there are non-canonical choices, one for each representative of the conformal structure.

On the contrary, in a Weyl frame, as well as in a Riemannian structure, a representative $g \in \mathfrak{g}$ of the conformal structure has been selected precisely to describe our operative protocols for measuring distances and time lapses. Accordingly, in these cases, one has a preferred notion of *g-proper clocks*.

By definition, a *g*-proper clock has a constant *rate*. Because of that, they will be assumed to model *uniform clocks*, i.e. they will provide us with a definition of *uniform clocks*.

Of course, here as in other contexts—as, e.g., in probability theory—*uniform* has no absolute meaning at all. Something is *uniform* with respect to a measure, here with respect to a metric $g \in \mathfrak{g}$, as it nicely rendered by the definition of *g*-proper clock.

Once the rate is constant, one often chooses a time unit so that the rate $\omega = 1$. The clocks with a unit rate are called *normalised proper clocks* or, for short *NP-clocks* with respect to *g*.

A *g*-proper clock comes with a unique parameterisation (modulo affine transformations) fixed on it.

Let $\gamma_{\circ} : I \to M$ be a $\mathfrak{g}$-clock and $\phi : I \to J : s' \mapsto s$ be a reparameterisation. The composition $\gamma = \gamma_{\circ} \circ \phi : \mathbb{R} \to M$ is a new clock $\gamma(s')$, which shares the trajectory with $\gamma_{\circ}$ though it tells a different time. Given a metric g, the rate of the new clock compares with the rate of the first one as

$$g(\dot{\gamma}(s'), \dot{\gamma}(s')) = (\dot{\phi}(s'))^2 g(\dot{\gamma}_{\circ}, \dot{\gamma}_{\circ}) = -(\dot{\phi}(s'))^2 \omega^2(\phi(s')) \tag{10.2.1}$$

If we want the new clock to be *g*-proper, we need to require that $(\dot{\phi})^2 \omega^2$ is constant, i.e.

$$2\dot{\phi}\ddot{\phi}\omega^2 + 2(\dot{\phi})^3 \omega\dot{\omega} = 0 \qquad \Rightarrow \ddot{\phi} = -\frac{\dot{\omega}(\phi)}{\omega(\phi)}(\dot{\phi})^2 \tag{10.2.2}$$

Being $\omega(\phi)$ a known function, which depends on the old clock and on the metric, one can associate to this equation and equivalent first order system

$$\begin{cases} \dot{\phi} = \psi \\ \dot{\psi} = -\dfrac{\dot{\omega}(\phi)}{\omega(\phi)}\psi^2 \end{cases} \qquad \Rightarrow X = \psi\partial_{\phi} + \frac{\dot{\omega}(\phi)}{\omega(\phi)}\psi^2 \partial_{\psi} \tag{10.2.3}$$

The parameterisations which make the new clock a *g*-proper clock are associated with integral curves of this vector field X, which hence exist as an initial condition $s_0 = \phi(s'_0)$ is provided. Accordingly, any $\mathfrak{g}$-clock $\gamma_{\circ}$ can be reparameterised to obtain a new clock γ which is proper with respect to any metric $g \in \mathfrak{g}$ one could select.

Now, if $\gamma : I \to M$ is a *g*-proper clock and $\gamma_{*} := \gamma \circ \phi : J \to M$ is a reparameterisation of it for some isomorphism $\phi : I \to J : s' \mapsto s$, then one has $\dot{\gamma}_{*}(s') = \dot{\phi}(s')\dot{\gamma}(s)$. Accordingly, γ_{*} is also *g*-proper iff

$$g(\dot{\gamma}_{*}, \dot{\gamma}_{*}) = \dot{\phi}^2 g(\dot{\gamma}, \dot{\gamma}) = -\dot{\phi}^2 = -\omega^2 \qquad \Rightarrow \quad \dot{\phi}^2 = \omega^2 \qquad \Rightarrow \quad s = \omega s' + s_0 \tag{10.2.4}$$

Thus, if we want to preserve *g*-proper clocks, their parameterisation is frozen up to an affine transformation $s = \omega s' + s_0$.

Accordingly, an *NP*-clock defines a unique way of assigning time lapses.

Given two events $p, q \subset \gamma(I)$, there exist two times s_p, s_q of γ such that $\gamma(s_p) = p$ and $\gamma(s_q) = q$. We define the *time lapse* with respect to the clock γ between the events p and q as the number $s_q - s_p$. The time lapse between p and q is invariant with respect to origin translations, so two *NP*-clocks supported on the same trajectory $\gamma(I) \subset M$ define the same time lapse between p and q.

This is not the case for a general clock. In general, the time lapse along an arbitrary clock is totally conventional as it depends entirely on the clock.

The time measured by a *g*-proper clock is called the *g-proper time*.

Notice that we did not ask the g-proper clock to be freely falling, with respect to $\{g\}$ or to any other connection. If the g-proper clock is affected by any non-gravitational force, it will not be freely falling. However, at any event $p \in \gamma(I)$ on it, there will be a free falling g-proper clock γ_*, passing through p with the same covariant velocity of γ at p (i.e. the worldline of γ and γ_* share the same tangent vector at p) which, hence, has the same rate at p.

> Thus the g-proper time is measured instantaneously at any event by a freely falling g-proper clock, even though, if the clock motion is accelerated, the free falling g-proper clock is always different at any event.
>
> [I beg your pardon, it is clear that this note would not be of any use, if not to explain the description of proper time for accelerated clocks which is sometimes found in the literature, especially discussing SR. In fact, we did not even use free fall in the definition of proper time and that should be enough to *"extend"* to accelerated clocks.]

So the only issues about g-proper clocks are which metric to use to define them (especially, though, as we argued, not only, in a theory with more than one metric) and how to check whether a clock is g-proper by means of a suitable operational protocol based only on light rays and particles.

Before proceeding, we can define a similar class of clocks, which is though well defined on a Weyl geometry, i.e. a class depending only on the conformal structure, not on a representative for it.

When a clock γ is given in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, it is natural to define its covariant velocity and its covariant acceleration as

$$v(s) = \dot{\gamma}^\mu(s)\partial_\mu \qquad a_{\tilde{\Gamma}}(s) = \left(\ddot{\gamma}^\mu(s) + \tilde{\Gamma}^\mu_{\alpha\beta}(\gamma(s))\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s)\right)\partial_\mu \tag{10.2.5}$$

respectively.

Let us stress that, while the covariant velocity v depends on the curve only, the covariant acceleration $a_{\tilde{\Gamma}}$ depends on a connection which describes the free fall, as well. They are both tangent vectors to M at the points $\gamma(s)$.

> We have to check how they transform under spacetime diffeomorphisms. That is trivial for the covariant velocity, while it deserves some details for the covariant acceleration.
>
> Let us consider a new coordinate system $x'^\mu = x'^\mu(x)$; the local expression for the clock $\gamma : s \mapsto \gamma^\mu(s)$ in the old coordinates x^μ, becomes $\gamma : s \mapsto \gamma'^\mu(s) = x'^\mu(\gamma(s))$ in the new coordinates x'^μ.
>
> The covariant velocity and the covariant acceleration are
>
> $$\dot{\gamma}'^\mu = J^\mu_\nu \dot{\gamma}^\nu \qquad \ddot{\gamma}'^\mu = J^\mu_\nu \ddot{\gamma}^\nu + J^\mu_{\lambda\nu}\dot{\gamma}^\lambda\dot{\gamma}^\nu \qquad \Rightarrow \qquad a'^\mu_{\tilde{\Gamma}} = \ddot{\gamma}'^\mu + \tilde{\Gamma}'^\mu_{\alpha\beta}\dot{\gamma}'^\alpha\dot{\gamma}'^\beta = J^\mu_\nu \ddot{\gamma}^\nu + \left(J^\mu_{\alpha\beta} + \tilde{\Gamma}'^\mu_{\rho\sigma}J^\rho_\alpha J^\sigma_\beta\right)\dot{\gamma}^\alpha\dot{\gamma}^\beta = J^\mu_\nu a^\nu_{\tilde{\Gamma}} \tag{10.2.6}$$
>
> Accordingly, also the covariant acceleration is a tangent vector and it depends on the connection exactly for it to happen.

Definition: *a clock $\gamma : I \to M$ is said to be a* standard clock *with respect to the connection $\tilde{\Gamma}$ and a conformal structure $\mathfrak{g}$, if its covariant acceleration $a_{\tilde{\Gamma}}$ is $\mathfrak{g}$-orthogonal to the covariant velocity v, i.e. for any representative $g \in \mathfrak{g}$ one has $g(a_{\tilde{\Gamma}}, v) = 0$ for any s.*

> Notice that, if $g(a_{\tilde{\Gamma}}, v) = 0$ for one representative g of the conformal structure, then $\tilde{g}(a_{\tilde{\Gamma}}, v) = 0$ for any other representative. In fact, if one wants a condition which is conformally invariant, there is not much else to try. Accordingly, the notion of a standard clock is somehow natural on a Weyl geometry.

The main good property of $(\mathfrak{g}, \tilde{\Gamma})$-standard clocks, as of proper clocks, is that they come with a unique parameterisation (modulo affine transformations).

Two clocks $\gamma : I \to M$ and $\tilde{\gamma} : J \to M$ are said to be *equivalent* iff they are two representatives of the same (oriented) trajectory, i.e. they share the same worldline and, in fact, they differ by a (orientation preserving) diffeomorphism $\phi : I \to J$ that maps I into J, i.e. iff $\gamma = \tilde{\gamma} \circ \phi$.

Given a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$ and a conformal representative $g \in \mathfrak{g}$, one can always find a parameterisation for any worldline such that it is a standard clock. Consider, in fact, two equivalent clocks

$$\gamma(s) = \tilde{\gamma} \circ \phi(s) \qquad\qquad (s' = \phi(s)) \tag{10.2.7}$$

Their velocities and accelerations (with respect to a connection $\tilde{\Gamma}$) are related by

$$v^\lambda = \dot{\phi}\, \tilde{v}^\lambda \qquad\qquad a_{\tilde{\Gamma}}^\lambda = \dot{\phi}^2 \tilde{a}_{\tilde{\Gamma}}^\lambda + \ddot{\phi} \tilde{v}^\lambda \tag{10.2.8}$$

where the dots denote derivatives with respect to s. The evaluation at s or s' is understood, as it is clear from the context.

Then $\tilde{\gamma}$ is a standard clock with respect to $\tilde{\Gamma}$ and $\mathfrak{g}$ iff, for $g \in \mathfrak{g}$, we have:

$$g(\tilde{a}_{\tilde{\Gamma}}, \tilde{v}) = \dot{\phi}^{-3} g(\dot{\phi}^2 \tilde{a}_{\tilde{\Gamma}}, \dot{\phi}\tilde{v}) = \dot{\phi}^{-3} g(a_{\tilde{\Gamma}} - \ddot{\phi}\tilde{v}, v) = \dot{\phi}^{-4}\left(\dot{\phi}g(a_{\tilde{\Gamma}}, v) - \ddot{\phi}g(v,v)\right) = 0 \quad \Longleftrightarrow \quad \ddot{\phi} = \frac{g(a_{\tilde{\Gamma}}, v)}{g(v,v)}\,\dot{\phi} \tag{10.2.9}$$

which has to be regarded as a differential equation for the unknown function $\phi(s)$. It can be equivalently regarded as a first order system, in turn, regarded as the equation for integral curves of a vector field X on the space of (s, ϕ, ψ)

$$\begin{cases} \dot{s} = 1 \\ \dot{\phi} = \psi \\ \dot{\psi} = \dfrac{g(a_{\tilde{\Gamma}}, v)}{g(v,v)}\,\psi = F(s, \psi) \end{cases} \qquad\qquad X = \frac{\partial}{\partial s} + \psi\frac{\partial}{\partial \phi} + F(s, \psi)\frac{\partial}{\partial \psi} \tag{10.2.10}$$

The function $F(s, \psi)$ is regular enough, since $g(v,v)$ never vanishes. Then the integral curve $(s, \phi(s), \psi(s))$ is uniquely determined once the initial condition $s_0' = \phi(s_0)$ is given, and the function $\phi(s)$ defines the parameterisation on the standard clock $\tilde{\gamma}$.

Notice that the vector field X, hence also the reparameterisation it defines, is invariant under Weyl conformal transformation. Accordingly, the standard parameterisation is in fact determined by $(\mathfrak{g}, \tilde{\Gamma})$, not by $(g, \tilde{\Gamma})$.

The solution is unique up to an affine transformation $s \mapsto \omega s + s_0'$ with constants $\omega, s_0' \in \mathbb{R}$. In fact, if we consider a standard clock γ, we change parameterisation and we want the new clock $\tilde{\gamma}$ to be standard again, then equation (10.2.9) still hold true, this time with $g(a_{\tilde{\Gamma}}, v) = 0$, since the first clock was originally standard. Thus we have $\ddot{\phi} = 0$, the solutions of which are precisely $s' = \phi(s) = \omega s + s_0'$.

Let us consider a clock $\gamma : I \to M$, in a general Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$; for a representative $g \in \mathfrak{g}$, one has

$$\frac{d}{ds}\left(g(v,v)\right) = (\partial_\lambda g_{\mu\nu})\dot{\gamma}^\mu\dot{\gamma}^\nu\dot{\gamma}^\lambda + 2\ddot{\gamma}^\mu\dot{\gamma}^\nu g_{\mu\nu} = (\tilde{\nabla}_\lambda g_{\mu\nu} + 2g_{\mu\rho}\tilde{\Gamma}_{\nu\lambda}^\rho)\dot{\gamma}^\mu\dot{\gamma}^\nu\dot{\gamma}^\lambda + 2\ddot{\gamma}^\mu\dot{\gamma}^\nu g_{\mu\nu} = 2g(a_{\tilde{\Gamma}}, v) + \boldsymbol{\alpha}(v)g(v,v) \tag{10.2.11}$$

If we restrict to a Riemannian structure (M, g), then $\boldsymbol{\alpha} = 0$. In this case, a clock is standard with respect to $[g]$ and $\{g\}$ iff it is g-proper. Accordingly, the concept of being standard can be regarded as a generalisation of proper clocks in Riemannian structures to Weyl geometries.

If the Weyl geometry $(M, \mathfrak{g} = [g], \tilde{\Gamma})$ is integrable, i.e. if $\tilde{\Gamma}$ is metric, i.e. one has $d\boldsymbol{\alpha} = 0$, then $\alpha_\mu = -d_\mu \ln \varphi$ and $\boldsymbol{\alpha}(v) = -\dot{\gamma}^\mu d_\mu \ln \varphi = \frac{d}{ds}\ln(\varphi)$.

Of course, because of that one has $\tilde{\Gamma} = \{\tilde{g}\}$, where we set $\tilde{g} = \varphi g$ for the usual conformal metric. Accordingly, we have

$$\frac{d}{ds}\ln\left(g(v,v)\right) = 2\frac{g(a_{\tilde{\Gamma}}, v)}{g(v,v)} - \frac{d}{ds}\left(\ln\varphi\right) \quad \Longleftrightarrow \quad \frac{d}{ds}\ln\left(\varphi g(v,v)\right) = 2\frac{g(a_{\tilde{\Gamma}}, v)}{g(v,v)} \quad \Longleftrightarrow \quad \frac{d}{ds}\tilde{g}(v,v) = 2\frac{\tilde{g}(v,v)}{g(v,v)}g(a_{\tilde{\Gamma}}, v) = 2\tilde{g}(a_{\tilde{\Gamma}}, v) \tag{10.2.12}$$

Since $\tilde{g} \in [g]$, we still have that the clock standard with respect to $\mathfrak{g}$ and $\tilde{\Gamma} = \{\tilde{g}\}$ iff it is proper, this time with respect to $\tilde{g}$.

Accordingly, the concept of standard clock can be regarded as a generalisation of proper clocks in Riemannian structures (and integrable Weyl structures) to a general Weyl geometries.

The parameterisation of a standard clock is called *standard time* and it coincides with $\tilde{g}$-proper time in integrable Weyl geometries. That is an interesting, somehow unexpected, result, since it extends proper time to a general Weyl geometry where there is no preferred metric. However, in integrable Weyl geometries (and, in particular, in Riemannian structures) there is no difference for clocks between being proper and being standard.

One can also regard the issue in a different way.

Consider a clock γ, which is freely falling in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, i.e. the worldline of γ is a geodesic trajectory of $\tilde{\Gamma}$, i.e. it obeys, for some function $\lambda(s)$, the equation

$$\ddot{\gamma}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = \lambda\dot{\gamma}^\mu \tag{10.2.13}$$

We already know that there is always a special parameterisation for which γ is a geodesic motion for $\tilde{\Gamma}$.

Now we know, also, that there is a special parameterisation in which γ is standard with respect to $\mathfrak{g}$ and $\tilde{\Gamma}$.

Thus one can ask whether these two parameterisations of γ are different. For, let us start over from equation (10.2.12). If the clock is freely falling we have $a_{\tilde{\Gamma}} = \lambda v$ so that the clock is standard iff $g(a_{\tilde{\Gamma}}, v) = \lambda g(v, v) = 0$, iff $\lambda = 0$. That is, γ is standard iff it is a geodesic motion.

Let us remark, however, that standard clock are defined also for not freely falling clocks, while, of course, geodesic motion applies to freely falling clocks only.

The notion of standard clock, as we said, applies also to non-freely falling clocks. For example, the clock $\gamma : s \mapsto (s, r_0, \theta_0, \varphi_0)$ is standard in a Schwarzschild metric though it is not free falling.

The covariant velocity is $v = \partial_0$ while the covariant acceleration is

$$a_g = \{g\}^\alpha_{00}\partial_\alpha = \tfrac{1}{2}g^{\alpha\lambda}\left(-\partial_\lambda g_{00} + 2\partial_0 g_{0\lambda}\right)\partial_\alpha = -\tfrac{1}{2}AA'(r)\partial_r \tag{10.2.14}$$

Of course, we have $g(a_g, v) = 0$, thus the clock is standard.

Also, in this case, of course, the tangent vector v has constant length $A(r_0)$ along the worldline and the clock is parameterised by its proper time.

We have to remark that nothing of this makes any physical sense yet, it is just geometry. Here, we searched for classes of clocks which are well defined in a Weyl geometry. If we want to define uniform clocks as a special class of clocks which are physically relevant, we need them, in view of general covariance principle, to be covariant with respect to diffeomorphisms and reparameterisations.

For example, it makes sense to try to define them as standard clocks, though at this stage being standard (or proper) is not particularly a physical property and it has not much to do with being *"uniform"* in any physical sense one could think of. We showed that proper clocks have a constant rate, though here the rate enters as a mere mathematical property, not as a physical rate. So, if we want to be clear, we have to better ground these definitions in a physical stance.

Let us finally restore a simpler notation for clocks, since we shall always use clocks in a specific Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, a Weyl frame $(M, g, \tilde{\Gamma})$, or a Riemannian structure (M, g).

In a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$ a *standard clock* is understood to be standard with respect to $\mathfrak{g}$ and $\tilde{\Gamma}$. Since we have no preferred metric selected on M there is no notion of proper clock. The corresponding parameterisation of the standard clocks is called *standard time*.

If the Weyl geometry is instead a Riemannian structure (M, g), it is understood to come from the Weyl geometry $(M, [g], \{g\})$ and a standard clock is also g-proper and it is also called a *proper clock*. In this case, *standard* and *proper clocks are synonyms* and they define *standard time* which coincides with *proper time*.

Once again, Weyl frames are richer in structure.

When dealing with a Weyl frame $(M, g, \tilde{\Gamma})$, a *standard clock* is understood to be standard with respect to $[g]$ and $\tilde{\Gamma}$, again in general as in the Weyl geometry $(M, [g], \tilde{\Gamma})$ which the Weyl frame comes from. However, now we have also selected a canonical metric g on M, so, accordingly, a *proper clock* is understood to be proper with respect to g. The 1-form $\boldsymbol{\alpha}$ measures by what extent a standard clock fails to be proper, as well as, vice versa, to what extent a proper clock fails to be standard.

When the Weyl frame is integrable, one has $\tilde{\Gamma} = \{\tilde{g}\}$ for the connection and $[g] = [\tilde{g}]$ for the conformal structure. A *standard clock* is standard with respect to $[g] = [\tilde{g}]$ and $\tilde{\Gamma} = \{\tilde{g}\}$, hence it is also $\tilde{g}$-proper. On the contrary, we define a *proper clock* in an integrable Weyl frame to be a g-proper clock. In this case, we have a *standard time* and a *proper time* which are two different parameterisations.

The 1-form $\boldsymbol{\alpha}$ still measures by what extent a standard clock fails to be proper, and vice versa. In this case, however, $d\boldsymbol{\alpha} = 0$, $\boldsymbol{\alpha} = -d\ln(\varphi)$ allows a potential (namely, $-\ln(\varphi)$), which is essentially the conformal factor. Thus the conformal factor has a natural interpretation as the field which measures the difference between standard and proper times in integrable Weyl frames. Of course, in a purely metric Weyl frame, standard and proper clocks coincide, also reproducing Riemannian structures.

The case of integrable Weyl frames is specially important for Palatini $f(\mathcal{R})$-theories. In Palatini $f(\mathcal{R})$-theories, we have to declare whether atomic clocks (or any other class of physical clocks) are proper or standard. Unless we say differently, we shall always assume atomic clocks to be proper, i.e. to tell the g-time, not $\tilde{g}$-time. That is in agreement with the motivation we declared to originally single out a representative g in the conformal structure $\mathfrak{g}$.

> That is a potential conflict about standard GR, which is both a purely metric theory and a special Palatini $f(\mathcal{R})$-theory. In it, an atomic clock should tell $\tilde{g}$-proper time since we are in a Riemannian geometry, but also proper time (which is g-proper time) as is understood in an integrable Weyl frame.
> However, this is not a conflict for standard GR, since the standard dynamics constrains the conformal factor to be $\varphi = 1$, so $g = \tilde{g}$ and one has just one proper time.

> This also gives a pretty simple and clear viewpoint on what novelty Palatini $f(\mathcal{R})$-theories introduce with respect to standard GR. They are precisely assuming that the one metric $\tilde{g}$ that describes free fall may be not the one metric g which makes the coupling with matter simple and accounts for the rate of the atomic clocks. Here, we are just learning to live with this.

In a Weyl frame $(M, g, \tilde{\Gamma})$, two proper clocks γ_1 and γ_2 are said to be *identical clocks* if they have the same (constant) rate. Notice that the rate is defined only with respect to a metric, i.e. it is defined in Weyl frames and Riemannian structures only. Accordingly, all *NP*-clocks are identical to each other.

Synchronisation protocols

Let us start, by telling a short story.

Eve lives in an isolated farm, few kilometres away from a small village. She has only one clock in her farm, stuck on the kitchen wall. One day the clock stops, since Eve forgot to wind it. That would not be a big deal, if it were not the only clock in the farm and there is no way to adjust it to the correct time, if not synchronising it with the clock on the village's council hall.

However, no telephone or radio communication is available in the area, nor anyone can move either of the clocks. Eve just has a scooter. She does not know the exact speed of the scooter, nor the distance from the farm to the village, though the road connecting the farm to the village is straight, plane, and with no traffic, so that Eve is pretty confident to be able to keep a constant speed for the scooter during her way to the village.

How can Eve synchronise her clock to the village one?

At noon, she cannot be at the village to check it is noon and at the same time in her kitchen to start her clock. And she would have no way to measure how much time it takes to go back home, since she has no other watch. She does not know the distance to go and the speed of the scooter, so she cannot compute that time either. However, there is a way of doing it.

She can wind her clock, set it to an arbitrary time, say $6:00$, notice it, take the scooter and go to the village. When she get there, she notice the village clock time, say $13:12$, and go back to her farm. When she gets there, her clock will tell the time, say $7:20$.

It means that it took Eve $1h20min$ to go back and forth to the village, *allegedly* $40min$ to go, $40min$ to come back. So now the clock of the village will probably tell it is $13:52$. Thus Eve can adjust her clock to $13:52$ and have lunch.

Let us notice that we should doubt about the assumption that going back and forth takes the same time. It may be that the road is in a slight slope so that it takes $35min$ to go and $45min$ to come back. In that case, it seems obvious that Eve should have set her clock to $13:57$, instead.

On the other hand, we need to remark that while we can measure the *two ways time* to be $1h20min$, there is no way to measure the time to return only (which is called the *one way time*), if not assuming that the two clocks are synchronised.

For, we should need a watch to bring with us during the trip, or we should already know that the two clocks are synchronised (so to read the village clock when we start coming back and the kitchen clock when we arrive). If the two clocks are not already synchronised we cannot measure the one way time.

Even if it seems obvious that the one way time is a better concept for synchronisation, the fact that it cannot be measured directly should ring a warning bell. As a matter of fact, assuming that the one way time is half of the two ways time is a *convention* which cannot be falsified, and which defines a synchronisation protocol. Without that assumption, there is no synchronisation and no time out of the clock worldline.

If we had a signal which travels faster than the scooter, then the synchronisation could be falsified.

To check that the village clock and the kitchen one are *not* synchronised we just need some help by someone at the village. We can agree that we shall send a signal to the village (for example, ring a gong, which is about 20 times faster than the scooter) at $15:00$, if the signal arrives to the village at $14:59:45$, it is clear that the two clocks are not synchronised. That is true whatever the speed of sound is. However, if the signal is received at $15:01$ we cannot really know whether the extra minute is, in fact, due to the time it takes to the signal to go from the farm to the village.

[Well, actually we could, *if we knew* the speed of the signal, the speed of sound in this case. Or, more precisely, if we knew how long does it take to sound to go from the village to the farm, or vice versa. However, that is a one way time of sound, which cannot be measured without assuming a synchronisation. What we can easily measure, with only one clock, is the two ways time of sound. However, this is useless for our purposes, since we would need to assume that the one way time is $1/2$ of the two ways time, which may be the case, though it is precisely what we are trying to prove.

How about if sound had two different velocities to come and go? Could we exclude it?

No, as a matter of fact, we cannot exclude it based on an experiment, not *before* we define a synchronisation protocol. Of course, it seems a safe assumption, once we excluded wind effect. Still it cannot be tested. Moreover, it is a useless assumption, since we can learn to live without it.

Try and remember it, when people tell you to relax, that one needs to do *some* assumptions, that assuming it is a *simpler* description of Nature, or that there is no experimental evidence to force us to assume it is not the case.]

Since with a message from the farm to the village we can falsify that the clocks are synchronised, if the kitchen clock goes ahead with respect to the clock of village, we can also agree that a signal is sent from the village to the farm at 15 : 10. If it is received at 15 : 09 is is pretty clear that the two clocks are not synchronised properly. If it is received at 15 : 11 we cannot decide, since we do not know the one way velocity of sound.

So, we say that the two clocks are synchronised, if signals in both directions are received at a later local time than the local time they have been sent.

Now, if the sound took $90s$ to go from the farm to the village (something which is easy to test only by using radio signals, which drastically reduces the uncertainties about synchronisation), as well as from the village to the farm, and the clock were off synchronisation by more than $90s$, either the signal in one direction, or in the other, would be received at an earlier local time than the local time they have been sent and we would conclude that the two clocks are not properly synchronised. However, it is clear that, if the clocks are off synchronisation by less that $90s$, the experience would be inconclusive and the synchronisation would be confirmed. In other words, if our fastest signal takes Δt to travel between the two clocks, we can spot mis-synchronisations by more than Δt, while mis-synchronisations by less than Δt cannot be detected. For this reason, in a Newtonian world, where there is no bound to the speed of signals, synchronisations are, in principle, perfectly precise. In a relativistic world, where no signal can travel faster that light, synchronisations are affected by an uncertainty which is theoretically impossible to reduce to zero. In Eve's world, where the scooter is the fastest signal available, only synchronisations up to an error of $40min$ are available. One can go below that limit only by convention, for example by fixing the one way time.

Let us stress once again that this has nothing to do with the fact that the speed of light is constant for all observers. One just needs to know that it is the fastest signal available. The only difference between light and Eve's scooter, is that there is a physical reason why the speed of light cannot be exceeded, while there is no reason to assume that tomorrow Eve could not buy a faster scooter and consequently, improve her synchronisation.

If the one way time of light seems to be a reasonable assumption in every day life, it is because we know that, if we could send a radio signal (which is approximately instantaneous on that scale), we could easily check that the two clocks strike 14 : 00 at the same time. That, precisely, means that if the village clock broadcasted a radio signal when it strikes 14 : 00, Eve could receive it in her kitchen, assume that she is receiving it *approximately* when it has been sent, and check whether her clock is striking 14 : 00 as well (by assuming that the light received by her eyes is received *approximately* at the same time it was reflected by the clock on the wall).

> Of course, these approximations are, in that case, very reasonable. The time for light to go from the village to the farm is approximately $10^{-4}s$, the time for light to go from the wall to the eyes is approximately $10^{-8}s$. Both are way shorter than the time for neural signal to reach the brain, for the radio to transform the electromagnetic signal in an audio signal, for the sound to go from the radio to Eve's ear.
>
> That is why, originally Newton assumption on absolute time seemed so natural. However, our protocol will be eventually applied to astrophysics and cosmology, and there is no signal which can be considered approximately instantaneous from a galaxy to another. That is why we initially mentioned that Eve has no way to travel or signal at a speed faster than the scooter.

Fig. 9.1: *Isochronous surface through a point B.*

Eve's experiment, with the scooter replaced by a light signal, is exactly the Einstein convention for synchronisation of two far away clocks. Let $\gamma : I \to M$ be a clock and B be an event on M not on the clock trajectory. We can send an echo from the clock on B, which has been sent at $A \in \gamma(I)$, reflected at B and received at $C \in \gamma(I)$.

Since A and C are events on the clock, they correspond to clock time-readings, say s_A and s_C. That is the two ways time of light measured by the clock γ. We can assume, as we did for defining a synchronisation protocol, that the one way time of light is $1/2$ of the two ways time. We can define the quantities

$$T = \frac{s_C - s_A}{2} \qquad s_D = s_A + T = \frac{s_A + s_C}{2} \tag{10.2.15}$$

Thus the signal takes a time T both to go from A to B and to come back from B to C, so that we can assume B to be synchronous to the event $D \in \gamma(I)$ which corresponds to the time s_D. Since we assigned to B a time $s_B = s_D$, and that can be done at any point in a neighbourhood of the clock worldline, we see that the clock γ defines a family of local hypersurfaces

$$\Sigma_D = \{B \in M : s_B = s_D, \ D \in \gamma(I)\} \tag{10.2.16}$$

which are called the *isochronous surfaces* induced by the clock γ, which locally foliate spacetime.

This is completely general. Any clock induces isochronous surfaces, at least locally. If the clock is standard (or proper) the surfaces are called *standard (or proper) isochronous surfaces.*

That is the closest we can go, in general, to Newton spacetime.

Now, for any clock γ, we have, at least locally, a family of hypersurfaces Σ_p, with $p \in \gamma(I)$. Each such surface cuts the clock's worldline at one point p, which corresponds to a time-reading s_p on γ, so that we can extend the parameter on the clock to a neighbourhood, by defining a local function $t : U \to \mathbb{R} : q \mapsto s_p$, iff $q \in \Sigma_p$. Of course, the surfaces Σ_p are also level surfaces of the function t.

However, unlike in Newton spacetime, where any (*uniform*) clock defines the same absolute time, here, instead, each clock defines a different time and even different isochronous surfaces, which is precisely what one means by saying that contemporaneity is *relative*. Two events which are contemporary for a clock can be seen at different times by another.

> Let us stress that, until now, we did not even mentioned SR, we have derived this directly in a general spacetime, actually more general than the standard GR spacetime.

Another difference with Newton spacetime is that the isochronous surfaces Σ_p are not necessarily global, nor they, in general, define a global foliation. That is a direct consequence of the locality assumed by EPS in Axioms AE and ECHO. In fact, in Minkowski, were Axioms AE and ECHO apply globally, one, as we shall see, gets back a global foliation, though still depending on the clock.

Let us now introduce an ordering of events in spacetime.

Let us define a *signal* as any worldline (i.e. any trajectory which is time-like or light-like with respect to the conformal structure $\mathfrak{g}$ defined on spacetime). Given two events $x, y \in M$, we say that x *causally precedes* y, and write it as $x \leq y$, if one can send a signal from x to y.

If we restrict to a worldline γ, this ordering is a total order.

> If we consider a worldline γ (being it a freely falling particle or not, it does not really matter here) the events on the worldline are totally causally ordered. That, if you want, is a definition of the pointwise causal structure of spacetime, which is also encoded in EPS axioms. Events on the worldline are experienced by P in a sequence, and P can be elected to judge about their order. According to P, they are happening at the same space point and no message is needed to be exchanged.
> The essential point is that, for any pair of events $x, y \in \gamma$, one can always send either a signal from x to y or a signal from y to x, since γ itself is a signal through x and y.

Out of the clock, say in a suitable neighbourhood U, the causal ordering is only a partial ordering.

> Of course, there is always a message from x to itself, thus $x \leq x$. The concatenation of two signals is a signal, so if $x \leq y$ and $y \leq z$, then $x \leq z$. Thus the causal ordering is a preorder.
> The events prior to x, i.e. $P_x = \{y \in U : y \leq x\}$, are the ones in the closed past nappe of the wavefront centred at x. In an open neighbourhood U, if we have $x \leq y$ and $y \leq x$, then $x = y$. Thus the causal order is also a partial order.

Since the causal order is a partial order, there are points x, y, which are different and such that not ($x \le y$ or $y \le x$), which are called *contemporary events*.

The difference between a Newton spacetime and a relativistic one is not in the properties of causal ordering; in both cases, the causal ordering is a partial order and it is not a total order. The difference is in the structure of contemporary events $C_x = \{y \in M : \text{not } (x \le y \text{ or } y \le x)\}$.

In a Newtonian spacetime, C_x are hypersurfaces, while in EPS the set C_x is the outer region of the wavefront, which is an open set in spacetime. We can say that the *present* is *thin* in Newton, while it is *thick* in relativistic theories.

> This is happening every time signals have a maximal finite speed. It has not much to do with the speed of light being constant or absolute, it has to do with the fact that no signal can travel faster than light. Every time there is a maximal speed for signals, one has a thick present.
>
> For, let us consider again the situation of Fig. 9.1, this time from the point of view of B. Imagine B is trying to order events on the worldline through A and C. It receives a light signal form A, thus A and all the events on the worldline before A are before B. It can also send a light signal to C, so C and all the events on the worldline after it are after B. However, being light signals of maximal speed, B cannot say whether the events between A and C are before or after it; to do that, it should be able to receive or send a signal to them, which would contradict that light signals are of maximal speed.
>
> Thus all events on the worldline between A and C are neither before nor after B so they are contemporary to B. Thus, the farer one goes, the thicker the present becomes.
>
> If, as in Newton, we had no bound on the speed of signals, we could receive a signal from, or send a signal to, any event out of the isochronous surface through B. Thus in Newton spacetime the present is reduced to the isochronous surface.

Let us finally remark that, while we showed that a clock locally defines a coordinate time, we did not define an *observer* out of it. We are able to say *when* an event happens, we are not able to say *where*, yet. For that, we need to define a relativistic positioning system (rPS).

Synchronisations in Minkowski spacetime

We want here to give examples of synchronisation in a simple case, namely a 2 dimensional Minkowski spacetime. Thus let us fix $M = \mathbb{R}^2$, a metric $\eta = -c^2\, dt^2 + dx^2$ and Cartesian coordinates (x^0, x) with respect to the metric η, where, as usual, we also set $x^0 := ct$.

We can consider the Riemannian structure (M, η).

Let us consider a clock $\gamma_0 : \mathbb{R} \to M : s \mapsto (x^0 = ct = s, 0)$.

> The curve γ_0 has a covariant velocity $u = \partial_0$ and a covariant acceleration $a = 0$. Then the curve is time-like with respect to the conformal structure $[\eta]$ defined by the Riemannian structure, since $\eta(u, u) = -1$. Then γ_0 are clocks in (M, η).
>
> It is also a η-proper clock, since $\eta(u, u) = -1$, as well as standard for the geometry (M, η), since $a = 0$ (as well as since it is η-proper).
>
> Since $a = 0$, the clock is in free fall in (M, η), and it is also a η-geodesic motion.

Let us consider an event $B = (t_B, x_B)$ (say $x_B > 0$, for simplicity) and consider the echo on B made of the two light rays

$$r_- : \sigma \mapsto (t_B + \sigma, x_B + c\sigma) \qquad r_+ : \sigma \mapsto (t_B + \sigma, x_B - c\sigma) \tag{10.2.17}$$

Those hit the clock at $s_- = t_B - x_B$ and $s_+ = t_B + x_B$ so the time s for B synchronisation is

$$s = \frac{s_+ + s_-}{2} = t_B \tag{10.2.18}$$

If we want the equation for the isochronous surface, we can look for all (t, x) such that $s = s_0$. We obtain the straight lines Σ which solve the equation

$$t_B = s_0 \tag{10.2.19}$$

These, of course, can be extended smoothly to $x_B \leq 0$ and they provide a (in this case global) foliation of Minkowski spacetime; see Fig.9.2.a.

In the same geometry (M, η), we can also consider a different clock ${}^\beta\gamma : \mathbb{R} \to M : s \mapsto (t = \alpha s, x = c\beta\alpha s)$, with $-c < c\beta =: w < c$. That is again a standard (hence proper) freely falling clock.

The curve ${}^\beta\gamma$ is a clock, since it has a covariant velocity $u = \alpha(\partial_t + w\partial_x)$ and a covariant acceleration $a = 0$. Then the curve is time-like with respect to $[\eta]$, since $\eta(u, u) = \alpha^2(w^2 - c^2) < 0$.

It is also a η-proper clock, since $\eta(u, u) = -(c^2 - w^2)\alpha^2$, which is constant, as well as standard for the geometry (M, η).

Since $a = 0$, the clocks are in free fall in (M, η), and they are also η-geodesic motions.

Notice that the clocks γ_0 and ${}^\beta\gamma$, even if they are both standard, have different rates, unless we set $\alpha = c^{-1}(1 - \beta^2)^{-\frac{1}{2}}$. By setting $c^2\alpha^2(1 - \beta^2) = 1$, instead, the two clocks become *identical proper clocks.*

Let us consider an event $B = (t_B, x_B)$ (say, with $x_B - wt_B > 0$, i.e. to the right of the clock) and consider the echo on B made of the two light rays

$$r_- : \sigma \mapsto (t_B + \sigma, x_B + c\sigma) \qquad r_+ : \sigma \mapsto (t_B + \sigma, x_B - c\sigma) \tag{10.2.20}$$

Those hit the clock at $\alpha s_- = t_B + \frac{wt_B - x_B}{c - w} = \frac{ct_B - x_B}{c - w}$ and $\alpha s_+ = t_B + \frac{x_B - wt_B}{c + w} = \frac{ct_B + x_B}{c + w}$ so the time s for B synchronisation is

$$\begin{aligned}
s &= \frac{s_+ + s_-}{2} = \frac{1}{2\alpha}\left(\frac{ct_B - x_B}{c - w} + \frac{ct_B + x_B}{c + w}\right) = \frac{c^2t_B + cwt_B - cx_B - wx_B + c^2t_B - cwt_B + cx_B - wx_B}{2\alpha(c^2 - w^2)} = \\
&= \frac{ct_B - \beta x_B}{1 - \beta^2}\sqrt{1 - \beta^2} = \frac{ct_B - \beta x_B}{\sqrt{1 - \beta^2}}
\end{aligned} \tag{10.2.21}$$

If we want the equation for the isochronous surfaces, we can look for all (t_B, x_B) such that $s = s_0$. We obtain the straight lines ${}^w\Sigma$ which solve the equation

$$ct_B = \beta x_B + \sqrt{1 - \beta^2}\, s_0 \tag{10.2.22}$$

These, of course, can be extended smoothly to the left of the clock and they provide a (in this case global) foliation of Minkowski spacetime; see Fig.9.2.a.

Notice how, when the blue clock ticks, that event is considered by the red clock in the future of the corresponding tick of the red clock. Vice versa, when the red clock ticks that event is considered by the blue clock in the future of the corresponding tick of the blue clock. Hence both clocks see the other clock slowed down by the relative motion, which is in fact a symmetric situation.

Let us consider a clock $\tilde{\gamma} : \mathbb{R} \to M : s \mapsto \left(c^{-1}x_0 \sinh\left(\frac{s}{x_0}\right), x_0\left(\cosh\left(\frac{s}{x_0}\right) - 1\right)\right)$.

Its covariant velocity and acceleration are

$$u = \mathbf{c}^{-1} \cosh\left(\tfrac{s}{x_0}\right) \partial_t + \sinh\left(\tfrac{s}{x_0}\right) \partial_x \qquad a = \tfrac{1}{x_0}\left(\mathbf{c}^{-1}\sinh\left(\tfrac{s}{x_0}\right)\partial_t + \cosh\left(\tfrac{s}{x_0}\right)\partial_x\right) \qquad (10.2.23)$$

It is a clock since its worldline is time-like

$$\eta(u,u) = \left(-\cosh^2\left(\tfrac{s}{x_0}\right) + \sinh^2\left(\tfrac{s}{x_0}\right)\right) = -1 < 0 \qquad \eta(u,a) = 0 \qquad (10.2.24)$$

and it is η-proper and, equivalently, standard.

Of course, $\tilde{\gamma}$ is not a geodesic trajectory, since the acceleration is not parallel to the velocity.

This clock, which is called a *Rindler clock*, is a non-freely falling proper clock, which, by the way, is identical to γ_0.
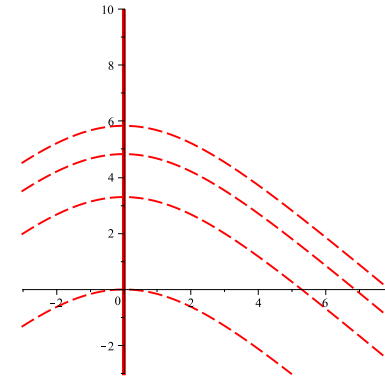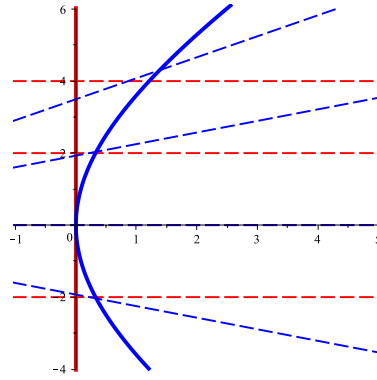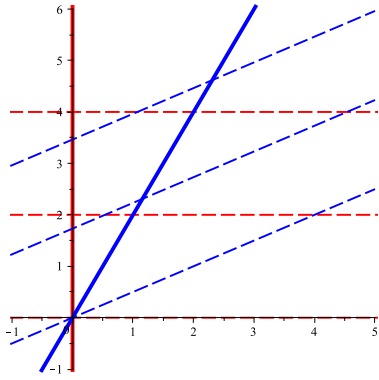
Fig. 9.2: Isochronous surfaces

a) for the clocks γ_0 (red) and $^w\gamma$ (blue)   b) for the clocks γ_0 (red) and $\tilde{\gamma}$ (blue) ($x_0 = ...$)   c) for the clock $\hat{\gamma}$ (blue) ($t_0 = ...$)

Let us consider an event $B = (t_B, x_B)$ (say, in the region $x_B > 0$ and $\mathbf{c}|t_B| < x_B$) and consider the echo on B made of the two light rays

$$r_- : \sigma \mapsto (t_B + \sigma, x_B + \mathbf{c}\sigma) \qquad r_+ : \sigma \mapsto (t_B + \sigma, x_B - \mathbf{c}\sigma) \qquad (10.2.25)$$

Those hit the clock at $s_{\mp} = \pm x_0 \ln\left(\frac{x_0}{x_0 + x_B \mp \mathbf{c}t_B}\right)$ and so the time s for B synchronisation is

$$s = \frac{s_+ + s_-}{2} = \frac{x_0}{2}\ln\left(\frac{x_0 + x_B + \mathbf{c}t_B}{x_0 + x_B - \mathbf{c}t_B}\right) \qquad (10.2.26)$$

If we want the equation for the isochronous surfaces, we can look for all (t_B, x_B) (see Fig. 9.2.b) such that

$$s_0 = \frac{x_0}{2}\ln\left(\frac{x_0 + x_B + \mathbf{c}t_B}{x_0 + x_B - \mathbf{c}t_B}\right) \qquad (10.2.27)$$

which is the equation of the isochronous surfaces in implicit form.

Let us finally consider a clock $\hat{\gamma}_0 : \mathbb{R} \to M : s \mapsto (t_0 \ln(1+s), 0)$ which shares the same trajectory of γ_0, though it has a different parameterisation.

The covariant velocity and acceleration are

$$u = \frac{t_0}{1+s}\partial_t \qquad\qquad a = -\frac{t_0}{(1+s)^2}\partial_t \tag{10.2.28}$$

It is a clock since it is time-like

$$\eta(u,u) = -\mathrm{c}^2\frac{t_0^2}{(1+s)^2} < 0 \qquad\qquad \eta(u,a) = \frac{\mathrm{c}^2 t_0^2}{(1+s)^3} \neq 0 \tag{10.2.29}$$

though it is not proper or standard.

Of course, $\hat{\gamma}_0$ is a geodesic trajectory, since it is a reparameterisation of the geodesic motion γ_0. The clock is hence freely falling, though it is not parameterised to be a geodesic motion, otherwise it would be a proper clock, which it is not.

Let us stress that, not being a proper clock, we do not even have a notion of being identical to another clock, to γ_0 in particular. Even if they share the same worldline, the two clocks are different because of the parameterisation. It is precisely in view of this difference that it is interesting to compare the isochronous surfaces.

Finally, let us remark that the rate of the clock $\hat{\gamma}_0$ become singular at $s = -1$, which however, corresponds to the event $(t = \infty, x = 0)$.

Let us consider an event $B = (t_B, x_B)$ (say, with $x_B > 0$, i.e. to the right of the clock) and consider the echo on B made of the two light rays

$$r_- : \sigma \mapsto (t_B + \sigma, x_B + \mathrm{c}\sigma) \qquad\qquad r_+ : \sigma \mapsto (t_B + \sigma, x_B - \mathrm{c}\sigma) \tag{10.2.30}$$

Those hit the clock at $s_- = \exp\left(\frac{\mathrm{c}t_B + x_B}{\mathrm{c}t_0}\right) - 1$ and $s_+ = \exp\left(\frac{\mathrm{c}t_B - x_B}{\mathrm{c}t_0}\right) - 1$ so the time s for B synchronisation is

$$s = \frac{s_+ + s_-}{2} = \exp\left(\frac{t_B}{t_0}\right)\cosh\left(\frac{x_B}{\mathrm{c}t_0}\right) - 1 \tag{10.2.31}$$

If we want the equation for the isochronous surfaces, we can look for all (t_B, x_B) such that $s = s_0$. We obtain the lines $\hat{\Sigma}$ which solve the equation

$$t_B = t_0 \ln\left(\frac{s_0 + 1}{\cosh\left(\frac{x_B}{\mathrm{c}t_0}\right)}\right) \tag{10.2.32}$$

These, of course, can be extended smoothly to the left of the clock; see Fig.9.2.c.

We have shown that we can compute isochronous surfaces out of a clock and we showed it for some proper and non-proper clocks. As a consequence locally a *coordinate time* is defined in a neighbourhood of a clock. That is not quite the definition of an observer, it is somehow induced by the observer, yet. In fact we defined a time coordinate but we have no way of defining spatial coordinates, yet.

We have, however, to stress that the isochronous surfaces, as well as the coordinate time which comes with them, are a *convention*, they are based on the Einstein's synchronisation convention. They have no fundamental interest, even if they eventually will provide us with a framework that is similar to Newtonian legacy (and its bad habits).

Before going on, let us consider also the same construction in a Weyl frame.

Let us also consider two (inverse) conformal factors $\varphi_1 = e^{2\frac{ct}{x_0}}$ and $\varphi_2 = \frac{x_0}{4ct}$ (say, φ_2 in the upper half plane $t > 0$), which define two metric $g_1 = \varphi_1 \eta$ and $g_2 = \varphi_2 \eta$, which, in turn, define two integral Weyl frames $W_1 = (M, g_1, \{\eta\})$ and $W_2 = (M, g_2, \{\eta\})$.

Let us consider a clock $\gamma_0 : \mathbb{R} \to M : s \mapsto (t = c^{-1}s, 0)$, which we know is η-proper and hence standard.

> The curve is time-like with respect to the conformal structure $\mathfrak{g} = [g_i]$, since $g_i(u, u) = -\varphi_i$, then it is a clock, both in W_1 and in W_2. It is also a η-proper clock, since $\eta(u, u) = -1$. The clock γ_0 is standard for both W_1 and W_2, since $a = 0$, as well as since it is η-proper.

Of course, the clock γ_0 is not g_i-proper. However, we can reparameterise it to be g_i-proper, namely defining two other clocks

$$\gamma_1 : s \mapsto \left(t = \frac{x_0}{c} \ln\left(\frac{x_0 + s}{x_0} \right), x = 0 \right) \qquad \gamma_2 : s \mapsto \left(t = \frac{s^2}{cx_0}, x = 0 \right) \tag{10.2.33}$$

> In fact, we have the velocities

$$u_1 = \frac{c^{-1}x_0}{x_0 + s} \partial_t \qquad u_2 = \frac{2s}{cx_0} \partial_t \tag{10.2.34}$$

> and we have then

$$g_1(u_1, u_1) = -c^2 e^{2\frac{cx_0}{cx_0} \ln\left(\frac{x_0+s}{x_0}\right)} \frac{c^{-2}x_0^2}{(x_0+s)^2} = -1 \qquad g_2(u_2, u_2) = \frac{x_0}{4c}\frac{cx_0}{s^2}\left(-c^2 \frac{4s^2}{c^2 x_0^2} \right) = -1 \tag{10.2.35}$$

> i.e. the clock γ_i is g_i-proper.

Since the acceleration a_i is parallel to the velocity u_i, as well as they are reparameterisations of a η-geodesic motions γ, the clocks γ_i are in free fall, both in W_1 and W_2, though they are not η-geodesic motions. All that is in complete agreement with the theory, of course.

Let us consider an event $B = (t_B, x_B)$ and compute the isochronous surfaces of the clocks γ_i. We shall show the isochronous surfaces at the events $(1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, in red for the clock γ_0, in blue for the clock γ_1, in green for the γ_2.

For the clock γ_1, we can do the usual computation for the message to obtain

$$s_B = x_0 \left(\cosh\left(\frac{x_B}{x_0} \right) e^{\frac{ct_B}{x_0}} - 1 \right) \tag{10.2.36}$$

which provides the implicit equation for the isochronous surfaces, which reproduce the results for the clock $\tilde{\gamma}$.

For the clock γ_2, we obtain

$$s_B = \sqrt{x_0(ct_B - x_B)} + \sqrt{x_0(ct_B + x_B)} \tag{10.2.37}$$

which accounts for the green lines in Fig. 9.3.

Let us stress that the green lines intersect and they do not define a foliation. However, if we consider a minimum $s_m > 0$ and we consider only surfaces for $s > s_m$, the region within the parabola corresponding to the value $s = s_m$ is free of intersection. Hence, the isochronous surfaces define a good foliation in the neighbourhood of the clock, defined inside the parabola corresponding to $s = s_m$.

As the point s_m gets near the axis $t = 0$ the corresponding parabola gets tighten up around the clock and the region becomes smaller and smaller.
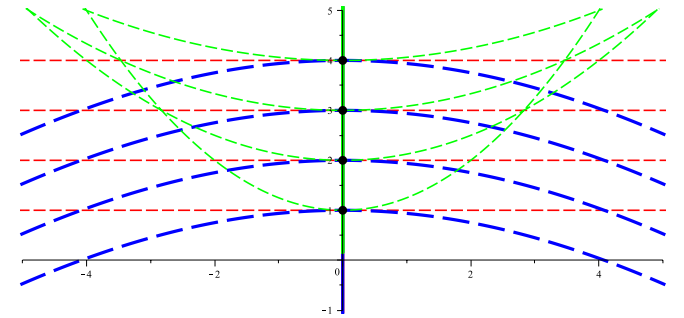
Fig. 9.3: Isochronous surfaces for γ_0 (red), γ_1 blue, γ_2 (green)

Synchronisations in Schwarzschild spacetime

We want to compute the isochronous surfaces in a (2d) Schwarzschild induced by a freely falling NP-clock. We already discussed time-like geodesics in a two dimensional Schwarzschild, which are obtained by setting $k := 0$ and $A = B^{-1} := 1 - \frac{\sigma}{r}$ in equation (9.2.15). Let us choose as a clock the time-like geodesic which arrives at $r = \infty$ with zero kinetic energy (which corresponds to $e = -1$) and through the event $(t_c = 0, r_c = 5\sigma)$. Although in Schwarzschild the orbit is not a parabola, we call it a *parabolic trajectory*, meaning the first escaping trajectory.

We also have an explicit formula for light-like geodesics through an event (t_0, r_0), given by equation (9.5.17).

The parameterisation of the clock is not proper with respect to the Schwarzschild metric. In fact, the clock rate is $g(\gamma', \gamma') = -r$. However, we can reparameterise $r = r(s)$ by choosing

$$\dot\gamma = \gamma' \frac{dr}{ds} \qquad \Rightarrow g(\dot\gamma, \dot\gamma) = -r \left(\frac{dr}{ds}\right)^2 = -1 \qquad \Rightarrow s = \int_{r_c}^{r} \sqrt{r}\, dr = \tfrac{2}{3}\left(r^{\frac{3}{2}} - r_c^{\frac{3}{2}}\right) \qquad (10.2.38)$$

where the dots denote the derivative with respect to proper time s, the primes the derivatives with respect to r. The parameter r_c enters as an integration constant and it corresponds to the parameter at the event in which the clock is set to $s = 0$.

Fig. 9.4: Isochronous surfaces for Schwarzschild (t, r)

> It is definitely inconvenient to try to use proper time parameterisation from the beginning. To obtain geodesics, one needs to integrate the corresponding Weierstrass equation (9.2.15). Being a Weierstrass equation, it is an easy differential equation. It is separable, thus it reduces to an integral. Still, if one wants to integrate it explicitly, one parameterisation is not as good as any other one.
>
> If you write down the equation using proper time s, instead of r, the integral one gets is considerably more difficult to do (meaning to do it otherwise than changing integration variable back to r, of course). The parameter r is as beautiful as it allows an explicit integration. Accordingly, one has a parameterisation of the geodesics $\gamma(s) = [t(r), s(r)]$ in its proper time. Moreover, the worldline parameterised according to proper time—as one would get doing all of it in terms of proper time—would be $t(s) = t(r(s))$ which, in fact, depends on the *inverse* of the parameterisation $s(r)$, which is not always even possible to express in terms of elementary functions. In this case, the function $s(r)$, which is given by the condition (10.2.38), is simple and can be inverted to
>
> $$r = r(s) = \left(\tfrac{3}{2}s + r_0^{\frac{3}{2}}\right)^{\frac{2}{3}} \qquad (10.2.39)$$
>
> However, we shall not use the inverse function $r(s)$, but always the direct one $s(r)$.

It is relatively easy to fix an event $b = (t_b, r_b)$, not on the clock worldline, computing light rays through it and their intersection with the clock. That gives the values of the parameter $r = r_\pm$ at intersection, as well as, by using the expression for the clock worldline, the corresponding coordinate times $t_\pm$. Finally, equation (10.2.38) gives the corresponding proper times $s_\pm$ at the intersections. Then one has the synchronisation time $s_b = \frac{1}{2}(s_+ + s_-)$ which, in fact, selects the isochronous surface on which b is.

However, drawing the surface itself is a completely different story. To do that, one needs to be able to do the integrals for a generic event (leaving its coordinates unknown, so no numeric method is allowed). Then, one has to fix a value for the synchronisation time s_0 and find all points b for which $s_b = s_0$.
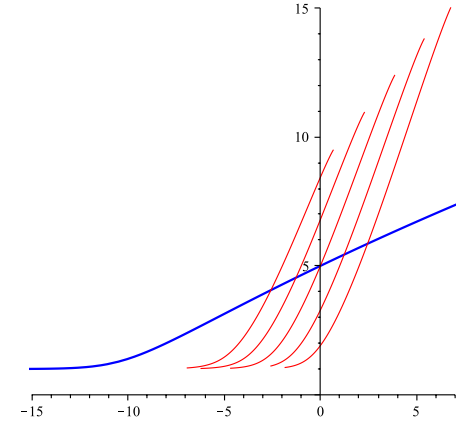
In this specific case, leaving (t_b, r_b) unknown, the equation for the intersection is too hard to be solved analytically and the intersection with the clock cannot be determined (since, in the example, the zeros $r_\pm$ were found numerically).

For that reason, one needs a certain amount of *prestidigitation* to go around this issue. As a matter of fact, if we want to find the surface for $s = s_0$, we already know a point of it, namely the event on the clock worldline corresponding to the proper time $s = s_0$, call it $p_0 = (t(s_0), r(s_0))$. Then, whatever a point b on the surface is, the intersections of an echo with the clock will be at $s_\pm = s_0 \pm \rho$ (otherwise b would not be on the surface corresponding to the proper time s_0) and ρ is a parameter along the surface (which, of course, in this case is a line).

Hence we can proceed backwards, from $s_\pm$ to the intersections $p_\pm$, back to the light rays through them, and to their intersection b. In this way, one finds $b = (t_b(\rho), r_b(\rho))$ which is a parameterisation of the isochronous surface s_0.

From the Fig. 9.4, one can see that the isochronous surfaces are almost like the ones in Minkowski (which are straight lines, not the vertical ones, since the clock is moving). One can see that they significantly deviate from the straight lines near the horizon and, less significantly, away from the clock on the side away of the horizon. One also expects the surfaces to approach vertical lines as the clock goes away from the horizon, since its speed approaches zero.

As a second thought to this issue, one should notice that comparing what happens on Schwarzschild with what *would happen* in Minkowski is much less trivial than it seems. The two spacetimes $(\mathbb{R}^2, \eta)$ and (M, g) are two different manifolds. Also if one regards M as a subset of $\mathbb{R}^2$, it is definitely unclear how to compare what happens in the two different models.

On the other hand, the difference between the two models has to be physically relevant, otherwise we could model the solar system by Minkowski as well as by Schwarzschild. The only way in which one can mathematically compare two different spacetimes (M, g_1) and (M', g') is by selecting a map $\Phi : M \to M'$ and consider the spacetime $(M, g_2 = \Phi^* g)$, i.e. comparing a single manifold M with two different metrics (g_1 and g_2) on it.

Once we have two different metrics on the same manifold, then we can fix coordinates, e.g. (t, r), and have two local expressions of the two metrics in the *same coordinates*, namely

$$g_1 = -A(r)\, dt^2 + B(r)\, dr^2 \qquad\qquad g_2 = -\, dt^2 +\, dr^2 \tag{10.2.40}$$

Out of this setting, one cannot even define what happens to a test particle, one single test particle, assuming either of the two spacetimes. If we compare two different manifolds, one cannot know what it means for a test particle to be prepared with the same initial conditions. It becomes hence impossible to compare the observations in the two models.

Of course, there may be more then one map $\Phi : M \to M'$ which makes the trick. And in those cases, one has different models to compare, one for each of such maps. By selecting a different map, one induces a different correspondence between initial conditions of the test particles and ends up comparing different pairs of test particles. Also in view of this remark, one should also notice that the freely falling clock in Schwarzschild is not freely falling in Minkowski, it is not even an NP-clock in Minkowski spacetime, the light rays are definitely different in the two cases. Depending on the experimental setting, one may mean that the clock is left freely falling staring form the same (i.e. the *corresponding* with respect to the map Φ) initial conditions or that it moves along the same worldline (and, of course, in this case it may be freely falling in one model, not in the other, so we need to imagine to apply forces to maintain it on the desired worldline).

Synchronisations in FLRW spacetimes

In a spatially homogeneous and isotropic spacetime, one can set coordinates (t, χ, θ, ϕ) as in Section 9.4 and restrict to the radial direction, so that the metric becomes

$$g = -c^2\, dt^2 + a^2(t)\, d\chi^2 \tag{10.2.41}$$

The scale factor $a(t)$ is determined by Friedman equation and it depends on the matter content of the universe.

Here we shall discuss, in particular, a spatially flat universe with dust or cosmological constant. The corresponding Friedman equation being

$$\dot{a}^2 = \alpha a^{-1} + \beta a^2 = \frac{\alpha + \beta a^3}{a} =: \Phi(a) \qquad (10.2.42)$$

The methods discussed here, however, generally apply to cosmological solutions, except that computations soon becomes quite difficult to be carried over analytically.

Timelike geodesics are described by equation (9.4.7), the comoving case being obtained by setting $J = 0$. Light rays through the event (t_0, χ_0) are described by equation (9.4.8).

Let us consider a comoving NP-clock $\gamma : \mathbb{R} \to M : s \mapsto (t_0 + s, \chi_0)$.

There is no loss of generality in setting $t_0 = 0$ and $\chi_0 = 0$ so that the clock is set to zero here and now.
One has $\dot{\gamma} = \partial_0$ and $g(\dot{\gamma}, \dot{\gamma}) = -1$, so in fact it is an NP-clock.

Let us then consider an event $b = (t_b, \chi_b)$ and light rays through it.

The intersections of light rays and the comoving clock (in an expanding universe) are

$$\begin{cases} \chi(t) = \chi_b \pm \displaystyle\int_{t_b}^{t} \frac{dt}{a(t)} \\ \chi(t) = \chi_0 = 0 \end{cases} \Rightarrow \chi_b = \mp \int_{t_b}^{t_\mp} \frac{dt}{a(t)} = \mp \int_{a_b}^{a_\mp} \frac{da}{a\sqrt{\Phi(a)}} \qquad (10.2.43)$$

Let us assume $\chi_b > 0$; thus, as usual, the upper sign is for outgoing light rays, the lower sign is for incoming.
If we consider an event b closed enough to the clock, we have $\chi_b \simeq 0$, $t_\pm = t_b \pm \epsilon$, and $a_\pm = a(t_\pm) = a_b \pm \dot{a}_b\epsilon + O(\epsilon^2)$.
Accordingly, we have

$$\chi_b = \mp \int_{a_b}^{a_b \mp \dot{a}_b\epsilon} \frac{da}{a\sqrt{\Phi(a)}} = \frac{\dot{a}_b\epsilon}{a_b\sqrt{\Phi(a_b)}} + O(\epsilon^2) \qquad (10.2.44)$$

Thus, at first order, one has

$$\epsilon \simeq \sqrt{\Phi(a_b)}\frac{\chi_b}{H_b} \qquad (10.2.45)$$

where we set $H_b := \frac{\dot{a}_b}{a_b}$ for the *Hubble parameter* at time t_b. Hence we have

$$s_\pm = t_0 + s_\pm = t_\pm = t_b \pm \epsilon \simeq t_b \pm \sqrt{\Phi(a_b)}\frac{\chi_b}{H_b} \qquad \Rightarrow s_b \simeq \frac{s_+ + s_-}{2} = t_b \qquad (10.2.46)$$

a) dust

b) cosmological constant

Fig. 9.5: Isochronous surfaces for a FLRW spacetime.

The isochronous surface through b is tangent to the coordinate surface $t = t_b$, obviously in b.

Let us, however, stress that this is valid *at first order* and, unless some (really) magic cancellation occurs, in general, it is lost as we are allowed to go far away from the clock. Moreover, once again, it seems reasonable to assume that the way the isochronous surfaces get away from the surfaces $t = t_b$ is something which remembers about the functional form of the scale factor $a(t)$, i.e. of the matter content of the Friedman model.

If we are considering a Palatini $f(\mathcal{R})$-cosmology, that means to depend on the specific model as well, i.e. that is a potential way to observe the effects of the function $f(\mathcal{R})$ one selects.

For showing that cancellations do not occur in general, all we have to do is computing an example exactly. In Fig. 9.5 some isochronous surfaces are presented for a spatially flat, dust universe (see Fig. 9.5a) and for an empty universe with a cosmological constant (see Fig. 9.5b). We see that near the clock at $\chi \sim 0$ they are tangent to the coordinate surfaces $t = t_b$, though, going away from the clock, events in coordinate past are regarded as synchronous.

> Practically speaking, the Einstein synchronisation in cosmology is only of theoretical interest. To actually synchronise an event on a far galaxy we should echo that event with a comoving NP-clock here, which means we should wait few billion years for the message to go and some further billion years for it to come back. At that point, and only *at that point*, we would be able to synchronise the event to the clock.
>
> However, it is interesting to notice that, since the universe in the meanwhile is expanding, it is reasonable to assume that the message takes less time to go than to come back. However, in Einstein synchronisation, we assume the one way time to be half of the two ways time, anyway.

Also notice that, if we had two comoving clocks quite far away, they see each other going slower than when they are closed. That will be the basis for discussing Hubble law.

Spatial distances by a clock

As we discussed in the previous Subsection, a clock induces, through its isochronous surfaces, a (local) notion of time lapse.

We define the *time lapse* between two events a, b in a clock neighbourhood to be the number $s_b - s_b$, if the events a and b lie on the isochronous surfaces labelled by s_a and s_b, respectively. In particular, that applies if the two events lie on a nearby worldline of a particle P.

Accordingly, by assuming a reference clock, we have a specific (local) time ordering of events. That is similar to what absolute time does in a Newtonian setting, though here it is just local and it depends on the reference clock. Even two identical NP-clocks, which are moving one with respect to the other or they simply lie in two different positions, thus in general experiencing different gravitational fields, they do define different time lapses between the same events. In particular, the time lapse between two different ticks of two identical NP-clocks, reciprocally determined on the other clock, are not generally the same.

That may sound strange, though it is something one should learn to live with. It is the main (or one of the main) facts to get along in a relativistic theory.

A similar situation occurs for spatial distances, though, if possible, there the issue is even more involved. In fact, as long as times are considered, at least in some cases, judging the time lapse between two events on a particle is something that one does at a spatial point. The particle on which the events lie does that without referring to anywhere away. If it carries a clock (or receives a clock signal) it has all it takes. On the contrary, defining a spatial distance always extends in space and that makes everything more difficult.

We shall consider two situations: a first simple situation in which one has to judge the spatial distance between two particles, P and Q, at a given time; the second, considerably trickier, in which we want to define the spatial distance between two events.

In the first setting, we have two worldlines, σ_P and σ_Q, and a reference clock γ near them and we want to define the spatial distance between P and Q at a time t_0 with respect to the reference clock γ. Here the most relevant remark is that the very same notion of *spatial distance* is inextricably tied to time: the spatial distance is a distance between two points *at the same time*.

If we want to measure the length of a train during its motion, we have to single out the position of its extremal points *at the same given time* and then measure the distance between these two points.

It is exactly because of this fact that the spatial distance will eventually depend of the observer (or, here, on the reference clock). Two different clocks define two different families of isochronous surfaces and, consequently, they single out two different pairs of events on the worldlines. Then, even though we define the spatial distance by a geometric quantity, still the result is not absolute because each observer measures a different geometric quantity.

That is not different from what happens with covariant conserved quantities. When one defines the *energy* as the integral of the superpotential associated to the symmetry generator $\xi = \partial_0$, the integral *is a geometric quantity*, still different observers define different generators ξ, thus different energies.

Hence it is clear how to proceed in our setting: we have two worldlines and a family of isochronous surfaces defined by the reference clock γ. If we fix a isochronous surface Σ_{t_0} for the time t_0 of the reference clock, the two worldlines single out two points p_0, q_0 on the surface Σ_{t_0}. Then we have two possibilities:

i) if we have a metric g (as we have in a Weyl frame) that induces a spatial metric ${}^*g$ on the isochronous surface Σ_{t_0}. We can consider a geodesic σ on the surface with respect to the induced metric ${}^*g$, which is space-like, and compute its length.

If there are more than one geodesic on the surface joining the points p_0, q_0, we select the one (or one of the ones) for which the distance is minimal. This is called the *geometric distance*.

ii) Since the two points are on Σ_{t_0}, it means they echo a message from the reference clock. The two echoes identify the clock readings $s_\pm(p_0)$ for p_0 and $s_\pm(q_0)$ for q_0 such that

$$t_0 = \frac{s_+(p_0) + s_-(p_0)}{2} \qquad\qquad t_0 = \frac{s_+(q_0) + s_-(q_0)}{2} \tag{10.2.47}$$

Thus let us define

$$\rho(p_0) := \frac{s_+(p_0) - s_-(p_0)}{2} \qquad \rho(q_0) := \frac{s_+(q_0) - s_-(q_0)}{2} \qquad \Rightarrow s_\pm(p_0) = t_0 \pm \rho(p_0) \qquad s_\pm(q_0) = t_0 \pm \rho(q_0) \tag{10.2.48}$$

Accordingly, $2\rho(p_0)$ is the two ways time for p_0 and, by convention, $\rho(p_0)$ is its one way time. Of course, if $\rho(p_0)$ is a *time*, then $d = c\rho(p_0)$ is a distance: it is the distance light would travel in vacuum (in Minkowski spacetime) in a time $\rho(p_0)$. Thus it is (somehow conventional) distance associated to p_0 from the reference clock γ.

At this point, in dimension 2, it is not a surprise to define the *relative distance* between p_0 and q_0 as $d(p_0, q_0) = c|\rho(q_0) - \rho(p_0)|$.

While the geometric distance depends on a metric, the relative distance seems, at first glance, more physically founded. It depends on a reference clock, which, at least, is a physical device. It is conventional but, al least, it depends on conventions on the physical stance, not on a mathematical convention as choosing a metric. Although *any clock* defines (locally) a distance, of course, one can restrict to proper clocks—even better to NP-clocks.

On the other hand, following the EPS discussion, the physical clock does determine a metric. Hence one can ask what is the relation between the relative distance and geometric distance with respect to the metric induced by the reference clock.

Go back to EPS, to equation (6.6.7), and use that to show that, if we use a clock to define the metric following EPS prescription, then the clock is proper with respect to such a metric. This now makes it clear, how a clock defines, in EPS framework, a metric (i.e. a specific representative of the conformal structure) and what representative is selected.

In other words, EPS always produces metrics with respect to which the clock is proper. (up to a factor to be corrected in EPS, an NP-clock)

We would like to discuss whether the relative distance with respect to a clock and the geometric distance with respect to the corresponding metric define the same quantity. Of course, we cannot be too demanding. In EPS, given a clock, one can define a metric on its worldline, only (in fact, to define the metric everywhere, one needs a congruence of clocks). Accordingly, given a clock, one can modify the metric at will out of the worldline, for example by a conformal factor which is 1 on the worldline, still having a NP-clock. This affects the geometric distance, not the relative distance. Hence it seems unreasonable to try to identify the two distances. However, it may be that relative distance and geometric distance of two *infinitesimally closed* particles can be identified (which is perfectly fine to determine for example the relative speed).

Let us consider a clock $\gamma(t)$, its isochronous surfaces Σ_{t_0}, and a local vector field ζ which is tangent to the isochronous surfaces.

> Anything which lives on a isochronous surface, as ζ or the geometric distance, is the closest thing we can have in homogeneous formalism to a *spatial quantity*. One can regard to the isochronous surface as a copy of space which flows in spacetime, the only differences with Newtonian framework being that it is local and each observer defines space in a different way.
>
> Because of this reason, all spatial quantities are not absolute, they are irrelevant at a fundamental level (that is why they come here in Chapter 10 after we covered most of the covariant description of a relativistic world). We have to stress that they come for satisfying our relative inability to get rid of Newtonian legacy and embrace a covariant reality. In particular, we still need a (pseudo)-Newtonian framework to project and interpret experiments and observations.

Let P be a time-like worldline, which cuts each isochronous surface (once and only once). Hence the clock induces a parameterisation through its isochronous surfaces and we have a parameterised curve $\sigma(t)$ which, by the way, is a clock and it is synchronised to the clock γ.

> One could think to be able to fill in spacetime with such synchronised clocks and replace Newtonian absolute time after all. That is partially true: we can fill in a region of spacetime with a fleet of synchronised clocks. However, that is nothing like Newtonian absolute time.
>
> In particular, the clock $\sigma(t)$ is not proper, even when $\gamma(t)$ was. Moreover, even in Minkowski spacetimes, the isochronous surface defined by $\sigma(t)$ do not coincide with the ones defined by γ, as one could tend to believe.
>
> For example, in Minkowski, one can consider $\gamma : t \mapsto (t, 0)$ (which is a proper freely falling clock and it defines $t = t_0$ as isochronous surfaces) and $\sigma : t \mapsto (t, x + vt)$ as a synchronised clock.
>
> The rate of the clock σ is
>
> $$-\omega^2 = -c^2 + v^2 = -c^2 \left(1 - \left(\tfrac{v}{c}\right)^2\right) \tag{10.2.49}$$
>
> Thus σ is not proper (unless $v = 0$). Moreover, one can easily show (by using what we did in Subsection 10.2.2) that the isochronous surfaces of σ are the lines
>
> $$\Sigma'_{t_0} = \{(t, x) : c^2 t = vx + \sqrt{c^2 - v^2}\, t_0\} \tag{10.2.50}$$
>
> which differ from the isochronous surfaces defined by γ.
>
> Thus, unfortunately, two synchronised clocks do not share the same notion of contemporary events. That is true in particular in Minkowski, thus more generally in any EPS spacetime. We need extra conditions for the clock for being synchronised and share the same contemporaneity.

If Φ_ϵ denotes the flow associated to ζ, then we can define the time-like worldlines $\sigma(t; \epsilon) = \Phi_\epsilon \circ \sigma(t)$. Consider a family of time-like curves $\sigma(t; \epsilon)$ and define the two generators of the family

$$\dot{\sigma} = \frac{d\sigma}{dt} \qquad \zeta = \frac{d\sigma}{d\epsilon} \tag{10.2.51}$$

The curves $\sigma(t; \epsilon)$ are integral curves of $\dot{\sigma}$ and the flow of ζ sends one curve of the family into the others. The family foliates a two dimensional surface and both the generators $\dot{\sigma}$ and ζ are vector fields on that surface.

One can expand (possibly, though not necessarily, setting $\epsilon_0 = 0$ and hence $\sigma(t; \epsilon_0) = \sigma(t)$)

$$\sigma(t; \epsilon_0 + \epsilon) = \sigma(t; \epsilon_0) + \epsilon\zeta(t; \epsilon_0) + O(\epsilon^2) \tag{10.2.52}$$

and we understand $\zeta(t; \epsilon_0) := \zeta(\sigma(t; \epsilon_0))$. The quantity $\vec{D} := \epsilon\zeta(t; \epsilon_0)$ is a tangent vector at the point $\sigma(t; \epsilon_0)$, it is tangent to the isochronous surface, and it is called the *infinitesimal displacement* at time t.

> Unlike in Newtonian framework, in general spacetime has no structure of an affine space, thus *finite* displacements are considerably less trivial to be defined. In a sense, in a relativistic setting, it is more fundamental to define finite spatial distances (as the integral of the norm of infinitesimal displacements), while in a Newtonian setting one obtains them as the norm of finite displacements.
>
> Again, one just needs to get rid of bad habits coming from a Newtonian worldview.

There is a geodesic on the isochronous surface Σ_t (with respect to the induced metric ${}^*g$ induced on the surface) joining the points $\sigma(t; \epsilon_0)$ and $\sigma(t; \epsilon_0 + \epsilon)$. Its initial conditions are not $(\sigma(t; \epsilon_0), \zeta(t; \epsilon_0))$, though the difference with $(\sigma(t; \epsilon_0), \zeta(t; \epsilon_0))$ is infinitesimal.

> Essentially, we have two curves, the curve $\sigma(\epsilon) = \sigma(t; \epsilon_0 + \epsilon)$ and the geodesic, joining the two points. The curve $\sigma(\epsilon)$ has initial conditions $(\sigma(t; \epsilon_0), \zeta(t; \epsilon_0))$ and as long as we let $\epsilon \to 0$, the two points get nearer and the tangent to the geodesics approaches the tangent to the curve $\sigma(\epsilon)$.

The length of the geodesic at first order is

$$d := \epsilon\sqrt{g_{\mu\nu}\left(\sigma(t; \epsilon_0)\right)\zeta^\mu(t; \epsilon_0)\zeta^\nu(t; \epsilon_0)} + O(\epsilon^2) \simeq \epsilon|\zeta(t; \epsilon_0)| = |\vec{D}| \tag{10.2.53}$$

> The fact that we are integrating along a geodesics is irrelevant since the geodesic equation constrains second derivatives of the curve, though the length, at first order, only depends on the first derivatives.

Thus the quantity $D := \epsilon|\zeta(t; \epsilon_0)|$ is the *infinitesimal geometric distance* and it coincides with the length of the infinitesimal displacement.

On the other hand, if we set $\epsilon_0 = 0$ and $\sigma(t; 0) = \gamma(t)$, i.e. we consider $\sigma(t; 0)$ as the reference clock, then we have the infinitesimal displacement between $\sigma(t; 0)$ and $\sigma(t, \epsilon)$ and we can stay around the point $\sigma(0; 0)$, where for example normal coordinates exists.

> In this situation we have that the relative distance is a first order approximation of the geometric distance.
>
> To see that, we can fix normal coordinates around the point $\sigma(0; 0)$. The two light rays echoing on the event $p = \sigma(0; \epsilon)$ are approximately as the light rays in Minkowski. They approximately hit the clock $\gamma = \sigma$ at $p_\pm = (\pm\epsilon|\zeta|, 0)$. Consequently, the relative distance will be approximately $\rho = \epsilon|\zeta|$.
>
> To be rigorous one only needs to show that this result is true at order 1, i.e., for example, the *"difference"* between the clock readings in Minkowski and in the original spacetime are infinitesimal of order 2, as $\epsilon \to 0$.

Hence d is also the infinitesimal relative distance D. Accordingly, the relative distance is to be identified with the first order geometric distance. The two distances are the same for infinitesimally closed particles, while the geometric distance is defined for finite distances, as far as one can define a isochronous surface foliation.

> Even for having Cauchy theorem, one sometimes restricts to globally hyperbolic spacetimes, which allow by definition a global ADM foliation.
>
> Most of the construction above can be repeated for any space-like foliation, being it or not the isochronous foliation for a clock. Also in that case we define a spatial distance (with respect to the ADM foliation) which is global, although one may loose contact with an operational protocol associated to some physical clock. In fact,

an ADM foliation does transform any worldline in a clock, although the ADM leaves are not necessarily isochronous surfaces for the endowed clock, and in general, the clock is not a proper clock.

If this is the case, even though we do define a good notion of spatial distance with respect to an ADM foliation, it is not clear its relation with what one does in practice when everything is done by using an atomic clock.

In SR all the subtleties about definition of time lapses and distances are replaced by a an observer-dependent definition which at least is easy to compute thanks to the affine structure of spacetime. In a more general spacetime, unfortunately, the corresponding definitions are in general quite difficult to compute, since they require quite a precise account of how light rays move around. Just for that, thanks to normal coordinates, one recovered the same notion for infinitesimal displacements. Unfortunately, one cannot simply integrate the infinitesimal result to obtain a finite result; in fact, when we consider an infinitesimal displacement which is finitely far away from a clock γ (as we would do above considering the displacement from $\sigma(t; \epsilon_0)$ and $\sigma(t; \epsilon_0 + \epsilon)$, though not setting $\epsilon_0 = 0$ or idetifying $\gamma(t) = \sigma(t; \epsilon_0)$) We still should follow echoes from $\sigma(t; \epsilon_0 + \epsilon)$ to γ, i.e. following light rays form a finite step. Or equivalently, one would need to change the reference clock from step to step, thus obtaining a quantity which does not even depend on a single clock.

3. Spatial kinematics

Now that we have a definition for spatial distance we can define the covariant counterparts of speed, velocity and acceleration. Once again the issue is not straight forward, indicating once again how fundamentally poor is our spatial intuition.

First of all, one can easily get convinced that as long, as velocities are concerned, infinitesimal displacements are more relevant than distances. Unfortunately, *easily* does not mean *truly*; the displacements itself is a distance measurement after all (with an extra information about the direction).

Here after we shall define the displacement itself as the vector field which infinitesimally accounts for dragging one worldline into another and such that its norms accounts for the distances as defined above.

As a notation, we devote vectors as $\vec{v}$ whenever they happen to be tangent to isochronal surfaces. They are spacetime vectors, though quite close to what a spatial vector would be.

Derivative of a scalar function

Let us consider a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, a scalar function $f : M \to \mathbb{R}$, and a clock $\gamma(t)$. The function can be restricted to the clock and define a function $f_\gamma : \mathbb{R} \to \mathbb{R} : t \mapsto f(\gamma(t))$.

For later convenience, we want to expand the function f_γ in Taylor series around the value $t = 0$.

For the coefficients of the Taylor expansion we need to compute the time derivatives of the scalar function f_γ. Since f_γ is a scalar the time derivative is definitely given by $\dot\gamma^\alpha(t)\overset{*}{\tilde\nabla}_\alpha$, where the covariant derivative can be consider with respect to any connection, depending on convenience. We have

$$
\begin{aligned}
f_\gamma(t) &= f(\gamma(t)) \\
\dot f_\gamma(t) &= \dot\gamma^\alpha(t)\partial_\alpha f(\gamma(t)) = \dot\gamma^\alpha(t)\overset{*}{\tilde\nabla}_\alpha f(\gamma(t)) \\
\ddot f_\gamma(t) &= \ddot\gamma^\alpha(t)\partial_\alpha f(\gamma(t)) + \dot\gamma^\alpha(t)\dot\gamma^\beta(t)\partial_{\alpha\beta} f(\gamma(t)) = \left(\ddot\gamma^\lambda(t) + \tilde\Gamma^\lambda_{\alpha\beta}\dot\gamma^\alpha(t)\dot\gamma^\beta(t)\right)\tilde\nabla_\lambda f(\gamma(t)) + \dot\gamma^\alpha(t)\dot\gamma^\beta(t)\tilde\nabla_{\alpha\beta} f(\gamma(t))
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow\quad f_\gamma(0) = f(x) \\
&\Rightarrow\quad \dot f_\gamma(0) = v^\alpha\tilde\nabla_\alpha f(x) \\
&\Rightarrow\quad \ddot f_\gamma(0) = a^\lambda_{\tilde\Gamma}\tilde\nabla_\lambda f(x) + v^\alpha v^\beta\tilde\nabla_{\alpha\beta} f(x)
\end{aligned}
\tag{10.3.1}
$$

where we set $x = \gamma(0)$ and $v = \dot\gamma := \dot\gamma(x)$.

Thus the Taylor expansion of the function f_γ reads as

$$
f_\gamma(t) = f(x) + v^\alpha\tilde\nabla_\alpha f(x)t + \tfrac{1}{2}\left(a^\lambda_{\tilde\Gamma}\tilde\nabla_\lambda f(x) + v^\alpha v^\beta\tilde\nabla_{\alpha\beta} f(x)\right)t^2 + O(t^3)
\tag{10.3.2}
$$

A special case of interest is expanding the norm of a vector field ζ, i.e. choosing the scalar function $f_\gamma(t) = g(\zeta(t),\zeta(t)) = g_{\mu\nu}(\gamma(t))\zeta^\mu(\gamma(t))\zeta^\nu(\gamma(t))$.

Let us here introduce some short notation. We shall set $\tilde\nabla := v^\alpha\tilde\nabla_\alpha$, $\tilde\nabla^2 := v^\alpha\tilde\nabla_\alpha\left(v^\beta\tilde\nabla_\beta\right)$, and $\alpha := \boldsymbol\alpha(v)$. Also remember that, because of EPS-compatibility, $\tilde\nabla g_{\mu\nu} = \alpha g_{\mu\nu}$. In this case, we have

$$
\begin{aligned}
f(x) &= g(\zeta,\zeta) \\
\tilde\nabla f(x) &= 2g\left(\tilde\nabla\zeta,\zeta\right) + \alpha g(\zeta,\zeta) = 2g\left(\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta,\zeta\right)
\end{aligned}
\qquad \Rightarrow D\zeta := \tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta
\tag{10.3.3}
$$

where we set $x = \gamma(0)$ and $\zeta = \zeta(\gamma(0)) = \zeta(x)$.

As long as the second coefficient is concerned, we have that $a^\lambda = v^\alpha\tilde\nabla_\alpha v^\lambda = \ddot\gamma^\lambda + \tilde\Gamma^\lambda_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta$ so that one has

$$
a^\beta\tilde\nabla_\beta f + v^\alpha\tilde v^\beta\tilde\nabla_{\alpha\beta} f = v^\alpha\tilde\nabla_\alpha v^\beta\tilde\nabla_\beta f + v^\alpha\tilde v^\beta\tilde\nabla_{\alpha\beta} f = v^\alpha\tilde\nabla_\alpha\left(v^\beta\tilde\nabla_\beta f\right) = \tilde\nabla^2 f
\tag{10.3.4}
$$

Hence the second coefficient is

$$
\begin{aligned}
\tilde\nabla^2 f(x) &= 2g\left(\tilde\nabla^2\zeta + \tfrac{1}{2}\alpha\tilde\nabla\zeta + \tfrac{1}{2}\tilde\nabla\alpha\zeta,\zeta\right) + 2g\left(\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta,\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta\right) + 2\alpha g\left(\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta,\zeta\right) - \alpha g\left(\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta,\zeta\right) = \\
&= 2g\left(\tilde\nabla^2\zeta + \alpha\tilde\nabla\zeta + \tfrac{1}{2}\tilde\nabla\alpha\zeta + \tfrac{1}{4}\alpha^2\zeta,\zeta\right) + 2g\left(D\zeta, D\zeta\right)
\end{aligned}
\qquad \Rightarrow D^2\zeta := \tilde\nabla^2\zeta + \alpha\tilde\nabla\zeta + \tfrac{1}{4}\alpha^2\zeta + \tfrac{1}{2}\tilde\nabla\alpha\zeta
\tag{10.3.5}
$$

Since the second coefficient reads as $\tilde\nabla^2 f(x)$, the third one is obtained from

$$
\begin{aligned}
\dot{\ddot f}_\gamma(t) = \tilde\nabla\left(\tilde\nabla^2 f(x)\right) &= 2g\left(\tilde\nabla^3\zeta + \tilde\nabla\alpha\tilde\nabla\zeta + \alpha\tilde\nabla^2\zeta + \tfrac{1}{2}\tilde\nabla^2\alpha\zeta + \tfrac{1}{2}\tilde\nabla\alpha\tilde\nabla\zeta + \tfrac{1}{4}\alpha^2\tilde\nabla\zeta + \tfrac{1}{2}\alpha\tilde\nabla\alpha\tilde\nabla\zeta,\zeta\right) + 6g\left(\tilde\nabla^2\zeta + \alpha\tilde\nabla\zeta + \tfrac{1}{2}\tilde\nabla\alpha\zeta + \tfrac{1}{4}\alpha^2\zeta,\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta\right) + \\
&\quad {\color{red}{-2g\left(\alpha\tilde\nabla\zeta + \tfrac{1}{2}\alpha^2\zeta,\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta\right)}} + g\left(\alpha\tilde\nabla^2\zeta + \tfrac{1}{2}\alpha\tilde\nabla\alpha\zeta + \alpha^2\tilde\nabla\zeta + \tfrac{1}{4}\alpha^3\zeta,\zeta\right) + {\color{red}{2g\left(\alpha\tilde\nabla\zeta + \tfrac{1}{2}\alpha^2\zeta,\tilde\nabla\zeta + \tfrac{1}{2}\alpha\zeta\right)}} = \\
&= 2g\left(\tilde\nabla^3\zeta + \tfrac{3}{2}\tilde\nabla\alpha\tilde\nabla\zeta + \tfrac{3}{2}\alpha\tilde\nabla^2\zeta + \tfrac{1}{2}\tilde\nabla^2\alpha\zeta + \tfrac{3}{4}\alpha^2\tilde\nabla\zeta + \tfrac{3}{2}\alpha\tilde\nabla\alpha\zeta + \tfrac{1}{8}\alpha^3\zeta,\zeta\right) + 6g\left(D^2\zeta, D\zeta\right)
\end{aligned}
$$
$$
\Rightarrow D^3\zeta := \tilde\nabla^3\zeta + \tfrac{3}{2}\tilde\nabla\alpha\tilde\nabla\zeta + \tfrac{3}{2}\alpha\tilde\nabla^2\zeta + \tfrac{1}{2}\tilde\nabla^2\alpha\zeta + \tfrac{3}{4}\alpha^2\tilde\nabla\zeta + \tfrac{3}{2}\alpha\tilde\nabla\alpha\zeta + \tfrac{1}{8}\alpha^3\zeta
\tag{10.3.6}
$$

Thus we can define the Taylor expansion of the vector field

$$\zeta(\gamma(t)) := \zeta + tD\zeta + \tfrac{1}{2}t^2 D^2\zeta + \tfrac{1}{3!}t^3 D^3\zeta + O(t^4) \tag{10.3.7}$$

so that

$$
\begin{aligned}
g(\zeta(t),\zeta(t)) =& g_{\mu\nu}(x)\zeta^\mu(t)\zeta^\nu(t) = g\left(\zeta + tD\zeta + \tfrac{1}{2}t^2 D^2\zeta + \tfrac{1}{3!}t^3 D^3\zeta, \zeta + tD\zeta + \tfrac{1}{2}t^2 D^2\zeta + \tfrac{1}{3!}t^3 D^3\zeta\right) + O(t^4) = \\
=& g(\zeta,\zeta) + 2g(D\zeta,\zeta)t + \tfrac{1}{2}t^2\left(2g(D^2\zeta,\zeta) + 2g(D\zeta,D\zeta)\right) + \tfrac{1}{3!}t^3\left(2g(D^3\zeta,\zeta) + 6g(D^2\zeta,D\zeta)\right) + O(t^4)
\end{aligned} \tag{10.3.8}
$$

Actually, we are more interested in the distance $D = \epsilon|\zeta|$, rather than the norm $|\zeta|^2$. Of course, we have

$$
\begin{aligned}
D =& \epsilon|\zeta(t)| = \epsilon\sqrt{g(\zeta(t),\zeta(t))} = \epsilon|\zeta|\bigg(1 + \frac{g(D\zeta,\zeta)}{|\zeta|^2}t + \frac{4\left(g(D^2\zeta,\zeta) + g(D\zeta,D\zeta)\right)|\zeta|^2 - (g(D\zeta,\zeta))^2}{8|\zeta|^4}t^2 + \\
&+ \frac{8|\zeta|^4\left(g(D^3\zeta,\zeta) + 3g(D^2\zeta,D\zeta)\right) - 12g(D\zeta,\zeta)\left(g(D^2\zeta,\zeta) + g(D\zeta,D\zeta)\right)|\zeta|^2 + 3\left(g(D\zeta,\zeta)\right)^3}{48|\zeta|^6}t^3\bigg) + O(t^4)
\end{aligned} \tag{10.3.9}
$$

In the case of a Riemannian structure, i.e. when $\boldsymbol{\alpha} = 0$, that reduces to

$$\zeta(t) := \zeta(\gamma(t)) := \zeta + t\nabla\zeta + \tfrac{1}{2}t^2\nabla^2\zeta + \tfrac{1}{3!}t^3\nabla^3\zeta + O(t^4) \tag{10.3.10}$$

and

$$g(\zeta(t),\zeta(t)) = g(\zeta,\zeta) + 2g(\nabla\zeta,\zeta)t + \tfrac{1}{2}t^2\left(2g(\nabla^2\zeta,\zeta) + 2g(\nabla\zeta,\nabla\zeta)\right) + \tfrac{1}{3!}t^3\left(2g(\nabla^3\zeta,\zeta) + 6g(\nabla^2\zeta,\nabla\zeta)\right) + O(t^4) \tag{10.3.11}$$

so that the corrections in α which appear in the definition of $D^k\zeta$ take full care of Weyl structure. Such corrections may be a bit tedious to compute though it is clear that they can be obtained by iteration.

In fact, the whole computation can be regarded as a way to define the expansion of the displacement ζ from the expansion of its norm $|\zeta|^2$, which in some sense, is physically more fundamental. In particular, we shall obtain a covariant expression for velocity and acceleration from the expansion of the displacement.

Relative velocity and acceleration

Let γ be a reference clock in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, Σ_{t_0} its isochronous surfaces, $\sigma : t \mapsto \sigma(t; \epsilon_0)$ a particle P, ζ a vector field tangent to the surfaces Σ_{t_0} and $\sigma(t; \epsilon_0 + \epsilon)$ a dragging of the particle P along the vector field ζ.

The particles $\sigma(t; \epsilon_0 + \epsilon)$ are parameterised by t which, by construction, is the clock reading of the reference clock γ. As we discussed, the spatial infinitesimal displacement between two particles $\sigma(t; \epsilon_0)$ and $\sigma(t; \epsilon_0 + \epsilon)$ is $\vec{D}(t) = \epsilon\zeta(t)$. As we discussed, that is the closest thing to a spatial distance we can define, it depends on the reference clock, though it is canonically constructed with it.

We have the infinitesimal displacement (which is a spatial length) in time (which is a clock-defined time).

Thus we can define the *relative velocity* as the derivative of space with respect to time, nemaley

$$\vec{V}(t) = \epsilon D\zeta(t) = \epsilon \left(\dot{\gamma}^\lambda \tilde{\nabla}_\lambda \zeta + \tfrac{1}{2}\boldsymbol{\alpha}(\dot{\gamma})\zeta(t) \right) \tag{10.3.12}$$

The *relative acceleration* is defined as

$$\begin{aligned}
\vec{A}(t) =& \epsilon \left(\dot{\gamma}^\alpha \tilde{\nabla}_\alpha \left(\dot{\gamma}^\beta \tilde{\nabla}_\beta \zeta \right) + \boldsymbol{\alpha}(\dot{\gamma})\dot{\gamma}^\alpha \tilde{\nabla}_\alpha \zeta + \tfrac{1}{4}\boldsymbol{\alpha}(\dot{\gamma})^2 \zeta + \tfrac{1}{2}\dot{\gamma}^\alpha \tilde{\nabla}_\alpha \left(\boldsymbol{\alpha}(\dot{\gamma}) \right) \zeta \right) = \\
=& \epsilon \left(\dot{\gamma}^\alpha \dot{\gamma}^\beta \tilde{\nabla}_{\alpha\beta} \zeta + \left(\ddot{\gamma}^\alpha + \tilde{\Gamma}^\alpha_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu \right) \tilde{\nabla}_\alpha \zeta + \boldsymbol{\alpha}(\dot{\gamma})\dot{\gamma}^\alpha \tilde{\nabla}_\alpha \zeta + \tfrac{1}{4}\boldsymbol{\alpha}(\dot{\gamma})^2 \zeta + \tfrac{1}{2}\dot{\gamma}^\alpha \tilde{\nabla}_\alpha \left(\boldsymbol{\alpha}(\dot{\gamma}) \right) \zeta \right)
\end{aligned} \tag{10.3.13}$$

Both the relative velocity and the relative acceleration are vectors tangent to the surfaces Σ_t.

Notice that, if the clock is not freely falling, it is affected by external forces which affects the acceleration $a^\alpha_{\tilde{\Gamma}} = \ddot{\gamma}^\alpha + \tilde{\Gamma}^\alpha_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu$ and consequently the relative acceleration. The contribution from external forces is of the same type of the one coming from the possible effects of the clock not being standard, namely the terms depending on $\boldsymbol{\alpha}(\dot{\gamma})$, which in fact vanishes in a Riemannian structure, in particular in standard GR. It will not be a surprise to see that the fact that the clock is not standard could be modelled as the action of an effective external force acting universally on all particles.

In a Palatini $f(\mathcal{R})$-theory, the exact expression of the effective force acting depends on the conformal factor—which encodes $\boldsymbol{\alpha}$—i.e., ultimately, depends on the specific model $f(\mathcal{R})$ one is considering.

In a Riemannian structure, i.e. when $\boldsymbol{\alpha} = 0$ and, consequently, $\tilde{\Gamma} = \{g\}$, both the relative velocity and acceleration reduce to the corresponding quantities used in standard GR, namely

$$\vec{V}(t) = \epsilon \dot{\gamma}^\lambda \tilde{\nabla}_\lambda \zeta \qquad\qquad \vec{A}(t) = \epsilon \left(\dot{\gamma}^\alpha \dot{\gamma}^\beta \tilde{\nabla}_{\alpha\beta} \zeta + a^\alpha_{\tilde{\Gamma}} \tilde{\nabla}_\alpha \zeta \right) = \dot{\gamma}^\alpha \nabla_\alpha \left(\vec{V}(t) \right) \tag{10.3.14}$$

These further specialise to SR, if one considers Minkowski spacetime.

The relative velocity and acceleration defined in (10.3.13) accounts for the spatial quantities measured between two nearby particles. The corrections in $\boldsymbol{\alpha}$ accounts for the effect of atomic reference clock possibly being not standard.

As it happens in Newtonian framework, one can also define a *velocity* and an *acceleration* for one particle, by using the clock itself as a reference point, i.e. specialising for $\gamma(t) = \sigma(t; \epsilon_0)$.

Let us remark that in a Newtonian reference frame, one defines the velocity and the acceleration *with respect to the (inertial) frame*, which are the quantities with respect to a particles which did not experience forces. In an electromagnetic situation one would define the acceleration of a charge with resect to a neutral particle which, not being affected by the electromagnetic field, would move in a uniform straight motion.

That is possible because one has neutral particles in the first place, i.e. particles which are not affected by the electromagnetic field. That is not what happens with the gravitational field, which is *universal*, i.e. it affects any particle—and any particle in the same way.

On the other hand, in a gravitational theory, one has the gravitational field which acts on any particle, no neutral charges. However, we can take advantage of the other part of the weak equivalence principle: the gravitational field acts *in the same way* on any particle, any particle has the same free fall. Hence also in a gravitational theory one has a well-defined family of motions (actually better defined than *inertial* motions): the free falling motions. Accordingly, one can define the velocity and acceleration of a particle *with respect to a freely falling one*, i.e. a particles with $a_{\tilde{\gamma}} = \lambda v$, which in fact moves along a geodesic motion.

Hence we have the definition of the *velocity* and *acceleration* of a particle near a reference clock γ

$$\vec{V}(t) = \epsilon \dot{\gamma}^\lambda \tilde{\nabla}_\lambda \zeta \qquad \vec{A}(t) = \epsilon \left(\dot{\gamma}^\alpha \dot{\gamma}^\beta \tilde{\nabla}_{\alpha\beta} \zeta \right) \tag{10.3.15}$$

where now ζ is the infinitesimal displacement from the clock to the particle.

The acceleration is related to the *"external forces"* (all forces other than the gravitational field) acting on the particle (though not on the freely falling reference clock).
Of course, the gravitational field itself cannot be associated with an external forces. That is what they mean saying that gravity is not a force.
Of course, different clocks define different external forces, thus external forces are not an absolute notion, they are conventional. That is what they mean when they say that *forces* (as well as the electric field, velocities, accelerations) are vectors *in space*, not in spacetime, hence they do not exist in a covariant theory.

Let us remark that both external forces and deviation from proper clocks produce non-zero acceleration, so that it will not be a surprise that regarding a clock as a proper clock as a standard one can be confused with an extra effective force acting on the clock.

Again, notice that here we did not even mention SR or Minkowski spacetime (if not as a source of counter examples which are easy to compute). Traditionally, these protocols specialise to the definition of spatial distance with resect to an inertial observer in SR, and then are generalised somehow to standard GR. Here we are giving the protocols essentially in a general Weyl frame (and then specialise them to simple situations) as one should always do.

From an equivalent viewpoint, we considered a clock γ an event b (out of its worldline) and defined two quantities

$$s_b = \frac{s_+ + s_-}{2} \qquad \rho_b = \frac{s_+ - s_-}{2} \tag{10.3.16}$$

These are somehow associated to a time and a distance.

Let us remark that we can compute the quantity

$$-s_b^2 + \rho_b^2 = \tfrac{1}{4} \left(-(s_+ + s_-)^2 + (s_+ - s_-)^2 \right) = \tfrac{1}{4} \left(-4 s_+ s_- \right) = -s_+ s_- = G(p) \tag{10.3.17}$$

which somehow clarifies the definition of the function G defined in EPS, as well as why one should expect a relation with the metric.

In general, the two quantities (s_+, s_-), as well (s_b, ρ_b) are not coordinates on spacetime; they are on two dimensional spacetimes. Accordingly, a clock defines an observer on two dimensional spacetimes, not on general spacetimes, yet. On the other hand, from EPS we know that in general, using n-clocks one can define a coordinate system, hence an obvserver. In order to do that in detail, we need to discuss relativistic positioning systems.

Examples of accelerated motions

We already considered special test particles in some spacetimes and, as an example, we are now able to discuss their velocity and acceleration.

We considered a Minkowski spacetime $(\mathbb{R}^m, \eta)$ and *freely falling test particles* (also known as *uniform straight motions*) on it. In Cartesian coordinates $x^\mu = (t, x^i)$, straight motions are $\sigma : \mathbb{R} \to M : s \mapsto (s + s_0, x_0^i + w^i s)$ for some $0 \le |\vec{w}| < c$.

In dimension $m = 2$, if we consider an NP-clock $\gamma : \mathbb{R} \to M : s \mapsto (s, 0)$, it induces isochronous surfaces $\Sigma_{t_0} = \{(t, x) : t = t_0\}$. Let us consider a different straight motion $\sigma : t \mapsto (t, wt)$ as the worldline of a particle P. One can regard $\sigma(t)$ as a deformation of the clock worldline, i.e. embed it into a 1-parameter family of particles

$$\sigma_\epsilon : \mathbb{R} \to M : t \mapsto (t, \epsilon w t) \tag{10.3.18}$$

This dragging induces space curves

$$\sigma_t : \mathbb{R} \to M : \epsilon \mapsto (t, \epsilon wt) \tag{10.3.19}$$

which should be integral curves of the infinitesimal displacement ζ to be defined. In fact, defining the dragging is equivalent to defining the infinitesimal displacement $\zeta(t, x) = wt \partial_x$.

The curves (10.3.19) foliates a region in spacetime. Let us consider a point (t_0, x_0). There is a curve $\sigma_t(\epsilon)$ through it, namely the one with $t = t_0$ and $\epsilon = \epsilon_0 = \frac{x_0}{wt_0}$. Its tangent vector $\dot{\sigma}_t(\epsilon_0)$, which coincides with the generator $\zeta(t_0, x_0)$, is

$$\zeta(t_0, x_0) = \dot{\sigma}_t(\epsilon_0) = wt_0 \partial_{x_0} \tag{10.3.20}$$

By focusing on the region $t \geq 0$, $x \geq 0$, assuming $w \geq 0$ and $\epsilon \geq 0$, the relative distance, the relative velocity, the relative acceleration of P from the clock γ are defined as

$$D(t) = \epsilon wt \qquad \vec{V}(t) = \partial_0(\epsilon wt) = \epsilon w \qquad \vec{A}(t) = 0 \tag{10.3.21}$$

In this case, the geometric distance between the points $\gamma(t)$ and $\sigma(t)$ is easy to be determined

$$d(t) = wt \qquad \Rightarrow \quad v(t) = \partial_0(wt) = w \qquad a(t) = 0 \tag{10.3.22}$$

Taking into account that the infinitesimal displace is defined to measure the distance between $\gamma(t)$ and $\sigma_\epsilon(t)$ the two notions, in this case, essentially coincide and they both coincide with what we are used to take as the physical distance. Moreover, they certainly account for the name *uniform straight motions*.

However, before considering less trivial cases, let us point out that the agreement between infinitesimal displacement and geometric distance is just apparent in general. In fact, unlike the geometric distance, the infinitesimal displacement depends on the dragging one chooses in the beginning to connect the two particles.

To see that, let us consider a different dragging

$$\sigma_{\hat{\epsilon}} : \mathbb{R} \to M : t \mapsto \left(t, \left(e^{\hat{\epsilon}} - 1\right) wt\right) = \sigma_t(\hat{\epsilon}) \tag{10.3.23}$$

which also connects $\gamma(t)$ and $\sigma(t)$, though, instead of with $\epsilon \in [0, 1]$, this time with $\hat{\epsilon} \in [0, \ln(2)]$.
As in the previous case, one can fix a point (t, x) and find an integral curve $\sigma_{(t,x)}(\hat{\epsilon})$ through it, namely

$$\sigma_{(t,x)}(\hat{\epsilon}) = \left(t, x + \left(e^{\hat{\epsilon}} - 1\right) wt\right) \qquad \sigma_{(t,x)}(0) = (t, x) \tag{10.3.24}$$

These integral curves define a different generator $\hat{\zeta}(t, x) := (x + wt) \partial_x$, which is still tangent to the isochronous surfaces. Its integral curves define the flow

$$\Phi_{\hat{\epsilon}} : M \to M : (t, x) \mapsto \left(t, e^{\hat{\epsilon}} x + \left(e^{\hat{\epsilon}} - 1\right) wt\right) \tag{10.3.25}$$

which in fact is a flow and it drags the reference clock γ into the curves $\sigma_{\hat{\epsilon}}(t) = \left(t, \left(e^{\hat{\epsilon}} - 1\right) wt\right)$ which are (uniform straight) motions (through the origin) with speed $\hat{w} := \left(e^{\hat{\epsilon}} - 1\right) w$.
Thus we have the infinitesimal displacement $D(t)$ to be

$$\vec{D}(t) = \hat{\epsilon} \hat{\zeta} \circ \gamma \qquad \Rightarrow D(t) = \hat{\epsilon} |wt| \tag{10.3.26}$$

The corresponding *relative velocity* and *relative acceleration* are

$$\vec{V}(t) = \hat{\epsilon}|w| \qquad\qquad \Rightarrow \vec{A}(t) = 0 \tag{10.3.27}$$

We also have the *geometric distance* between $\gamma(t)$ and $\sigma_{\hat{\epsilon}}(t)$, namely

$$d(t) = |\sigma_{\epsilon}(t) - \gamma(t)| = \left(e^{\hat{\epsilon}} - 1\right)wt = \hat{w}t \simeq \hat{\epsilon}wt + \tfrac{1}{2}\hat{\epsilon}^2 wt + O(\hat{\epsilon}^3) = D(t) + O(\hat{\epsilon}^2) \tag{10.3.28}$$

Hence, in this case, geometric distance and infinitesimal displacement coincide only at first order (i.e. with small $\hat{\epsilon}$).

Let us stress that the motion we are considering, which is $x(t) = \hat{w}t$, supports the fact that the geometric distance is, in fact, a better physical distance, also in SR. In the first example, the two distances agree, just because the dragging is linear in the parameter ϵ meaning that ϵ is the ratio between the geometric distance between $\gamma(t)$ and $\sigma_\epsilon(t)$ and the geometric distance between $\gamma(t)$ and $\sigma(t)$. The dragging is defined taking ϵ constant with respect to time. Once again, geometric distance is more fundamental than the infinitesimal displacement.

The second example we would like to discuss is the kinematics of Rindler particles

$$\sigma_\alpha : \mathbb{R} \to M : t \mapsto \left(t, \tfrac{1}{\alpha}\left(\sqrt{1 + \alpha^2 t^2} - 1\right)\right) = \sigma_t(\alpha) \tag{10.3.29}$$

with respect to the same freely falling NP-clock $\gamma(t)$.

The Rindler particle through the point (t_0, x_0) is obtained for $t = t_0$ and $\alpha = \alpha_0 = \frac{2x_0}{t_0^2 - x_0^2}$. The tangent vector is

$$\frac{d\sigma_t}{d\alpha}(\alpha_0) = \frac{1}{2}\frac{(t_0^2 - x_0^2)^2}{t_0^2 + x_0^2}\partial_x \qquad\qquad \Rightarrow \zeta(t, x) = \frac{1}{2}\frac{(t^2 - x^2)^2}{t^2 + x^2}\partial_x \tag{10.3.30}$$

The infinitesimal displacement and the relative quantities are

$$D(t) = \tfrac{1}{2}\alpha t^2 \qquad\qquad \vec{V}(t) = \alpha t \qquad\qquad \vec{A}(t) = \alpha \tag{10.3.31}$$

The corresponding geometric quantities are

$$d = \frac{1}{\alpha}\left(\sqrt{1 + \alpha^2 t^2} - 1\right) \simeq \tfrac{1}{2}\alpha t^2 + O(\alpha^3) \qquad v = \frac{\alpha t}{\sqrt{1 + \alpha^2 t^2}} \simeq \alpha t + O(\alpha^3) \qquad a = \frac{\alpha}{(1 + \alpha^2 t^2)^{3/2}} \simeq \alpha + O(\alpha^3) \tag{10.3.32}$$

Again the infinitesimal quantities agree with the geometric ones at first order in α. In Minkowski, a uniformly accelerated motion does not make sense (a constant acceleration would imply a linear velocity, which sooner or later will bring the velocity to exceed the speed of light). In any event, the Rindler particle is always time-like, which supports once again that the geometric quantities are better physical quantities than the infinitesimal quantities.

Then we can assume the Rindler particle

$$\sigma(s) : \mathbb{R} \to M : s \mapsto \left(\tfrac{1}{\alpha}\sinh(\alpha s), \tfrac{1}{\alpha}\left(\cosh(\alpha s) - 1\right)\right) \tag{10.3.33}$$

as a reference clock and discuss the kinematics of the particle γ in the reference frame defined by σ.

> Notice that this is not SR any longer. In SR only inertial observers are allowed. The observer defined by the Rindler clock is not inertial; if it were with would move along a uniform straight motion with respect to γ which, as we discussed, is not the case.
>
> Let us also remark that $\sigma(s)$ is an NP-clock, identical to the clock γ we used until now.

The isochronous surfaces for the clock σ are not the same though. We already computed them (see (10.2.27)) to be

$$\alpha \sinh(\alpha s)x = \alpha \cosh(\alpha s)\mathrm{c}t - \sinh(\alpha s) \tag{10.3.34}$$

This intersects the axis γ at

$$t(s) = \frac{\sinh(\alpha s)}{\mathrm{c}\alpha \cosh(\alpha s)} = s - \tfrac{1}{3}s^3\alpha^2 + O(\alpha^4) \tag{10.3.35}$$

The geometric distance is then

$$d^2(s) = -\frac{\sinh^2(\alpha s)}{\alpha^2}\left(\frac{\cosh(\alpha s) - 1}{\cosh(\alpha s)}\right)^2 + \frac{(\cosh(\alpha s) - 1)^2}{\alpha^2} = \frac{(\cosh(\alpha s) - 1)^2}{\alpha^2 \cosh^2(\alpha s)} \qquad \Rightarrow d = \tfrac{1}{2}\alpha s^2 + O(\alpha^3) \tag{10.3.36}$$

so that the geometric velocity and acceleration are

$$v(s) = \frac{\sinh(\alpha s)}{\cosh^2(\alpha s)} = \alpha s + O(\alpha^3) \qquad\qquad a(s) = -\frac{\alpha\left(\sinh^2(\alpha s) - 1\right)}{\cosh^3(\alpha s)} = \alpha + O(\alpha^3) \tag{10.3.37}$$

As one could expect, they do agree as the leading term is concerned though they differ at higher orders. Hence if we measured the acceleration (for different values of α) we could detect if the clock is inertial or the not by considering how the geometric quantities depart from the leading term.

Now let us consider what happens in a Weyl frame (which is not trivially a Riemanniann structure). That is a toy example, though it contains the most important aspect of the whole point we are making here. To make computation easier, let us consider the integrable Weyl frame $(M, g, \tilde{\Gamma})$ with $\tilde{\Gamma} = \{\eta\}$ and $g = \varphi^{-1}(t)\eta$.

> We already discussed that we can look at these examples in two equivalent ways.
>
> From a mathematical perspective, we can say that the geometry of spacetime is described by a Weyl frame, not by a Riemannian structure. The metric accounts for distances, the connection describes the free fall. These two structures needs to be compatible, which partially fixes the connection in terms of the metric. The residual degrees of freedom are described in general by a 1-form $\boldsymbol{\alpha}$ or, if the Weyl frame is integrable, by a conformal factor φ so that $\boldsymbol{\alpha} = d\ln(\varphi)$. Given the conformal factor we can define the parameterisation on the reference clock so that it turns out to be proper with respect to the metric g.
>
> Equivalently, from a physical perspective, we can say that atomic clocks are not standard. As we shall see in a while they though define a representative g in the conformal structure with respect to they are proper. Accordingly, we can start the examples below, either by assigning the conformal factor φ, or by assigning the clock parameterisation so that the clock is not standard and then define the conformal factor from the parameterisation.
>
> The difference between the two scenarios is just a matter of tastes.

We already considered few such cases. We know that isochronous surfaces become curvilinear in these cases.

For example, for $\varphi = e^{2\frac{ct}{x_0}}$, we have a proper clock (see Subsection 10.2.2, here and here)

$$\gamma : s \mapsto \left(t = \frac{x_0}{\mathsf{c}} \ln\left(\frac{x_0 + s}{x_0}\right), x = 0 \right) \tag{10.3.38}$$

The isochronous surfaces defined by γ are

$$s = x_0 \left(\cosh\left(\frac{x}{x_0}\right) e^{\frac{ct}{x_0}} - 1 \right) \tag{10.3.39}$$

Again we consider γ and $\sigma : s \mapsto (s, ws)$ for some $0 < |w| < 1$. Both are freely falling with respect to $\tilde{\Gamma}$. Let us fix $\hat{x}$ on the position axis. To that it corresponds the time $\hat{t} = \frac{\hat{x}}{w}$ so that the point $(\hat{t}, \hat{x})$ is on the particle σ. Hence by using the isochronal surface (10.3.39), we have a clock parameter

$$\hat{s}(\hat{x}; w, x_0) = x_0 \left(\cosh\left(\frac{\hat{x}}{x_0}\right) e^{\frac{c\hat{t}}{x_0}} - 1 \right) \tag{10.3.40}$$

singled out. Our duty is to compute the length of the isochronal line from the event corresponding to the reading $\hat{s}$ to the event $(\hat{t}, \hat{x})$ which lies on the same isochronous line.

Let us first consider an event (t, x) along the isochronous line. Because of the equation (10.3.39) we have

$$t(x; \hat{x}, w, x_0) = \frac{x_0}{\mathsf{c}} \ln\left(\frac{x_0 + \hat{s}}{x_0 \cosh\left(\frac{\hat{x}}{x_0}\right)} \right), \tag{10.3.41}$$
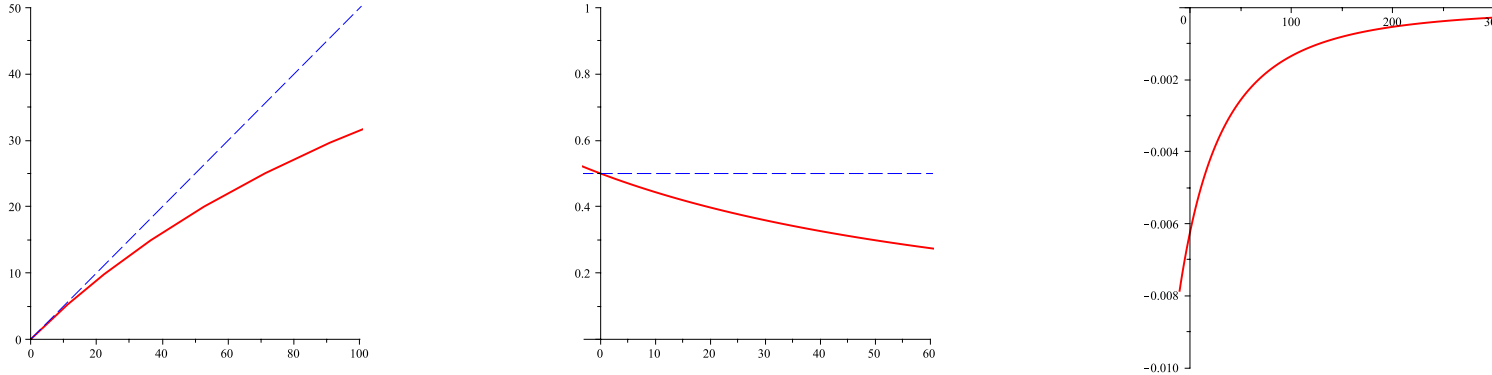
Fig. 9.6: *The geometric distance (a), geometric velocity (b)), geometric acceleration (c) ($x_0 = 100$, $w = 0.5$) as functions of the clock reading.*

Accordingly, the length of the line is

$$d(\hat{x}) = \int_0^{\hat{x}} \sqrt{1 - \left(\frac{dt}{dx}\right)^2} \, dx = \int_0^{\hat{x}} \frac{dx}{\cosh\left(\frac{x}{x_0}\right)} \tag{10.3.42}$$

That is the geometric distance between the reference clock γ and the particle σ, though as a function of $\hat{x}$ instead of as a function of the clock reading $\hat{s}$, which is known as a function of $\hat{x}$ as well. Accordingly, the pair $(\hat{s}(\hat{x}), d(\hat{x}))$ is a parametric representation of the function $d(\hat{s})$.

Then we have

$$\frac{dd}{ds} = \frac{dd}{d\hat{x}}\frac{d\hat{x}}{ds} =: v(\hat{x}) \qquad\qquad \frac{dv}{ds} = \frac{dv}{d\hat{x}}\frac{d\hat{x}}{ds} =: a(\hat{x}) \tag{10.3.43}$$

from which we obtain a parametric representation of velocity $(\hat{s}(\hat{x}), v(\hat{x}))$ and of acceleration $(\hat{s}(\hat{x}), a(\hat{x}))$.

In this case, the analytic expressions of these functions can be computed, though they are quite complicated and uninformative. More interesting are their series expansions

$$\begin{aligned}
d(\hat{s}) =& w\hat{s} - \frac{w(w^2+1)}{2x_0}\hat{s}^2 + O(\hat{s}^3) \\
v(\hat{s}) =& w - \frac{w(w^2+1)}{x_0}\hat{s} + \frac{w(3w^4+2w^2+2)}{2x_0^2}\hat{s}^2 + O(\hat{s}^3) \\
a(\hat{s}) =& -\frac{w(w^2+1)}{x_0} + \frac{w(3w^4+2w^2+2)}{x_0^2}\hat{s} - \frac{w(15w^6+10w^4+5w^2+6)}{x_0^3}\hat{s}^2 + O(\hat{s}^3)
\end{aligned} \tag{10.3.44}$$

From these we see that, as an effect of the clock being non-standard, the particles gets an effective acceleration. The effective acceleration vanishes in the limit $x_0 \to +\infty$, in which the clock becomes standard. Hence for a clock which is slightly non-standard the particle gets a small effective acceleration (as in MOND theories, though here the effects is near the observer, not away from it).

> The current example is a toy model exactly because the conformal factor is never well approximated by $\varphi = 1$, as one would expect to happen locally.

Finally, let us point out that the effective acceleration is *universal*. It effects in the same way any test particle, it depends on position only. Thus, for us by definition, it will be confused with an extra gravitational field acting. If we insist that standard GR is valid, an extra effective gravitational field will need an effective source which will be defined through its gravitational effects alone. Let us call these effective sources *dark sources*.

> Depending on the model, these effective dark sources can be all al just contribute to dark matter and dark energy. Here, at this stage we are just claiming we need to consider this effect, not that it explains all dark matter and dark energy phenomenology.
> Of course in Palatini $f(\mathcal{R})$-theories, this effect is there whenever $f(\mathcal{R})$ is not $\alpha\mathcal{R} + \beta$. On the other hand, since these models depend on a function $f(\mathcal{R})$ it is hard to think of a prove that can exclude that *whatever* function one selects then some extra dark matter (and or dark energy) will be needed. That is true at least to the same extent in which it is hard to find such a particular function which would account for all dark sources as a purely metrological effect.

Anyway, it is quite encouraging that, at least in principle we are able to compute these effects exactly.

Rigid rulers

Rigid rulers in relativistic theories had a long and difficult life. The Newtonian idea of reference frame follows Descartes and imagines to use a rigid ruler to establish a coordinate system in space, not differently from what one does on the floor of the room. Galileo transformations fix the length of the ruler so that different inertial observers define different though equivalent coordinate systems.

A rigid ruler has been used to keep a sample of the unit of length for a long time.

Now the official sample is defined in terms of the space travelled by light in a certain time. Originally it was defined as some fraction of a specific meridian on Earth. It is quite interesting to notice that originally, when they had to select a unit of length, they discussed a while whether to define the fundamental unit as a fraction of a meridian or as the length of a pendulum which has a period of a second (of course in some fix location). One of the arguments used in favour of the geodesic definition was not define length as a unit derived from time but keep length and time fundamental on an equal footing.

That is quite peculiar in view of a modern perspective. In homogeneous systems units are derived from universal constants so that essentially any unit is derived from time. That is true also because we have grown damn good in measuring times.

Metrologists are working in optical atomic clocks which are expected to be precise to the order of $10^{-18}s$. That is enough to appreciate the gravitational red shift on Earth surface for different altitudes of the order of few centimetres. It corresponds to the time in which light travels for $0.3 \cdot 10^{-9}m = 0.3 \cdot 10^{-6}mm$.

In SR it was soon realised that rigid bodies have a problem in definition.

If one considers a stick of $1m$, travelling (along its axis) at a speed $1ms^{-1}$, and hitting a thick wall, then, in view of rigidity, the two endpoints of the stick should stop simultaneously. Unfortunately, there is a upper bound in propagation of a physical signal, so that the endpoint far from the wall cannot be informed that the other endpoint hit the wall and stop accordingly.

At most it can stop after $3.3 \cdot 10^{-9}s$ that the other end point hit the wall. In this time it travels unaware for $3.3 \cdot 10^{-9}m$, in fact affecting the length of the ruler which, hence in principle, cannot be rigid.

From a theoretical viewpoint, the definition of a rigid ruler has the same difficulties of defining a uniform clock. To know what *rigid* means one should be able to measure distances, thus one cannot use rulers to define distances to avoid loopholes. Said that, after one define distances, e.g. geometric distance, it is possible to define a rigid ruler as a ruler having a constant length.

Let us define a *ruler* as a worldsheet in spacetime, i.e. the image of an embedded time-like surface $i : \mathbb{R} \times [0,1] \to M$. The boundaries $\gamma_0 : t \mapsto i(t,0)$ and $\gamma_1 : t \mapsto i(t,1)$ are called *endpoints* and they are assumed to be time-like worldlines.

If we now fix a clock χ, it defines isochronous surfaces and we can define the *length of the ruler* to be the geometric distance computed on the surface corresponding to time t. The ruler is *rigid* with respect to the reference clock χ if its length, *measured by* χ, is constant in time.

That has nothing to do with (non-)existence of rigid bodies. The endpoints can be two satellites which are maneuvered so that their distance stays constant, so that they define a rigid ruler.

The argument about non-existence of rigid bodies in SR just shows that a stick is not a physical device which realises a rigid ruler.

Of course, we should remark that, for a ruler, being rigid with respect to a reference clock does not imply being rigid with respect to other clocks.

That was true also in Minkowski spacetime if we considered accelerated clocks. At different times, the clock moves at different speed, seen different length contractions. A ruler which is rigid for an inertial clock is not for an accelerated one.

This definition makes sense in a Weyl frame $(M, g, \tilde{\Gamma})$. Of course, in that case distances are computed by g, by construction.

Distances of two events

If we now consider two events, instead of two worldlines, given a reference clock γ they may or may not lie on the same isochronous surface. If they do, we can proceed as we did above and define the relative distance as well as the geometric distance between them.

If they do not, there is not much we can do. There is no canonical way to map one isochronous surface into another. In order to define the relative distance between two general events one needs to select a further structure which enables to do that, defining isomorphisms between different isochronous surfaces.

Now, whatever continuous family of isomorphisms $\Phi_{t_0}^{t_1} : \Sigma_{t_0} \to \Sigma_{t_1}$ one had, whatever point $x_0 \in \Sigma_{t_0}$ one fixed, then a family of curves $\sigma_{x_0} : \mathbb{R} \to M : t \mapsto \Phi_{t_0}^t(x_0)$ would be identified and each curve σ_{x_0} in the family is transverse to all isochronous surfaces. The isomorphisms $\Phi_{t_0}^{t_1}$ can be further restricted so that the associated curves σ_{x_0} are time-like (with respect to some Weyl frame $(M, \mathfrak{g}, \tilde{\Gamma})$) and, accordingly, each of them can be considered as the motion of a test particle.

Hence we have a congruence of worldlines which foliates a region of spacetime M. Of course, giving such a congruence of worldlines is, in fact, equivalent to give the original family of isomorphisms, since one can set $\Phi_{t_0}^{t_1} : \Sigma_{t_0} \to \Sigma_{t_1} : x_0 \mapsto x_1$ iff the two events $x_0 \in \Sigma_{t_0}$ and $x_1 \in \Sigma_{t_1}$ belong to the same worldline.

Moreover, we can immagine an observer which considers all test particles in the curve congruence to be at rest, i.e. using them as a (sort of curvilinear) reference frame. Hence we call such a congruence $\sigma_{x_0} : \mathbb{R} \to M$ of time-like curves a *rest motion* in the Weyl frame $(M, \mathfrak{g}, \tilde{\Gamma})$. A rest motion can be made, in particular, of time-like geodesics, in which case it is called a *free fall*.

Of course, fixing a rest motion on a spacetime is a convention, it is not canonical (even free falls are not), different observers in general fix different rest motions. That is not a surprise any longer.

In particular, in a spacetime with a particular rest motion fixed on it, one can define the distance between two different events. One can use the rest motions to map both events on a particular isochronous surface Σ_{t_*} and define the geometric distance of them on that surface. Such a notion depends on the clock which defines the isochronous surfaces, on the rest motion fixed on them, and on the selected surface Σ_{t_*}.

> As one might comment, these definitions are getting less and less absolute. They depend on more and more structures, conventional structures to be fixed on a spacetime. We agree. Let us however, stress that it is important to keep track of all structures one needs to fix to reproduce a Newtonian-like framework. It is important both to stress that Newtonian frameworks are a bad thing (even though we might be forced to use them in real science) and to be able to bridge the covariant theory with observations.

On the other hand, rest motions may be naturally selected on some spacetimes, due to extra structure they may have. For example, we already selected one rest motion on spatially isotropic and homogeneous spaces. The comoving motions are in fact a rest motion in FLRW spacetimes.

4. Congruences of clocks

We are now ready for a further step. Until now, we considered a single clock and we used it to define a local foliation of isochronous surfaces and their (local) coordinate time.

We now want to define a *congruence of clocks*, i.e. a family of clocks which locally foliates an open region and which is *regular*, so that the time-reading is smooth even when we go from one clock to another.

In fact, we have a canonical prescription to build such a structure in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, which is eventually used as a definition. Let us consider $U \subset M$ an open set in spacetime, and X a local time-like vector field defined on U (not necessarily outside U) which never vanishes (in U). Let γ be an integral curve of X, which is called the *reference clock*. As we discussed, the reference clock γ defines a family of space-like isochronous hypersurfaces Σ_{t_0} which foliates a neighbourhood of the clock. There is no loss in generality by assuming that the isochronous surfaces foliate U.

> Since the vector field X is time-like and the hypersurfaces Σ_{t_0} are space-like, then X is necessarily transverse to the surfaces. Being space-like means that all tangent vectors to Σ_{t_0} are space-like, hence X cannot be tangent to Σ_{t_0}.

This construction is uniquely singled out by the data $(U, \hat{X}, x_0)$, where x_0 are initial conditions for the reference clock γ.

Now let us consider another integral curve $\hat{\sigma}$ of $\hat{X}$ and consider its trajectory $[\hat{\sigma}]$. That is also a time-like curve so it is also transverse to the isochronous surfaces. This trajectory gets a parameterisation by synchronisation with the reference clock: any event $y \in [\hat{\sigma}]$ on the trajectory also lies on an isochronous surface $y \in \Sigma_{t_0}$ and we can choose a parameterisation $\sigma : \mathbb{R} \to M : t_0 \mapsto y$, which is only another representative of the trajectory $\sigma \in [\hat{\sigma}]$. Hence we have a family of clocks σ which also foliates U and such clocks are synchronised, by construction, with the reference clock γ.

Accordingly, we have an open set U which comes with two transverse foliations on it: one is a space-like hypersurface foliation, the other is a clock foliation, as well as a rest motion.

> The clocks σ defined in the structure are synchronised to the reference clock γ. However, as we pointed out, they may not coincide with the integral curves $\hat{\sigma}$ of $\hat{X}$. On the other hand (as well as just because of that), there are different vector fields $\hat{X}$ which produce the same congruences. For example, any $X = \varphi(x)\hat{X}$ share with $\hat{X}$ the same integral trajectories.
>
> Thus, once we defined the clock congruence σ, we can go back and define X as the generator of foliation, so that now σ are integral curves of the vector field X. Accordingly, there is no loss in generality assuming that, when we select (U, X, x_0), the clocks σ are synchronised to γ *and* integral curves of X.

Definition: *let us define a congruence of clocks on U as a family of clocks for which there exists (U, X, x_0) as above.*

When we do it in a Weyl frame (though having a metric structure $g \in \mathfrak{g}$ is, in fact, enough), the quantity $\omega = \sqrt{-g(X, X)}$ coincides with the rate of the clock through the point. Accordingly, the field X is also called the *rate field*.

Although that may sound satisfactory, we already noticed that the isochronous surfaces of a clock σ in the congruence do not need to agree with the isochronous surfaces of the reference clock.

> Hence if we considered a different reference clock in the same congruence, the hypersurface foliation would end up to be different. This happens also in SR, so it has nothing to do with curvature, it is just a consequence of relativity of contemporaneity.
>
> However, in Minkowski spacetime, one can consider orthonormal coordinates (t, x), and the clock congruence defined by $(\mathbb{R}^m, \partial_0, 0)$ in which the hypersurfaces $\Sigma_{t_0} = \{(t, x) : t = t_0\}$ are isochronous surfaces of all the clocks of the congruence. Hence, at least in that case there are better behaving clock congruences, which are called *isochronal congruences*.

Also in a general spacetime, *isochronal counguences* do exist. In a spacetime of dimension 2, a reference clock $\gamma(t)$ defines two quantities

$$s(p) = \frac{s_+(p) + s_-(p)}{2} \qquad \rho(p) = \frac{s_+(p) - s_-(p)}{2} \tag{10.4.1}$$

which are in fact coordinates around the reference clock.

We can use ρ to define a congruence of clocks σ out of the reference clock γ, by requiring $\rho(\sigma(t))$ being constant. We can show that in a congruence so defined, any clock shares with γ its isochronous surfaces, i.e. if two events P and Q are isochronous for γ, then thay also are for the congruence clock passing through P.

Let us consider the reference clock $\gamma(s)$ and two points p, q on the isochronal surface Σ_{s_0} (in green) defined by γ through a point $x = \gamma(s_0)$.

Being p and q on the same isochronal surface, means that the echo a message from the reference clock. The message echoed by p is emitted at $s_-(p)$ and received back at $s_+(p)$. Analogously, the message echoed by q is emitted at $s_-(q)$ and received back at $s_+(q)$. Since $p, q \in \Sigma_{s_0}$ we have

$$s(p) = \frac{s_+(p) + s_-(p)}{2} = s_0 = \frac{s_+(q) + s_-(q)}{2} = s(q) \tag{10.4.2}$$

When we define the clock congruence, in particular, we have a clock σ_p through p and a clock σ_q through q. The message echoed by q singles out on the clock σ_p two further events $p_\pm$.

Both $p_\pm$ also echo a message from the reference clock, emitted from γ at $s_-(p_\pm)$ and received back to γ at $s_+(p_\pm)$. Of course we have $s_+(p_+) = s_+(q)$ and $s_-(p_-) = s_-(q)$.

Since the clocks in the congruence are synchronised, the readings of the clock σ_p at the events $p_\pm$ are

$$s(p_+) = \frac{s_+(p_+) + s_-(p_+)}{2} = \frac{s_+(q) + s_-(p_+)}{2} \qquad s(p_-) = \frac{s_+(p_-) + s_-(p_-)}{2} = \frac{s_+(p_-) + s_-(q)}{2} \tag{10.4.3}$$

and when the clock σ_p tries to give a time to the event q the result is

$$s_p(q) = \frac{s(p_+) + s(p_-)}{2} = \frac{s_+(q) + s_-(p_+) + s_+(p_-) + s_-(q)}{4} = \frac{s_-(p_+) + s_+(p_-)}{4} + \frac{1}{2}s(q) \tag{10.4.4}$$

Again in view of the synchronisation, the congruence is isochronal iff one has $s_p(q) = s_p(x) = s(p) = s_0$, which is in general false. However, if we require $\rho(\sigma_p(s))$ to be constant we also have $\rho(p) = \rho(p_\pm)$, i.e.

$$\rho(p_+) = \frac{s_+(p_+) - s_-(p_+)}{2} = \rho(p) = \frac{s_+(p_-) - s_-(p_-)}{2} = \rho(p_-) \tag{10.4.5}$$

from which one gets $s_-(p_+) = s_+(q) - 2\rho(p)$ and $s_+(p_-) = s_-(q) + 2\rho(p)$, hence $s_-(p_+) + s_+(p_-) = s_+(q) + s_-(q)$. Then in this case we have

$$s_p(q) = \frac{s_-(p_+) + s_+(p_-)}{4} + \frac{1}{2}s(q) = \frac{s_+(q) + s_-(q)}{4} + \frac{1}{2}s(q) = \frac{1}{2}s(q) + \frac{1}{2}s(q) = s(q) \tag{10.4.6}$$

and the congruence is isochronal.

In this case we can also easily obtain the distance $\rho_p(q)$ of q measured by the clock σ_p as

$$\rho_p(q) = \frac{s(p_+) - s_-(p_-)}{2} = \frac{s_+(q) + s_-(p_+) - s_+(p_-) - s_-(q)}{4} = \frac{1}{2}\rho(q) + \frac{s_-(p_+) - s_+(p_-)}{4} = \frac{1}{2}\rho(q) + \frac{s_+(q) - s_-(q) - 4\rho(p)}{4} = \rho(q) - \rho(p) \tag{10.4.7}$$

Fig. 9.7: *Construction of an isochronal clock congruence.*

This foliation grid is similar to the one in Minkowski spacetime obtained by clock at rest (though, of course, it is obtained in general on a curve spacetime—of dimension 2). In higher dimensions, the situation gets trickier. The two quantities $s(p)$ and $\rho(p)$ are not enough to define a coordinate system, one needs more than one clock for that. Accordingly, the system becomes more similar to a relativistic positioning system rather than to

a radar as it is here. The good news with it is that, we remark that once again, in astrophysics and cosmology one cannot realistically use echoes, anyway, so positioning systems need to be preferred.

Breaking the conformal invariance

We already proved that, if we fix $\mathfrak{g}$ and $\tilde{\Gamma}$, then there exists a parameterisation for which any worldline is a standard clock with respect to $(\mathfrak{g}, \tilde{\Gamma})$. Thus given a clock congruence on U we can always reparameterise the clocks so that they are standard clocks with respect $(\mathfrak{g}, \tilde{\Gamma})$. A clock congruence in which each clock is standard is called a *standard clock congruence*.

> Of course, in general, there is nothing like a standard isochronal congruence. The condition $\rho(\sigma(s)) = \rho_0$, used to define isochronal congruences, defines the clock worldline out of γ. There is no reason, even in standard GR, for these trajectories to be geodetics (of g, even less of $\tilde{\Gamma}$). ????

Let us now show that, if we consider a clock congruence and we fix the parameterisation and the conformal structure $\mathfrak{g}$, then there exists a connection $\tilde{\Gamma}$ which is EPS-compatible to $\mathfrak{g}$ and for which the clocks (with *their fixed parameterisations*) are standard with respect to $(\mathfrak{g}, \tilde{\Gamma})$.

> A connection which is EPS-compatible to $\mathfrak{g}$ is necessarily in the form (6.9.13), namely, for any representative $g \in \mathfrak{g}$ there exists a 1-form $\boldsymbol{\alpha} = \alpha_\epsilon \, dx^\epsilon$ such that

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + \tfrac{1}{2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\alpha_\epsilon \tag{10.4.8}$$

> One can select any representative $g \in \mathfrak{g}$ of the conformal structure, though, of course, the 1-form $\boldsymbol{\alpha}$ will depend on the choice. Thus we claim that we can fix $\boldsymbol{\alpha}$ so that any clock $\gamma : I \to M : s \mapsto \gamma^\mu(s)$ in the congruence is standard with respect to $\tilde{\Gamma}$ and $\mathfrak{g}$. For, the covariant acceleration of the clock is

$$a^\mu_{\tilde{\Gamma}} = \ddot{\gamma}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta \tag{10.4.9}$$

> so the clock is standard with respect to $\tilde{\Gamma}$ and $\mathfrak{g}$ iff

$$g(a_{\tilde{\Gamma}}, v) = \ddot{\gamma}^\mu\dot{\gamma}_\mu + \tilde{\Gamma}^\mu_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta\dot{\gamma}_\mu = \ddot{\gamma}^\mu\dot{\gamma}_\mu + \left(\{g\}^\alpha_{\beta\mu} + \tfrac{1}{2}\left(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\alpha_\epsilon\right)\dot{\gamma}^\beta\dot{\gamma}^\mu\dot{\gamma}_\alpha = g(a_g, v) - \tfrac{1}{2}\boldsymbol{\alpha}(v)g(v, v) = 0 \tag{10.4.10}$$

> i.e. iff

$$\boldsymbol{\alpha}(v) = 2\frac{g(a_g, v)}{g(v, v)} \qquad \Leftarrow \quad \alpha_\epsilon = \frac{2}{g(v, v)}(a_g)^\lambda g_{\lambda\epsilon} \tag{10.4.11}$$

> Since we have a clock of the congruence through any point in U, that is sufficient to fix $\boldsymbol{\alpha}$ in U so that (locally) there always exists a connection $\tilde{\Gamma}$, EPS-compatible with $\mathfrak{g}$, for which the whole clock congruence is standard.
>
> That, essentially, also shows that there is not much essential with standard clocks, since any family of clocks can be made standard by changing the connection in a Weyl geometry. It also points out that, if there is something special about standard clocks, it must be in relation with their free fall. For example, if we have a clock congruence *and* we fix their free fall, then we are not allowed any longer to change the connection (since that would change the free fall worldlines). Thus what is really special are clocks which are standard with respect to the connection which describe their free fall.

Now let us show that, given a clock congruence, we can select a representative $g \in \mathfrak{g}$ of the conformal structure such that the congruence is g-proper, i.e. each clock in the congruence is g-proper.

Let us first consider a clock γ in the congruence and a representative $\hat{g} \in \mathfrak{g}$. In view of equation (10.2.12), the clock γ is proper with respect to the metric $\hat{g}$ iff $\hat{g}(v, v)$ is constant along the curve γ, which in general is not the case. However, one can always choose a conformal factor Φ and, consequently, a conformal metric g by

$$\Phi(\gamma(s)) = -\frac{a^2}{\hat{g}(v,v)} \quad \Rightarrow g = \Phi\, \hat{g} \in \mathfrak{g}$$

so that $g(v, v) = -a^2$ along the curve γ. Then, for such a conformal representative $g \in \mathfrak{g}$, the clock γ is proper with respect to g. Let us stress that this result is achieved without changing the clock parameterisation.

> Without loss of generality, one can also fix $a = 1$ and obtain NP-clocks.

Of course, this procedure does not *fix* the representative g of the conformal structure, since we have no information about the value of the conformal factor outside the worldline of γ. However, we have a congruence of clocks which fills (an open set of) the spacetime M, then the conformal factor and the metric g will be (locally) fixed uniquely (up to affine transformations) by the requirement of the clocks to be proper (or NP-) clocks.

Thus the main idea is that, if we start from a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$, possibly a solution in a Palatini $f(\mathcal{R})$-theory or a more general ETG, and we want to provide a class of clocks (e.g. atomic clocks) with a special physical status we can always promote the congruence to be proper with respect to a suitably chosen representative $g \in \mathfrak{g}$ of the conformal structure.

If the connection $\tilde{\Gamma}$ is metric, i.e. $\tilde{\Gamma} = \{\tilde{g}\}$ for some metric $\tilde{g}$ which, because of EPS-compatibility, is hence a conformal representative $\tilde{g} \in \mathfrak{g}$, and the atomic clocks are standard, then they are also $\tilde{g}$-proper. Then, by the procedure describe above, the conformal invariance is broken by selecting $\tilde{g}$ as a representative for the conformal structure $\mathfrak{g}$.

> That is exactly what is assumed to happen in standard GR: an atomic clock is standard (hence proper) with respect to the metric which describes free fall. EPS did discuss this possibility in their original project, just they called *gravitational time* that measured by standard clock, and *atomic time* that measured by atomic clocks. They questioned how strong is the evidence that atomic clocks are standard.
>
> We have to remark that in standard GR this is assumed *in principle*, i.e. it extends to all scales. It means that we imagine to check that an atomic clock is standard on the scale up to $15 BY$. No need to say that nobody ever observed an atomic clock for $15 BY$ to check it is exactly standard.
>
> Notice that it is not even clear what exactly one needs to observe and how (if not whether) a non-standard behaviour of a clock can be observed in principle. Thus we need to go in greater detail and consider the issue from scratch.

However, it may not be the case. If atomic clocks are not standard, we can easily imagine to consider a congruence of atomic clocks, and this congruence do break the conformal invariance by selecting a conformal representative $g \in \mathfrak{g}$ with respect to which the atomic clocks are g-proper.

> This is what is assumed in ETG: we go from a Weyl geometry to a Weyl frame $(M, g, \tilde{\Gamma})$ by selecting a representative $g \in \mathfrak{g}$ which is precisely selected to described a preferred class of clocks, e.g. atomic clocks as proper clocks.
>
> As a special case, it may happen that the atomic clocks are already standard, hence the selected representative will be $g = \tilde{g}$. However, in general, we assume they are not necessarily standard, so to be able to discuss the issue on a physical stance.

Once again, Weyl geometries show to be less strict than Lorentzian geometries. We have just shown that considered a representative of the conformal structure g, a connection for free fall $\tilde{\Gamma}$ and a clock congruence γ, one can keep fixed two of them and modify the third to obtain a proper clock congruence.

5. Observing Curvature

As it happens in Riemannian geometries, in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$ the curvature cannot be observed directly, though it becomes manifest through its action on test particles. Tracing how these effects appear is important to give a physical interpretation of the Weyl frame and the conformal factor when the Weyl frame is integrable.

Let us consider a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$ and two non-parallel tangent vectors (v, w) at $x_0 \in M$.

Let us first consider a geodesic motion $\gamma(t)$ (obviously, with respect to $\tilde{\Gamma}$) with initial conditions $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v$. Accordingly, we have

$$\ddot{\gamma}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta = 0 \qquad \gamma^\mu(0) = x_0^\mu, \quad \dot{\gamma}^\mu(0) = v^\mu \tag{10.5.1}$$

where dots denote derivative with respect to t. Let us also consider a geodesic motion $x(s)$

$$x''^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} x'^\alpha x'^\beta = 0 \qquad x^\mu(0) = x_0^\mu, \quad x'^\mu(0) = w^\mu \tag{10.5.2}$$

where primes denote derivative with respect to s.

Then let us define a family of vectors $v(s)$ defined along the curve $x(s)$, which is parallely transported along $x(s)$ with initial condition $v(0) = v$

$$v'^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} x'^\alpha v^\beta = 0 \qquad v^\mu(0) = v^\mu, \tag{10.5.3}$$

as well as a family of vectors $w(t)$ defined along the curve $\gamma(t)$, which is parallely transported along $\gamma(t)$ with initial condition $w(0) = w$

$$\dot{w}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} \dot{\gamma}^\alpha w^\beta = 0 \qquad w^\mu(0) = w^\mu, \tag{10.5.4}$$

Fig. 9.8: Curvature from geodesic circuits.

Then we have two families of initial conditions $(\gamma(t), w(t))$ and $(x(s), v(s))$ which can be used to define two families of geodesics, namely $\gamma(t, s)$ and $x(s, t)$ such that

$$\ddot{\gamma}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta = 0 \qquad \gamma^\mu(0, s) = x^\mu(s), \ \dot{\gamma}^\mu(0, s) = v^\mu(s), \tag{10.5.5}$$

and

$$x''^\mu + \tilde{\Gamma}^\mu_{\alpha\beta} x'^\alpha x'^\beta = 0 \qquad x^\mu(0, t) = \gamma^\mu(t), \ x'^\mu(0, t) = w^\mu(t), \tag{10.5.6}$$

Let us remark that there is no reason to assume that the curves $\gamma(t, s)$ and $x(s, t)$ lie (exactly) on the same surface; as far as one knows, the two lines $\gamma(t, s)$ and $x(s, t)$ can be skew. Also, in any case, there is no reason to have that $\gamma(t, s) = x(s, t)$ for their endpoints. Thus we (pretend not to notice that such a quantity does not make any sense and we) wish to express $\gamma(t, s) - x(s, t)$, to measure how much the curves fail to be closed. Since we do not have a general analytic expression for the curves, but we only know the equations they obey, it is a good technique to expand in Taylor series and check for the first non-zero term, which unfortunately is the third order term.

> We could try to simplify the computation a bit, for example by somehow restricting the initial directions. However, this computation is as beautiful as it is general. So take it easy and do it in general.

Let us start with γ. Keep constantly in mind that $\gamma(0,s) = x(s)$, $\dot\gamma(0,s) = v(s)$.

$$\begin{aligned}
\gamma^\mu(t,s) =& x^\mu(s) + v^\mu(s)t + \tfrac{1}{2}\ddot\gamma^\mu(0,s)t^2 + \tfrac{1}{6}\dddot\gamma^\mu(0,s)t^3 + O(t^4) = \\
=& \left(x_0^\mu + w^\mu s + \tfrac{1}{2}x''^\mu(0)s^2 + \tfrac{1}{6}x'''^\mu(0)s^3\right) + \left(v^\mu + v'^\mu(0)s + \tfrac{1}{2}v''^\mu(0)s^2\right)t - \tfrac{1}{2}\tilde\Gamma^\mu_{\alpha\beta}(x(s))v^\alpha(s)v^\beta(s)t^2 + \\
& + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}(x(s))\tilde\Gamma^\epsilon_{\alpha\lambda}(x(s)) - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}(x(s))\right)v^\lambda(s)v^\alpha(s)v^\beta(s)t^3 + O^4 = \\
=& \left(x_0^\mu + w^\mu s - \tfrac{1}{2}\tilde\Gamma^\mu_{\alpha\beta}w^\alpha w^\beta s^2 + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta s^3\right) + \left(v^\mu - \tilde\Gamma^\mu_{\alpha\beta}w^\alpha v^\beta s + \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} + \tilde\Gamma^\mu_{\alpha\epsilon}\tilde\Gamma^\epsilon_{\beta\lambda} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda}\right)w^\beta w^\alpha v^\lambda s^2\right)t + \\
& - \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\alpha\beta}v^\alpha v^\beta + \left(-2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} + \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda v^\alpha v^\beta s\right)t^2 + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda v^\alpha v^\beta t^3 + O^4
\end{aligned} \tag{10.5.7}$$

where we used the derivative of the geodesic equation $x''^\mu(0) = -\tilde\Gamma^\mu_{\alpha\beta}w^\alpha w^\beta$ and

$$x'''^\mu(0) = \left(2\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta \qquad v''^\mu(0) = \left(\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} + \tilde\Gamma^\mu_{\epsilon\alpha}\tilde\Gamma^\epsilon_{\lambda\beta} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda}\right)v^\lambda w^\alpha w^\beta \tag{10.5.8}$$

Thus we obtain

$$\begin{aligned}
\gamma^\mu(t,s) =& x_0^\mu + (w^\mu s + v^\mu t) - \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\alpha\beta}w^\alpha w^\beta s^2 + 2\tilde\Gamma^\mu_{\alpha\beta}w^\alpha v^\beta st + \tilde\Gamma^\mu_{\alpha\beta}v^\alpha v^\beta t^2\right) + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta s^3 + \\
& + \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} + \tilde\Gamma^\mu_{\alpha\epsilon}\tilde\Gamma^\epsilon_{\beta\lambda} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda}\right)w^\alpha w^\beta v^\lambda ts^2 + \tfrac{1}{2}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda v^\alpha v^\beta t^2 s + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda v^\alpha v^\beta t^3 + O^4
\end{aligned} \tag{10.5.9}$$

Analogously, for $x^\mu(s,t)$ we get

$$\begin{aligned}
x^\mu(t,s) =& x_0^\mu + (v^\mu t + w^\mu s) - \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\alpha\beta}v^\alpha v^\beta t^2 + 2\tilde\Gamma^\mu_{\alpha\beta}v^\alpha w^\beta ts + \tilde\Gamma^\mu_{\alpha\beta}w^\alpha w^\beta s^2\right) + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda v^\alpha v^\beta t^3 + \\
& + \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} + \tilde\Gamma^\mu_{\alpha\epsilon}\tilde\Gamma^\epsilon_{\beta\lambda} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda}\right)v^\alpha v^\beta w^\lambda st^2 + \tfrac{1}{2}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda w^\alpha w^\beta s^2 t + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta s^3 + O^4
\end{aligned} \tag{10.5.10}$$

For the difference, which makes sense (only) in a fixed coordinate system, we get

$$\begin{aligned}
\gamma^\mu(t,s) - x^\mu(t,s) =& \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta s^3 - \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)w^\lambda w^\alpha w^\beta s^3 - \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda v^\alpha v^\beta t^3 + \tfrac{1}{6}\left(2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda v^\alpha v^\beta t^3 + \\
& + \tfrac{1}{2}\left(\tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} + \underline{\tilde\Gamma^\mu_{\alpha\epsilon}\tilde\Gamma^\epsilon_{\beta\lambda}} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda} - 2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} + \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta}\right)v^\lambda w^\alpha w^\beta s^2 t + \tfrac{1}{2}\left(\underline{\underline{2\tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha}}} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta} - \tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} - \underline{\underline{\tilde\Gamma^\mu_{\alpha\epsilon}\tilde\Gamma^\epsilon_{\beta\lambda}}} + \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda}\right)v^\alpha v^\beta w^\lambda st^2 + O^4 = \\
=& \tfrac{1}{2}\left(\partial_\lambda\tilde\Gamma^\mu_{\alpha\beta} - \partial_\beta\tilde\Gamma^\mu_{\alpha\lambda} + \tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta} - \tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha}\right)v^\lambda w^\alpha w^\beta s^2 t + \tfrac{1}{2}\left(\partial_\beta\tilde\Gamma^\mu_{\alpha\lambda} - \partial_\lambda\tilde\Gamma^\mu_{\alpha\beta} + \tilde\Gamma^\mu_{\epsilon\beta}\tilde\Gamma^\epsilon_{\lambda\alpha} - \tilde\Gamma^\mu_{\epsilon\lambda}\tilde\Gamma^\epsilon_{\alpha\beta}\right)v^\alpha v^\beta w^\lambda st^2 + O^4 = \\
=& \tfrac{1}{2}\tilde R^\mu_{\alpha\beta\lambda}\left(w^\lambda v^\alpha v^\beta t^2 s - v^\lambda w^\alpha w^\beta ts^2\right) + O^4
\end{aligned} \tag{10.5.11}$$

Thus, as a first result, the evolution of the "displacement" $\gamma - x$ (if it made any sense as a finite quantity) is controlled by the curvature, and it is in fact a definition of the curvature—or, at least, of its symmetric part $\tilde R^\mu_{(\alpha\beta)\lambda}$.

Unfortunately, the (finite) displacement is not particularly relevant since, in general, spacetime has no affine structure. What is definite the *infinitesimal displacement* though it is a tangent vector. That is not even the worse thing about this example! Since we constrained the sides of the *"quadrilateral"* to be geodesics, in general, the two families of curves do not even lie on the same surface.

So let us rewind and do it again in a slightly different way.

Jacobi fields

Let ζ be a vector field and let $x(s;p)$ be the integral curve of ζ through the point p. Of course, the integral curves define the flow $\Phi_s(p) = x(s;p)$ of ζ.

One can drag the geodesic γ along the vector field ζ and define a family of curves

$$\gamma(t;s) := \Phi_s \circ \gamma(t) = x(s;\gamma(t)) \tag{10.5.12}$$

Notice that the curves $\gamma(t;s)$, for how we defined them, do not foliate the manifold M, though, this time, they foliate a two dimensional surface in it.

First of all, let us consider two curves in the family, namely $\gamma(t,s)$ and $\gamma(t)$, and consider the *infinitesimal displacement* $D = s\zeta$. One has that $\gamma(t;s) - \gamma(t) = s\zeta(\gamma(t)) + O(s^2)$, i.e. the displacement is a tangent vector. It is not a physical distance as we would like, though, however, it is geometrically well defined, as a tangent vector, and, in Minkowski, it reduces (as other quantities do) to the usual displacement.

We define the *relative covariant velocity* as well as the *relative covariant acceleration*, as

$$U(t) = s\dot{\gamma}^\mu \tilde{\nabla}_\mu \zeta^\epsilon \partial_\epsilon \qquad\qquad A(t) = s\left(\dot{\gamma}^\nu \dot{\gamma}^\mu \tilde{\nabla}_\nu \tilde{\nabla}_\mu \zeta^\epsilon\right)\partial_\epsilon \tag{10.5.13}$$

Again, one should not be too confident in the physical meaning of these quantities. However, they are covariant and they reduce to the usual physical quantities on Minkowski spacetime.

Let us stress that these can be interpreted as defining acceleration of a motion $\gamma(t,s)$ with respect to a freely falling motion $\gamma(t)$. That makes sense since gravity is universal, i.e. it acts on anything.

For electromagnetism, since we know there are neutral particles, we can define the acceleration of a charge relative to a neutral particle with the same initial conditions. In gravity, this would not be a good idea, though, just in view of universality of gravity, it makes a better sense to define the acceleration of a particle with respect to a freely falling one.

Fig. 9.9: Geodesic deviation.

Of course, because of that, the acceleration of a freely falling particle is zero, and, as we shall see in a while, there is a relative acceleration between two nearby freely falling particles, which is called the *tidal force* and it is related to the curvature. That is the only remnant of gravity as a force and it is what people mean by saying that in GR gravity is not a force (understanding that it is exactly true at a point only).

In general, there is no reason to believe that the dragged curves $\gamma(s;s)$ are also geodesic motions. If that is the case, then the vector field ζ is called a *Jacobi field*.

We have $\gamma_s(t) := \gamma(t;s) = \gamma(t) + s\zeta(\gamma(t)) + O(s^2)$ and

$$\dot{\gamma}_s^\mu = \dot{\gamma}^\mu + s\dot{\gamma}^\alpha \partial_\alpha \zeta^\mu(\gamma(t)) + O(s^2) \qquad \ddot{\gamma}_s^\mu = \ddot{\gamma}^\mu + s\left(\ddot{\gamma}^\alpha \partial_\alpha \zeta^\mu + \dot{\gamma}^\alpha \dot{\gamma}^\beta \partial_{\alpha\beta}\zeta^\mu\right) + O(s^2) \tag{10.5.14}$$

Hence we have

$$\left(\ddot{\gamma}_s^\mu(t) + \tilde{\Gamma}_{\alpha\beta}^\mu(\gamma_s(t))\dot{\gamma}_s^\alpha(t)\dot{\gamma}_s^\beta(t)\right) - \left(\ddot{\gamma}^\mu + \tilde{\Gamma}_{\alpha\beta}^\mu \dot{\gamma}^\alpha \dot{\gamma}^\beta\right) = s\left(\left(\ddot{\gamma}^\alpha + \tilde{\Gamma}_{\rho\sigma}^\alpha \dot{\gamma}^\rho \dot{\gamma}^\sigma\right)\partial_\alpha \zeta^\mu + \dot{\gamma}^\alpha \dot{\gamma}^\beta\left(\partial_{\alpha\beta}\zeta^\mu + 2\tilde{\Gamma}_{\alpha\rho}^\mu \partial_\beta \zeta^\rho - \tilde{\Gamma}_{\alpha\beta}^\rho \partial_\rho \zeta^\mu + \partial_\lambda \tilde{\Gamma}_{\alpha\beta}^\mu \zeta^\lambda\right)\right) + O(s^2) =$$

$$= s\dot{\gamma}^\alpha \dot{\gamma}^\beta\left(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \zeta^\mu + \partial_\sigma \tilde{\Gamma}_{\alpha\beta}^\mu \zeta^\sigma - \partial_\alpha \tilde{\Gamma}_{\sigma\beta}^\mu \zeta^\sigma - \tilde{\Gamma}_{\sigma\beta}^\mu \tilde{\nabla}_\alpha \zeta^\sigma + \tilde{\Gamma}_{\alpha\rho}^\mu \tilde{\nabla}_\beta \zeta^\rho - \tilde{\Gamma}_{\epsilon\alpha}^\mu \tilde{\Gamma}_{\sigma\beta}^\epsilon \zeta^\sigma + \tilde{\Gamma}_{\sigma\epsilon}^\mu \tilde{\Gamma}_{\alpha\beta}^\epsilon \zeta^\sigma\right) + O(s^2) = \tag{10.5.15}$$

$$= s\dot{\gamma}^\alpha \dot{\gamma}^\beta\left(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \zeta^\mu - \tilde{R}^\mu{}_{\alpha\beta\sigma}\zeta^\sigma\right) + O(s^2)$$

Then ζ is a Jacobi field iff it satisfies the equation:

$$\xi^\alpha \xi^\beta \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \zeta^\mu = \tilde{R}^\mu{}_{\alpha\beta\lambda} \xi^\alpha \xi^\beta \zeta^\lambda \tag{10.5.16}$$

This equation is also called the *geodesic deviation equation*.

The left hand side is the relative covariant acceleration, thus the right hand side is the force acting which is called the *tidal force*.

Notice that the tidal force is a function of the curvature.

Of course, Jacobi fields do exist. If we consider a geodesic flow $\gamma(t; s)$, the infinitesimal generators are

$$\zeta(\gamma(t;s)) = \frac{d\gamma^\alpha}{ds}(t;s)\partial_\alpha \qquad\qquad \xi(\gamma(t;s)) = \frac{d\gamma^\alpha}{dt}(t;s)\partial_\alpha \tag{10.5.17}$$

The field ζ is, by construction, a Jacobi field and ξ is a geodesic field, $\xi^\mu \tilde{\nabla}_\mu \xi^\alpha = 0$.

Sectional curvature

Let us now consider a family of $\mathfrak{g}$-time-like $\tilde{\Gamma}$-geodesic motions $\gamma(t; \epsilon)$ though an event $x_0 \in M$, so that we have $\gamma(0, \epsilon) = x_0$ for all ϵ. We can define the *"vector fields"*

$$\xi(\gamma(t,\epsilon)) = \frac{d\gamma^\alpha}{dt}(t;\epsilon)\partial_\alpha \qquad\qquad \zeta(\gamma(t,\epsilon)) = \frac{d\gamma^\alpha}{d\epsilon}(t;\epsilon)\partial_\alpha \tag{10.5.18}$$

The geodesics $\gamma(t; \epsilon)$ are integral curves of ξ and ζ is a Jacobi field which commutes with ξ, i.e. $[\xi, \zeta] = 0$.

Then we can compute the expansion of the infinitesimal displacement $D = \epsilon\sqrt{|g(\zeta, \zeta)|}$ which, informally, is the separation between the two geodesics.

Let us consider the geodesic $\gamma(t) = \gamma(t; 0)$, restrict to it the Jacobi field $\zeta(t) = \zeta(\gamma(t))$, and set $\xi(\epsilon) = \xi(\gamma(0; \epsilon)) =: w$; then we have

$$\zeta^\mu = \zeta^\mu(\gamma(0)) = \frac{d\gamma^\mu}{d\epsilon}(0;\epsilon) = 0$$
$$D\zeta^\mu := \tilde{\nabla}\zeta^\mu + \tfrac{1}{2}\alpha\zeta^\mu = \tilde{\nabla}\zeta^\mu = \zeta^\alpha \tilde{\nabla}_\alpha \xi^\mu = \xi^\alpha \tilde{\nabla}_\alpha \zeta^\mu = \dot{\zeta}^\mu(0) =: u^\mu \tag{10.5.19}$$

Here we considered that $\gamma(t; \epsilon) = \gamma(t) + \epsilon\zeta(t) + O(\epsilon^2)$, taking the derivative with respect to t at $t = 0$, dividing by ϵ, and taking the limit to define $u := \lim_{\epsilon \to 0} \frac{w - \xi}{\epsilon} = \dot{\zeta}(0) = \xi^\alpha \tilde{\nabla}_\alpha \zeta$, i.e. $w = \xi + \epsilon u + O(\epsilon^2)$. Then we can consider the following derivatives:

$$D^2\zeta^\mu := \tilde{\nabla}^2\zeta^\mu + \alpha\tilde{\nabla}\zeta^\mu + \tfrac{1}{4}\alpha^2\zeta^\mu + \tfrac{1}{2}\tilde{\nabla}\alpha\zeta^\mu = \tilde{\nabla}^2\zeta^\mu + \alpha\tilde{\nabla}\zeta^\mu = \tilde{R}^\mu{}_{\alpha\beta\lambda}\xi^\alpha\xi^\beta\zeta^\lambda + \alpha u^\mu = \alpha u^\mu$$
$$D^3\zeta^\mu := \tilde{\nabla}^3\zeta^\mu + \tfrac{3}{2}\tilde{\nabla}\alpha\tilde{\nabla}\zeta^\mu + \tfrac{3}{2}\alpha\tilde{\nabla}^2\zeta^\mu + \tfrac{1}{2}\tilde{\nabla}^2\alpha\zeta^\mu + \tfrac{3}{4}\alpha^2\tilde{\nabla}\zeta^\mu + \tfrac{3}{2}\alpha\tilde{\nabla}\alpha\zeta^\mu + \tfrac{1}{8}\alpha^3\zeta^\mu = \tilde{R}^\mu{}_{\alpha\beta\lambda}\xi^\alpha\xi^\beta u^\lambda + \tfrac{3}{2}\tilde{\nabla}\alpha u^\mu + \tfrac{3}{4}\alpha^2 u^\mu \tag{10.5.20}$$

where we set $\alpha = \boldsymbol{\alpha}(\dot{\gamma})$ and $\tilde{\nabla} = \dot{\gamma}^\alpha \tilde{\nabla}_\alpha$ as in Subsection .

Then the field ζ can be expanded as

$$\zeta^\mu(t) = tu^\mu + \tfrac{1}{2}t^2\alpha u^\mu + \tfrac{1}{3!}t^3\left(\tilde{R}^\mu{}_{\alpha\beta\lambda}\xi^\alpha\xi^\beta u^\lambda + \tfrac{3}{2}\tilde{\nabla}\alpha u^\mu + \tfrac{3}{4}\alpha^2 u^\mu\right) + O(t^4) \tag{10.5.21}$$
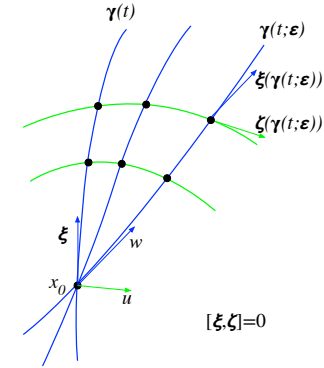
Fig. 9.10: Sectional curvature.

and its norm is expanded as

$$|\zeta|^2 = g(\zeta(t), \zeta(t)) = t^2 \left(1 + t\alpha\right) |u|^2 + \left(\tfrac{1}{3} u_\mu \tilde{R}^\mu{}_{\alpha\beta\lambda} \xi^\alpha \xi^\beta u^\lambda + \tfrac{1}{2} \left(\tilde{\nabla}\alpha + \alpha^2\right) |u|^2\right) t^4 + O(t^5) \tag{10.5.22}$$

Let us set

$$\mathcal{K}(u, \xi) := -\frac{u^\mu g_{\mu\nu} \tilde{R}^\nu{}_{\alpha\beta\lambda} \xi^\alpha \xi^\beta u^\lambda}{|u|^2 |\xi|^2 - g(u, \xi)^2} \tag{10.5.23}$$

for the *sectional curvature* in the plane (u, ξ)—which is a plane in the tangent space.

> The sectional curvature in fact depends on the plane only and it is independent of the particular vectors used to span it. For example, let us define $u' = au + b\xi$ and $\xi' = c\xi$; then, generically, (u', ξ') spans the same plane as (u, ξ). Then we have
>
> $$\mathcal{K}(u', \xi') = a^2 c^2 \frac{|u|^2 |\xi|^2 - g(u, \xi)^2}{|u'|^2 |\xi'|^2 - g(u', \xi')^2} \mathcal{K}(u, \xi) = a^2 \frac{|u|^2 |\xi|^2 - g(u, \xi)^2}{(a^2|u|^2 + 2abg(u,\xi') + b^2c^2|\xi|^2)|\xi|^2 - (a^2 g(u,\xi)^2 + b^2c^2|\xi|^4 + 2abg(u,\xi')|\xi|^2)} \mathcal{K}(u, \xi) = \mathcal{K}(u, \xi) \tag{10.5.24}$$
>
> Accordingly, we can always consider u' to be a unit vector orthogonal to ξ and ξ' the unit vector parallel to ξ.

Then the infinitesimal displacement $D = \epsilon|\zeta|$ is expanded as

$$\begin{aligned} D = \epsilon|\zeta| &= \epsilon|u| t \sqrt{1 + \alpha t + \left(-\frac{|u|^2|\xi|^2 - g(u,\xi)^2}{3|u|^2} \mathcal{K}(u, \xi) + \tfrac{1}{2}\left(\tilde{\nabla}\alpha + \alpha^2\right)\right) t^2 + O(t^3)} = \\ &= \epsilon|u| \left(t + \tfrac{1}{2}\alpha t^2 + \tfrac{1}{2}\left(-\frac{|u|^2|\xi|^2 - g(u,\xi)^2}{3|u|^2} \mathcal{K}(u, \xi) + \tfrac{1}{2}\tilde{\nabla}\alpha + \tfrac{1}{4}\alpha^2\right) t^3\right) + O(t^4) \end{aligned} \tag{10.5.25}$$

That is pretty general, though let us now specialise a bit the family of geodesics. Let us fix x_0 and the time-like geodesic through it from initial conditions $(x_0, \dot{\gamma}(0) = \xi)$. One can always use the affine freedom of parameterisation along geodesic motions to set $|\xi|^2 = -1$.

> In a Riemannian structure the norm of the tangent vector to a geodesic motion is constant so one would have $|\xi(t)|^2 = -1$ as well. That is not the case in a Weyl frame though, so we require $\xi(0)$ to be a unit vector; $\xi(t)$ will be not unit in general.

The we can fix u to be a unit vector normal to ξ and set $w = \xi + \epsilon u$. Thus we can define the geodesic flow by $\gamma(t; \epsilon)$ being the geodesic corresponding to the initial conditions (x_0, w). That flow defines the geodesic vector field ξ and the Jacobi field ζ as above. Accordingly, we can specialise the discussion above to thel case in which $w = \xi + \epsilon u$ with $|\xi| = -1$, $|u| = 1$, and $\xi \cdot u = 0$. Consequently, the infinitesimal displacement is

$$D = \epsilon \left(t + \tfrac{1}{2}\alpha t^2 + \left(\tfrac{1}{6}\mathcal{K}(u, \xi) + \tfrac{1}{4}\tilde{\nabla}\alpha + \tfrac{1}{8}\alpha^2\right) t^3\right) + O(t^4) \tag{10.5.26}$$

In the Riemannian case, we have $\alpha = 0$ and $D = \epsilon \left(t + \tfrac{1}{6}K(u, \xi)t^3\right) + O(t^4)$, where $K(u, \xi)$ is now the sectional curvature of the Levi Civita connection $\{g\}$. The corrections to the displacements are controlled by the Riemannian sectional curvature alone.

Thus in a Weyl frame, the corrections in α are further corrections to the displacement which makes potentially observable the fact that the clock is not standard. Once again we see that they enter at second order in t, i.e. as an acceleration, even when there is curvature at all. That accounts once

again of the effective acceleration we already discussed above in a special case, also showing that to be a general effect. The initial relative acceleration is controlled *purely* by the clock being not standard, it is a signature of this effect.

Somewhere in between a general Weyl frame and a Riemannian structure we have an integrable Weyl frame, which is an interesting case for the Palatini $f(\mathcal{R})$-theories. In an integrable Weyl frame $(M, g, \tilde{\Gamma})$ we have $\alpha = -\dot{\gamma}^\alpha \overset{*}{\tilde{\nabla}}_\alpha \ln(\varphi) = -\frac{d}{dt}\ln(\varphi \circ \gamma(t))$, where φ is the conformal factor. Hence α measures the change of the conformal factor φ along the geodesic γ.

Even if the normalise the conformal factor to be $\varphi(x_0) = 1$, we still have $\alpha = -\frac{d\ln(\varphi \circ \gamma)}{dt}(0) = -\xi^\alpha \overset{*}{\tilde{\nabla}}_\alpha \varphi$. Then we have $\alpha^2 = \xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi)$ and

$$\tilde{\nabla}\alpha = -\left(\xi^\beta \tilde{\nabla}_\beta \xi^\alpha\right) \overset{*}{\tilde{\nabla}}_\alpha \ln(\varphi) - \xi^\alpha \xi^\beta \tilde{\nabla}_{\alpha\beta} \ln(\varphi) = -\xi^\alpha \xi^\beta \tilde{\nabla}_{\alpha\beta} \ln(\varphi) = -\xi^\rho \xi^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) + \frac{1}{2}\nabla_\rho \ln(\varphi) \nabla^\rho \ln(\varphi) \tag{10.5.27}$$

For the curvature (see (2.7.150)), we know that

$$\tilde{R}_{\alpha\beta\mu\nu} = \varphi \left(R_{\alpha\beta\mu\nu} + 2\delta^\epsilon_{[\alpha} g_{\beta][\mu} \nabla_{\nu]\epsilon} \ln(\varphi) + \left(\delta^\rho_{[\alpha} g_{\beta][\nu} \delta^\sigma_{\mu]} - \frac{1}{2}g^{\rho\sigma} g_{\nu[\beta} g_{\alpha]\mu}\right) \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi)\right) \tag{10.5.28}$$

and, consequently, (always in the case $\varphi(x_0) = 1$, $g(\xi, \xi) = -1$, $g(u, u) = 1$, and $g(u, \xi) = 0$) the sectional curvature is

$$\mathcal{K}(\xi, u) = u^\lambda g_{\lambda\alpha} \tilde{R}^\lambda_{\beta\mu\nu} \xi^\beta \xi^\mu u^\nu = \varphi^{-1} u^\lambda \tilde{g}_{\lambda\alpha} \tilde{R}^\lambda_{\beta\mu\nu} \xi^\beta \xi^\mu u^\nu = \varphi^{-1}\left(u^\alpha \tilde{R}_{\alpha\beta\mu\nu} \xi^\beta \xi^\mu u^\nu\right) = \varphi\left(\tilde{u}^\alpha \tilde{R}_{\alpha\beta\mu\nu} \tilde{\xi}^\beta \tilde{\xi}^\mu \tilde{u}^\nu\right) = \varphi \tilde{K}(\tilde{\xi}, \tilde{u}) \tag{10.5.29}$$

where $\tilde{K}(\tilde{\xi}, \tilde{u})$ is the sectional curvature for the metric $\tilde{g}$.

> The unit vectors $\tilde{u} = \varphi^{-\frac{1}{2}}u$, $\tilde{\xi} = \varphi^{-\frac{1}{2}}\xi$ are now normalised with respect to the metric $\tilde{g}$.
> Let us, however, remark that eventually we shall compute everything at x_0 where $\varphi(x_0) = 1$.

We can also write the sectional curvature in terms of the metric g:

$$\mathcal{K}(\xi, u) = \varphi^{-1}\left(u^\alpha \tilde{R}_{\alpha\beta\mu\nu} \xi^\beta \xi^\mu u^\nu\right) = K(\xi, u) + u^\alpha \left(2\delta^\epsilon_{[\alpha} g_{\beta][\mu} \nabla_{\nu]\epsilon} \ln(\varphi) + \left(\delta^\rho_{[\alpha} g_{\beta][\nu} \delta^\sigma_{\mu]} - \frac{1}{2}g^{\rho\sigma} g_{\nu[\beta} g_{\alpha]\mu}\right) \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi)\right) \xi^\beta \xi^\mu u^\nu =$$
$$= K(\xi, u) - \frac{1}{2}u^\rho u^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \frac{1}{2}\xi^\rho \xi^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \frac{1}{4}u^\rho u^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) - \frac{1}{4}\xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) - \frac{1}{4}\nabla_\rho \ln(\varphi) \nabla^\rho \ln(\varphi) \tag{10.5.30}$$

Hence the infinitesimal displacement is

$$D = \epsilon \left(t - \frac{1}{2}\xi^\alpha \nabla_\alpha \ln(\varphi)t^2 + \left(\frac{1}{6}\mathcal{K}(u, \xi) + \frac{1}{4}\tilde{\nabla}\alpha + \frac{1}{8}\alpha^2\right)t^3\right) + O(t^4) =$$
$$= \epsilon\left(t - \frac{1}{2}\xi^\alpha \nabla_\alpha \ln(\varphi)t^2 + \left(-\frac{1}{4}\xi^\rho \xi^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \frac{1}{4}\xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) + \frac{1}{8}\nabla_\rho \ln(\varphi)\nabla^\rho \ln(\varphi) + \frac{1}{8}\xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi)\right)t^3 +$$
$$+ \frac{1}{6}\left(K(\xi, u) - \frac{1}{2}u^\rho u^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \frac{1}{2}\xi^\rho \xi^\sigma \nabla_{\rho\sigma} \ln(\varphi) + \frac{1}{4}u^\rho u^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) - \frac{1}{4}\xi^\rho \xi^\sigma \nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) - \frac{1}{4}\nabla_\rho \ln(\varphi)\nabla^\rho \ln(\varphi)\right)t^3 + O(t^4)\right) =$$
$$= \epsilon\left(t - \frac{1}{2}\xi^\alpha \nabla_\alpha \ln(\varphi)t^2 + \frac{1}{6}\left(K(\xi, u) + \left(-\xi^\rho \xi^\sigma - \frac{1}{2}u^\rho u^\sigma\right)\nabla_{\rho\sigma} \ln(\varphi) + \left(\frac{1}{4}u^\rho u^\sigma + 2\xi^\rho \xi^\sigma\right)\nabla_\rho \ln(\varphi) \nabla_\sigma \ln(\varphi) + \frac{1}{2}\nabla_\rho \ln(\varphi)\nabla^\rho \ln(\varphi)\right)t^3 + O(t^4)\right) \tag{10.5.31}$$

> And that is where the discussion is getting interesting. In a Palatini $f(\mathcal{R})$-theory, we obtain from dynamics an integral Weyl frame $(M, g, \tilde{\Gamma})$.

If we show that $|\epsilon\zeta|$ has anything to do with acceleration of freely falling particles starting at the same event with different velocities, then $\alpha = \boldsymbol{\alpha}(\xi) = -\xi^\mu d_\mu \ln(\varphi) = \frac{d}{dt}\ln(\varphi(\gamma(t)))$ measures the variation of the conformal factor experienced by the particles during their motion.

That term kicks in quadratically with t, before the curvature effects which kick at third order in t. Of course the variation of the conformal factor could easily be tiny on astronomical scales, e.g. in the solar system, which may though become relevant at galactic up to cosmological scales.

That will be our main road to discuss the observability of conformal factor.

We can also try to express, for future reference (and just to check we can), D in terms of the metric $\tilde{g}$, obtaining

$$
\begin{aligned}
D =& \epsilon\left(t - \tfrac{1}{2}\xi^\alpha\nabla_\alpha\ln(\varphi)t^2 + \left(\tfrac{1}{6}\mathcal{K}(u,\xi) + \tfrac{1}{4}\tilde{\nabla}\alpha + \tfrac{1}{8}\alpha^2\right)t^3\right) + O(t^4) = \\
=& \epsilon\left(\tilde{t} - \tfrac{1}{2}\tilde{\xi}^\alpha\tilde{\nabla}_\alpha\ln(\varphi)\tilde{t}^2 + \left(\tfrac{1}{6}\tilde{K}(\tilde{u},\tilde{\xi}) - \tfrac{1}{4}\tilde{\xi}^\rho\tilde{\xi}^\sigma\tilde{\nabla}_{\rho\sigma}\ln(\varphi) + \tfrac{1}{8}\tilde{\xi}^\rho\tilde{\xi}^\sigma\tilde{\nabla}_\rho\ln(\varphi)\tilde{\nabla}_\sigma\ln(\varphi)\right)\tilde{t}^3\right) + O(\tilde{t}^4) =
\end{aligned}
\tag{10.5.32}
$$

where we set $\tilde{t} = \sqrt{\varphi}t$.

Let us remark that these quantities are computed at x_0 where the conformal factor is normalised to $\varphi(x_0) = 1$ so that there we have $\tilde{t} = t$, $\tilde{\xi} = \xi$, and $\tilde{u} = u$.

Let us also stress that D is anyway the infinitesimal displacement in the (integral) Weyl frame $(M, g, \{\tilde{g}\})$, even if we decided to express everything in terms of $\tilde{g}$ or g.

In other words, the clock is anyway considered non-standard.

Focusing of geodesics

Let us here consider another setting to connect the curvature to physical quantities in a Weyl geometry $(M, \mathfrak{g}, \tilde{\Gamma})$.

To be explicit, the attempt is for implementing more physical setting requirements.

We would like to express mathematically the following setting: an observer prepares two test particles, leaving them at the same time, initially at rest one with respect to the other, from two nearby spatial points and it observes them freely falling, measuring their relative displacement in time, so measuring their relative velocity and acceleration.

If the initial displacement is infinitesimal, relative quantities are a good indication at first order of the corresponding geometric quantities.

We still cannot identify an observer with a clock, unless we restrict to dimension 2. So we consider a two dimensional spacetime and a clock $\gamma(t)$. The first test particle will be the clock itself. Accordingly, let us consider a $\mathfrak{g}$-time-like $\tilde{\Gamma}$-geodesic motion $\gamma(t)$.

The clock defines a foliation of isochronous surfaces and be Σ_0 the leaf where the clock is set to $t = 0$. Let $x(\epsilon)$ be a line on Σ_0, hence $x(\epsilon)$ is $\mathfrak{g}$-space-like.

Let us denote by v the initial covariant velocity at $x_0 = \gamma(0) = x(0)$ for the geodesic $\gamma(t)$ and by $(x(\epsilon), v(\epsilon))$ the initial condition for the $\tilde{\Gamma}$-geodesic $\gamma(t; \epsilon)$. This defines a family $\gamma(t; \epsilon)$ of time-like geodesics starting from initial conditions $(x(\epsilon), v(\epsilon))$. Let us define as usual the generators

$$
\xi(\gamma(t;\epsilon)) = \frac{d\gamma}{dt}(t;\epsilon) \qquad \zeta(\gamma(t;\epsilon)) = \frac{d\gamma}{d\epsilon}(t;\epsilon)
\tag{10.5.33}
$$
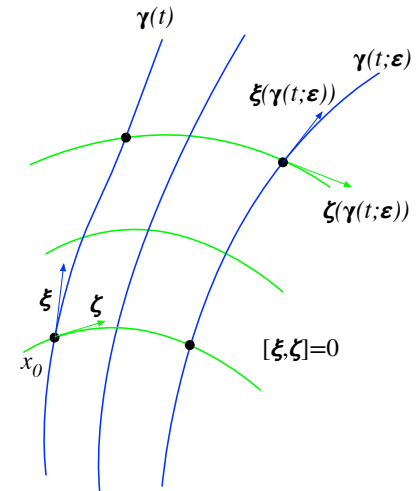
Fig. 9.11: Tidal forces.

The geodesics $\gamma(t;\epsilon)$ are integral curves of ξ, which is hence a geodesic field ($\xi^\alpha \tilde{\nabla}_\alpha \xi^\mu = 0$), ζ is a Jacobi field $\xi^\alpha \xi^\beta \tilde{\nabla}_{\alpha\beta} \zeta^\mu = \tilde{R}^\mu_{\ \alpha\beta\lambda} \xi^\alpha \xi^\beta \zeta^\lambda$. Since we want the particle $\gamma(t;\epsilon)$ to be initially comoving with the particle $\gamma(t)$ their relative velocity should be initially zero, i.e. we should have $V(0) = \xi^\alpha \tilde{\nabla}_\alpha \zeta + \frac{1}{2}\alpha\zeta = 0$ at x_0.

> Let us remark that in standard GR, one usually say that two particles are initially comoving if their initial conditions are parallelly transpronted. That us because in a Riemannian frame $\alpha = 0$ and the condition becomes $V(0) = \xi^\alpha \tilde{\nabla}_\alpha \zeta = 0$ which is precisely the condition for which ζ is parallelly transported along $x(\epsilon)$. Since the relative velocity gets corrections in a Weyl frame, the conditions needs to be generalised properly in that case.

We want now again to expand the quantity $D = |\epsilon\zeta|$, this time to second order in t, so let us start to expand the field $\zeta(t;0) = \zeta(\gamma(t;0))$.

In this case, we have

$$\zeta = \frac{d\gamma}{d\epsilon}(0;0) = x'(0) =: w \qquad\qquad D\zeta = \tilde{\nabla}\zeta + \tfrac{1}{2}\alpha\zeta = 0 \qquad\qquad D^2\zeta^\mu = \tilde{R}^\mu_{\ \alpha\beta\lambda}\xi^\alpha\xi^\beta w^\lambda + \tfrac{1}{4}\alpha^2 w^\mu + \tfrac{1}{2}\tilde{\nabla}\alpha w^\mu \tag{10.5.34}$$

The expansion of the vector field is

$$\zeta^\mu(t;0) = w^\mu + \tfrac{1}{2}\left(\tilde{R}^\mu_{\ \alpha\beta\lambda}\xi^\alpha\xi^\beta w^\lambda + \tfrac{1}{4}\alpha^2 w^\mu + \tfrac{1}{2}\tilde{\nabla}\alpha w^\mu \right)t^2 + O(t^3) \tag{10.5.35}$$

and, expanding the infinitesimal displacement, we have

$$
\begin{aligned}
D^2 =&|\epsilon\zeta|^2 = \epsilon^2|w|^2 \left(1 + \left(\frac{w^\lambda g_{\lambda\mu}\tilde{R}^\mu_{\ \alpha\beta\lambda}\xi^\alpha\xi^\beta w^\lambda}{|w|^2} + \tfrac{1}{4}\alpha^2 + \tfrac{1}{2}\tilde{\nabla}\alpha \right)t^2 + O(t^3) \right) = \\
=&\epsilon^2|w|^2 \left(1 + \left(-\frac{|w|^2|\xi|^2 - g(w,\xi)^2}{|w|^2}\mathcal{K}(w,\xi) + \tfrac{1}{4}\alpha^2 + \tfrac{1}{2}\tilde{\nabla}\alpha \right)t^2 + O(t^3) \right) \\
D =&|\epsilon\zeta| = \epsilon|w| \left(1 + \left(-\frac{|w|^2|\xi|^2 - g(w,\xi)^2}{2|w|^2}\mathcal{K}(w,\xi) + \tfrac{1}{4}\tilde{\nabla}\alpha + \tfrac{1}{8}\alpha^2 \right)t^2 + O(t^3) \right)
\end{aligned}
\tag{10.5.36}
$$

To simplify the setting we can further specialise it: the isochronous surface is orthogonal to the clock at x_0 (i.e. $g(\xi,\zeta) = 0$) and we can use the affine freedom for geodesic motions to set $|\xi|^2 = -1$, again at x_0. Accordingly, we have

$$D(t) = \epsilon|w| \left(1 + \left(\tfrac{1}{2}\mathcal{K}(w,\xi) + \tfrac{1}{4}\tilde{\nabla}\alpha + \tfrac{1}{8}\alpha^2 \right)t^2 + O(t^3) \right) \tag{10.5.37}$$

The infinitesimal displacement is initially $D = |\epsilon\zeta(0)| = \epsilon|w|$ and later on it becomes

$$D(t) = D + \Delta D \qquad\qquad \Delta D(t) = \frac{\epsilon|w|}{2}\left(\mathcal{K}(w,\xi) + \tfrac{1}{2}\tilde{\nabla}\alpha + \tfrac{1}{4}\alpha^2 \right)t^2 + O(t^3) \tag{10.5.38}$$

Thus

$$\frac{\Delta D}{D} = \left(\mathcal{K}(w,\xi) + \tfrac{1}{2}\tilde{\nabla}\alpha + \tfrac{1}{4}\alpha^2 \right)t^2 + O(t^3) \tag{10.5.39}$$

which is once again controlled by the sectional curvature, with corrections in α, which, in integrable Weyl frames, are ruled by variations of the conformal factor.

> Notice that in $\frac{\Delta D}{D}$ there is no linear term in t (initially comoving particles) and that the second derivative $2\left(\mathcal{K}(w,\xi) + \frac{1}{2}\tilde{\nabla}\alpha + \frac{1}{4}\alpha^2\right)$ is the initial acceleration, which is a force (called the *tidal force*) acting between the two particles.
>
> The tidal force gets a contribution from the curvature, which is there in standard GR as well. This is pretty easy to guess. Consider two particles left freely falling, initially at rest, from a height near the Earth, i.e. in a Schwarzschild spacetime. The two particles will fall radially, thus they will get closer as they fall. Tidal force is responsible for the relative acceleration that will get the two particles to collide as they arrive to the center of the the the Earth.
>
> In a Weyl frame though, the tidal force also gets a contribution from α, i.e. just because the reference clock is non-standard but an NP-clock with respect to g.

6. Cosmological distances

As we said, some of the protocols discussed above to define distances have an application to astrophysics and cosmology which is academic, only. Though in principle we could wait for a light signal to go back and forth to the center of the galaxy, it is not very practical, to be euphemistic.

In these areas, we often rely on what we receive only, with no possibility of sending messages. Accordingly, we use different protocols to define synchronisation and distances, which often depend on the relativistic theory one assumes. Cosmology is a paradigmatic example.

The new protocols are similar to the old ones, though they use different foliations. We already discussed that spatially homogeneous and isotropic spacetimes come to us with two foliations singled out on them by the Killing algebra: the space foliation of the hypersurfaces $t = t_0$ (in adapted coordinates, i.e. the ones in which the metric is written in FLRW form) and the foliation of comoving motions $(r = r_*, \theta = \theta_*, \phi = \phi_*)$.

Comoving motions can be parameterised so to be NP-clocks (with respect to the metric g on spacetime). Since in adapted coordinates (t, r, θ, ϕ) the metric is in FLRW form, the comoving NP-clocks are $\gamma_* : s \mapsto (t = s, r = r_*, \theta = \theta_*, \phi = \phi_*)$ and they do foliate the spacetime. However, if we consider a space surface $t = t_0$, that is not isochronous with respect to a comoving clock, although it is *approximately*, in a region near the clock. As we discussed, when we go away from the clock, the sphere $(t = t_0, r = r_*)$ is not isochronous to $(t = t_0, r = 0)$, it stays in its future by an offset which is dynamically determined by the Friedman equation, i.e. by the matter content of the universe.

On the other hand, the comoving clocks stay synchronised with respect to the space foliation, i.e., if they are synchronised on the surface $t = t_0$, they are on any other space surface $t = t_1$.

> Notice that an observer sitting at $r = 0$, having its clock could define synchronisations using Einstein convention in a neighbourhood of itself. Such a synchronisation agrees only approximately with the one used by comoving clocks. For such an observer, a far away comoving clock is speeding up with respect to its clock.
>
> Is it a surprise if such a mismatch could be interpreted as an extra gravitational field which needs extra sources to be created?
>
> That has nothing to do with extended theories, though it is similar to what happens in a Weyl frame due to the conformal factor. The effect we mentioned here exists only if one defines the distance on the surface $t = t_0$ and erroneously mixes that with the geometric distance defined on the isochronous surfaces. Unlike in extended theories, where the mismatch between g and $\tilde{g}$ is from a physical hypothesis and hence it is determined by the dynamics, here the effect is produced by a mistake and can be removed by doing things properly.

In view of this remark, consider an event (t_0, χ_0) and a comoving observer sitting at the origin with an atomic clock which is proper with respect to g; let us define the *proper distance* $\delta(t_0, \chi_0)$ (with respect to a metric g) to be the geometric distance on the surface $t = t_0$ with respect to the metric $^*g$ induced by g on that surface.

Finally, we need to show that the geometric distance and the proper distance are different, i.e. that the foliation used matters.

For, let us consider spatially homogenous and isotropic spacetime (M, g). In adapted coordinates (t, χ, θ, ϕ) the metric is in FLRW form. For the sake of simplicity, let us consider a universe containing only dust and evolving in standard GR. Friedman equation determines the evolution of the scale factor as

$$\left(\frac{da}{dt}\right)^2 = \frac{1}{3}\frac{\rho_0}{a^3}a^2 =: \Phi(a) \qquad \Rightarrow t - t_0 = \int_{a_0}^a \frac{da}{\sqrt{\Phi(a)}} \qquad \Rightarrow t = t(a) \qquad (10.6.1)$$

where we can fix $t_0 = 0$ (today) and $a_0 = 1$ for the usual scale factor normalisation. The function $t(a)$ provides us with a parametric representation of the evolution of the scale factor.

Now let us consider an NP-clock at the origin, $\gamma : \mathbb{R} \to M$, where $\gamma(s)$ is represented by $(t = s, \chi = 0)$, as well as an event p at $(t = t_*, \chi = \chi_*, \theta = \frac{\pi}{2}, \phi = 0)$. For the proper distance between the clock γ and p, we consider the surface Σ at $t = t_*$, and a line $\sigma : \mathbb{R} \to \Sigma : s \mapsto (t_*, \chi = s, \frac{\pi}{2}, 0)$ which, in fact, connects the origin $(t = t_*, \chi = 0)$ with the event p at $(t = t_*, \chi = \chi_*)$, for $s \in [0, \chi_*]$.

The curve σ is a geodesic trajectory for the metric $^*g$ induced on Σ by g.
We have $\dot\sigma = \partial_\chi$ and $\ddot\sigma = 0$ so that the covariant acceleration reads as

$$a = \ddot\sigma^\alpha + \{^*g\}^\alpha_{\mu\nu}\dot\sigma^\mu\dot\sigma^\nu = \{^*g\}^\alpha_{\chi\chi} \qquad (10.6.2)$$

That is parallel to the covariant velocity $\dot\sigma$; in fact, for $i = (\theta, \phi)$ one has

$$a^i = \{^*g\}^i_{\chi\chi} = \tfrac{1}{2}g^{ij}\left(-\partial_j g_{\chi\chi} + 2\partial_\chi g_{j\chi}\right) = 0 \qquad (10.6.3)$$

The radial component is

$$a^\chi = \tfrac{1}{2}g^{\chi\chi}\partial_\chi g_{\chi\chi} = 0 \qquad (10.6.4)$$

so, parameterised in χ, that is also a geodesic motion. Since the second derivative a, as well as the connection's coefficients $\{^*g\}^\alpha_{\chi\chi}$, are not a vector, this, of course, does not mean that, for example, $a^r = 0$, which in fact is false.

If we repeat the same discussion in the spacetime M with its metric g, the χ-component of the geodesic equation is still satisfied, however, the geodesic equations has also a t-component to be satisfied. The time component $a^t = 0$ also vanishes, so the equation reads as

$$0 = a^t = \{g\}^t_{\chi\chi} = -\tfrac{1}{2}g^{tt}\partial_t g_{\chi\chi} = a\dot a \neq 0 \qquad (10.6.5)$$

Accordingly, the curve σ is a geodesics in $(\Sigma, ^*g)$, not in (M, g).

The length of the curve σ is hence

$$\delta(t_*, \chi_*) = a(t_*)\int_0^{\chi_*} d\chi = a(t_*)\chi_* \qquad (10.6.6)$$

which is the (exact) proper distance at time t_*. Let us remark that no approximations have been made.

One can see that proper distance of comoving points changes in time solely due to the scale factor. The quantity

$$\chi_* = \int_0^{\chi_*} d\chi = \int_0^{r_*} \frac{dr}{\sqrt{1-kr^2}} = \frac{\delta(t_*, \chi_*)}{a(t_*)} \tag{10.6.7}$$

is called the *comoving distance* and it is independent of time for comoving particles at $\chi = \chi_*$.

As far as the geometric distance is concerned, we need to define the isochronous surfaces $\hat{\Sigma}$; see (10.2.43). For an event b at (t_b, χ_b), we have two radial light rays, which hit the clock γ at $t_\pm$; the clock readings $t_\pm$ (and t_b, respectively) correspond, in view of the solution $t(a)$ of Friedman equation, to the scale factors $a_\pm$ (and a_b, respectively).

Then, by using equation (10.2.43) for light rays in FLRW, we have

$$\chi_b = -\int_{a_b}^{a_-} \frac{da}{a\sqrt{\Phi(a)}} = \int_{a_b}^{a_+} \frac{da}{a\sqrt{\Phi(a)}} \tag{10.6.8}$$

From these two equations, we can solve for $a_b = a_b(a_+, a_-)$ as well we can define $\chi_b = \chi_b(a_+, a_-)$.

In other words, fix $a_\pm$ and one can find a_b (and, consequently, $t_b = t(a_b)$) and χ_b, i.e. determining the event (t_b, χ_b) at which the light rays hitting the clock γ at $t_\pm$ intersect points.

The echo event (t_b, χ_b) determined in this way corresponds to the clock readings $s_\pm = t_\pm$, i.e. it is synchronised with

$$s_b = \frac{t(a_+) + t(a_-)}{2} =: s_b(a_+, a_-) \tag{10.6.9}$$

If we fix the surface $\hat{\Sigma}$ passing here through today, its equation $s_b(a_+, a_-) = 0$ determines $a_+ = a_+(a_-)$.

Now, if we fix a_-, then we can compute a_+ so that the echo event singled out by $a_\pm$ is synchronised to here today. Then, in view of the previous remark, we can determine the event (t_b, χ_b) which, by construction, is on $\hat{\Sigma}$.

The quantity a_- then parameterises the radial line σ on $\hat{\Sigma}$ linking $\gamma(0)$ to $(t_b(a_-, t_0), \chi_b(a_-, t_0))$ on the comoving particle P. The length of σ is then given by

$$d(t_0) = \int_{a_0}^{a_*} \sqrt{-\left(\frac{dt_b}{da_-}\right)^2 + a^2(t_0)\left(\frac{d\chi_b}{da_-}\right)^2}\, da_- \tag{10.6.10}$$

and it is the geometric distance.

In the case of a dust, spatially flat universe, we can compute this all. For $a_* = \frac{1}{2}$, $a_0 = 1$, $t_* = t_0 = 0$, we obtain the proper distance $\delta \simeq 0.733$ and the geometric distance $d \simeq 0.6838$. Hence the geometric distance is different, in general, from the proper distance.

While the geometric distance is perhaps more covariant, the proper distance is more practical, as well as easier to compute. Let us remark once again, we did no approximation until now. To summarise, we defined defined proper distance and showed that it is $\delta(t_*, \chi_*) = a(t_*)\chi_*$.

For later convenience, let us remind that, depending on the spatial curvature k, we defined a specific relation between r and χ, namely

$$r_* = \begin{cases} \frac{1}{\lambda}\sin(\lambda\chi_*) & k > 0 \\ \chi_* & k = 0 \\ \frac{1}{\lambda}\sinh(\lambda\chi_*) & k < 0 \end{cases} \tag{10.6.11}$$

where we set $\lambda = \sqrt{|k|}$.

Of course, for small comoving distances χ_*, all three right hand sides expand to χ_*, meaning that for nearby objects $r_* \simeq \chi_*$ independently of the the space curvature k. However, if we consider far away objects, as we do sometimes in cosmology, one needs to keep track of the deviation of χ_* from r_*, which is precisely ruled by the spatial curvature k.

The Hubble parameter and red shift

In an integrable Weyl frame $(M, g, \tilde{\Gamma} = \{\tilde{g}\})$ one has two conformal metrics. If one metric is spatially homogeneous and isotropic, i.e. it obeys cosmological principle, then the other is as well, though with respect to a different time coordinate $\tilde{t}$ and scale factor $\tilde{a}$. The two conformal metrics have two scale factors, a and $\tilde{a}$, and one can define two different *Hubble parameters*, namely:

$$H(a) := \frac{1}{a}\frac{da}{dt} = \frac{\sqrt{\Phi(a)}}{a} \qquad\qquad \tilde{H}(\tilde{a}) := \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}} = \frac{\sqrt{\tilde{\Phi}(\tilde{a})}}{\tilde{a}} \tag{10.6.12}$$

where we assumed an expanding universe, i.e. $\dot{a} > 0$ or, equivalently, $\dot{\tilde{a}} > 0$.

These two quantities are different.

In fact, we have

$$\tilde{H}(\tilde{t}) = \frac{\dot{\tilde{a}}(\tilde{t})}{\tilde{a}(\tilde{t})} = \frac{d\tilde{a}(\tilde{t})}{dt}\frac{dt}{d\tilde{t}}\frac{1}{\sqrt{\varphi(t)}a(t)} = \frac{d(\sqrt{\varphi(t)}a(t))}{dt}\frac{1}{\sqrt{\varphi(t)}}\frac{1}{\sqrt{\varphi(t)}a(t)} = \left(\sqrt{\varphi(t)}\dot{a} + \frac{\dot{\varphi}(t)}{2\sqrt{\varphi(t)}}a(t)\right)\frac{1}{\varphi(t)a(t)} = \frac{1}{\sqrt{\varphi(t)}}H(t) + \frac{\dot{\varphi}(t)}{2\varphi(t)^{3/2}} \tag{10.6.13}$$

Even if we evaluate that today (when $\varphi(t_0) = 1$), the difference is still non-zero

$$\tilde{H}(\tilde{t}_0) = H(t_0) + \tfrac{1}{2}\dot{\varphi}(t_0) \tag{10.6.14}$$

and it is ruled by the rate of change of the conformal factor which, in a generic Palatini $f(\mathcal{R})$-theory, is non-zero.

Of course, if we measure the red shift of a far away, comoving source, then that is either associated to H or to $\tilde{H}$. It cannot fit both, just because they are different. Thus we need to discuss which Hubble parameter is correlated with red shift of distant sources and how. According to our interpretation of Weyl frames, we shall here assume that free fall is described by $\tilde{g}$ and atomic clocks are described by g.

Let us consider a light ray starting at comoving coordinates $(\tilde{t}_e, \chi_e, \theta_*, \phi_*)$ and travelling radially towards the origin, arriving at $(\tilde{t}_o, \chi_o = 0, \theta_*, \phi_*)$. Its trajectory in spacetime is parameterised as

$$\tilde{t} = \tilde{t}(s) \qquad \chi = \chi(s) \tag{10.6.15}$$

and, being a light ray, its covariant velocity must be light like, i.e.

$$\mathrm{c}^2 \left(\dot{\tilde{t}}\right)^2 = \tilde{a}^2(\tilde{t}) \frac{(\dot{r})^2}{1 - kr^2} = \tilde{a}^2(\tilde{t}) \left(\dot{\chi}\right)^2 \qquad \Rightarrow -\mathrm{c}d\tilde{t} = \tilde{a}(\tilde{t}) \frac{dr}{\sqrt{1 - kr^2}} = \tilde{a}(\tilde{t})d\chi \qquad (10.6.16)$$

Since g is conformal to $\tilde{g}$, let us remark that the light ray is a light-like geodesics trajectory for g as well, since

$$-\mathrm{c}\sqrt{\varphi(t)}dt = \sqrt{\varphi(t)}a(t)\frac{dr}{\sqrt{1 - kr^2}} = \sqrt{\varphi(t)}a(t)d\chi \qquad \Rightarrow -\mathrm{c}dt = a(t)\frac{dr}{\sqrt{1 - kr^2}} = a(t)d\chi \qquad (10.6.17)$$

Now let us consider a second light ray starting at the same comoving spatial coordinates at a time $\tilde{t}_e + \delta\tilde{t}_e$ and being received at the origin at time $\tilde{t}_o + \delta\tilde{t}_o$. It follows the geodesics trajectory

$$\mathrm{c}\int_{\tilde{t}_e + \delta\tilde{t}_e}^{\tilde{t}_o + \delta\tilde{t}_o} \frac{d\tilde{t}}{\tilde{a}(\tilde{t})} = \int_0^{\chi_e} d\chi \stackrel{(10.6.17)}{=} \mathrm{c}\int_{t_e}^{t_o} \frac{dt}{a(t)} \qquad (10.6.18)$$

Setting $F(\tilde{t})$ for a primitive of $\mathrm{c}\tilde{a}^{-1}(\tilde{t})$, one has

$$\mathrm{c}\int_{\tilde{t}_e + \delta\tilde{t}_e}^{\tilde{t}_o + \delta\tilde{t}_o} \frac{d\tilde{t}}{\tilde{a}(\tilde{t})} = F(\tilde{t}_o + \delta\tilde{t}_o) - F(\tilde{t}_e + \delta\tilde{t}_e) \simeq F(\tilde{t}_o) - F(\tilde{t}_e) + F'(\tilde{t}_o)\delta\tilde{t}_o - F'(\tilde{t}_e)\delta\tilde{t}_e = \int_0^{\chi_e} d\chi + \mathrm{c}\left(\frac{\delta\tilde{t}_o}{\tilde{a}(\tilde{t}_o)} - \frac{\delta\tilde{t}_e}{\tilde{a}(\tilde{t}_e)}\right) \qquad (10.6.19)$$

This *is*, of course, an approximation. We are expanding in series the primitive F between $\tilde{t}_o$ and $\tilde{t}_o + \delta\tilde{t}_o$ and between $\tilde{t}_e$ and $\tilde{t}_e + \delta\tilde{t}_e$. These intervals are for as long as the time between a front wave and the following one at emission or observation. Unless we are talking about wavelengths of light years, that seems really a safe approximation.

It is *quantitatively different* with respect to the approximation that we shall consider later for Hubble law, in which we shall expand between $\tilde{t}_o$ and $\tilde{t}_e$. If we are observing the closest galaxies, such an interval can easily span $2 \cdot 10^6$ years (e.g. for Andromeda galaxy). And it can be as far as $13 \cdot 10^9$ years for the farthest objects, where the linear approximation certainly fails.

Of course, one should refer to adimensional variables when expanding. In this case we should compare the time scale with the timescale at which one can appreciate the second derivative of $a(t)$, which is what rules the estimate of the approximation error. That scale, in fact, depends on the model. What we are saying is that in any reasonable model such approximations are good or bad.

In view of equations (10.6.18) and (10.6.19), one obtains

$$\frac{\delta\tilde{t}_o}{\tilde{a}(\tilde{t}_o)} \simeq \frac{\delta\tilde{t}_e}{\tilde{a}(\tilde{t}_e)} \qquad (10.6.20)$$

which, in fact, determines the delay $\delta\tilde{t}_o$ at receiving as a function of the delay $\delta\tilde{t}_e$ at emission.

Let us stress that both the quantities $\delta\tilde{t}_o$ and $\delta\tilde{t}_e$ are not physical quantities, they are coordinate differences, not physical times.

In order to transform them into physical times we need to introduce atomic clocks and use them to measure the corresponding relevant distances on spacetime. Of course, the difference can be tiny, though there is always time to neglect it later. Even when the difference is in fact negligible, it should not make any difference to keep track of it and then neglect it later.

An atomic clock is measuring the interval $\delta \tilde{t}_e$ at emission and the corresponding frequency is

$$\nu_e = \tfrac{1}{\Delta t_e} \tag{10.6.21}$$

where $\mathbf{c}\Delta t_e$ is the length of the vector $\xi_e = \delta \tilde{t}_e \tilde{\partial}_0$ measured by g, namely

$$-\mathbf{c}^2 \Delta t_e^2 = g(\xi_e, \xi_e) = \frac{\tilde{g}(\xi_e, \xi_e)}{\varphi(t_1)} = -\mathbf{c}^2 \frac{\delta \tilde{t}_e^2}{\varphi(t_e)} \quad \Rightarrow \Delta t_e = \frac{\delta \tilde{t}_e}{\sqrt{\varphi(t_e)}} \tag{10.6.22}$$

Analogously, the frequency at absorption (or observation) is

$$\nu_o = \tfrac{1}{\Delta t_o} \qquad\qquad \Delta t_o = \frac{\delta \tilde{t}_o}{\sqrt{\varphi(t_o)}} = \frac{\delta \tilde{t}_e}{\sqrt{\varphi(t_o)}} \frac{\delta \tilde{t}_o}{\delta \tilde{t}_e} = \frac{\delta \tilde{t}_e}{\sqrt{\varphi(t_e)}} \frac{\sqrt{\varphi(t_e)}}{\sqrt{\varphi(t_o)}} \frac{\tilde{a}_o}{\tilde{a}_e} = \frac{a_o}{a_e} \Delta t_e \tag{10.6.23}$$

Thus the observed wavelength shift z is

$$z := \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\frac{c}{\nu_o} - \frac{c}{\nu_e}}{\frac{c}{\nu_e}} = \left(\frac{\nu_e - \nu_o}{\nu_o \nu_e}\right) \nu_e = \frac{\nu_e - \nu_o}{\nu_o} = \frac{\Delta\nu}{\nu_o} = \frac{\frac{1}{\Delta t_e} - \frac{1}{\Delta t_o}}{\frac{1}{\Delta t_o}} = \frac{\Delta t_o - \Delta t_e}{\Delta t_e} = \frac{a_o}{a_e} - 1 = \frac{a_o - a_e}{a_e} \tag{10.6.24}$$

Thus we measure the wavelength shift z for some line of hydrogen and this is the prediction of our model.

> For an expanding universe, $a_o > a_e$, since absorption is at later time than emission, one has $t_o - t_e > 0$, thus $z > 0$ meaning that $\Delta\lambda > 0$, i.e. the light ray is absorbed with a longer wavelength than the wavelength at emission. Accordingly, one has *red shift*.

The point here is that each model is capable of deciding if the observed red shift is represented by z or by $\tilde{z}$, provided one declares which metric describes atomic clocks. One should only repeat the argument to decide, for each quantity, if it has to be computed with g or $\tilde{g}$ (or never mind).

> Let us stress again that we made no approximation (with the exception of keeping wavelengths reasonably short). If we consider an observer measuring red shifts of different sources at a given time t_o, then the measured red shifts z define a function of emission time, and consequently of the source distance. Once again, one can use emission time, emission scale factor, or the red shift, to quantify distances. In any event, one has

$$z(a) = \frac{a_o - a}{a} \tag{10.6.25}$$

> This relation between the red shift and the scale factor is relatively independent of the model; it certainly does not depend on the dynamics (i.e. on Friedmann equation). Still it relies on the assumptions that spacetime is spatially homogeneous and isotropic and that light rays move along light like geodesics of FLRW metric.

Luminosity distance

Let us consider a star (as a comoving test particle, since we do not have an exact solution for a pointlike source in an asymptotically FLRW spacetime, furthermore emitting an electromagnetic radiation preserving spatial isotropy) in the origin of a FLRW spacetime, emitting photons isotropically, namely the number of photons emitted is equal in each direction.

Let us also consider a sphere $S = \{t = t_*, r = r_*\}$. Since the FLRW metric is pulled-back on S to ${}^*g = a^2 r^2 d\Omega^2$, the sub-manifold $S \subset M$ is a metric sphere of radius $\rho = a(t_*)r_*$, just embedded in a curved manifold M instead of in an Euclidean space. The induced metric ${}^*g$ defines a volume element on S and its area is $A_S = 4\pi\rho^2$.

Emitted photons go away along light like worldlines and they reach a (spatial) sphere S at $r = r_*$ (or, equivalently, $\chi = \chi_*$); in view of symmetries, two photons, emitted at the same time, reach the sphere S at the same time $t = t_*$ each carrying its energy.

In view of what we discussed in the previous Subsection, we can argue that n photons are emitted during the time Δt_e, each carrying an energy $E = \hbar\nu_e$, for a total emitted power

$$P_e = \frac{n\hbar\nu_e}{\Delta t_e} \tag{10.6.26}$$

The same photons go through the sphere S in a time Δt_o each carrying an energy $E' = \hbar\nu_o$. The power flow through the sphere is

$$P_o = \frac{n\hbar\nu_o}{\Delta t_o} = \frac{n\hbar\nu_e}{\Delta t_e}\frac{\Delta t_e}{\Delta t_o}\frac{\nu_o}{\nu_e} = \left(\frac{\Delta t_e}{\Delta t_o}\right)^2 P_e = \left(\frac{a_e}{a_o}\right)^2 P_e \tag{10.6.27}$$

If we consider a detector on the sphere S measuring the energy flow per unit of time through a small area A_D, we know that the energy through the detector is proportional to its area, that is

$$P_D = \frac{A_D}{A_S}P_o = \frac{A_D}{4\pi a_o^2 r_*^2}\left(\frac{a_e}{a_o}\right)^2 P_e = \frac{A_D P_e}{4\pi r_*^2}\frac{a_e^2}{a_o^4} = \frac{A_D}{4\pi d_L^2}P_e \qquad \Rightarrow \qquad d_L := \frac{a_o}{a_e}\,a_o r_* = (1+z)a_o r_* \tag{10.6.28}$$

where $d_L := \sqrt{\frac{P_e}{4\pi}\frac{A_D}{P_D}}$ is here defined and it is called the *luminosity distance*. The luminosity distance would be the length of the radius of the sphere in an Euclidean space in which the detected power flow were the same as observed, assuming of course the same emitted power flow.

> I know it is not particularly geometric or elegant as a definition, though it is a standard distance used in astrophysics or, more generally, whenever one has a *standard candle*, i.e. a source of known emitted power flow P_e.
>
> There are different classes of astrophysical objects which are considered good standard candles, among which the Cepheids and supernovae Ia. Of course, standard candles introduce *a posteriori* scales, due simply to the fact that (luckily) we have no Cepheids or supernovae on the scale of kilometers, or we cannot see Cepheids on the scales of 10^9 light years when they become to faint to be observed. Accordingly, we practically cannot measure distances on Earth with standard Cepheids, even if in principle we could.

We also have the proper distance, which for a comoving source is $\delta(r_e, t_o) = a_o\chi_*$, as well as the relation between r and χ, namely $r_* = S(\chi_*; k)$. These provide us with a relation between proper distance, red shift, and luminosity distance:

$$\frac{d_L}{(1+z)a_o} = r_* = S(\chi_*; k) = S\left(\frac{\delta(r_e, t_o)}{a_o}; k\right) \tag{10.6.29}$$

> Again there are a lot of very reasonable approximations, which simplify this expression in some regimes, making difficult to keep track of hypotheses under which easy formulae are valid. This is one of the reason why we decided to keep approximation separate from the fundamental theory.

For example, in spatially flat case $k = 0$, we have *exactly*

$$d_L = (1 + z)\delta(r_e, t_o) \tag{10.6.30}$$

Moreover, if we restrict to not too far away objects, then this relation is *approximately* true, since $S(\chi_*; k) \simeq \chi$. How far the approximation goes, it depends on how small k is.

On the other hand, the spatial curvature is measured by observations, which also rely on some estimate of distances, so one needs to watch out for circular reasoning.

Or one avoids approximations and embraces the exact, more complicated formulae.

To summarise, we introduced (the Hubble parameter as well as) the red shift and the luminosity distance, which are easily (in the sense of *clearly*) measured and precisely correlate to theoretical quantities (as the emission time, the scale factor, the proper distance) which are instead difficult to be measured directly, though they have a clearer and direct geometric meaning.

As we did in Palatini $f(\mathcal{R})$-cosmologies, the game we shall play will be to find parametric relations among these quantities, precisely finding them as functions of the scale factor a. This will also clearly single out which quantities and relations depend on the dynamics (i.e. on Friedmann equation), and which do not.

On the other hand, these relations depending on dynamics contain through Friedmann equation, the possible deviations from standard GR, so they will allow to discuss the possibility of distinguishing modified theories from standard GR.

Of course, in Palatini $f(\mathcal{R})$-theories, one has to pay attention to how the quantities are measured. One could also define a tilde proper distance

$$\tilde{\delta}(r_e, \tilde{t}) := \tilde{a}(\tilde{t}) \int_0^{r_e} \frac{dr}{\sqrt{1 - kr^2}} = \sqrt{\varphi(t)} a(t) \int_0^{r_e} \frac{dr}{\sqrt{1 - kr^2}} = \sqrt{\varphi(t)} \delta(r_e, t) \tag{10.6.31}$$

a tilde red shift

$$\tilde{z} = \frac{\tilde{a}_o}{\tilde{a}_e} - 1 = \frac{\sqrt{\varphi_o}}{\sqrt{\varphi_e}}(z + 1) - 1 \qquad \Rightarrow \qquad \sqrt{\varphi_e}(1 + \tilde{z}) = \sqrt{\varphi_o}(z + 1) \tag{10.6.32}$$

a tilde luminosity distance

$$\tilde{d}_L = (1 + \tilde{z})\tilde{a}_o r_* = \frac{\sqrt{\varphi_o}}{\sqrt{\varphi_e}}(1 + z)\sqrt{\varphi_o} a_o r_* = \frac{\varphi_o}{\sqrt{\varphi_e}} d_L \tag{10.6.33}$$

as well as a tilde Hubble parameter

$$\tilde{H}(\tilde{t}) = \frac{1}{\sqrt{\varphi(t)}} H(t) + \frac{\dot{\varphi}(t)}{2\varphi(t)^{3/2}} \tag{10.6.34}$$

As a general rule, the conformal factor rules on the transformations between quantities related to g and $\tilde{g}$.

Hubble law and approximations

All the quantities introduced above correlate with distances, thus also among each other. *Hubble law* is the relation between proper distance and red shift.

Let us consider a source and measure its red shift. If the source is too near we have two issues. First, at small scales, the proper velocity predicted by the model is very small and any peculiar velocity an object may have is not negligible if the object is too near. In other words, it is difficult to check that a specific source is comoving. As the distance increases, the proper velocity increases and soon any peculiar velocity can be neglected.

The second reason, is even more important: at small scales (e.g. the scale of solar system) the actual gravitational field is not well modelled by a FLRW metric. The sun gravitational field, seen from the Earth, is not at all isotropic or homogeneous, since the sun is actually in some direction at a given distance. The same applies to the Milky Way galaxy which has a preferred center in a given direction.

The universe is spatially homogeneous and isotropic only at a big enough scale, i.e. the Killing vectors are locally only approximated. This is not a mathematical approximation, it is a physical one. The cosmological model is a well-defined and precise mathematical framework which sometimes can be exactly solved. *At the same time*, the mathematical model is an approximated account of the real gravitational fields just because the symmetry assumptions are only approximated in practice.

> That is not different of what happens with quantum gravity. Of course, the gravitational field will sooner or later exhibit a quantum behaviour which is not caught by the classical theory. However, at all scales we actually see, the gravitational field is a classical object, the classical gravitational theory, being it standard GR or some extended theory, is a self contained, well-defined and mathematically exact framework, which has a definite behaviour we slowly unravel analysing field equations and the geometry of solutions.
>
> In fact, for example, one can prove as a theorem that the classical model, for a wide class or physically meaningful initial conditions, predicts singularities, i.e. the appearance of actual infinites in classical quantities. Now, besides the fact that one may have some hard time to figure out what does even mean for a gas density (or temperature, or curvature) to become actually infinite at a point (not *very big*, infinite!), these are real predictions of the model and one should argue what they say about the real world.
>
> And once and for all, we should say: not much. As a matter of fact, today we do not know if the universe was born in the Big Bang. Of course we do not know it because we are not absolutely certain that standard GR is a good physical model (actually we know it is not since it predicts infinite quantities). *But* we do not know it, *also* because, even if standard GR were a good description of classical regime, it neglects other effects, e.g. quantum effects.
>
> In our experience, quantum effects from gravitational field *are negligible*, the classical framework matching perfectly fine with observations (all the observations we are currently able to make), which, however, does not prove that they are irrelevant also near the Big Bang. Accordingly, we cannot exclude (and somehow we expect) that getting near the Big Bang, sooner or later the quantum effects will become relevant and, even if classical model is still well defined mathematically, it stops be physically accurate.
>
> So, does the infinite actually develop? We do not know until we have a better theory able to deal with the effect we classically neglected. As a matter of fact, the universe can be still emerged from nothing at the Big Bang, even if the process may be completely finite, or may come from a pre-existing universe, possibly a universe which has always existed, as predicted by the models of Big Bounce.
>
> So, please, let us embrace our ignorance about it. Maybe it will not last for long, though we are very much in the dark now.

More or less the same argument applies to cosmology on the small scale. Does the expansion of the universe mean that we are all getting taller even after puberty? Does it mean that the area of our houses is increasing with time? And the answer is, we do not know. On human scale, the approximation of cosmology are certainly violated, and in the theory we have no implementation of what is a rigid body, should it maintain spatial distances with respect to the spatial metric or the comoving spatial metric. That is something simply beyond the possibility of out mathematical framework, just as matter is beyond the framework of SR.

The bug is not (necessarily) in the mathematics of the model, it may not be something one could correct by changing the Lagrangian to kick off. Almost certainly (i.e. *maybe*), it is (or it is also) in the relation of mathematical models with real world:

> *Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.*
>
> *In so far as theories of mathematics speak about reality, they are not certain, and in so far as they are certain, they do not speak about reality.*
>
> Geometrie and Erfahrung (1921), (A.Einstein)

So let us consider a source far enough to be described as a comoving test particle in a FLRW spacetime. It emits a light ray at time t_e which travels to us and it is received at time t_o. The scale factor at time t_e is (usually) analytical and it can be expanded in Taylor series around t_o.

If the linear approximation does not break before the cosmological principle becomes accurate, there exists a scale at which one has

$$a(t_e) \simeq a(t_o) + \dot{a}(t_o)(t_e - t_o) = a(t_o)\Big(1 + H(t_o)(t_e - t_o)\Big) = a(t_o)\Big(1 - H(t_o)(t_o - t_e)\Big) \tag{10.6.35}$$

> Compute in the ΛCDM model how much the scale factor has changed in the last million and billion years.
>
> ...
>
> This is just to show that one needs a model to check if the approximation is accurate.

If the linear approximation holds true at some scale then, for sources at that scale, one has

$$1 + z \simeq \frac{a(t_o)}{a(t_o)\left(1 - H_o(t_o - t_e)\right)} = \frac{1}{1 - H_o(t_o - t_e)} \simeq 1 + H_o(t_o - t_e) \qquad z \simeq H_o(t_o - t_e) \tag{10.6.36}$$

so that z is proportional to the *look back time* $t_o - t_e$.

In the same regime, one has

$$\begin{aligned}
\delta(t_o, \chi_e) &= a(t_o)\int_0^{\chi_e} d\chi = \mathrm{c}\int_{t_e}^{t_o}\frac{dt}{a(t)} \simeq \mathrm{c}\int_{t_e}^{t_o}\frac{dt}{1 + H_o(t - t_o)} \simeq -\mathrm{c}\int_0^{t_e - t_o}(1 - H_o t)dt = -\mathrm{c}(t - \tfrac{1}{2}H_o t^2)\Big|_0^{t_e - t_o} = \\
&= -\mathrm{c}\left((t_e - t_o) - \tfrac{1}{2}H_o(t_e - t_o)^2\right) = \mathrm{c}(t_o - t_e) + \tfrac{\mathrm{c}}{2}H_o(t_o - t_e)^2
\end{aligned} \tag{10.6.37}$$

Thus, in this approximation, one has

$$z \simeq \tfrac{1}{\mathrm{c}}H_o\delta(r_e, t_o) \tag{10.6.38}$$

which is a form of the *Hubble law: the red shift is proportional to the proper distance and the proportionality coefficient is the Hubble parameter at observation time.*

> This is more or less how the value of the Hubble parameter today was first established. One measures the red shift and the proper distance of some not-too-far sources, and computes the linear fit restricting to the first data.
>
> On the other hand, in the higher order terms there is information about how the linear approximation eventually fails thus, equivalently, about the evolution of the scale factor. Let us remark that discussing whether the linear approximation holds true at some scales is quite difficult, since it depends on at what scale there are structures that spoil homogeneity and isotropy. The prediction of structure formation is not exactly easy to obtain: it depends on perturbation theories, as well as many details about the model.

If we want to compute tilde quantities in a Palatini $f(\mathcal{R})$-cosmology, we can remark that, today, the conformal factor is exactly $\varphi(t_0) = 1$, since its value has been hidden within the value of the universal constants. If we consider observation today (i.e. $t_o = t_0$) and emissions far enough so that the scale factor growth becomes relevant (and, e.g., it is not masked by the peculiar velocities the sources may have) but near enough so that the linear approximation holds true, then we can imagine the conformal factor to be almost constant, thus

$$\varphi(t_e) \simeq \varphi(t_o) + \dot{\varphi}(t_o)(t_e - t_o) = 1 + \dot{\varphi}(t_o)(t_e - t_o) = 1 - \dot{\varphi}(t_o)(t_o - t_e) \tag{10.6.39}$$

Of course, today one has $\sqrt{\varphi(t)} = 1$ and $\tilde{\delta}(r_e, \tilde{t}_0) = \delta(r_e, t_0)$. However, at later times one has

$$\tilde{\delta}(r_e, \tilde{t}) = \sqrt{\varphi(t)}\delta(r_e, t) \simeq c\left(1 + \dot{\varphi}(t_o)(t - t_o)\right)\left((t - t_o) + \tfrac{1}{2}H(t)(t - t_0)^2\right) \simeq c\left((t - t_o) + \left(\dot{\varphi}(t_o) + \tfrac{1}{2}H(t_o)\right)(t - t_o)^2\right) \tag{10.6.40}$$

At small distances, one also has

$$\tilde{\delta}(r_e, \tilde{t}) \simeq c(t - t_o) \tag{10.6.41}$$

thus the difference with standard GR being encoded in mid distances, in the rate of changes of the relation between proper distance and $(t_e - t_o)$. One needs to be extra careful to discuss whether different estimates of proper distance in GR need to be represented by δ or $\tilde{\delta}$.

If the linear approximation is in general quite doubtful, we can go after the *exact Hubble law*, i.e. the exact relation between z and the proper distance $\delta(t_o, \chi_e)$. As now usual, we can look for expressing both z and $\delta(t_o, \chi_e)$ as functions of the scale factor, so to have a parametric representation of the exact Hubble law and, at the same time, avoiding formal inversion of functions.

The red shift z is already known as a function of the scale factor a; see equation (10.6.25). As for the proper distance we have

$$\delta(t_o, \chi_e) = a(t_o)\int_0^{\chi_e} d\chi = c\int_{t_e}^{t_o} \frac{dt}{a(t)} = c\int_a^{a_o} \frac{da}{a\sqrt{\Phi(a)}} =: \delta(a) \tag{10.6.42}$$

If we expand $\delta(a)$ as a Taylor series around $a = a_o = 1$ we obtain

$$\delta(a) \simeq \delta(a_o) - \frac{1}{a_o\sqrt{\Phi(a_0)}}(a - a_o) = \frac{1}{H_o a_o^2}(a_o - a) = \frac{1}{H_0}(1 - a) \tag{10.6.43}$$

For the redshift we have

$$z(a) \simeq z(a_o) - \frac{1}{a_o}(a - a_o) = \frac{1}{a_o}(a_o - a) = (1 - a) \tag{10.6.44}$$

Thus we have

$$z \simeq 1 - a \simeq H_0\delta(a) \tag{10.6.45}$$

which reproduces the approximated Hubble law.

Thus the exact Hubble law contains the approximated one. Moreover, it contains extra information about the evolution of the scale factor, through the dynamics which is encoded by $\Phi(a)$ which, we remark, *is* the Friedmann equation.

As a matter of fact, if we consider a particular Palatini $f(\mathcal{R})$-theory, the exact Hubble law (as well as the evolution of the Hubble parameter $H(a)$) contains information about the $f(\mathcal{R})$ used to define the dynamics and consequently observations put constraints to the parameters which parameterise the deviation from standard GR.

Cosmographic parameters

As we remarked, the evolution of the scale factor contains all informations about the theory. We already considered the Hubble parameter today H_0 as a partial description of the scale factor, in the sense it is the first coefficient of the Taylor expansion of $a(t)$ around t_0.

Hubble parameter is the first in a sequence of quantities which altogether contain information about the evolution of the scale factor, called the *cosmographic parameters*. After the Hubble parameter $H = \frac{\dot{a}}{a}$ we define the *deceleration parameter*

$$q := -\frac{1}{H^2}\frac{\ddot{a}}{a} \tag{10.6.46}$$

the *jerk* (or *statefinder*) *parameter*

$$j := \frac{1}{H^3}\frac{\dddot{a}}{a} \tag{10.6.47}$$

the *snap parameter*

$$s := \frac{1}{H^4}\frac{\ddddot{a}}{a} \tag{10.6.48}$$

and so on.

Here the pattern is clear: besides some manipulations, the cosmographic parameters are the derivatives of the scale factor, i.e., if we measure them today, they are the Taylor coefficients of its expansion. Each of them gives a better accurate description of the evolution of the scale factor. Measuring the cosmographic parameters is equivalent to measuring the derivatives of the scale factor today, which determines the Taylor expansion, so in principle the evolution $a(t)$.

7. Relativistic positioning system

It is time now to show how a number of clocks (depending on the dimension of spacetime) can define an observer. Actually, to some extent, everything is already contained in EPS, in axiom DTS. Given m clocks, whatever near enough clocks, they define through their time-readings of messages sent to an event the coordinates of such an event in a region of spacetime, i.e. they define an observer. That is more or less the end of the story, in a sense.

On the other hand, they define GPS coordinates (s_i), which have not much to do with the ordinary coordinates one usually uses, for example (t, x^i) in Minkowski or (t, r, θ, ϕ) in Schwarzschild spacetimes. In GPS coordinates, just to mention one odd feature, one determines the position of an event in spacetime by means of 4 times, not 1 time and 3 positions.

One can decide, in fact with some right, that GPS coordinates are more fundamental than the ones we are used to, define the metric directly in that coordinates and keep stuck to them everywhere. Or, one can ask the GPS to determine the transition functions from GPS coordinates to ordinary coordinates.

Currently, there are a number of positioning systems: the two bigger systems are NAVSTAR-GPS and Galileo Global Navigation Satellite System (Galileo-GNSS) which is becoming fully operational in the next years.

> Actually, there also exists a Russian Global Navigation Satellite System (GLONASS), which is not maintained since the year 2000 so that it fails to cover the whole surface of Earth, as well as other local positioning systems covering specific regions on the surface, which to some extent are similar to GPS in design.

All these systems are designed to work near the Earth surface, and they all take advantage of the specific conditions one has in that weak-field region. To some extent, they are all essentially built to be compliant to our Newtonian intuition: they define position in space of events assuming that we can put a synchronised clock anywhere.

> On Earth, we have an international system of atomic clocks which maintains and broadcasts an international atomic standard time, called *Coordinated Universal Time* (UTC), to which, once in a while, we add spare seconds to match astronomical (UT0) and solar motion (UT1) (so that daytime first and then seasons do not shift as time passes by): that is called *Universal Time*. The universal time is then adjusted to *local times* in different time zones, sometimes adding daylight saving time (DST) in certain seasons.
>
> The definition of such UTC is quite ugly, plagued with historical motivations, and approximations.

Besides that, NAVSTAR–GPS is based of a fleet of atomic clocks on satellites which broadcast their time to a client so that it can determine its coordinates, usually latitude, longitude, sometimes altitude, sometimes the velocity, sometimes just an offset position.

Actually, the NAVSTAR–GPS is based on the assumption that the time signals received by the client from different satellites have been emitted at the same time; they are synchronised at emission. The client receives them at different times (because of different distances to travel) and it determines its position by using the time delay of one signal with respect to the other. The system in fact does not deal with times, only with differences of time which are easier to be measured.

> As a rough idea, by measuring the time delay of two signals from two different satellites, the client has information about the difference in the distance of the two satellites; thus it is able to get a hyperboloid which it lies on (which in fact is the locus on which the difference of distances from two *"fixed"* points stays constant) with foci in the broadcasting satellites. By considering three pairs of satellites, it knows it lies in the intersection of three hyperboloids, and such an intersection is generically made of a finite number of points. If we add one more pair, always generically, each client is able to select at which point of the intersection it is.

One could not insist enough on how exquisitely such a derivation relies on Euclidean geometry, which we know is, in principle, violated (slightly violated) in the space around the Earth, actually in spacetime around the Earth. Moreover, such a construction relies on two other assumptions (besides the assumption of the gravitational field being weak so that the space is approximately Euclidean): that the signals are synchronously emitted, and that the clients knows where the satellites are in space.

> We know that both of these assumptions are quite unnatural in a relativistic theory: the satellite clocks are moving around, they are quite fast with respect to the surface, as well as one with respect to the other, and, moreover, in a time dependent gravitational potential (they are not geostationary satellites), usually at different altitudes and along quasi-elliptic orbits. The real orbits have usually a small eccentricity, so they are quasi-circular, though, because of *"relativistic corrections"*, they are not even closed curves; they are not even exactly ellipses.
>
> Then the satellites broadcast a fake signal which is based on their local atomic time, though corrected so that the various satellites in the fleet are artificially kept synchronised by correcting all sort of "relativistic effects" which should force them to loose synchronisation.
>
> As far as the satellites positions are concerned, essentially each satellite maintains its orbital data which are transmitted to the clients. Again, the orbital parameters are maintained in a fairly traditional way as "relativistic corrections" to the Keplerian orbit. Moreover, of course each satellites is not precisely along a Keplerian orbit (due to the Earth not being a point-like source, perfectly spherically symmetric, as well as, e.g. because of gravitational perturbations from the other astronomical bodies, ...), it is not exactly freely falling (due to remnants of the atmosphere, solar wind, manoeuvring, ...).

Because of these and other limitations, the NAVSTAR–GPS clients can determine their position with a precision ranging from $3m$ down to something of the order of $15m$.

The Galileo system, is not qualitatively different in design, though at least it deals a relativistic model of gravitational field. Hence, Galileo system is expected to be about 10 times more precise than NAVSTAR–GPS.

All existing GPS are essentially Newtonian in design. They work quite well near the surface of the Earth and are based on a long series of approximations. Here we want to discuss the issue of *relativistic positioning systems* (rPS) from a more fundamental viewpoint. The more fundamental our attitude, less approximations we are allowed. From a fundamental viewpoint a positioning system precedes any observation other than coincidences (i.e. events happening at the same event in spacetime). When one designs it, in principle, one has no way of obtaining specific knowledge of the physical situation in the desired spacetime region. On the contrary, in many cases a GPS is a prerequisite to make observations, to obtain detailed descriptions of the physics in a region.

As usual, we shall assume we have a theoretical idea of how gravity works in general, and we plan to use the GPS to obtain information about client positions, the gravitational field, the motion of satellites. Our rPS should fully be compliant to relativistic theories, possibly not restricted to standard GR. It should work near a planet surface as well near a black hole, it should be able in principle to cover (patchwise) the whole spacetime. It should learn to live with the facts we learnt about spacetime geometry: it should not assume synchronisations at a distance at all, less than ever it should not assume the synchronisation is artificially maintained among clocks at a distance. It should define the position of events in spacetime, not in space. If we guess, as we shall do, a geometry of spacetime, the system should be able to detect if such an assumption turns out not to be accurate, which is as to say that a rPS can be used to measure the gravitational field.

Since we do not want to model too precisely the motion of the satellites, their orbital data should be part of the unknowns of the problem. A client should be able to determine its positions as well as the orbits of the satellites, detecting for example if they happen not to be freely falling. Also, we should assume a model for gravity (in the form of a specific relativistic theory) and the rPS should be able to detect if the model fails to be a good description of gravity as well as, in principle, to fix parameters in the theory. After all, a rPS is a way of producing observables in a gravitational theory, not absolute observables though observables, anyway. They are the natural framework to discuss equivalence and observability of different theories.

The design we have in mind is not particularly original. It is similar to models which have been proposed, for reasons which have been discussed. Here we just want to repeat the discussion, obtaining maximal simplifications (after all we are interested in discussing feasibility in principle more than in practice). We shall also focus on some aspects of the systems which, to the best of our knowledge, are original.

The basic composition is that we have a fleet of (identical) atomic clocks going around in spacetime, e.g. freely falling. They broadcast their time, e.g. by means of an encoded radio signal.

If we want the client to be able to determine the trajectories of the satellites together with its position in spacetime, we need to allow the satellites to receive messages from other satellites.

If we are in a spacetime of dimension m, we have n satellites, and all that the client receives is the time-reading of each satellite, then the client will have n messages at any time, which may even be dependent.

On the other hand, we have m unknowns for the position of the client, mn unknowns for the positions of the satellites, $(m-1)n$ for the velocity of the satellites (only the direction of the covariant velocity matter for the trajectory), for a total of $m + (2m-1)n$ unknowns. That is essentially always greater than the number of conditions (n) that we have.

Of course, the client could remember a sequence of past readings, so having kn messages. That is fair and should be investigated. Actually, it has been investigated. For the sake of simplicity, we could think that the client has a clock itself and describes the messages it receives as functions $\psi_i(s)$ meaning that it received the reading

s from the clock i at client's local time $\sigma = \phi_i(s)$. Still we will have $m + (2m - 1)n$ functions to determine with n conditions. The conditions are not linear, so some miracle may happen ($x^2 + y^2 = 0$ is one condition with allows to determine two real variables), though in practice it does not.

In view of these reasons, we have to say that there are many designs worth discussing, with different features. We are just choosing one. Again we are interested in feasibility.

Hence we imagine the satellites also receive the broadcasted signals from the other clocks and mirror them to the clients, so that a client receives a signal from each satellite, encoding the time-reading of the clock on the satellite, as well as the time-reading that the satellite received from the other clocks around when the message was sent, for a total of $(n - 1) + 1 = n$ time-readings per message. This procedure can be nested at will: now each satellite receives from the others $n(n-1)$ time-readings, instead of only $(n-1)$. Then the client will receive $n^2(n-1) + n = n^3 - n^2 + n = n(n^2 - n + 1)$ messages.

Going back one more step: each satellites will receive $(n - 1)((n - 1)^2 - (n - 1) + 1) = (n - 1)(n^2 - 3n + 3)$ messages to mirror together with its current time. Then the client will have $n[(n - 1)(n^2 - 3n + 3) + 1] = n[n^3 - 4n^2 + 6n - 2]$ time-readings to play with. It will have the time-readings of the clocks at emission, together with the emission time-readings of the clocks for the messages which were received by the satellites when they emitted the message to the client, together with the emission time-readings of the clocks when they sent the message, received by the other satellites, then mirrored to the other satellites, then mirrored to the clients.

Hence the time-reading received by a client are naturally divided in generations: the first generation of data are time-readings at emission. The second generations are data which were mirrored once. The nth-generation messages are the ones which have been mirrored $(n - 1)$-th times.

Thus, at each step, the client potentially receives an infinite sequence of times reading bringing information about the past of the system. The number of data grows as n^k if we go back $(k - 1)$ generations so that we may have enough conditions to determine all unknowns we may have.

It is not difficult to guess that as the number of clocks and the dimension of spacetime grows we need a handy notation to trace where each data comes from, its generation, among which satellites it has been mirrored to us.

What we need is a systematic notation to trace messages of different generations and their way to the client. Let us suppose we are dealing with $n + 1$ clocks χ_i, with $i = 0..n$, and a client C, which is an event, not a worldline. There are a number of light worldlines bouncing back and forth between them.

We represent the whole exchange history by a graph: each node represents an event p_a on the clock χ_i or the client C, each (oriented) link in the graph represents a light signal. A link from χ_i to χ_j (or C) represents a light signal emitted by the clock χ_i and received by the clock χ_j (or the client C). Of course, the client receives signals though it does not emit anything; clients are passive and the node C is *final* in the graph.

Each link singles out an emission event p_a on the worldline χ_i, as well as an absorption event p_b on the worldline χ_j. The absorption events on the clients are at C. The client has no memory other than the one which is continuously refreshed by the messages mirrored to it. The time-reading of the clock χ_i at the event p_a which is on the worldline χ_i is denoted by s_a.

Thus, on the other hand, we have an (infinite) sequence of events $(p_{a_1}, p_{a_2}, \ldots, p_{a_h}, \ldots)$ on the worldline χ_i, and, consequently, an (infinite) sequence of time-readings $(s_{a_1}, s_{a_2}, \ldots, s_{a_h}, \ldots)$ on the clock χ_i. Since we have one infinite sequence of events for each clock, we need to know which event is in which worldline: let us agree upon the notation that p_a is on χ_i iff $a \bmod n = i$, $n + 1$ being the number of clocks around.

For example, if we have 3 clocks around, then the events $(p_1, p_4, p_7, \ldots)$ are on χ_1, $(p_0, p_3, p_6, \ldots)$ are events on χ_0. Thus the graph accounts for the information we need. Always for 3 clocks we have
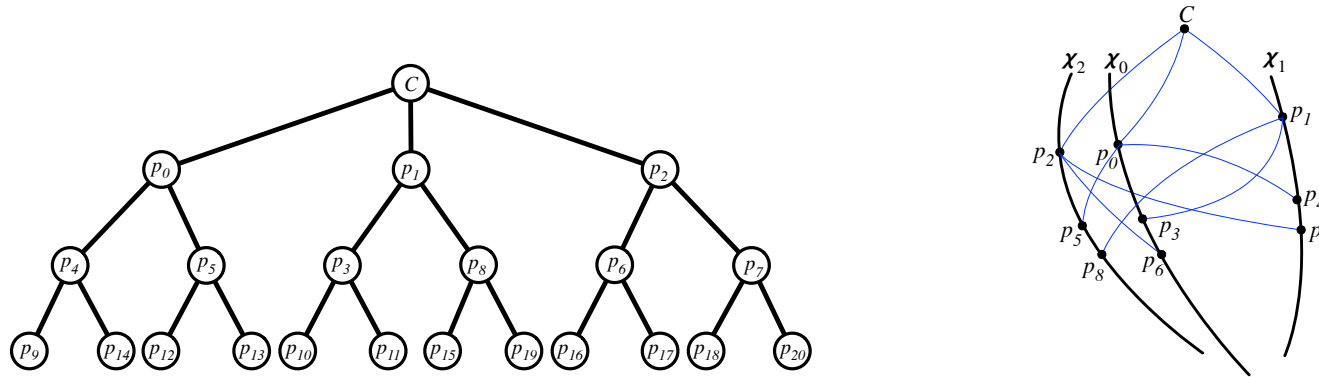
Fig. 9.12: *GPS setting with 3 flying clocks: a) by graph b) spacetime diagram (up to second generation).*

In general, we also need to notice that we can know in advance that the positioning in some cases has no unique solution. If our spacetime allows Killing vectors, it is easy to get that an isometry dragging the clients, the clocks, as well as the light signals, will preserve the whole sequence of time-readings. Accordingly, for example in Schwarzschild, we could never determine the absolute angles of rotation of the system or the time offset, since the solution is static and spatially spherically symmetric.

This can be cured by imagining that some of the clocks are partially known: for example, on Earth, one of the clocks can be set on the surface at known coordinates. It can be regarded at rest and the others are determined with respect to it. In the specific case of Schwarzschild spacetime, also after fixing a clock on the surface, one is left with isometries (rotations) which fix the reference clock, so that, to fix the system, one needs *two* clocks fixed on the surface.

When one introduces fixed clocks, we can imagine that the client still receives messages from them, or that, as it reasonable on the Earth, that the ground clocks cannot in general be seen by the client though they can exchange messages with flying satellites (and the client gets information about the reference clocks only in messages of generation higher that 1). The number of messages is slightly diminished, in the sense that one has less messages of a given generation, though of course the total sequence of messages is still infinite.

The position of the client is determined by $1st$ generation messages, so we need to assume that the client sees at least m flying clocks at any moment.

If χ_0 is a reference clock and it is allowed to exchange messages with χ_1 and χ_2 but not with the client, we have that we have to remove the message from p_0 to C, the graph disconnects in two components. One component has the final event in p_0, and the messages represented in it are irrelevant for the client, they will be broadcasted to the satellites and used for later clients.

We could rename the events on the other clocks to account for the cancelled events, though we actually did not require a specific order for p_a on a clock, hence we can leave them unchanged as well so that comparison between the two situations are easier.

Thus with one clock on the ground the situation is described as
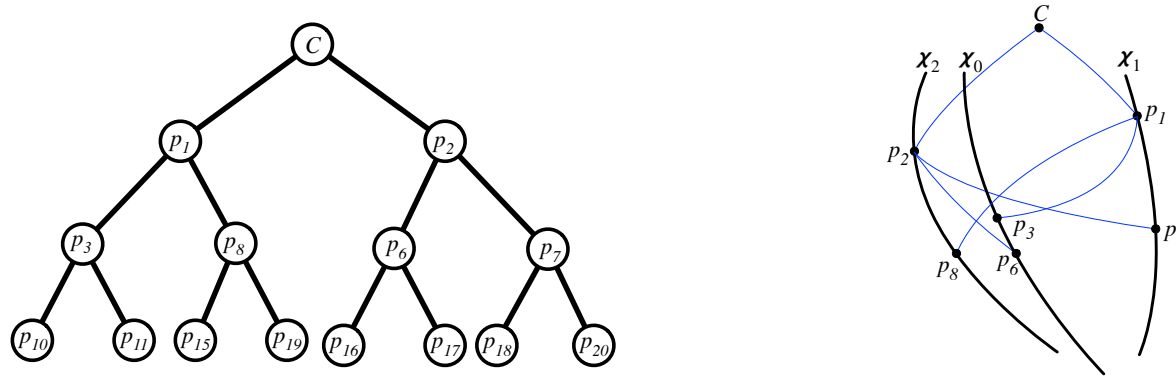
Fig. 9.13: GPS setting with 2 flying clocks and 1 at rest: a) by graph b) spacetime diagram (up to second generation).

 Due to all these remarks, we can guess that a general discussion of these systems becomes difficult, due to varying details. Thus let us present some simple examples. After that we shall discuss the features of this model of positioning and compare with some other proposals.

 As usual, before going into higher dimensions and more general geometries, let us consider a quite trivial example: Minkowski in dimension 2. In this case, we already know that one clock defines an observer, though that is a radar observer, i.e. to find the position of an event we need an echo message to go back and forth from te clock to the event. If we are considering an event on another galaxy, waiting for a message to come back may be a bit boring, so in general, using more clocks, allows the client to determine everything as they receive the messages, without waiting any longer.

Minkowski case in dimension 2

 Let us assume the spacetime M to be in dimension 2 and flat. Although what we shall discuss is intrinsic, let us use a system of Cartesian coordinates (t, x) to sketch objects.

 Since there is no gravitational field, particles move along geodesics which are straight lines

$$x - x_* = c\beta(t - t_*) \qquad -1 < \beta < 1 \tag{10.7.1}$$

while light rays have $|\beta| = 1$, i.e. they move along

$$x - x_* = \pm c(t - t_*) \tag{10.7.2}$$

 In view of covariance, what we are saying is that free fall is expressed by first order polynomials *in the given coordinates* (t, x). If we used polar coordinates (r, θ) free fall would not be given by first order polynomials (such as $r - r_* = \gamma(\theta - \theta_*)$). It would be rather given by the *same* straight lines (e.g. $x - x_* = \beta(t - t_*)$) expressed in the new coordinates, i.e.

$$r\sin(\theta) - r_*\sin(\theta_*) = \beta\left(r\cos(\theta) - r_*\cos(\theta_*)\right) \tag{10.7.3}$$

 which are in fact the *same* curves.

It is precisely because of this that in fact here we are not using coordinates in an essential way (and thus not spoiling covariance). We instead are just selecting a class of intrinsic curves to represent free fall.

A *standard clock* is a clock $\chi : \mathbb{R} \to M : s \mapsto (t(s), x(s))$ for which the covariant acceleration $a^\mu := \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$ is perpendicular to its covariant velocity $\dot{x}^\alpha$; in this case, and in Cartesian coordinates, the acceleration is given by the second derivative (since $\Gamma^\mu_{\alpha\beta} = \{g\}^\mu_{\alpha\beta}$ and Christoffel symbols are vanishing). Since a freely falling clock is moving along a straight line, then it is standard iff the functions $t(s), x(s)$ are linear in s. Hence the most general standard clock is

$$\chi_* : \mathbb{R} \to M : s \mapsto (t = t_* + \alpha s, x = x_* + \alpha\beta s) \tag{10.7.4}$$

For later convenience, let us set $\zeta := \alpha\beta$. Since we are in a Riemannian geometry, a standard clock is also proper and we can set α as

$$\alpha = \frac{1}{\sqrt{1 - \zeta^2}} \tag{10.7.5}$$

so that is a NP-clock.

An NP-proper clock has 3 degrees of freedom, since it is uniquely determined by 4 parameters $(t_*, x_*, \alpha, \zeta)$ with the relation (10.7.5).

Let us consider two proper clocks (χ_0, χ_1) in M corresponding to the parameters $(t_0, x_0, \alpha_0, \zeta_0)$ and $(t_1, x_1, \alpha_1, \zeta_1)$. As we anticipated above the whole system has a Poincaré invariance which can be fixed by setting $\chi_0 : \mathbb{R} \to M : s \mapsto (t = s, x = 0)$ (which still leaves an invariance with respect to spatial reflections, which will be eventually used) and consequently, $\chi_1 : \mathbb{R} \to M : s \mapsto (t = t_1 + \alpha s, x = x_1 + \zeta s)$.

Before proceeding on, let us once again explain which problem we want to consider herefter. Usual PS would assume $(t_1, x_1, \alpha, \zeta)$ to be known parameters fixed during the calibration of the system. A *client* receiving the values of s_i (with $i = 0, 1$) from the clocks at an event $c = (t_c, x_c)$ is able to compute its position (t_c, x_c) as a function of the signals (s_0, s_1).

That is essentially assumed as an axiom in EPS framework. The signals (s_0, s_1) are assumed to be coordinates on the spacetime manifold and one can prove that the transition functions $\varphi : (s_0, s_1) \mapsto (t, x)$ are smooth, so that also (t, x) are good coordinates on spacetime as well .

This case is particularly simple. The situation is described in Fig. 9.14.
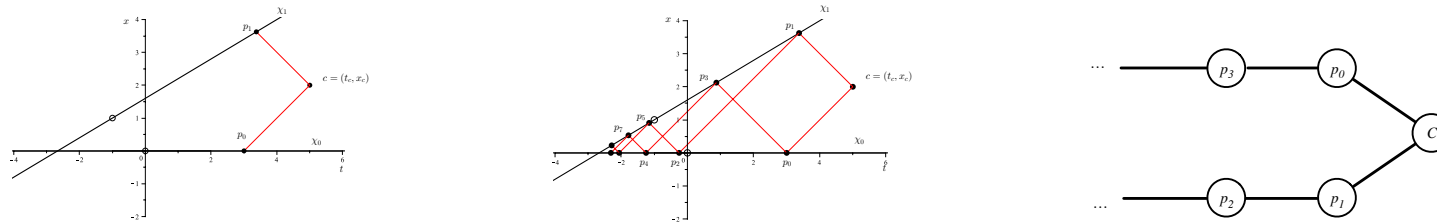
Fig. 9.14: rPS in Minkowski 2d.

a) messages (s_0, s_1) exchanged from the clocks to the client. The empty circles are the events when the clocks have been reset.

b) past messages $(s_0, s_1, s_2, s_3, s_4, \ldots)$ exchanged from the clocks to the client.

c) Graph representing the message exchange history.

One can readily compute that

$$s_0 = t_c - x_c \qquad s_1 = \frac{(x_1 - x_c) + (t_1 - t_c)}{\alpha + \zeta} \tag{10.7.6}$$

which can be readily solved for (t_c, x_c) to get

$$t_c = \tfrac{1}{2}\left((\alpha + \zeta)s_1 + s_0 + t_1 + x_1\right) \qquad x_c = \tfrac{1}{2}\left((\alpha + \zeta)s_1 - s_0 + t_1 + x_1\right) \tag{10.7.7}$$

Accordingly, one can *define* the coordinates $(t, x) := (t_c, x_c)$ as above. The maps (10.7.7) are in fact transition functions between coordinates (s_0, s_1) and (t, c), which are regular being polynomial.

Let us remark that here $(t_1, x_1, \alpha, \zeta)$ are treated as known parameters.

Our problem hereafter, will be to show that, if we promote $(t_1, x_1, \alpha, \zeta)$ to be unknowns of the problem together with (t_c, x_c) and we add a whole past sequence of readings (see Fig. 9.14.b), then we are still able to solve the system and use the infinite redundancy to check the assumptions of the model (e.g. that the gravitational field is vanishing, that the clocks are free falling, that the clocks are identical NP-clocks, . . .).

Before sketching the solution, we are better to introduce some notation which will be useful later in higher dimensions when drawing diagrams as in Fig. 9.14.a and 9.14.b will become difficult. We shall use the affine structure on Minkowski space, so that the difference $P - Q$ of two points $P, Q \in M$ denotes a vector (tangent to M at the point Q) leading from Q to P. On the tangent space the Minkowski metric induces inner products so that we can define a pseudonorm $(P - Q) \cdot (P - Q) = |P - Q|^2$ so that the vector $P - Q$ is *light-like* iff $|P - Q|^2 = 0$.

Then the segments $c - p_0$, $c - p_1$, $p_2 - p_1$, $p_3 - p_0, \ldots$ are all light rays so that one has

$$|c - p_0|^2 = 0 \quad |c - p_1|^2 = 0 \quad |p_2 - p_1|^2 = 0 \quad |p_3 - p_0|^2 = 0 \quad \ldots \tag{10.7.8}$$

as dictated by Fig. 9.14.c.

Now one can check that equations

$$|p_2 - p_1|^2 = 0 \quad |p_3 - p_0|^2 = 0 \quad |p_4 - p_3|^2 = 0 \quad |p_5 - p_2|^2 = 0 \tag{10.7.9}$$

allows to solve for

$$\alpha = \alpha(s_i) \quad \zeta = \zeta(s_i) \quad t_1 = t_1(s_i) \quad x_1 = x_1(s_i) \tag{10.7.10}$$

with $i = 0, 1, 2, 3, 4, 5$ and that one has $\alpha^2 - \zeta^2 = 1$. Then, one can use the first two equations

$$|c - p_0|^2 = 0 \quad |c - p_1|^2 = 0 \tag{10.7.11}$$

to solve for

$$t_c = t_c(s_i) \quad x_c = x_c(s_i) \tag{10.7.12}$$

Actually, by solving the system, one gets 8 solutions. If our assumptions (e.g. about free fall) are accurate, the remaining equations are identically satisfied.

Four of the 8 solutions are spurious solutions since they do not satisfy the equations

$$|p_4 - p_7|^2 = 0 \qquad |p_5 - p_6|^2 = 0 \tag{10.7.13}$$

The remaining 4 solutions then identically satisfy all the following equations.

To be precise one obtains multiple solutions (as we discussed above we are left with 4 solutions) but the correct solution can be selected by checking that all the vectors

$$c - p_0, \quad c - p_1, \quad p_1 - p_2, \quad p_0 - p_3, \quad p_3 - p_4, \quad p_2 - p_5, \quad \ldots \tag{10.7.14}$$

are future directed. This reduces the solutions to 2.

To find the unique solution then, we can resort back to Poincarè invariance. We already used the boost invariance to adjust the clock χ_0. However, we have a residual invariance with respect to spatial reflections. We could have originally used that to keep $x_1 \geq 0$. This condition can be effectively used to select a unique solution between the 2 residual solutions of the system.

In other words, by using the signals $(s_0, \ldots, s_5)$, one is able to uniquely determine both the clock parameters $(\alpha, \zeta, t_1, x_1)$ and the client position in spacetime (t_c, x_c). No need of clocks calibration or synchronisation. Of course, this means that the infinite sequence $(s_0, s_1, s_2, \ldots)$ is not independent. Actually, we can compute allowed sequences in a *simulation phase* in which we assume $(\alpha, \zeta, t_1, x_1, t_c, x_c)$ as parameters, and we compute the signals $(s_0, s_1, s_2, \ldots)$ by using equations (10.7.9), (10.7.11), (10.7.13).

Then we start the *positioning phase*, in which we consider those time-readings as parameters, and we determine the unknowns $(\alpha, \zeta, t_1, x_1, t_c, x_c)$, so that the positioning phase essentially deals with the inversion of what is done in simulation mode.

Let us finally remark that when we prove that the particular system of inertial coordinates (t, x) is allowed, then consequently, any other inertial coordinate system is allowed as well.

> We can also add an unexpected acceleration a to the clock $\chi_1 = (t_1 + \alpha s, x_1 + \zeta s + \frac{1}{2} a s^2)$ while computing the signals to be transmitted to the client. If the client does not know about the acceleration and it keeps assuming (wrongly this time) that the clocks are freely falling, then one can show that the client can still determine the parameters of the clock, but this time the constraint $\alpha^2 - \zeta^2 = 1$ and the redundant equations cannot be identically satisfied. This shows that the client is potentially able to test the assumptions we made and to self-diagnose their break down.
>
> If the acceleration dies out, as soon as the transmitters exchange few signals the system manage to satisfy the constraints and it becomes operational again. That shows the rPS to be *self-calibrating*.

Minkowski case in dimension 3

As far as we go to Minkowski spacetime in dimension 3, the situation becomes more complicated and one needs to think about what is going on in order to apply the simple program we presented in dimension 2.

In dimension 3, we consider 3 proper clocks χ_i, with $i = 0, 1, 2$. Each has 5 degrees of freedom, an initial position (t_i, x_i, y_i) and an initial direction given by $(\alpha_i, \zeta_i, \xi_i)$ obeying the constraint $\alpha_i^2 - \left(\zeta_i^2 + \xi_i^2\right) = 1$.

One can still use the Poincaré invariance to set the first clock to be $\chi_0 : s \mapsto (s, 0, 0)$, though one still has two clocks χ_1, χ_2 and will have to deal with signals back and forth between them (which will turn out to be coupled quadratic equations, while in dimension 2 each equation contained only the parameters of one clock at a time). Moreover, while in dimension 2 one has only two light rays through any event and each of them goes to a clock, in dimension 3, one has infinitely many light rays through an event and one has to select the ones hitting a clock.

Finally, in dimension 2, when Poincaré invariance is used to fix the 0th clock, we are left with a discrete residual invariance with respect to spatial reflections. On the contrary, in dimension 3, one is left with a 1-parameter rotation group (as well as spatial reflections), i.e. with $O(2)$, which can be used to set $y_1 = 0$ and $x_1 \geq 0$.

However, by applying extra care to these facts, one can still show that the unknowns which in our case are now $(t_1, x_1, y_1, \alpha_1, \zeta_1, \xi_1, t_2, x_2, y_2, \alpha_2, \zeta_2, \xi_2)$ can be computed out of signals and the constraints $\alpha_i^2 - \left(\zeta_i^2 + \xi_i^2\right) = 1$. The redundancy and the Poincaré fixing described above can be used to select a unique solution and further redundancy to check assumptions.

To do that, let us use the scheme of signals shown in Fig. 9.12.a. We first use the equations

$$
\begin{aligned}
&|p_0 - p_4|^2 = 0 \quad |p_1 - p_3|^2 = 0 \quad |p_4 - p_9|^2 = 0 \quad |p_3 - p_{10}|^2 = 0 \\
&|p_6 - p_{16}|^2 = 0 \quad |p_7 - p_{18}|^2 = 0 \qquad \alpha_1^2 - \left(\zeta_1^2 + \xi_1^2\right) = 1
\end{aligned}
\tag{10.7.15}
$$

which just depend on $(t_1, x_1, y_1, \alpha_1, \zeta_1, \xi_1)$, to determine the parameters of the first clock. Since three of them are not independent, one has two extra parameters $\epsilon_1 = \pm 1$ and an angle ω_1 which are left undetermined and will be fixed later on. Similarly, we used equations

$$
\begin{aligned}
&|p_0 - p_5|^2 = 0 \quad |p_2 - p_6|^2 = 0 \quad |p_5 - p_{12}|^2 = 0 \quad |p_3 - p_{11}|^2 = 0 \\
&|p_8 - p_{15}|^2 = 0 \quad |p_6 - p_{17}|^2 = 0 \qquad \alpha_2^2 - \left(\zeta_2^2 + \xi_2^2\right) = 1
\end{aligned}
\tag{10.7.16}
$$

which just depend on $(t_2, x_2, y_2, \alpha_2, \zeta_2, \xi_2)$, to determine the parameters of the second clock, leaving two parameters $\epsilon_2 = \pm 1$ and an angle ω_2 undetermined due to relations of the equations.

Then one has the extra equations representing signals between the clocks χ_1 and χ_2

$$
\begin{aligned}
&|p_1 - p_8|^2 = 0 \quad |p_2 - p_7|^2 = 0 \quad |p_4 - p_{14}|^2 = 0 \\
&|p_5 - p_{13}|^2 = 0 \quad |p_8 - p_{19}|^2 = 0 \quad |p_7 - p_{20}|^2 = 0
\end{aligned}
\tag{10.7.17}
$$

Using these equations (and depending on the four possibilities for (ϵ_1, ϵ_2)), we can determine $\omega_1 - \omega_2$, thus leaving only ω_2 undetermined, which in fact accounts for the residual Poincaré invariance and can be fixed to have $y_1 = 0$ and $x_1 \geq 0$. In the meanwhile, two of the four possibilities for (ϵ_1, ϵ_2) needs to be abandoned because they are incompatible with these equations.

Eventually, two solutions are found, only one of which agrees with the gauge fixing $x_1 \geq 0$ we did for fixing spatial reflection invariance. Accordingly, also in this case the client is able to determine the parameters of the clocks uniquely.

Then, once clocks are known, one can use the equations

$$
|C - p_0|^2 = 0 \qquad |C - p_1|^2 = 0 \qquad |C - p_2|^2 = 0
\tag{10.7.18}
$$

to determine the client position (t_c, x_c, y_c). Using Maple, one needs to solve equations on a one by one basis to control the process in details.

Minkowski case in general dimension

:Notation: :Symbols: :AIndex: :Index:

In a Minkowski spacetime of dimension m we consider m proper clocks $(\chi_0, \ldots, \chi_{m-1})$ and we fix Poincaré invariance setting the first clock to be $\chi_0 = (s, 0, \ldots, 0)$. We are left with a spatial residual invariance parameterised by $O(m-1)$ which we still have to gauge fix somehow.

Each clock exchanges $(m-1)$ signals with the other clocks and the graph analogous to that in LinkAnchorGPS1Fig. 9.12.a becomes of order m (i.e. each node receives $m-1$ edges, plus one edge out, except the client node which receives m signals from the clocks).

Thus the graph accounts for

$$\sigma := m + m(m-1) + m(m-1)^2 = m(m^2 - m + 1) \tag{10.7.19}$$

signals. To each of these signals there is associated an equation.

The signals exchanged between χ_0 and χ_i are in fact $2m$ (for any $i = 1..m$) which, together with the constraint $\alpha_i^2 - \left(\zeta_{i1}^2 + \zeta_{i2}^2 + \ldots + \zeta_{i(m-1)}^2 \right) = 1$, partially fixes the parameters of the clock χ_i.

Then one has $2m$ equations for any pair (χ_i, χ_j) to freeze the relative parameters.

Thus one has

$$m + 2m \binom{m}{2} = m + 2m \frac{m(m-1)}{2} = m(m^2 - m + 1) \equiv \sigma \tag{10.7.20}$$

signals as discussed above.

The client position will be computed once the parameters of the clocks have been determined by the equations corresponding to the m signals emitted from χ_i to the client C.

Measuring the conformal factor

Let us now consider a Weyl frame $(M, g, \tilde{\Gamma} = \{\eta\})$ where (M, η) is Minkowski spacetime in dimension 2, (t, x) are Cartesian coordinates for it, and $g = \varphi^{-1}(t)\eta$ is a conformal metric for the conformal factor $\varphi(t) = \exp\left(\frac{2ct}{x_0}\right)$. We already did some of the computation in Subsection 10.2.2 to which we refer.

If one restricts to now $(t = 0)$ the conformal factor is 1. If we consider big values for x_0 the interval in which the conformal factor is almost 1 is expanded.

Let us repeat the setting in Subsection 10.7.1, with two freely falling clocks χ_0 and χ_1, which moves along the $\tilde{\Gamma}$-geodesics $x = 0$ and $x = x_* + c\beta(t - t_*)$, respectively, this time though parameterised to be proper clocks for g, i.e.

$$\chi_0 : \mathbb{R} \to M : \sigma \mapsto (s(\sigma), 0) \qquad \chi_1 : \mathbb{R} \to M : \sigma \mapsto (\alpha s(\sigma) + t_*, \zeta s(\sigma) + x_*) \tag{10.7.21}$$

where we set $s(\sigma) := x_0 \ln\left(\frac{\sigma + x_0}{x_0}\right)$, $\zeta := \alpha\beta$, and $\alpha := (1 - \beta^2)^{-\frac{1}{2}}$ in order of having two identical NP-clocks.

By comparing with the Riemannian case, one sees that even though the parameterisation here is different, due to the conformal factor, the parameterisation in s used in the Riemannian case arises spontaneously. One can compute events along the clocks by using s and then associate to them their time-reading as $\sigma = x_0 \left(\exp\left(\frac{s}{x_0}\right) - 1 \right)$. That gives a simple parametric expression for the clocks $\chi_i : \sigma \mapsto (t(\sigma), x(\sigma))$.

Simulation phase is identical to the Riemannian case, except that in the end one computes the signal as $\sigma_n = x_0 \left(\exp\left(\frac{s_n}{x_0} \right) - 1 \right)$ and forgets about s_n, which would be the signal in the Riemannian case.

In the positioning phase, we have unknowns $(t_c, x_c, \alpha, \zeta, t_*, x_*, x_0)$, i.e. one more than in the Riemannian case. As in that case, we used some of the equations to determine the parameters

$$|p_0 - p_3|^2 = 0 \qquad |p_2 - p_5|^2 = 0 \qquad |p_1 - p_2|^2 = 0 \qquad |p_3 - p_4|^2 = 0 \tag{10.7.22}$$

which in fact determine $(t_*, x_*, \alpha, \zeta)$ as (quite complicated) functions of x_0. Actually, one gets 4 solutions.

For one of these, we can use the equation $|p_4 - p_7|^2 = 0$ to determine the value of x_0 and check that the constraint as well as the other equations

$$\alpha^2 - \zeta^2 = 1 \qquad |p_5 - p_6|^2 = 0 \qquad |p_6 - p_9|^2 = 0 \qquad |p_7 - p_8|^2 = 0 \tag{10.7.23}$$

are satisfied. Once we know the value of the orbital parameters, we can use the equations

$$|C - p_0|^2 = 0 \qquad |C - p_1|^2 = 0 \tag{10.7.24}$$

to determine the client position (t_c, x_c).

> Actually, we get 2 solutions which are compatible with the equations, though only one of them has the vectors $C - p_0$ and $C - p_1$ which are future directed.
> Of course, the survivable solution is the one assumed in simulation mode.

Hence we see that in a Weyl frame, the rPS is also able to detect (and check) the conformal factor. In other words, that shows that the conformal factor (or, better, its variation) is, in fact, in principle, observable.

Let us remark that the parameter x_0 is, in fact essentially, the value of $\partial_0 \ln(\varphi)$ and it completely determines the variation of the conformal factor which, as we discussed above, then is responsible for the effective anomalous accelerations.

Once again, the rPS is self-calibrating; if, for some reasons, the approximations which fixes the conformal factor to be in the form $\varphi = \exp\left(\frac{2ct}{x_0} \right)$, or the value of x_0 is changed, the rPS actually can detect the breaking down of the assumptions by detecting violations in the redundant equations.

Let us also remark that, in a Palatini $f(\mathcal{R})$-theory, the conformal factor is determined by the choice of the function $f(\mathcal{R})$. Being able to check the value of the conformal factor means that the function $f(\mathcal{R})$ itself can be, in principle, observed. If there are parameters in it, they can be fixed by observations.

Schwarzschild case in dimension 2

As we discussed, the dimension should not introduce difficulties other than making computations more involved. However, we discussed Minkowski case only. Let us here try to introduce a rPS on a curved spacetime. In order to keep the discussion simple, we shall consider Schwarzschild spacetime in dimension 2. We are not endowing the model with a physical meaning since gravity in dimension 2 is trivial (Einstein equations are identically satisfied) even though probably one could argue for a meaning as radial solutions in Schwarzschild 4d.

In fact, the Minkowski cases we studied above are exposed to two different concerns:

    1) we extensively used the affine structure of $\mathbb{R}^n$ to write the equations to identify light rays;

    2) the metric is flat, thus, e.g. the Lagrangian for geodesics has an extra first integral (the conjugate momentum to x which is cyclic).

In both cases, we should check that we are able to perform the computation on a more general curved spacetime, otherwise what we have done above would be restricted to SR.

Let us try the metric

$$g = -A(r)dt^2 + \frac{dr^2}{A(r)} \qquad\qquad A(r) := 1 - \frac{2m}{r} \qquad\qquad (10.7.25)$$

We already covered a lot in this case: we already have a good description of time-like worldlines (see Subsection 9.5.1) to describe satellites, a good description of light rays (see equation (9.5.17))) to trace messages. The symplectic method we introduced in Subsection 9.5.2 suggests to use r as a parameter along worldlines and in fact it allows us to keep the result analytic and exact.

Thus we can fix two transmitters (let us consider, e.g., a bound clock χ_0 and an unbound outgoing clock χ_1) and a client in between them; e.g. as shown in Fig. 8.7. We can trace back light rays exchanged by clocks and in this way we are able to find *exactly* the points $(p_0, p_1, p_2, p_3, \ldots)$ at which the signals are fired by the clocks and the corresponding clock readings $(s_0, s_1, s_2, s_3, \ldots)$. In other words, we can, also in this case, exactly model the simulation phase in the 2d Schwarzschild case.

In the positioning phase, we can instruct the client that it is in a Schwarzschild spacetime (and to be nasty not to inform it of the actual value of m), that the first clock is bounded and the second is outgoing and unbounded, that the clock χ_0 was outgoing when it was reset the last time (while of course χ_1 is always outgoing). We need it to know for each signal generated by χ_0 whether it is generated on the outgoing or ingoing branch, which we can imagine to be an information broadcast by the transmitters together with the sequence of readings. As we discussed, we do not provide the client with more detailed information about the transmitters, in particular the position and energy of the transmitters worldlines are part of the unknowns.

> That is a bit too much for Maple to solve the system by brute force. We can still check that what described in simulation mode is a solution of the equations of positioning mode. Just the generic numeric method seems to be be unable to find it.
>
> Of course there are better numerical methods, more dedicated and adapted to the current situation, to try. Here, we are more or less satisfied with checking that one can write down the equations for positioning and that they have the solution found in simulation phase. Actually instructing the clients to find the solution is a numerical issue, which is interesting for application, less for a fundamental perspective.

The client is, in principle, able to find the parameters for the clocks, namely (t_0, r_0, E_0) and (t_1, r_1, E_1), its position in spacetime (t_c, r_c) and the value of m. It is also able to validate the result by verifying that the constraints are satisfied this time.

Features of relativistic positioning systems

Coll *et al.* analysed rPS and they proposed a classification for them depending on their characteristics. In their classification, a rPS is *generic* if can be built in any spacetime, it is *gravity free* if one does not need to know the metric field to built it, it is *immediate* if any event can compute its position in spacetime as soon as it receives the data from transmitters.

Another important characteristic of rPS is being *auto-locating*, meaning that the user is able not only to determine its position in spacetime, but also the position of the transmitters.

The basic design they investigated is a cluster of atomic clocks based on satellites which *continuously* broadcast the time-reading of their clock together with the readings they are receiving from the other clocks. Sometimes they argue the transmitters may be also equipped with accelerometers or the users can be equipped with a clock themselves. In these rPS the user receives the readings of the transmitters' clock, together with the readings which each transmitter received from the others, for a total of m^2 readings for m transmitters. In these systems, the user may have no *a priori* knowledge of transmitter trajectories which are determined by received data (sometimes assuming a qualitative knowledge of the kind of gravitational field in which they move or whether they are free falling or subject to other forces). As a matter of fact, one can define many different settings and investigate what can be computed by the user depending on its *a priori* knowledge and assumptions. More generally, it has been shown that these rPS can be used to measure the gravitational field.

These rPS define a family of basic null coordinate systems (in dimension m an event receiving the readings of m clocks can directly use, in some regions of spacetimes, these readings as local coordinates). Coll et collaborators showed that one can consider settings so that the rPS is at the same time generic, gravitational free and immediate. The user in these rPS is potentially able to define familiar (i.e. more or less related to the Earth) coordinates, as well.

In view of what we did, we can propose to extend the classification of rPS. In particular we say that a rPS is *chronometric* if it only uses clocks, *simple* if it is chronometric and users have no clock but they uniquely rely on transmitters' clocks.

> We can here argue that being *simple* is important from a foundational viewpoint. Atomic clocks are already complicated objects from a theoretical perspective. They can be accepted as an extra structure but that does not mean that one should accept other apparata (e.g. accelerometers or rulers) as well. They sometimes can be defined in terms of clocks, sometimes are even more difficult to be described theoretically, sometimes, finally, they are simply ill-defined in a relativistic setting (as rulers are). Moreover, atomic clocks are complex (as well as expensive) technological systems; while it is reasonable to disseminate a small number of them keeping their quality high, it is not reasonable to impose each client to maintain one of them without increasing costs and worsening quality.

We also say that a rPS is *instantaneous* if a user is regarded as an event, not as a worldline, and it is still able to determine immediately its position in spacetime. In an instantaneous rPS the clients have no memory, they need to determine their position and other parameters by relying on the data that they receive in that precise moment.

> There is another reason why one should like instantaneous rPS. They do not depend on what the gravitational field was in the past. Earth gravitational field is not Schwarzschild, it is a quite complicated perturbation of it, instead, which in fact depends on the exact shape of the Earth, as well as its inhomogeneities. There are quite precise numerical models fo the exact Earth gravitational field, which in fact are a small though accurate perturbations of the spherically symmetric solution.
> If we rely on a long series of data in positioning phase, the gravitational model should be accurate, since small perturbations produce small deviations which though accumulate in time. If they are neglected, soon the rPS will produce inaccurate results. On the contrary, if we know that positioning uses data more recent that some time, than we know that small effects have not much time to accumulate. In the design we described above, the clients uses up to 3rd generation messages; on Earth one could imagine that it takes less than a second, in the very worse scenario.
> If we decide to neglect non-sphericity of Earth's gravitational field, we need to discuss if we are even able to see the error due to the approximation. If we are, it means that we can measure the perturbations of Earth gravitational field with less than a second of observation.
> In any event, as we said, here are have a more fundamental attitude, so we are more interested in computability in principle to be used to compare different gravitational theories.

Moreover, we say that a rPS is *discrete* if the signals used by the user are a discrete set of clock readings (as opposite to a *continuous* stream of them). Of course, not being discrete implies not be instantaneous.

> Studying *instantaneous* and *discrete* rPS is interesting because it keeps the information used to define the system finite. In other settings, the clock readings are described by functions. This does not really affect the analysis as long as the positioning is done in null coordinates, but it essentially enters in the game when one wants to transform null coordinates in more familiar ones (e.g. inertial coordinates in Minkowski spacetime). We have to remark that clocks are essentially and intrinsically discrete objects. Regarding them as continuous objects can be done by interpolation, which partially spoils their physical direct meaning, as well as it introduces approximation biases. As long as it is possible, also in this case as for simple rPS, we prefer to keep stuck to simplicity.

Finally, we say that a rPS is *self-calibrating* if it can function without a calibrating phase, just as soon as it has broadcasted some few generations of clock readings, in particular without synchronising clocks sometime in the past, without tuning initial conditions, without tracing transmitter geodesics by a mission control.

> Finally, *self-calibrating* rPS are a natural extension of self-positioning systems. If one has a self-positioning systems there should be no need to trace back the trajectories of transmitters back in time. We believe it is interesting to explicitly keep track of how long back the user needs to know the transmitters, both for a fundamental viewpoint and for later error estimate, e.g. in case one wanted or needed to keep into account anisotropies of Earth gravitational field. Of course, a self-calibrating rPS is also auto-locating. We can assume though that a self-calibrating system, if temporarily disturbed by a perturbation (e.g. a transient force acting for a while), will detect the perturbation and go back operational automatically and with no external action as soon as the perturbation has gone.

The settings we discussed are simple, instantaneous, discrete, and self-calibrating (as well as general, immediate, and self-locating) rPS. We also discussed how the users can explicitly find the coordinate transformation to familiar systems (e.g. inertial coordinates in Minkowski, or (t, r) in Schwarzschild) since, even though null coordinates are more fundamental one cannot pretend users to adapt to them (as well as, doing that, one also proves that those classes of coordinates are also admitted by spacetime differential structure axiomatised in EPS).

References

add

Chapter 11. Loop Quantum Gravity Approach

1. Holst gravity

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Holst Dynamics

Dynamical equivalence with standard GR

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2. Constraint equations

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3. Spin networks

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Operators on spin networks

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Operators on spin knots

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5. Spin foams

Part III debrief

Part IV

Mathematical background

Kale pe a

Go slowly, if you want to be back

(*Tibetan farewell to the expeditions to the highest picks*)

IV. Introduction to Part IV

We shall here briefly review the mathematical background and fix notation. As usual, for applications in physics, mathematics is a language and one has to decide how to be strict on it.

The relation between differential topology, differential geometry, and GR are however stricter than usual. Most of the topics treated in the last century are really entangled and often it is difficult to say what came (or should have come) first. There is also a good reason for this to be: as a matter of fact, differential topology and relativistic theories did (and do) study the same problem.

In relativistic theories one is studying the properties of spacetime which are independent of the observer.

> For us an observer is a material point with a clock, i.e. a parameterized curve in spacetime. When we shall have a gravitational field on spacetime the curve will turn out to be timelike. As a matter of fact, it goes the other way around, as we discussed in Chapter 6, in particular Section 6.6. Given an observer, gravitational fields are chosen among the metrics for which the observer worldline is timelike.
>
> As we discussed in Chapter 10, each observer sets up conventionally a local laboratory which is a coordinate chart on spacetime. More generally, a lab is a set of conventions needed to represent local physics in terms of numbers, however, in the case of natural relativistic theories a coordinate charts is enough, since the configuration bundle is a natural bundle, hence a coordinate chart induces also a local trivialization. Accordingly, also fields can be expressed in terms of numbers by using the induced fibered coordinates on configuration bundle.

Unfortunately, one cannot describe locally a spacetime without fixing an observer and its lab. Thus physical quantities must be covariant with respect to changes of observers. As a matter of fact, one can *define* physical objects as the equivalence class of observer-dependent descriptions. One has a quotient space with respect to such an equivalence relation which provides an absolute description of physics. In fact, physical quantities live in (or on) this quotient space (covariance is in fact the compatibility with the projection to the quotient) and observers only provide convenient and conventional local descriptions of the absolute physical world.

In differential (topology or) geometry, one is studying properties of surfaces (or higher dimensional manifolds). Unfortunately, to describe a surface one usually needs a parameterisation (which is just the inverse map with respect to a chart). For example, let $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ be a paraboloid surface. A parameterisation is a map

$$\hat{\varphi} : \mathbb{R}^2 \to S \subset \mathbb{R}^3 : (u, v) \mapsto (x = u, y = v, z = u^2 + v^2) \tag{11.1}$$

The parameters (u, v) can take any value in $V_1 := \mathbb{R}^2$. Notice how the parameterisation $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ factorises through the canonical embedding $i : S \to \mathbb{R}^3$ to define a one-to-one map $\hat{\varphi} : \mathbb{R}^2 \to S$ such that

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\ \varphi\ } & \mathbb{R}^3 \\
{\scriptstyle \hat{\varphi}}\downarrow & \nearrow{\scriptstyle i} & \\
S & &
\end{array}
\tag{11.2}
$$

The inverse map $\hat{\varphi}^{-1} : S \to \mathbb{R}^2$ is a chart on the surface.

> Here we cheated and we considered an example in which the parameterization is global on S.
> In general, a parameterization is local, namely something like $\hat{\varphi} : V \subset \mathbb{R}^2 \to U \subset S$, where V is an open set in $\mathbb{R}^2$ and U is an open set in S. Still, the map $\hat{\varphi} : V \to U$ is invertible, and its inverse $\hat{\varphi}^{-1} : U \to V$ is a chart on S.

A different parameterisation of the same surface is $\psi : \mathbb{R}^2 \to \mathbb{R}^3 : (r, \theta) \mapsto (x = r\cos\theta, y = r\sin\theta, z = r^2)$. To be precise, the map ψ is defined for $r > 0$ and $\theta \in (0, 2\pi)$, i.e. on $V_2 \subset \mathbb{R}^2$ out of a ray in $\mathbb{R}^2$. One has transition functions

$$\gamma : V_2 \to V_1 : (r, \theta) \mapsto (r\cos\theta, r\sin\theta) \tag{11.3}$$

and

$$\psi = \varphi \circ \gamma$$

$$\tag{11.4}$$

The same point $p = (x, y, z) \in S$ can be represented by coordinates $(u, v) \in V_1$ using the parameterisation φ or by $(r, \theta) \in V_2$ using the parameterisation ψ. By using the transition function, one can define an equivalence relation on $X = (\{1\} \times V_1) \cup (\{2\} \times V_2)$

$$(2, r, \theta) \sim (1, u, v) \iff (u, v) = \gamma(r, \theta) \tag{11.5}$$

One can consider the quotient $M = X/\sim$ and a point $[1, u, v] \in M$ has different representatives, namely $(1, u, v)$ and $(2, r, \theta)$ which both represent the same point on S in the two different representations. Actually, one can show that the quotient space M is diffeomorphic to S. Accordingly, one can use local representations of S to define an absolute representation of the surface, i.e. the class of S modulo diffeomorphisms.

This construction is general and provides an (almost) equivalent definition of manifolds modulo diffeomorphisms which is independent of the choice of a local parameterisation. One has then local parameterisations as representatives of absolute manifolds. Quantities defined using a parameterisation are, in fact, defined on the abstract manifold iff they are compatible with the quotient and they hence define quantities on the quotient.

This is quite the same as one defines physical quantities as the quantities measured by observers which do not depend on the observer, i.e. covariance principle. This analogy between relativistic theories and differential geometry is quite deep and provides a reason why methods developed in one context apply so well to the other.

In order to make this book self contained, we shall hereafter review basic notions in algebra, topology, differential geometry, bundle theory, as well as we shall fix notation. Most of what follows should be known to readers with some background. They can just go through what follows to check that notation agrees.

Physics can in principle avoid (most of) mathematics or defines mathematics along the way when it becomes necessary. This approach has some advantage, one for all mathematics is introduced once physical motivations for it are clear. However, this approach forces the readers to continuous detours. Moreover, it makes the language imprecise since often one discusses a topics without having first created the language needed to discuss about that properly.

We here prefer to introduce some basic mathematics and language on their own and took them for granted when discussing physics.

Chapter 12. Algebraic constructions

I'll always take the roundabout way
(Marillion, *Bitter Suite*)

1. Sets, maps and functions

'Obvious' is the most dangerous word in mathematics.
(E.T. Bell)

We shall take sets for granted as primitive objects. The essential thing for a set is that one can always decide whether something belongs or not to the set.

It is too bad that even this apparently innocent claim does not make sense.
In a town there is only one chef who cooks for all those who do not cook for themself. Does the *"set"* C of citizens who the chef cook for include the chef himself?

A *relation* between two sets A and B is any subset R in the Cartesian product $A \times B$ (i.e. the set of pairs (a, b) with $a \in A$ and $b \in B$). Given a relation $R \subset A \times B$ and a point $(a, b) \in R$ we also write aRb and we say that the element a is in relation R with the element b.

A relation f between A and B is called a *map* iff $\forall a \in A, \exists! b \in B$ such that $(a, b) \in f$, and in this case we write $f(a) = b$. The element $b = f(a) \in B$ is unique and it is called the image of $a \in A$. If f is a map between A and B, we write $f : A \to B : a \mapsto f(a)$.

The set A is called the *domain* of f, B the *codomain*. The *image* of f is the set $\mathrm{Im}(f) = \{b \in B : \exists a \in A : f(a) = b\} \subset B$.

A map is *surjective* iff $\mathrm{Im}(f) = B$, i.e. $\forall b \in B : \exists a \in A : b = f(a)$. A map is *injective* iff $f(a) = f(a') \Rightarrow a = a'$. A map f is *one-to-one* (or *bijective*) iff it is both injective and surjective, i.e. $\forall b \in B, \exists! a \in A$ such that $f(a) = b$.

One can always define the *identity map* $\mathrm{id} : A \to A : a \mapsto a$, which is one-to-one.

Given a map $f : A \to B$ and a map $g : B \to C$ one can define a map $g \circ f : A \to C : a \mapsto g(f(a))$ which is called the *composition map*. A map $f : A \to B$ is *invertible* iff there exists a map $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id} : B \to B$ and $f^{-1} \circ f = \mathrm{id} : A \to A$. In this case, the map f^{-1} is called the *inverse* of the map f. One can easily show that if the inverse of f exists then it is unique and that a map $f : A \to B$ is invertible iff it is one-to-one.

We shall reserve the term *function* for the maps between a set A and the special set $\mathbb{R}$ (or $\mathbb{C}$, as well as any other field). One can associate to any element $\alpha \in \mathbb{R}$ a *constant function* $\alpha : A \to \mathbb{R} : a \mapsto \alpha$. Given two real functions $f, g : A \to \mathbb{R}$ one can define the *sum function*

$$f + g : A \to \mathbb{R} : a \mapsto f(a) + g(a) \tag{12.1.1}$$

the *product function*

$$f \cdot g : A \to \mathbb{R} : a \mapsto f(a) \cdot g(a) \tag{12.1.2}$$

and, for any $\alpha \in \mathbb{R}$, the function

$$\alpha \cdot f : A \to \mathbb{R} : a \mapsto \alpha \cdot f(a) \tag{12.1.3}$$

The constant function 0 is the neutral element for the sum, the constant function 1 is the neutral element for the product. Being $\mathbb{R}$ a field, the set $\mathcal{F}(A)$ of all functions over A inherits from $\mathbb{R}$ the structure of a ring and a $\mathbb{R}$-module. Unfortunately, $\mathcal{F}(A)$ is not a field itself, since functions with zeroes do not allow inverse with respect to the product, even if they are not the constant 0 function.

If a map $f : A \to B$ is not surjective and a set C is such that $\mathrm{Im}(f) \subset C \subsetneq B$, we can define a map $f|^C : A \to C : a \mapsto f(a)$. By an abuse of notation the map $f|^C$, which is not even a restriction of f, is denoted by $f = f|^C$, anyway.

2. Disjoint union of sets

Consider a circle S^1 embedded in a plane $\mathbb{R}^2$ in the canonical way $i : S^1 \to \mathbb{R}^2$. Let $x \in S^1$ be a point and $T_x S^1$ the tangent line at x. The tangent
line $T_x S^1$ is canonically realized as a line in $\mathbb{R}^2$. Unfortunately, the tangent lines at two different points, namely $T_x S^1$ and $T_y S^1$, generically intersect. Let z be the intersection point $\{z\} = T_x S^1 \cap T_y S^1$. Even if z is one single point in $\mathbb{R}^2$, it also has two quite different meanings when considered as a tangent vector to S^1; it is tangent vector at x when considered as an element of $T_x S^1$ and a tangent vector at y when considered as an element of $T_y S^1$. Accordingly, if one considers the union of the two tangent lines, one has only one element z in the union to denote two different tangent vectors.

For this reason, we define the *disjoint union* of two sets A and B to be the union of the sets $\{1\} \times A$ and $\{2\} \times B$. The disjoint union is denoted by

$$A \coprod B := (\{1\} \times A) \cup (\{2\} \times B) \tag{12.2.1}$$

A point in the disjoint union is a pair (n, x). If $n = 1$ then $x \in A$, if $n = 2$ then $x \in B$.

> If the sets A and B are not disjoint sets and $x \in A \cap B$, then in the disjoint union $A \coprod B$ has two different elements $(1, x)$ and $(2, x)$. We say that $(1, x)$ represents the element x *as an element of A*, while $(2, x)$ represents the element x *as an element of B*.

Disjoint unions can be easily generalised to families of sets. Let A_i be a family of sets labelled by indices $i \in I$, in the label set I. The disjoint union of the family is defined as

$$\coprod_{i \in I} A_i := \cup_{i \in I} (\{i\} \times A_i) = \{(i, x) : x \in A_i, i \in I\} \tag{12.2.2}$$

In the example above, one has now two different elements (x, z) and (y, z) in the disjoint union $T_x S^1 \coprod T_y S^1$ to denote the two tangent vectors at x and y.

3. Quotient sets

An *equivalence relation* is a special relation $\sim_E \subset A \times A$, which obeys the following three properties:

(1) $\forall a \in A, a \sim_E a$ (reflexive property)

(2) $\forall a, b \in A, a \sim_E b \Rightarrow b \sim_E a$ (symmetric property)

(3) $\forall a, b, c \in A, a \sim_E b$ and $b \sim_E c \Rightarrow a \sim_E c$ (transitive property)

For any equivalence relation $\sim$ on A, one can define the equivalence class of an element $a \in A$ to be the subset $[a] = \{b \in A : b \sim a\}$ of all elements of A which are equivalent to a.

By means of the reflexive property, each equivalence class $[a]$ is not empty, it always contains a. By the symmetric property, if $b \in [a]$ then $a \in [b]$, and, by the transitive property, if $b \in [a] \Rightarrow [b] = [a]$. Then the original set A is partitioned into disjoint equivalence classes.

The *quotient* $Q = A/\sim$ is the set of all possible equivalence classes in A. One can define a surjective map $p : A \to Q : a \mapsto [a]$ which is called the *projection* to the quotient.

Given a map $\phi : A \to B$ and an equivalence relation $\sim$ on A, we say that ϕ is *compatible* with the equivalence relation $\sim$ iff

$$a \sim a' \Rightarrow \phi(a) = \phi(a') \tag{12.3.1}$$

Then one can define the quotient set $Q = A/\sim$ and a map $\hat{\phi} : Q \to B : [a] \mapsto \phi(a)$ such that

$$\tag{12.3.2}$$

Notice that the map $\hat{\phi}$ is well defined just as ϕ is compatible with the quotient.

Analogously, if one considers two sets A and B, each with an equivalence relation defined on it, and hence defines two quotient sets $Q_A = A/\sim_A$ and $Q_B = B/\sim_B$ and their projections $\pi_A : A \to Q_A$ and $\pi_B : B \to Q_B$, then a map $\phi : A \to B$ is said to be *compatible* with the equivalence relations iff

$$a \sim_A a' \Rightarrow \phi(a) \sim_B \phi(a') \tag{12.3.3}$$

In this case, one can define a map $\hat{\phi} : Q_A \to Q_B : [a] \mapsto [\phi(a)]$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\pi_A \downarrow & & \downarrow \pi_B \\
Q_A & \dashrightarrow{\hat{\phi}} & Q_B
\end{array}
\tag{12.3.4}
$$

4. Ordering relations

Maps and equivalence relations are not the only interesting relations.

A relation $(A, \leq)$ on the set A is called a *preorder order* iff it has the following properties:

   (i) $x \leq x$ (reflexive)

   (ii) $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive)

A preorder is called a *partial order* iff it has the property

   (iii) $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric)

A partial order is called a *(total) order* iff it has the property

   (iv) for any $x, y \in A$, $x \leq y$ or $y \leq x$ (comparability)

The subsets of a set A are partially ordered by inclusion. The set of real numbers is totally ordered.

Given two vectors $v, w \in V$ define the relation

$$
v < w \iff |v|^2 \leq |w|^2
\tag{12.4.1}
$$

where $|v|^2 = v \cdot v$ is the norm induced by a definite positive inner product. One can easily show that it defines a preorder which is not a partial order.

Cardinality

Given two sets S_1, S_2 we say they have the same *cardinality*, denoted by $S_1 \equiv S_2$, iff there exists a bijection $e : S_1 \to S_2$. That is an equivalence relation, an equivalence class $[S]$ contains all sets with the same cardinality of S.

The empty set $\varnothing$ has, by definition, cardinality 0 and it is the only set in $[\varnothing]$. The set $\{\varnothing\}$ has (precisely) one element. Its cardinality class is denoted by $1 = [\{\varnothing\}]$.

Given a representative S of cardinality k, we can create the disjoint union $S^+ := \{\varnothing\} \coprod S$, the cardinality class of which is made of all sets with $k+1$ elements. The cardinality class $[S^+]$ is called the *successor* of the cardinality class $k = [S]$ and it is denoted by $k + 1 = [S']$.

The we can define $\mathbb{N}$ by the Peano axioms:

    a) $0 = [\{\varnothing\}] \in \mathbb{N}$

    b) if $k = [S] \in \mathbb{N}$, then $k + 1 = [\{\varnothing\} \coprod S] \in \mathbb{N}$

The set $\mathbb{N}$ comes with total order. We say that $k = [S] \leq n = [T]$ if there is no surjective map $i : S \to T$. Prove this is compatible with the quotient defined by cardinality. Also show that it is a total order.

> Here we give the definition of the order in $\mathbb{N}$ although it can be generalized to all cardinalities, finite or infinite.
>
> In $\mathbb{N}$ we could equivalently define $k = [S] \leq n = [T]$ if there is an injection map $i : S \to T$. However, this does not extend to infinite sets, as we shall show in a while.

The elements in $\mathbb{N}$ are called *natural numbers*. A set S of cardinality $k = [S] \in \mathbb{N}$ is called a *finite set*, and we say that S has $k \in \mathbb{N}$ elements.

The set $\mathbb{N}$ is a set itself and we set the class $[\mathbb{N}] =: \aleph_0$ for its cardinality. Any set of cardinality $\aleph_0$ is called *countably infinite*. Any set which is finite or countably infinite is called *countable*.

Infinite cardinalities are weird if one tries to treat them as numbers. Consider the set of even natural numbers $2\mathbb{N} = \{0, 2, 4, \ldots\}$. That is a proper subset of $\mathbb{N}$, however, the map $2\cdot : \mathbb{N} \to 2\mathbb{N} : k \mapsto 2k$ is a bijection so that $\mathbb{N} \equiv 2\mathbb{N}$. We can also define the map $e : \mathbb{Z} \to \mathbb{N}$ as

$$e(k) = 2k \qquad\qquad e(-k) = 2k - 1 \qquad\qquad (12.4.2)$$

for any positive $k \in \mathbb{Z}$, which is also a bijection. Accordingly, even natural numbers, natural numbers, and integers are all the same cardinality.

> We can define integers $\mathbb{Z}$ as pair of natural numbers quotiented by the equivalence relation $(p_1, m_1) \sim (p_2, m_2)$ iff $p_1 + m_2 = p_2 + m_1$.
>
> The map $\varepsilon : \mathbb{N}^2/\sim \to \mathbb{Z} : [(p, m)] \mapsto p - m$ is well defined and one-to-one. An a matter of fact, we can define $\mathbb{Z}$ as $\mathbb{N}^2/\sim$.

All this strange properties of infinite sets have been discovered by Cantor, who called them *transfinite numbers*, and they are exemplified in some stories which are usually called the *Hilbert Hotel*. It is an important step in mathematics. It was the first time that mathematicians decided to deal with infinities as quantities as real as numbers themselves, just with their own properties to be discovered.

> The Hilbert Hotel is a mountain hotel with $\aleph_0$ rooms. One day, it was complete at dinner. In the middle of the stormy night a new client knocked on the door looking for a room to spend the night safe from the storm. At first the concierge said the hotel was complete. However, he was soon be moved to sympathy for the new client who would have been left in the storm. Then, after having thought about the issue for a while, he had an idea and he broadcast a message for all clients:
>
> *we are facing an emergency, we ask all clients to comply to the following instructions. If you are sleeping in room k please take your stuff and go to sleep to room $k+1$. In return, you will receive a discount of 10% on the agreed fare.*
>
> After that, room 0 was empty and the new client could go to sleep, not before he offered the concierge to pay the difference out of his pocket. By the way, the concierge pointed out that the fare of the Hotel was very small and the total income per night could not decrease if the Hotel was complete, or rather they have a finite number of rooms empty, or even with infinite rooms left empty provided that countably infinite rooms were taken.
>
> Later that night, another countably infinite number of new clients knocked on the door looking for a room to avoid the storm. Then the concierge broadcast a new message for all clients:
>
> *I'm sorry for bothering you again, but we are facing another emergency. We kindly ask all clients to comply to the following instructions. If you are sleeping in room k please take your stuff and go to sleep to room $2k + 1$. In return, you will receive a further discount of 10% on the agreed fare.*

That left all even-numbered rooms empty which were able to accomodate for the countably many new clients. As a matter of fact, it seems the Hilbert Hotel cannot rum out of available rooms. [But it can.]

Once we defined integers, we can define rational numbers as pairs of an integer and a non-zero natural number (n, d) modulo the equivalence relation

$$(n_1, d_1) \sim (n_2, d_2) \iff n_1 d_2 = n_2 d_1 \tag{12.4.3}$$

The corresponding quotient space is denoted by $\mathbb{Q}$ and its elements are called *rational numbers*. Notice that the representation of a rational number $[(n, d)]$ is not unique, since $(nk, dk) \sim (n, d)$ for any non-zero $k \in \mathbb{N}$.

In other words, we can define a canonical representative (n, d) for any rational number $[(n, d)] \in \mathbb{Q}$ when n and d have no common factor. Consequently, each rational number canonically corresponds to a definite point in the upper plane $\mathbb{Z}^2$ with integer coordinates (and $d > 0$). Not all integer points are taken (e.g. the point $(6, 2)$ is not the canonical representative of any rational number), but all rational numbers are accommodated somewhere in the plane.

Fig. x: listing rational numbers, here presented by red dots.

The one can list all rational numbers by following the green line as in Figura x. The list of rational numbers is effectively equivalent to a bijection $l : \mathbb{N} \to \mathbb{Q}$ since to any natural number $n \in \mathbb{N}$ it corresponds the rational number at *nth* position in the list. Therefore, rational numbers again are countably infinite.

One can think to see a pattern here: *any infinite set is countably infinite*. After all, infinite is infinite, right? Wrong!

Consider real numbers in $(0, 1] \subset \mathbb{R}$. Each element in $(0, 1]$ can be represented by its decimal expression as an infinite list of digits starting with 0. (e.g. 0.5, 0.023, 0.333..., 0.7071..., 0.999...)

Here we need to be careful, since the correspondence between lists of digits and numbers in $(0, 1]$ is not one-to-one. The number 0.9999... (ending with an infinite list of 9s) *is* the number 1, as well as 0.749999... *is* the number 3/4. As a matter of fact, the two decimal sequences 0.749999... and 0.750000... correspond to the same number 3/4. Thus, to avoid overcounting, we need to get rid of the sequences ending in an infinite list of 0s.

Once we have done that, if we assume that $(0, 1]$ is countable, it means we could form a list exhausting all allowed sequences of digits. Yet, we can show there are allowed sequences that are not in the list, whatever list we consider.

That is easy, start with 0 and consider the first decimal place of the first number of the list, say it is the digit n_1, the second decimal place of the second number of the list, say n_2, the third decimal place of the third number of the list, say n_3, and so on.

Now we can build a sequence that is not on the list, by the following rule: if n_k is a digits in the sequence $0.n_1 n_2 n_3 ...$ we replace it by $(n_k + 1)$ if $n_k = 0..8$, by 1 if $n_k = 9$.

The resulting sequence is allowed (it does not even contain 0s anywhere), it is not on the original list since it differs from any sequence on the list by at least one digit. That shows that the set $(0, 1]$ has cardinality grater than $\aleph_0$. Elements in $(0, 1]$ are uncountably infinitely many.

This is called the *Cantor diagonal argument*. It shows that there exists infinites which differ in cardinality. One may ask whether there are infinite sets with a cardinality greater than $\aleph_0$ but smaller than the continuum cardinality of $(0, 1]$. However, it turns out that the answer is undecidable, it is an independent axiom of number theory. Usually one can assume the answer in the negative in that case we set the cardinality of $(0, 1]$ to $\aleph_1$, which is called the *continuum hypothesis*. However, that is an axiom so that one obtains different number theories by assuming any different answer.

That shows we still are missing something about infinity although we progressed a lot with respect to our Greek masters.

Once we recognized that we can iterate that forever: maps $\phi : \mathbb{R} \to \mathbb{R}$ have a cardinality greater than points in $(0, 1]$ and so on.

Ordinality

Cardinality provides a rough scale to classify infinite sets, as the Hilbert Hotel story shows. Many incapsulated infinite sets turn out to be all ithe same cardinality. However, we see that cardinality is nothing more than *a definition* of an equivalence relation to identify sets to define that scale. There is no evidences that there are no other ways of obtaining a finer scale. In fact, there is a finer scale.

A *well-order* on a set X is a total order $\leq$, such that any non-empty subset $S \subset X$ has a least element. For example, $(\mathbb{N}, \leq)$ is a well-order, $(\mathbb{Z}, \leq)$ is not.

Let us consider 2 well-ordered sets $(A, \preceq)$ and $(B, \preceq)$. We say *they have the same length* (and denote it as $(A, \preceq) \sim (B, \preceq)$) iff there exists an *order-preserving map* $\phi : A \to B$ (i.e. $a_1 \preceq a_2 \iff \phi(a_1) \preceq \phi(a_2)$) with an inverse $\phi^{-1} : B \to A$ which is an order-preserving map as well.

That is an equivalence relation and an equivalence class is called an *ordinal*.

Ok, we should be more careful with set theory. This is not a book in number theory, though.

The set $\mathbb{N}$ is well-ordered, its ordinal is denoted by ω_0. Now consider $\mathbb{N}_{+1} := \mathbb{N} \cup \{a\}$ on which we extend the well-order we have on $\mathbb{N}$ by defining $n \prec a$ for all $n \in \mathbb{N}$. Then $(\mathbb{N}_{+1}, \preceq)$ is well-ordered and its ordinal is denoted by $\omega_0 + 1 = [\mathbb{N}_{+1}]$.

The sets $\mathbb{N}$ and $\mathbb{N}_{+1}$ have different lengths, they define different ordinals, although they are both countably infinite as a cardinality. In other words, we see that ordinals is a different, finer scale on infinite sets than the cardinality.

When we consider a well-ordered set $(A, \preceq)$ we can reverse the total order to obtain $(A, \succeq)$ which is again a total order, not necessarily a well-order. If $(A, \succeq)$ is again a well-ordered set, its ordinal is called *finite*, as it happens for 0, 1, ..., not for ω_0. Any ordinal α for which there exists no ordinal β such that $\alpha = \beta + 1$ is called *a limit ordinal*. For example, ω_0 is a limit ordinal (also in the sense that it is a limit point in the order topology).

The structure of ordinals in quite complicated. One can show directly that one can count

$$0, 1, 2, \ldots, \omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, 2\omega_0, 2\omega_2 + 1, 2\omega_2 + 2, \ldots, \ldots, \omega_0^2, \omega_0^2 + 1, \ldots \omega_0^3, \omega_0^3 + 1, \ldots, \ldots, \omega_0^{\omega_0}, \ldots \tag{12.4.4}$$

Of course, when α is countably infinite, also $\alpha + 1$ is countably infinite. But what happens at limit ordinals? Can one count to uncountably infinite?

It turns out one can, in some sense. By using the *axiom of choice* we can prove that on any set one can define a well-order.

The axiom of choice reads as: for all set X and any collection of non-empty subsets $\mathcal{A} := \{A_\alpha \subset X : \alpha \in I\}$ of X there exists a *selection map* $\sigma : I \to X : \sigma(\alpha) \in A_\alpha \subset X$ so that $\sigma(\alpha)$ is called a *representative* for the subset A_α. By an abuse of notation we also set $\sigma(A_\alpha) = \sigma(\alpha) \in A_\alpha$.

Proof: *suppose A is a set and we wish to define a well-order $\preceq$ on A. Let $\mathcal{P}_\varnothing(A)$ be the family of all non-empty subsets in A. By the axiom of choice there exists a selection map $\sigma : \mathcal{P}_\varnothing(A) \to A$.*

A pair $(U, \preceq)$ of subset $U \subset A$ with a well-order $\preceq$ on U is called a σ-ordered subset in A iff $\forall x \in U : x = \sigma(U - \{y : y \prec x\})$.

These subsets exist: $\varnothing$ is trivially a σ-ordered subset in A since there is nothing to order and nothing to check. Then consider $a = \sigma(A) \in A$, then $A_0 := \{a\}$ which is a σ-ordered subset in A. Then set $b = \sigma(A - A_0)$ and $A_1 := A_0 \cup \{b\}$. The element $b \in A$ is clearly different from a (since $b \in A - A_0$). We can define the well-order on A_1 by setting $a \prec b$. Then $(A_1, \preceq)$ is a σ-ordered subset in A. And so on.

Now let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be the family of all σ-ordered subsets in A. Define $\bar{A} = \cup_{\alpha \in I} A_\alpha \subset A$. One can define a well-order $\preceq$ on $\bar{A}$ by using the fact that $\forall a, b \in \bar{A} : \exists \alpha : a, b \in A_\alpha$. Then $\bar{A} \in \mathcal{A}$ is a maximal σ-ordered subset.

Now suppose $\bar{A} \neq A$, then $A - \bar{A} \neq \varnothing$ and we could build a σ-ordered subset $\bar{A} \cup \sigma(A - \bar{A})$ strictly larger than $\bar{A}$, which contradicts maximality.

Then $A = \bar{A}$ and A is well-ordered by $\preceq$. ∎

Then $\mathbb{R}$ is well-ordered (although since the axiom of the choice is not constructive we have no clue at all about how the selection function or the order look like). It has cardinality $\aleph_1$ (if the accept the continuum hypothesis) and still it corresponds to some ordinal.

If we exclude all ordinals which are finite or countably infinite, because of well-ordering we have a least ordinal that is uncountable, which is called the *first uncountable ordinal* and it is denoted by ω_1. Of course, $\omega_1 \lll [\mathbb{R}]$, since, for example, any uncountable subset of $\mathbb{R}$ is shorter than $[\mathbb{R}]$.

Of course, ordinales do not stop at ω_1, then one has

$$\ldots, \omega_1, \omega_1 + 1, \omega_1 + 2, \ldots, 2\omega_1, \ldots, 3\omega_1, \ldots, \omega_1^2, \ldots, \omega_1^3, \ldots, \omega_1^{\omega_1}, \ldots \tag{12.4.5}$$

At some point one breaks another wall and gets to a first ordinal of cardinalitry $\aleph_2$, denote it ω_2, and so on.

Welcome to the MultiHilbert Hotel.

Abstract simplices

Let us consider a set V. Elements of V are called *vertices*.

Given $n + 1$ (different) vertices x_i, with $i = 0, 1, \ldots, n$, an *(oriented) abstract n-simplex* is a *finite ordered set* which is denoted by $[x_0, x_1, x_2, \ldots, x_n]$. That means that we defined an order on the set $X = \{x_0, x_1, x_2, \ldots, x_n\}$ according to which $x_i < x_j$ when $(i < j)$.

An *n-chain* is a finite linear combination with integer coefficients of n-simplices. The set of all n-chains is denoted by C_n and it is a $\mathbb{Z}$-module (and in particular an abelian group). *Just look ahead for the definition of $\mathbb{Z}$-module and check it.*

We can define a linear operator $\partial : C_n \to C_{n-1}$, which is called the *boundary operator*, which is defined on abstract simplices $\sigma = [x_0, x_1, \ldots, x_n]$, and then extended by linearity to n-chains.

The boundary of the simplex σ is a $(n-1)$-chain $\partial\sigma$ given by

$$\partial\sigma = \sum_{k=0}^{n} (-1)^k [x_0, x_1, x_2, \ldots, \hat{x}_k, \ldots, x_n] \tag{12.4.6}$$

where $[x_0, x_1, x_2, \ldots, \hat{x}_k, \ldots, x_n]$ denotes the $(n-1)$-simplex obtained from σ by removing the vertex x_k, as it is indicated by the hat.

For example, given a 2-simplex $\sigma = [a, b, c]$, its boundary is the 1-chain

$$\partial\sigma = [b, c] - [a, c] + [ab] \tag{12.4.7}$$

By extending by linearity the boundary operator to the n-chains, we mean to define the boundary of a n-chain $\boldsymbol{\sigma} = \sum_i c_i \sigma_i$ as

$$\partial\boldsymbol{\sigma} = \sum_i c_i \,\partial\sigma_i \tag{12.4.8}$$

Hence we have a family of chain $\mathbb{Z}$-modules C_n and a family of operators $\partial : C_n \to C_{n-1}$, i.e. a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\;\partial\;} C_{n-1} \longrightarrow \cdots \longrightarrow C_3 \xrightarrow{\;\partial\;} C_2 \xrightarrow{\;\partial\;} C_1 \tag{12.4.9}$$

which is called a *chain complex*, since one has $\partial \circ \partial = 0$.

It is sufficient to prove that $\partial \circ \partial\sigma = 0$ for any n-simplex $\sigma = [x_0, x_1, \ldots, x_n]$. The general statement follows by linearity.

When computing $\partial \circ \partial\sigma$ we obtain an $(n-2)$-chain made of simplexes $\sigma_{ij} = [x_0, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n]$ in which we removed two vertices, x_i and x_j, from σ. Each of the simplex σ_{ij} appears in the boundary $\partial \circ \partial\sigma$ twice, once when we remove x_i first and then x_j, once when we remove x_j first and then x_i. Let us assume that $i < j$, then the two contributions are

$$(-1)^i(-1)^{j-1}[x_0, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n] \qquad \text{and} \qquad (-1)^j(-1)^i[x_0, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n] \tag{12.4.10}$$

respectively, i.e. the two contributions cancel out.

For example, let us consider a 3-simplex $\sigma = [a, b, c, d]$. Its boundary is

$$\partial\sigma = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c] \tag{12.4.11}$$

and the boundary of this 2-chain is

$$\partial \circ \partial\sigma = \partial[b,c,d] - \partial[a,c,d] + \partial[a,b,d] - \partial[a,b,c] = ([c,d] - [b,d] + [b,c]) - ([c,d] - [a,d] + [a,c]) + ([b,d] - [a,d] + [a,b]) - ([b,c] - [a,c] + [a,b]) = 0 \tag{12.4.12}$$

Given an n-simplex $\sigma = [x_0, x_1, \ldots, x_n]$, the $(n-1)$-simplex $\sigma_i = [x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n]$ is called the *ith face* of σ; hence an n-simplex has $(n+1)$ faces, and the boundary $\partial\sigma$ is a linear combination of the faces of σ. More generally, for any subset $\{y_0, y_1, \ldots, y_k\} \subset \{x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n\}$, the k-simplex $[y_0, y_1, \ldots, y_k]$ is a k-face of σ iff $i < j$, according to the ordering defined by σ, implies $y_i < y_j$. Thus a face of an n-simplex is a $(n-1)$-face.

Given two chain complexes, (C_n, ∂) and (D_n, ∂), a *morphism*, denoted by $f : (C_n, \partial) \to (D_n, \partial)$, is a family of maps $f_n : C_n \to D_n$ such that $f_{n-1} \circ \partial = \partial \circ f_n$, i.e. the following diagram is commutative

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_n & \xrightarrow{\;\partial\;} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_3 & \xrightarrow{\;\partial\;} & C_2 & \xrightarrow{\;\partial\;} & C_1 \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
\cdots \longrightarrow & D_n & \xrightarrow{\;\partial\;} & D_{n-1} & \longrightarrow \cdots \longrightarrow & D_3 & \xrightarrow{\;\partial\;} & D_2 & \xrightarrow{\;\partial\;} & D_1
\end{array}
\tag{12.4.13}
$$

A family Δ of simplices is called a *simplical complex* iff, for any $\sigma \in \Delta$ and any k-face ρ of σ, then $\rho \in \Delta$. One can adapt the definition of chains, boundary, and morphism to a simplicial complex. The corresponding chain complex is denoted by $(C_n(\Delta, \mathbb{Z}), \partial)$.

Given a simplicial complex Δ, a *n-cycle* is an n-chain $\sigma \in C_n(\Delta, \mathbb{Z})$ such that $\partial \sigma = 0$. The space of n-cycles is denoted by $Z_n(\Delta, \mathbb{Z}) \subset C_n(\Delta, \mathbb{Z})$ and it is an abelian group and a $\mathbb{Z}$-module.

An *n-boundary* σ is an n-cycle such that there exists a $(n+1)$-chain $\theta \in C_{n+1}(\Delta, \mathbb{Z})$ such that $\sigma = \partial \theta$. In view of the property $\partial^2 = 0$, a boundary is necessarily a cycle. The space of n-boundaries is denoted by $B_n(\Delta, \mathbb{Z}) \subset Z_n(\Delta, \mathbb{Z})$ and it is an abelian group and a $\mathbb{Z}$-module.

We can define an equivalence relation $\sigma \sim \sigma'$ between cycles iff $\sigma' - \sigma \in Z_n(\Delta, \mathbb{Z})$. The quotient

$$H_n(\Delta, \mathbb{Z}) = \frac{Z_n(\Delta, \mathbb{Z})}{B_n(\Delta, \mathbb{Z})} \tag{12.4.14}$$

is canonically an abelian group and a $\mathbb{Z}$-module which is called (the nth) *homology group*.

The most important property of homologies group, is that if two complexes are isomorphic $\Delta \simeq \Delta'$, then their homology groups are isomorphic, i.e. $H_n(\Delta, \mathbb{Z}) \simeq H_n(\Delta', \mathbb{Z})$. Accordingly, homology groups can be used to disprove isomorphisms between complexes.

We define a *n-cochain* as a linear map $\alpha : C_n \to G$ for some abelian group G. The space of n-cochains is denoted by $C^n(\Delta, G)$, which is also an abelian group (and $\mathbb{Z}$-module). We define the *coboundary operator* $\delta : C^n(\Delta, G) \to C^{m+1}(\Delta, G)$ as

$$(\delta \alpha)(\sigma) = \alpha(\partial \sigma) \tag{12.4.15}$$

Of course, one has $\delta^2 = 0$.

Cocycles are cochains such that $\delta \alpha = 0$ and the space of cocycles is denoted by $Z^n(\Delta, G)$. A *coboundary* is an n-cocycle α such that there exists a $(n-1)$-cochain β such that $\alpha = \delta \beta$. The space of coboundary is denoted by $B^n(\Delta, G) \subset Z^n(\Delta, G)$. The quotient group

$$H^n(\Delta, G) = \frac{Z^n(\Delta, G)}{B^n(\Delta, G)} \tag{12.4.16}$$

is called (the nth) *cohomology group* and it also is invariant under isomorphisms.

Given a $(p+q)$-simplex $\sigma = [x_0, x_1, \ldots, x_{p+q}]$ we define the *front p-face* of σ to be ${}^p(\sigma) = [x_0, x_1, \ldots, x_p]$ and the *back q-face* of σ to be $(\sigma)^q = [x_p, \ldots, x_{p+q-1}, x_{p+q}]$.

Then we can define the *cup product* of a p-cochain α and a q-cochain β to be the $(p+q)$-cochain $\alpha \cup \beta$ defined to act on the $(p+q)$-simplex σ as

$$(\alpha \cup \beta)(\sigma) = \alpha\left({}^p(\sigma)\right) \cdot \beta\left((\sigma)^q\right) \tag{12.4.17}$$

then extended by linearity to chains. The product $\alpha\left({}^p(\sigma)\right) \cdot \beta\left((\sigma)^q\right)$ is the product in the group G.

The cup product extends to $H^\bullet(\Delta, G) = \oplus_n H^n(\Delta, G)$ which can be shown to become a ring with it.

5. Categories and functors

A *category* is a set (though it can be relaxed to be a *class*) of *objects* together with a set of *morphisms*. A *morphism* $\phi : A \to B$ is an arrow between two objects and whenever one has two morphisms $\phi : A \to B$ and $\psi : B \to C$, then one also has the *composition morphism* $\psi \circ \phi : A \to C$. The composition is associative.

In any category, for any object A, there always exists a special morphism $\mathrm{id}_A : A \to A$ (which is the neutral element—both on the right and on the left—with respect to composition) called the *identity*.

It is standard to call *endomorphisms* the morphisms from an object into itself, *isomorphisms* the morphisms with an inverse (which needs to be a morphism as well), *automorphisms* the maps which are both isomorphisms and endomorphisms.

An example of category is $\mathfrak{Set}$, which is formed by the sets together with the maps. Despite categories are more general, we shall usually consider subcategories of the $\mathfrak{Set}$; accordingly, morphisms will be, for us, maps between objects, taking elements of one object and returning elements on the target object. In the category of sets and its subcategories, morphisms are called *epimorphisms* if they are surjective, *monomorphisms* if they are injective.

The reader already knows many categories besides $\mathfrak{Set}$. Let us considers vector spaces together with linear maps; they form a category which will be denoted by $\mathfrak{Vect}$. Groups together with group homomorphisms form a category denoted by $\mathfrak{Group}$.

Some more will be introduced below. At the very least, categories are a notation to be clear on which maps one is allowed to use with objects. Of course, a group, say $\mathbb{R}^2$ with the sum, is a group, but it is also a set and a vector space (as well as a topological space, a manifold, and a bundles over $\mathbb{R}$). Hence it is an object of $\mathfrak{Set}$, $\mathfrak{Vect}$ and $\mathfrak{Group}$, ... As a matter of fact, $\mathbb{R}^2$ must be regarded as a *different* object when regarded in different categories, even if it is always the same set $\mathbb{R}^2$. Here one should notice that on the set $\mathbb{R}^2$ one can use any map, on the vector space $\mathbb{R}^2$ one can use any linear map, on the group $\mathbb{R}^2$ one can use any group homomorphism. On the same object, one is allowed to use different classes of morphisms depending on the category one is considering.

Every time we shall hereafter introduce new classes of objects, they will always be introduced together with a class of allowed morphisms (with identities and closed under compositions) which, in fact, defines a category. Besides being a good habit, we shall show that sometimes very general results can be proven only in view of the properties of category (and functor below). Moreover, relativistic theories are quite deeply related to transformations among observers and covariance principle; accordingly, it will be good to have a control and general notation to consider *allowed* transformations which may change in different situations.

A *functor* is a kind of map between categories. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories; a *covariant functor* $\Phi : \mathfrak{A} \to \mathfrak{B}$ maps any object A of $\mathfrak{A}$ to an object $\Phi(A)$ of $\mathfrak{B}$ and any morphism $\phi : A \to A'$ to a morphism $\Phi(\phi) : \Phi(A) \to \Phi(A')$. A covariant functor must send identities into identities (i.e. $\Phi(\mathrm{id}_A) = \mathrm{id}_{\Phi(A)}$) and it preserves compositions, i.e.

$$\Phi(\phi \circ \psi) = \Phi(\phi) \circ \Phi(\psi) \tag{12.5.1}$$

Analogously, a a *controvariant functor* $\Phi : \mathfrak{A} \to \mathfrak{B}$ maps any object A of $\mathfrak{A}$ to an object $\Phi(A)$ of $\mathfrak{B}$ and any morphism $\phi : A \to A'$ to a morphism $\Phi(\phi) : \Phi(A') \to \Phi(A)$. (Notice the reversal of the morphism.) A controvariant functor must preserve identities (i.e. $\Phi(\mathrm{id}_A) = \mathrm{id}_{\Phi(A)}$) and it reverses compositions, i.e.

$$\Phi(\phi \circ \psi) = \Phi(\psi) \circ \Phi(\phi) \tag{12.5.2}$$

An example of covariant functor

A subcategory $\mathfrak{B}$ of a category $\mathfrak{A}$ can be embedded by a covariant functor $F : \mathfrak{B} \to \mathfrak{A}$ which sends any object of $\mathfrak{B}$ into the same object as an object of the category $\mathfrak{A}$, and each morphism $\Phi : B_1 \to B_2$ into the same morphism.

For example, the category $\mathfrak{Vect}$ of vector spaces with linear maps is a subcategory of the $\mathfrak{Set}$ category. The covariant functor $F : \mathfrak{Vect} \to \mathfrak{Set}$ *forgets* the linear structure and considers the objects as bare sets and the morphisms as general maps. For this reason, the covariant functor F which embeds a subcategory into a category is called the *forgetful functor*.

The dual functor

Let us give an example of a controvariant functor $()^* : \mathfrak{Vect} \to \mathfrak{Vect}$ from the category $\mathfrak{Vect}$ of vector spaces and linear maps into itself. We know from linear algebra that, for any (finite dimensional) vector space V (which is an object in $\mathfrak{Vect}$), one can define the dual vector space $(V)^*$ of linear forms $\alpha : V \to \mathbb{R}$.

Moreover, for any linear map $\phi : V_1 \to V_2$ one can define a dual map

$$(\phi)^* : (V_2)^* \to (V_1)^* : \alpha \mapsto (\phi)^*(\alpha) \qquad \forall v \in V_1 : (\phi)^*(\alpha)(v) = \alpha(\phi(v)) \tag{12.5.3}$$

One can easily check that (12.5.3) defines a good linear map $(\phi)^* : (V_2)^* \to (V_1)^*$. Moreover, one has $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$ and it reverses compositions, i.e. $(\phi \circ \psi)^* = (\psi)^* \circ (\phi)^*$. Hence $()^* : \mathfrak{Vect} \to \mathfrak{Vect}$ is a controvariant functor.

Of course, the bidual $()^{**} : \mathfrak{Vect} \to \mathfrak{Vect}$ is another example of a covariant functor. Show this example is rather trivial.

The categorical sum

We say a category $\mathfrak{A}$ has a *sum* (or a *coproduct*) iff for all families $\{A_\alpha : \alpha \in I\}$ of objects, there exists an object $A = \oplus_{\alpha \in I} A_\alpha$ and a family of maps $i_\alpha : A_\alpha \to A$ with the following universal property.

Universal property of the sum: for all objects B and all families of maps $f_\alpha : A_\alpha \to B$, then there exists a unique morphism $f := \oplus_{\alpha \in I} f_\alpha : A \to B$ which makes commutative the following diagram:

$$\begin{array}{ccc} & & B \\ & f_\alpha \nearrow & \uparrow f \\ A_\alpha & \xrightarrow{i_\alpha} & A \end{array} \tag{12.5.4}$$

The universal properties are a recurrent feature in category theory. It corresponds in defining structures in terms of their morphisms only.

We can easily prove that when the sum A exists, it is unique up to isomorphisms.

Let us suppose that there are 2 objects A and A' satisfying the universal property. By setting $B = A'$ in the universal property of A we obtain a map $i : A \to A'$ such that $i'_\alpha = i \circ i_\alpha$. We can also set $B = A$ in the universal property of A' we obtain a map $i' : A' \to A$ such that $i_\alpha = i' \circ i'_\alpha$.

Then we have the composition $i \circ i' : A' \to A'$ (as well as $i' \circ i : A \to A$) such that $i \circ i' \circ i'_\alpha = i \circ i_\alpha = i'_\alpha : A_\alpha \to A'$ (and $i' \circ i \circ i_\alpha = i' \circ i'_\alpha = i_\alpha : A_\alpha \to A$).

Compare with the universal property of A' setting $B = A'$ itself: by uniqueness we have to $i \circ i' = \mathrm{id}_{A'}$. Analougusly, we have $i' \circ i = \mathrm{id}_A$. In other words, $i : A \to A$ and $i' : A' \to A'$ are isomorphism and one the inverse of the other. Thus $A \simeq A'$. ∎

Let us remark that the proof of uniqueness up to isomorphisms has pretty much the same structure for any universal property.

In the category of sets, the sum is the disjoint union (just show the universal property holds true for $B = \coprod_{\alpha \in I} A_\alpha$). In the category of topological space the sum extends once we prove that the morphism f is continuous. In the category of vector spaces the sum is the *direct sum* as one can show by proving that the morphism f is linear when f_α are linear. In the category of groups the sum is the free product of groups.

Given two groups G, H their free product $G * H$ consists of strings of elements $g_1 h_1 g_2 h_2 \ldots g_n h_n$ with the product defined as

$$(g_1 h_1 g_2 h_2 \ldots g_n h_n) * (g'_1 h'_1 g'_2 h'_2 \ldots g'_m h'_m) = g_1 h_1 g_2 h_2 \ldots g_n h_n g'_1 h'_1 g'_2 h'_2 \ldots g'_m h'_m \tag{12.5.5}$$

It contains both G and H as subgroups. For example, $i : G \to G * H : g \mapsto g\, e_H$.

The categorical product

The constructions of the sum and the product are in some sense dual to each other.

We say a category $\mathfrak{A}$ has a *product* iff for all families $\{A_\alpha : \alpha \in I\}$ of objects, there exists an object $A = \otimes_{\alpha \in I} A_\alpha$ and a family of maps $p_\alpha : A \to A_\alpha$ with the following universal property.

Universal property of the product: for all objects B and all families of maps $f_\alpha : B \to A_\alpha$, then there exists a unique morphism $f := \otimes_{\alpha \in I} f_\alpha : B \to A$ which makes commutative the following diagram:

$$
\begin{array}{ccc}
 & & B \\
 & {\scriptstyle f_\alpha} \nearrow & \downarrow {\scriptstyle f} \\
A_\alpha & \xleftarrow{\ p_\alpha\ } & A
\end{array}
\tag{12.5.6}
$$

We can easily prove that when the product A exists, it is unique up to isomorphisms.

Try and repeat the proof for the sum taking care to the direction of morphisms.

In the category of sets, the product is the Cartesian product. In the category of topological space the product topology provides the categorical product. In the category of vector spaces the product is again the *Cartesian product*. In the category of groups the product is the direct product of groups.

As an object, it agrees with the categorical sum, although the same object satisfies 2 different universal properties. Again the call for different transformations to exist.

Categorical limits

In some categories, one can define objects as limits of a family of objects. As usual in category theory, the point is that the procedure can be abstracted from the details and then used in different contexts.

There are 2 kinds of limits that are dual (in the categorical sense, which is: *just revert the arrows*) to each other.

Definition*: let $(I, \leq)$ be an ordered set of labels.*

A family of objects A_i indexed by $\forall i \in I$, together with a family of morphisms $f_i^j : A_i \to A_j$ indexed by $\forall i, j \in I : i \leq j$ is called a *direct family* iff $\forall i, j, k \in I : i \leq j \leq k$

$$f_i^i = id_{A_i} : A_i \to A_i \qquad\qquad f_i^k := f_j^k \circ f_i^j : A_i \to A_k \qquad\qquad (12.5.7)$$

A direct family is denoted by $\mathcal{D} := \{(A_i, f_i^j) : i \leq j \in I\}$ or, by an abuse of notation simply by (A_i, f_i^j).

Definition*: let $(I, \leq)$ be an ordered set of labels.*

A family of objects A_i indexed by $\forall i \in I$, together with a family of morphisms $p_j^i : A_j \to A_i$ indexed by $\forall i, j \in I : i \leq j$ is called an *inverse family* iff $\forall i, j, k \in I : i \leq j \leq k$

$$p_i^i = id_{A_i} : A_i \to A_i \qquad\qquad p_i^k := p_i^j \circ p_j^k : A_k \to A_i \qquad\qquad (12.5.8)$$

An inverse family is denoted by $\mathcal{D} := \{(A_i, p_j^i) : i \leq j \in I\}$ or, by an abuse of notation simply by (A_i, p_j^i).

Let us give two examples. Consider the maps $i_n^m : \mathbb{R}^n \to \mathbb{R}^m$ defined for $n \leq m$ by $i_n^m : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots 0)$. One can check that $i_n^n = id$ and $i_n^m := i_k^m \circ i_n^k$ for $n \leq k \leq m$. Accordingly, (R^n, i_n^m) is a (countably infinite) direct family.

Consider instead the maps $p_m^n : \mathbb{R}^m \to \mathbb{R}^n$ defined for $n \leq m$ by $p_m^n : (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$. One can check that $p_n^n = id$ and $p_n^m := p_n^k \circ p_k^m$ for $n \leq k \leq m$. Accordingly, (R^n, p_m^n) is a (countably infinite) inverse family.

Given a direct family (A_i, f_i^j), its *direct limit* is a pair (A, f_i) where A is an object in the category and $f_i : A_i \to A$ are morphisms such that $f_i = f_j \circ f_i^j$ for $i \leq j$. The direct limit (A, f_i) has to satisfy the following universal property: given any object B together with a family of morphisms $\varphi_i : A_i \to B$ such that $\varphi_i = \varphi_j \circ f_i^j$, then there exists a unique (up to isomorphism) morphism $\varphi : A \to B$ such that $\varphi_i = \varphi \circ f_i$.

As we did for the categorical sum and product, this universal property can be given in form of the following diagram

$$
\begin{array}{ccc}
B & & \\
\varphi_i \uparrow & \diagdown\ \varphi & \\
A_i & \xrightarrow{\ f_i\ } & A
\end{array}
\qquad\qquad (12.5.9)
$$

If the direct limit exists we denote it as $\lim_{\to} A_i = A$.

Sometimes in the literature, one finds for the direct limit a more compliocated diagram as

$$(12.5.10)$$

which supposedly helps remembering the conditions $f_i = f_j \circ f_i^j$ for $i \leq j$ and $\varphi_i = \varphi_j \circ f_i^j$, provided one remembers that $i \leq j$.

Analogously, given an inverse family (A_i, p_j^i), its *inverse limit* is a pair (A, p_i) where A is an object in the category and $p_i : A \to A_i$ are morphisms such that $p_i = p_j^i \circ p_j$ for $i \leq j$. The inverse limit (A, p_i) has to satisfy the following universal property: given any object B together with a family of morphisms $\varphi_i : B \to A_i$ such that $\varphi_i = p_j^i \circ \varphi_j$, then there exists a unique (up to isomorphism) morphism $\varphi : B \to A$ such that $\varphi_i = p_i \circ \varphi$.

This universal property can be given in form of the following diagram

$$(i \leq j)$$

$$(12.5.11)$$

If the inverse limit exists we denote it as $\lim_{\leftarrow} A_i = A$.

By getting inspiration from the proof of uniqueness we did for sums, we can prove that, if the direct limit exists, it is unique up to isomorphisms. Analogously for the inverse limit.

Depending on the category, the existence of direct and inverse limits is guaranteed or not. For example, in the category of vector spaces one easily has infinite dimensional vector spaces since linearity is a stronger property than continuity or smoothness and better extends to infinite dimensional objects.

If you are not familiar with linear algebra, just refer ahead and jump back and forth to fill in the gaps.

We have a direct family $(\mathbb{R}^n, i_n^m)$ in the category of vector spaces. We know that if the direct limit exists it is unique (up to isomorphisms), thus we can simply exhibit a vector space that satisfies the universal property and claim it is the direct limit of that family, up to isomorphisms.

Let us define $\mathbb{R}_0^\infty := \{(x_1, x_2, \ldots, x_n, 0, \ldots)\}$ the vector space of infinite sequences of real numbers, eventually zero, i.e. for any $v \in \mathbb{R}_0^\infty$ there exists a $n_0 \in \mathbb{N}$ such that $\forall n_0 < k : x_k = 0$. The minimal n_0 for which that happens is called the *length* of v. All elements of $\mathbb{R}_0^\infty$ are finite of length, however there are arbitrary length vectors in $\mathbb{R}_0^\infty$.

We can define the linear maps $i_m : \mathbb{R}^m \to \mathbb{R}_0^\infty : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots)$ and these satisfy the condition $i_m = i_n \circ i_m^n$ for any $m \leq n$.

Also, for any other vector space B, given a family of maps $\varphi_i : A_i \to B$ such that $\varphi_m = \varphi_n \circ i_m^n$ for any $m \leq n$ one can define a unique map $\varphi : \mathbb{R}_0^\infty \to B$ such that $\varphi_n = \varphi \circ i_n$ for all n.

Proof: *we already notice that for any element $v \in \mathbb{R}_0^\infty$ there exists $n \in \mathbb{N} : \exists v_n \in \mathbb{R}^n$ such that $v = i_n(v_n) \in \mathbb{R}^n$. Hence, if the property of φ has to hold true, it necessarily has to be*

$$\varphi(v) = \varphi \circ i_n(v_n) = \varphi_n(v_n) \tag{12.5.12}$$

Then we see that the map φ so-defined has the property required and that the family of maps φ_n completely determines φ. ∎

Then we have the direct limit $\lim_\to \mathbb{R}^n = \mathbb{R}_0^\infty$.

We also have an inverse family $(\mathbb{R}^n, p_m^n)$, again in the category of vector spaces. We simply have exhibit a vector space that satisfies the universal property to claim it is the inverse limit of that family, up to isomorphisms.

Let us define $\mathbb{R}^\infty := \{(x_1, x_2, \ldots, x_n, \ldots)\}$ the vector space of infinite sequences of real numbers (no request about eventual behaviour). We can define the linear maps $p_m : \mathbb{R}^\infty \to \mathbb{R}^m : (x_1, \ldots, x_m, x_{m+1}, \ldots) \mapsto (x_1, \ldots, x_m)$ and these satisfy the condition $p_m = p_m^n \circ p_n$ for any $m \leq n$.

All elements of $\mathbb{R}^\infty$ can project onto $\mathbb{R}^m$ by just cutting their tail at m. Of course, $\mathbb{R}_0^\infty \subset \mathbb{R}^\infty$ and in $\mathbb{R}^\infty$ there are vectors of infinite length.

Also, for any other vector space B, given a family of maps $\varphi_i : B \to A_i$ such that $\varphi_m = p_m^n \circ \varphi_n$ for any $m \leq n$ one can define a unique map $\varphi : B \to \mathbb{R}^\infty$ such that $\varphi_n = p_n \circ \varphi$ for all n.

Proof: *if the property of φ has to hold true, it necessarily has to be*

$$\varphi_n(v) = p_n \circ \varphi(v) \qquad \forall n, \forall v \in B \tag{12.5.13}$$

which fixes the first n components of the vector $\varphi(v) \in \mathbb{R}^\infty$ in terms of the value of the map φ_n. And that is for an arbitrary n. Said differently, for any n the value of the nth-component of $\varphi(v)$ is fixed by the property for n and, in view of the compatibility conditions, confirmed by the property for any $m \geq n$.

Then we see that the map φ so-defined has the property required and that the family of maps φ_n completely determines φ. ∎

Then we have the inverse limit $\lim_\leftarrow \mathbb{R}^n = \mathbb{R}^\infty$.

6. Groups and actions

A *group* is a set G with an operation $\cdot : G \times G \to G : (h, k) \mapsto h \cdot k$ such that

    a) $\forall a, b, c \in G, \ a \cdot (b \cdot c) = (a \cdot b) \cdot c =: a \cdot b \cdot c$ (associative)

    b) $\exists e \in G, \ e \cdot a = a \cdot e = a$ (neutral element)

    c) $\forall a \in G, \exists b \in G, \ a \cdot b = b \cdot a = e$ (inverse)

If, for any $a, b \in G$, one has $a \cdot b = b \cdot a$, then the group $(G, \cdot)$ is called *commutative*.

The neutral element is also denoted by $e = \mathbb{I}$ and the inverse of $a \in G$ by a^{-1}. In general, a group will be denoted by $(G, \cdot)$.

> Sometimes, usually for commutative groups, one uses the so-called *additive notation*. In additive notation, the operation is denoted by $+$, the neutral element is denoted by 0 and the inverse of a is denoted by $-a$ and it is also called the *opposite* element.

A morphism $\Phi : (G, \cdot) \to (H, \cdot)$ is a map which preserved the group operation

    a) $\Phi(\mathbb{I}_G) = \mathbb{I}_H$

    b) $\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b)$

Group morphisms are also called *homomorphism*. The category of groups with homomorphisms is denoted by $\mathfrak{Group}$.

> Let us consider a finite set $A = \{a_1, a_2, \ldots, a_n\}$. A permutation of A is an (ordered) sequence $\sigma = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$ with no repetitions. Let us denote by $\Pi(A)$ the set of all permutations of A.
>
> Each permutation can be denoted by
>
> $$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} \tag{12.6.1}$$
>
> The composition of two permutations is defined as
>
> $$\sigma \circ \rho = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \ldots & n \\ \rho(1) & \rho(2) & \ldots & \rho(n) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(\rho(1)) & \sigma(\rho(2)) & \ldots & \sigma(\rho(n)) \end{pmatrix} \tag{12.6.2}$$
>
> which defines an operation in the set of permutations. This defines a group $(\Pi(A), \circ)$.
>
> For example, let us consider a set $A = \{a, b, c\}$. The group $\Pi(A)$ has 6 elements, for example
>
> $$() := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad (123) := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{12.6.3}$$
>
> The operation defined in (12.6.2) is associative and the permutation () is the neutral element.
>
> One can show that
>
> $$(123) \circ (123) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} =: (132) \qquad (123) \circ (132) = (132) \circ (123) = () \tag{12.6.4}$$
>
> so that, for example, $(123)^{-1} = (132)$. Let us define
>
> $$(12) := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tag{12.6.5}$$

One can easily compute

$$(12) \circ (123) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} =: (13) \qquad (123) \circ (12) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} =: (23) \tag{12.6.6}$$

so that the group $\Pi(A)$ is not commutative.

The set $\Pi(2) = \{().(12)\}$ is a subgroup and the canonical embedding is a group homomorphism. Notice that in $\Pi(2)$ one has

$$() \circ () = (12) \circ (12) = () \qquad () \circ (12) = (12) \circ () = (12) \tag{12.6.7}$$

Accordingly, $(12)^{-1} = (12)$ and the map $\Phi : \Pi(2) \to \mathbb{Z}_2$ defined by

$$\Phi() = 1 \qquad \Phi(12) = -1 \tag{12.6.8}$$

is a group homomorphism into the group $(\mathbb{Z}_2, \cdot)$.

The group $(\mathbb{Z}_2, \cdot)$ can be also given in additive notation by setting $\mathbb{Z}_2 = \{0, 1\}$ and defining the operation $+_2$ as the sum modulo 2, which is another example of isomorphism from $\mathbb{Z}_2$ into itself.

A *left action* of a group G on a set M is a map $\lambda : G \times M \to M : (g, x) \mapsto gx$ such that

a) $\lambda(\mathbb{I}, x) = x$ $\qquad\qquad$ $(\mathbb{I}x = x)$

b) $\lambda(k, \lambda(h, x)) = \lambda(k \cdot h, x)$ $\quad$ $(k(hx) = (k \cdot h)x)$

If M carries some structure (e.g. it is a vector space, a group, a manifold, ...) then the map $\lambda(g, \cdot) : M \to M$ is required to be a morphism of the corresponding category (i.e., e.g., it is required to be a linear map, a homomorphism, a smooth map, ...). For example, a linear left action is also called a *representation* of G on the vector space $V = M$.

The automorphisms of an object M in a category $\mathfrak{A}$ form a group with composition, denoted by $\mathrm{Aut}(M)$.

Accordingly, a left action can also be considered as a map $\hat{\lambda} : G \to \mathrm{Aut}(M) : g \mapsto \lambda(g, \cdot)$ which, in fact, is a group homomorphism. Often, by an abuse of language, the maps λ, $\lambda(g, \cdot)$, and $\hat{\lambda}$ will be all denoted by λ and they will all be referred to as a left action.

The map $L : G \times G \to G : (h, g) \mapsto h \cdot g$ is called the *left translation* and it defines a canonical left action of G onto itself. The maps $L(h, \cdot) =: L_h : G \to G$ are not group homomorphisms; accordingly, the left translation is an action of the group G on the *set* G, not on the *group* G.

Another example of left action is the map $\mathrm{Ad} : G \times G \to G : (g, h) \mapsto (g \cdot h \cdot g^{-1})$, which is called the *adjoint action*. The maps $\mathrm{Ad}(g, \cdot) := \mathrm{Ad}_g : G \to G$ are group homomorphisms and the adjoint action is an action of G onto the *group* G itself.

A subgroup $H \subset G$ is *normal* iff it is invariant with respect to the adjoint action, i.e. iff for all $h \in H$ and for all $g \in G$ one has

$$\mathrm{Ad}_g(h) := g \cdot h \cdot g^{-1} \in H \tag{12.6.9}$$

Similarly, a *right action* of a group G on a set M is a map $\rho : M \times G \to M : (x, g) \mapsto xg$ such that

a) $\rho(x, \mathbb{I}) = x$ $\qquad\qquad$ $(x\mathbb{I} = x)$

b) $\rho(\rho(x, h), k) = \rho(x, h \cdot k)$ $\quad$ $((xh)k = x(h \cdot k))$

For example, there is a canonical right action $R : G \times G \to G : (h, g) \mapsto h \cdot g = R_g(h)$ of the group G onto itself which is called the *translation on the right*. As it happens for the left translation, this action does not preserve the group structure since it does not act by group homomorphisms.

If $\lambda : G \times M \to M$ is a left action, then we can define a canonical right action $\rho : M \times G \to M : (x, g) \mapsto \lambda(g^{-1}, x)$. Vice versa, if $\rho : M \times G \to M$ is a right action one can define a canonical left action by $\lambda : G \times M \to M : (g, x) \mapsto \rho(x, g^{-1})$.

An action on M is called *trivial* if $\lambda : G \to \operatorname{Aut}(M) : g \mapsto \operatorname{id}_M$. It is called *faithful* if, whenever $\lambda(g) = \operatorname{id}_M$, then $g = e$. It is called *free* if whenever $\exists x \in M : \lambda(g, x) = x$ then $g = e$. It is called *transitive* iff for any $x, y \in M$ there always exists a $g \in G$: $\lambda(g, x) = y$.

We can define an equivalence relation in M associated to any action $\lambda : G \times M \to M$ as

$$x \sim y \iff \exists g \in G : gx = y \tag{12.6.10}$$

One can easily check that the relation $\sim$ is an equivalence relation (due exactly to the property of the action). The equivalence class $[x] \subset M$ of a point $x \in M$ is called an *orbit* of the action. The associated quotient space is also denoted by M/G for left actions (and $M \backslash G$ for right actions).

If M is an object of a category with structure (e.g. a group, a manifold, ...) the quotient space is not always in the same category. Usually, extra conditions have to be enforced on the action for it to be true.

Let us remark that an action is transitive iff the quotient is made of one single orbit.

7. Rings, modules, and algebras

A *ring* is an additive group $(R, +)$ equipped with a further operation called the *product*, denoted by $\cdot : R \times R \to R$, which is associative, i.e. for all $a, b, c \in R$ one has

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{(associative)} \tag{12.7.1}$$

it has a *unit element* $1 \in R$ such that

$$1 \cdot a = a \cdot 1 = a \tag{12.7.2}$$

and it is *distributive* with respect to the sum

$$a \cdot (b + c) = a \cdot b + a \cdot c \qquad (a + b) \cdot c = a \cdot c + b \cdot c \tag{12.7.3}$$

Smooth functions on a (smooth) manifold M form a ring denoted by $\mathcal{F}(M)$, as well as $\mathcal{C}^k(M)$ denotes the ring of $\mathcal{C}^k$-functions on a $\mathcal{C}^h$-manifold M for any $k \leq h$. Polynomials on $\mathbb{R}$ also form a ring, denoted by $\mathbb{R}[x]$.

Given two rings R and R', a homomorphism $\Phi : R \to R'$ is a map which preserves neutral elements

$$\Phi(0) = 0 \qquad \Phi(1) = 1 \tag{12.7.4}$$

and the operations

$$\Phi(a+b) = \Phi(a) + \Phi(b) \qquad\qquad \Phi(a \cdot b) = \Phi(a) \cdot \Phi(b) \tag{12.7.5}$$

Notice that even though we did not distinguish between the neutral elements on R and the neutral elements on R', nor between the operations on R and the operations on R', there is only one meaning for these properties which is syntactically correct.

> The homomorphism Φ takes elements in R and maps them in elements in R'. Hence, when we write $\Phi(a+b) = \Phi(a) + \Phi(b)$, a and b are necessarily elements in R and their sum in R is denoted by $a+b$ on the left hand side. The images $\Phi(a)$ and $\Phi(b)$ are necessarily in R' and $\Phi(a) + \Phi(b)$ necessarily denotes the sum in R'.

That defines the category of rings and all standard notation applies to it. An *isomorphism* $f : R \to R'$ is a homomorphism which has an inverse which is a homomorphism, a homomorphism $\Phi : R \to R$ is an *endomorphism*, and it is an *automorphism* iff it is also an isomorphism. Also for any subring $R' \subset R$ (which is a subset which is closed by sum and product, it contains the neutral elements, so that it is a ring on its own with the restrictions of the operations of R to $R' \subset R$) one has the canonical embedding $\iota : R' \to R$ which is a monomorphism.

Also a *representation* of a ring R is a homomorphism into another ring R', usually a ring $\mathrm{End}(V)$ or a ring of matrices, which is more or less the same thing, since matrices are local expressions of endomorphisms of a vector space (or and R-module).

An *ideal* in a ring R is a subring $I \subset R$ such that $\forall x \in R, \forall i \in I : ix \in I$ (while, to be a subring, one only requires $\forall i, j \in I : ij \in I$). For example, even numbers are an ideal in $\mathbb{Z}$, denoted by $2\mathbb{Z}$ (or by (2)). Analogously, let us set $(x+3) \subset \mathbb{R}[x]$ as $(x+3) = \{p(x) \in \mathbb{R}[x] : p(-3) = 0\}$, then $(x+3)$ is called the ideal generated by the element $x + 3 \in \mathbb{R}[3]$. Notice that there is no ideal in $\mathbb{R}[x]$ (other than trivially $\mathbb{R}[x]$ itself) that contains $(x+3)$. Hence the ideal $(x+3)$ is *maximal*. Also, let us remark that for any element $a \in \mathbb{R}$ we can define a maximal ideal $(x-a) \subset \mathbb{R}[x]$.

Also for any product of polynomials $p(x)q(x)$, if $p(x)q(x) \in (x+3)$ (namely, if $x+3$ divides $p(x)q(x)$ and we write $x+3|p(x)q(x)$) then either $x+3|p(x)$ or $x+3|q(x)$ (or, equivalently, either $p(x) \in (x+3)$ or $q(x) \in (x+3)$). Accordingly, $(x+3)$ is called a *prime ideal*.

> The modern definition of a prime number is in fact that $p \in \mathbb{Z}$ is prime iff $p \neq \pm 1$ and it has the property
>
> $$p|ab \quad \Rightarrow p|a \ \lor \ p|b \tag{12.7.6}$$
>
> That has not much to do with factorizations although one can prove that, in $\mathbb{Z}$, (p) is a prime ideal iff p has no non-trivial factorization.
> The main reason why one does not define primes as non-factorizable elements is that one can define pretty weird rings in which one can have non-zero elements with a zero product, or non-unique factorizations.

Also $(x^2 - 9)$ is the ideal (of polynomials which vanish on $x = \pm 3$) which is not a maximal ideal, nor a prime ideal.

One can prove directly that a maximal ideal is also prime.

> Let I be a maximal ideal of a ring R, be $x, y \in R$ two elements such that $xy \in I$, we should prove that either x or y are in I.
> Suppose, for the sake of argument that $x \notin I$. Then we can define a new ideal $I + (x) = \{i + z : i \in I, z = ax \in (x)\}$, which strictly contains I. However, since I was assumed to be maximal, it must be $I + (x) = R$. In particular, $\mathbb{I} \in I + (x)$, hence $\mathbb{I} = i + ax$ and for $i \in I$, $a \in R$. Then we have
>
> $$y = \mathbb{I}y = iy + axy = i' + axy = i'' \in I \tag{12.7.7}$$
>
> Then I is prime since necessarily $y \in I$.

On the other hand, there are prime ideals which are not maximal.

Consider $(x^2 + 1) \subset \mathbb{R}[x]$. That is a prime ideal, although it is strictly contained into $(3, x^2 + 1) = \{p \in \mathbb{R}[x] : p = 3q_1 + (x^2 + 1)q_2 \text{ with } q_1, q_2 \in \mathbb{R}[x] \}$. That is enough to see that $(x^2 + 1)$ is not maximal since, for example, $2 \notin (3, x^2 + 1)$ so that $(x^2 + 1)$ is also *non-trivially* contained in $(3, x^2 + 1)$.

When we consider an ideal $I \in R$, we can define an equivalence relation $\sim_I$ in R as

$$x \sim_I y \iff y = x + i \text{ for some } i \in I \tag{12.7.8}$$

Accordingly, we can consider the quotient set R/I of equivalence classes $[x] = \{x + i : i \in I\}$. One can easily define a sum and product on the quotient

$$[x] + [y] = [x + y] \qquad [x][y] = [xy] \tag{12.7.9}$$

and check they do not depend on the representatives, for the product, precisely because I is an ideal. This gives R/I the structure of ring. In other words, the quotient of a ring by one of its ideals is canonically a ring.

A *corp* is a ring $\mathbb{K}$ in which all elements (except 0) are invertible with respect to the product, i.e., for any $a \in \mathbb{K}$, there exists the *inverse element* a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 \tag{12.7.10}$$

A *field* is a commutative corp, i.e. $a \cdot b = b \cdot a$.

The real numbers $\mathbb{R}$, as well as the complex numbers $\mathbb{C}$, form a field. Quaternions $\mathbb{H}$ are a corp. The integers numbers $\mathbb{Z}$ form a ring. Its quotient by the ideal $p\mathbb{Z}$ of all multiples of a number p, is denoted by $\mathbb{Z}_p$. It inherits a sum and a product (mod p) and it is a field iff p is prime.

Theorem: *an ideal is maximal iff the quotient R/I is a field.*

A (left) *module* over a ring R (or an *R-module*) is an additive group V, together with an operation $* : R \times V \to V$ which obeys the following properties

    i) $a * (v + w) = a * v + a * w$

    ii) $(a + b) * v = a * v + b * v$

    iii) $a * (b * v) = (a \cdot b) * v$

    iv) $1 * v = v$

Given an R-module V, one can associate to any $a \in R$ a map $f_a : V \to V : v \mapsto a * v$. Consequently, the map is a group homomorphism in view of property *(i)*.

Since $\text{End}(V)$ forms a ring (with composition), the properties *(ii–iv)* simply state that the map $f : R \to \text{End}(V) : a \mapsto f_a$ is a ring homomorphism.

Accordingly, one often says that an R-module is a representation of the ring R over the commutative group V. In fact, given a representation $f : R \to \text{End}(V)$ of the ring R over the commutative group V, one can define the operation $* : R \times V \to V : (a, v) \mapsto a * v = f(a)(v)$ and V becomes with it an R-module.

A ring S is always an S-module. If we consider a ring homomorphism $f : R \to S$, we can define a product $* : R \times S \to S$ where we set $r * s = f(r) \cdot s$, with which S becomes an R-module.

Given a commutative ring R, an *associative algebra* (commutative, with unity) (also called an R-*algebra*) is a ring S together with a ring homomorphi9sm $f : R \to S$.

The set of polynomials $\mathbb{R}[x]$ is a ring and one has a ring homomorphism $f : \mathbb{R} \to \mathbb{R}[x] : a \mapsto a$, which maps a real number into a constant polynomial. Accordingly, $\mathbb{R}[x]$ is an $\mathbb{R}$-algebra.

Square matrices of order n also form a (non-commutative) ring M_n. One can consider $f : \mathbb{R} \times M_n \to M_n : a \mapsto a\mathbb{I}$ which is a ring homomorphism, so that M_n is an (associative, non-commutative, unital) algebra.

Real vector spaces

If the ring $R = \mathbb{K}$ is a field, then $\mathbb{K}$-modules are also called *vector spaces* over the field $\mathbb{K}$. The vector space which has only the zero vector is denoted by $V = \{0\}$, or simply by 0, is called *trivial*.

For $\mathbb{K} = \mathbb{R}$, we get a *real vector space* V. Elements of V are called *vectors*. Given a (possibly infinite) set of vectors $S = \{e_1, e_2, \ldots\}$ a *(finite) linear combination* is a vector v obtained as

$$v = \sum_{i=1}^{n} v^i e_i \qquad v^i \in \mathbb{R}, e_i \in S \tag{12.7.11}$$

> Let us stress that at this stage infinite linear combinations cannot be given a defined meaning. Infinite linear combinations are a kind of series and one needs a topology to give meaning to them. Here in an algebraic context, one can only make *finite* linear combinations.

A map $\Phi : V \to W$ between two vector spaces is *linear* if it preserves linear combinations, i.e. if

$$\Phi \left(\sum_{i=1}^{n} v^i e_i \right) = \sum_{i=1}^{n} v^i \Phi\left(e_i\right) \tag{12.7.12}$$

Real vector spaces together with linear maps defined the category $\mathfrak{Vect}(\mathbb{R})$.

Given a subset $U \subset V$ which is closed with respect to linear combinations is a vector space on its own and it is called a *subspace*. The canonical embedding $\iota : U \to V$ is a linear map. In a vector space V there always are two subspaces, namely 0 and V itself, which are called the *trivial subspaces*.

The set of all finite linear combinations of vectors in S is a subspace, denoted by $\mathrm{Span}(S) \subset V$. Since any vector in $\mathrm{Span}(S)$ can be obtained as a (finite) linear combination of vectors in S, we say that the vectors in S are *generators* of $\mathrm{Span}(S)$.

The set of all linear maps from V to W is denoted by $\mathrm{Hom}(V, W)$, the endomoprhisms are denoted by $\mathrm{End}(V)$, automorphisms are denoted by $\mathrm{Aut}(V)$ or $\mathrm{GL}(V)$. One can easily define linear combinations of linear maps, so that $\mathrm{End}(V)$ and $\mathrm{Aut}(V)$ are vector spaces themselves.

A set of vectors S is *independent* if any finite linear combination of them is zero iff all coefficients are zero, i.e.

$$\sum_{i=1}^{n} v^i e_i = 0 \qquad \Longleftrightarrow \quad v^i = 0 \tag{12.7.13}$$

A set of independent generators is called a *basis*. If S is a basis of V then any vector $v \in V$ has a *unique* decomposition along S. If $S = (e_0, e_i)$ is a basis of V, then e_0 is not a linear combination of the other e_i. If it were, then $e_0 = \sum v^i e_i$, and $(1, -v^i)$ would be non-zero coefficients for a linear combination of the elements of the basis which sum to the zero vector. That contradicts the independence of S.

If $v, w \in V$ are vectors, there is no way, without fixing an inner product, to compute the component of v along w. Such a component depends on a subspace transverse to w; you change the subspace, the components of v along w change as well.

If $(E_1, \ldots, E_n)$ is a basis of V and $v \in V$ a vector, we can look for the components v^i of v along vectors $S = (e_1, \ldots, e_k)$, apparently without fixing an inner product. That corresponds to look for solutions of the linear system

Fig. 12.1: *components of v along w depending on the complement space.*

$$\begin{cases} v^1 e_1^1 + v^2 e_2^1 + \ldots + v^k e_k^1 = V^1 \\ v^1 e_1^2 + v^2 e_2^2 + \ldots + v^k e_k^2 = V^2 \\ \ldots \\ v^1 e_1^n + v^2 e_2^n + \ldots + v^k e_k^n = V^n \end{cases} \qquad (12.7.14)$$

where $v = \sum V^i E_i$, and $e_i = \sum e_i^j E_j$. These are n equations for k unknowns v^i.

If $k = n$ and the coefficient matrix $|e_i^j|$ is non-degenerate there is one and only one solution. Accordingly, the vectors e_i form a basis and the decomposition of v along that basis is unique. In other words, one can find the components of v along n independent vectors, not along a single vector.

If the matrix is degenerate, generically there is no solution. The vectors e_i are dependent, some of them are redundant since they can be expressed as linear combinations of the others. There are vectors v which are not in the image $\mathrm{Span}(e_i)$, hence they are not generators.. When $v \in \mathrm{Span}(e_i)$, the system has infinite solutions, so the vectors e_i are not independent.

If $k < n$, there are less unknowns than equations. One can always find vectors which are not generated by e_i. The system S cannot generate the whole vector space V.

If $k > n$, there are more unknowns than equations, then solutions cannot be unique. The vectors e_i cannot be independent.

Accordingly, if there is a basis of n elements, then any basis has n elements as well.

A basis S has also a sort of *universal property*; if we consider a map $f : S \to W$ which associates a vector $w_i = f(e_i) \in W$ to any element of the basis $S \subset V$ then there exists a unique linear map $\hat{f} : V \to W$ such that

$$(12.7.15)$$

The map $\hat{f}$ so defined is called the *linear extension* of the map f. That means that if we defined a map on the elements of a basis S, then that map uniquely extends by linearity to the whole space V.

Theorem: *Any (non-trivial) vector space allows a basis.*

Proof: *In the finite dimensional case, one can prove the theorem directly. However, there is a general proof, which works on any vector space, and it uses the choice axiom.*

Consider subsets $S \subset V$ of independent vectors, partially ordered by inclusion. A subset made of a single non-zero vector is independent, so such subsets exist.

Then consider a totally ordered sequence S_α ($S_0 \subset S_1 \subset \ldots \subset S_i \subset \ldots$) of independent subsets. In view of being totally ordered, $S = \cup_\alpha S_\alpha$ is an independent subset of V as well. By construction, S is an upper bound of the totally ordered chain S_α. By Zorn lemma, the set of independent subsets has then a maximal element Σ.

Finally, we need to prove that Σ is a basis for V. It is independent, by construction. If Σ were not a generator system, then it would exist a vector $w \in V$ which is not spanned by vectors in Σ. Then $\Sigma \cup \{w\}$ would be independent, and Σ would not be maximal.

Then Σ does generate V and it is a basis. ∎

The proof relies on the choice axiom (through the Zorn lemma) and it is not constructive at all. For example, it shows that $\mathbb{R}$, which is a vector space over $\mathbb{Q}$, has a basis, i.e. one can find a set S of real numbers such that *any* real number can be uniquely expressed as a rational linear combination of numbers in $S \subset \mathbb{R}$. However, nobody has a characterisation of such a basis S, other than the proof of existence.

A vector space is called *finite dimensional* if it allows a finite basis. If a vector space V is finite dimensional, S is a finite basis and S' is another basis, then S' has the same number of vectors as S.

> As we already discussed, if S' had less elements than S it cannot generate the whole space. If S' had more elements than S, it cannot be independent.
> Since any other basis needs to be independent generators, it can only have n elements.

Thus we say that V has *dimension n* iff one of (hence all) its bases are made of n vectors. The trivial space 0 is finite-dimensional and it has dimension 0, by convention.

Operations and Subspaces

Given two subspaces $W_i \subset V$, we shall define the *intersection* as the subspace

$$W_1 \cap W_2 = \{w : w \in W_1 \text{ and } w \in W_2\} \tag{12.7.16}$$

i.e. the set intersection. However, if W_i are subspaces, then the intersection $W_1 \cap W_2$ is also a subspace. As such, it always contains the zero vector. Hence when the two subspaces intersection is trivial, namely when $W_1 \cap W_2 = \{0\}$, the two subspaces are said to be *disjoint*, by an abuse of notation.

Let us also define the sum $V_1 \oplus V_2$ of two vector spaces which is the vector space of pairs $(v, w) \in V_1 \oplus V_2$, on which a canonical sum and multiplication by a scalar are defined as

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \qquad \lambda(v_1, w_1) = (\lambda v_1, \lambda w_1) \tag{12.7.17}$$

By these definitions, $V_1 \oplus V_2$ is a vector space which is called the *direct sum*. We have two embeddings $j_i : V_i \to V_1 \oplus V_2$. For example, $j_1 : V_1 \to V_1 \oplus V_2 : v \mapsto (v, 0)$ and $j_2 : V_2 \to V_1 \oplus V_2 : v \mapsto (0, v)$. Thus both the original vector spaces can be regarded as two disjoint subspaces of their direct sum.

The universal property of direct sum holds true and it says that if we fix $f_i : V_i \to U$, then there exists a unique linear map $\hat{f} : V_1 \oplus V_2 \to U$ such that

$$
\begin{array}{ccc}
 & & U \\
 & \nearrow^{f_i} & \uparrow \hat{f} \\
V_i & \xrightarrow{\quad j_i \quad} & V_1 \oplus V_2
\end{array}
\tag{12.7.18}
$$

Again, universal property says that if we fix the maps $f_i : V_i \to U$ on V_i then there is a unique extension to the direct sum.

Later on we shall define other operations on vector spaces, namely, the dual $V^* := \mathrm{Hom}(V, \mathbb{R})$, the tensor product $V_1 \otimes V_2 := \mathrm{Hom}(V_1^*, V_2)$.

If we consider a vector space V and two subspaces $W_i \subset V$ then we can define the sum

$$
W_1 + W_2 = \{w = w_1 + w_2 : w_i \in W_i\} \subset V
\tag{12.7.19}
$$

The sum is naturally a subspace of V as well.

If $W_1 \cap W_2 = \{0\}$, the sum of two subspaces is called the direct sum, and it is denoted by $W_1 \oplus W_2$.

> Notice that the direct sum of two vector spaces is the direct sum of them regarded as disjoint subspaces of their direct sum and it coincides with it.

Given a subspace $W \subset V$, one can consider a basis e_i of it. Then we can iteratively complete the basis to a basis of V. In fact, either $W = V$ (so there is nothing to complete) or one can find a vector $\epsilon_1 \in V$ which is not in W. Then (e_i, ϵ_1) are independent. Thus either (e_i, ϵ_1) spans the whole V, or one can find ϵ_2 out of the span of that and $(e_i, \epsilon_1, \epsilon_2)$ are independent.

By iteration one can go on until a full basis of (e_i, ϵ_A) is obtained. In these bases, the first elements are a basis of the subspace, while the rest of the basis spans a complement of it. Thus these bases are said to be *adapted* to the subspace.

If we consider the intersection $W_1 \cap W_2$ we can adapt a bases to it, complete it to W_1, then complete it to $W_1 + W_2$ and finally to V. Analogously, if we consider $W_1 + W_2$, we can consider e_i a basis of W_1, f_i a basis of W_2. The system (e_i, f_i) generates $W_1 + W_2$, though they can be dependent. In fact, they are independent iff $W_1 \cap W_2 = 0$ and the sum is direct. Once we have a basis of $W_1 + W_2$, we can complete it to have a basis of V adapted to the sum.

Inner products

For us, an inner product is a bilinear, non-degenerate, symmetric, form η of signature (r, s).

For any inner product, one can define *orthonormal bases* which are proven by iteration. At each step, one can find a vector $e \in V$ such that $\eta(e, e)$ is not zero (otherwise η would be null, hence degenerate) and, without loss of generality, v can be normalised to be $\eta(e, e) = \pm 1$.

Then we define the orthogonal complement $V_1 = \{v \in V : \eta(v, e) = 0\}$ and restrict η to V_1. One can easily show that such a restriction is again an inner product η_1 on V_1 and repeat the process each time obtaining a space V_i of dimension one lower. If the initial space V is finite dimensional, after a finite number of steps one obtains a basis of V.

In an orthonormal basis, the inner product is expressed as

$$\eta(u,v) = \sum v^a u^b \eta_{ab} \qquad \eta_{ab} = \mathrm{diag}(\underbrace{-1,\dots,-1}_{s \text{ times}},\underbrace{1,\dots,1}_{r \text{ times}}) \tag{12.7.20}$$

Also the other way around, if we consider any basis e_a, we can define an inner product η of signature (r,s) by using (12.7.20).

If we have two vector spaces, each with their inner product, (V,η) and (W,η'), for any linear map $\Phi : V \to W$, we can define the *transpose map* $\Phi^\dagger : W \to V$ given by

$$\eta(v, \Phi^\dagger(w)) = \eta'(\Phi(v), w) \tag{12.7.21}$$

> Let us stress that, often, it is clear which inner product one needs to use in view of the arguments it has. Thus often, by an abuse of language, we use the same name for different inner products on different vector spaces. For example, above we could have used (W,η) instead of (W,η'), adding little confusion.

A linear map $\Phi : V \to V$ is called an *orthogonal transformation* iff

$$\eta(\Phi(v), \Phi(u)) = \eta(v,u) \tag{12.7.22}$$

i.e. iff it preserves inner products.

> The vector spaces with an inner product of a given signature (r,s) are denoted by (V,η).
> More generally, if we consider two vector spaces (V,η) and (V',η') with inner products (of the same signature) a map $\Phi : (V,\eta) \to (V',\eta')$ is called *orthogonal* (or an *isometry*) iff
> $$\eta'(\Phi(v), \Phi(u)) = \eta(v,u) \tag{12.7.23}$$
> Vector spaces with an inner product of a given signature (r,s) together with isometries form a category.
> From now on, by an abuse of notation, we denote by η the inner product, the signature (r,s) as well as the canonical form of the inner product η_{ab}.

Orthogonal transformations map orthonormal bases into orthonormal bases. Since during normalisation of vectors one cannot change the sign of the inner product $\eta(e,e)$, if $\Phi : (V,\eta) \to (V',\eta')$ is an isometry, two orthonormal bases e_i and e_i' share the same signature and the orthogonal transformation maps positive (negative, or zero) vectors into positive (negativee, or zero) vectors.

> Notice that in the category of vector spaces (V,η) with an inner product, sometimes we can do things that cannot be done in $\mathfrak{Vect}(\mathbb{R})$. For example, given two vectors v and w (with $\eta(w,w) \neq 0$) we can define the *component* of v along w as
> $$\alpha = \frac{\eta(v,w)}{\eta(w,w)} \tag{12.7.24}$$
> while, as we discussed above, there is no way of doing something like that with no inner product. This is because, using the inner product, we can select a single complement $V_w := \{v \in V : \eta(v,w) = 0\}$, called the *orthogonal complement*, to the space generated by w.

Let us also list here a number of abuses of notations which are usual when dealing with vector spaces with an inner product. The inner product of a vector with itself $\eta(v,v)$ is often called the *norm* or (the *norm squared*) and it is often denoted by $|v|^2$, even if it can be negative.

A vector of positive norm is often called a *spacelike* vector, a vector of zero norm is called *lightlike*, a vector of negative norm is often called *timelike*. Let us remark that a lightlike vector is orthogonal to itself, even when it is different from zero.

Let us also remark that one cannot define angles as usual, since, in general, the quantity

$$\frac{\eta(v, w)}{\sqrt{|\eta(v, v)|}\sqrt{|\eta(w, w)|}} \tag{12.7.25}$$

is not guaranteed to be in $(-1, 1)$ so that it cannot be generally set to $\cos(\alpha)$. Also the radicands are not guaranteed to be positive, so one usually consider their absolute values. Anyway, one can define only angles between positive (or spacelike) vectors.

Complex vector spaces

Under many viewpoints, the theory of vector spaces on a field $\mathbb{K}$ is analogous to the real case. All theorems about linear maps, bases, subspaces are unchanged on any vector space. The only difference about that is when one uses a finite field (e.g. $\mathbb{Z}_p$). In that case the situation is odd since a finite dimensional vector space contains a finite number of vectors. Other differences (which will not be discussed here) are introduced allowing infinite dimensional vector spaces, where a reasonable theory for bases needs a topology to be provided.

However, also in the finite-dimensional complex case some differences are introduced by the fact that one has an involution $^{\dagger} : \mathbb{C} \to \mathbb{C} : z \mapsto z^{\dagger}$ on the field given by the complex conjugation. This seems a small thing, though we shall see that it will grow to impact quite a lot of things in a not completely trivial way.

Let us consider a complex vector space V with a finite complex basis e_a. The vectors of the basis are independent over complex coefficients, i.e. any complex linear combination of them is zero iff the coefficients are all zero.

However, $\mathbb{C}$ itself is a real vector space of dimension 2; a basis of $\mathbb{C}$ over $\mathbb{R}$ is $(1, i)$ and any complex number can be written as $x + iy$. Accordingly, e_i and ie_i are dependent over complex numbers, independent over $\mathbb{R}$. Hence, if e_i is a complex basis, (e_i, ie_i) is a real basis. In other words, a complex vector space V of dimension n is also a real vector space of dimension $2n$, which is called the *underlying real vector space* and it is denoted by $V_{\mathbb{R}}$.

Also, if we start from a real vector space V of dimension n, with a basis e_i, we can define $V \otimes \mathbb{C}$ the space of complex linear combinations of e_i. That is a complex vector space of dimension n called the *complexification of V*.

> One could wish to go the other way around, i.e. starting from a complex vector space V of dimension n with a basis e_i and define a real vector space of dimension n (not $2n$ as for the underlying real vector space) considering real linear combinations of e_i.
> Unfortunately, this vector space depends on the basis, not only on the original vector space, as it happens in the previous cases.

Already when we define maps, one has two options: linear and anti-linear maps, namely

$$\Phi(\lambda v + \mu u) = \lambda \Phi(v) + \mu \Phi(u) \qquad\qquad \hat{\Phi}(\lambda v + \mu u) = \lambda^{\dagger} \hat{\Phi}(v) + \mu^{\dagger} \hat{\Phi}(u) \tag{12.7.26}$$

> That makes, for example, isomorphisms more complicated and richer. In real vector spaces, we have isomorphisms which can be canonical or not. Usually if two spaces are canonically isomorphic we identify their elements. That is why for example we identify polynomials with polynomial functions, convergent power series with

analytic functions. On the other hand, if two spaces are isomorphic though in a non canonical way we keep elements distinct. For example, since the matrix associated to linear map depends on the basis we do not identify matrices and linear maps.

Given two vector spaces V and W, we write $V \simeq W$ iff the are canonically isomorphic to each other. We write $V \approx W$ iff they are canonically anti-isomorphic and $V \sim W$ iff they are non-canonically isomorphic.

When it comes to define an inner product on V, one has 3 options: a bi-linear form, a sequi-linear (linear in the second argument, anti-linear in the first), or a bi-anti-linear form. In all cases, we ask the inner product to be non-degenerate. In the first and third case, we can ask it to be *symmetric*, in the second, we need to ask to be *hermitian*, i.e.

$$\delta(v,w) = \delta(w,v) \qquad\qquad \eta(v,w) = \eta(w,v)^{\dagger} \qquad\qquad \bar{\delta}(v,w) = \bar{\delta}(w,v) \tag{12.7.27}$$

When looking for an orthonormal basis for δ or $\bar{\delta}$, whaterver vector v can be normalized so that $\delta(v,v) = 1$ or $\bar{\delta}(v,v) = 1$. Hence the inner product in an orthonormal basis e_a is given by

$$\delta(v,w) = \sum v^a w^b \delta_{ab} \qquad\qquad \bar{\delta}(v,w) = \sum (v^a)^{\dagger}(w^b)^{\dagger}\delta_{ab} \tag{12.7.28}$$

In other words, for symmetric inner products there is no signature to be taken into account.

Instead, if we have a hermitian inner product η and a vector $v \in V$, we have

$$\eta(v,v) = \eta(v,v)^{\dagger} \tag{12.7.29}$$

i.e. $|v|^2 := \eta(v,v)$ is real. Thus if $\eta(v,v) < 0$ is negative, there is no way to make it positive. In other words, signatures are back for sesqui-linear inner products.

For traditional reasons, a *hermitian inner product* is a sesqui-linear, non-degenerate, definite positive, hermitian tensor $\eta : V \times V \to \mathbb{C}$. It is given by

$$\eta(v,w) = \sum (v^a)^{\dagger} w^b \delta_{ab} \tag{12.7.30}$$

while, more generally, a *pseudo-hermitian inner product* is

$$\eta(v,w) = \sum (v^a)^{\dagger} w^b \eta_{ab} \tag{12.7.31}$$

If we have two vector spaces, each with a hermitian inner product, (V,η) and (W,η), for any linear map $\Phi : V \to W$, we can define the *adjoint map* $\Phi^{\dagger} : W \to V$ given by

$$\eta(v, \Phi^{\dagger}(w)) = \eta(\Phi(v), w) \tag{12.7.32}$$

A linear automorphism $\Phi : V \to V$ is called *unitary* iff

$$\eta(\Phi(v), \Phi(w)) = \eta(v,w) \tag{12.7.33}$$

The group of unitary transformations is denoted by $U(V,\eta) \subset \mathrm{Aut}(V)$.

8. Exact sequences

A *sequence* is a (finite or infinite) chain of vector spaces and linear maps (or groups and homomorphisms, or objects in a category with kernels and images and its morphisms). By an abuse of language, we denote by 0 the vector space (or additive group) made of the only zero element. The map $0 : 0 \to V$ is the zero map which sends the zero element $\vec{0} \in 0$ into the zero element $\vec{0} \in V_1$ and it is usually understood.

$$0 \longrightarrow V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{k-2}} V_{k-1} \xrightarrow{\phi_{k-1}} V_k \xrightarrow{\phi_k} \cdots \tag{12.8.1}$$

Notice also that there exists only one map $0 : V \to 0$ which sends all vectors in V in the zero vector $\vec{0} \in 0$. By an abuse of language, it is also denoted by $0 : V \to 0$ and it is usually understood as well. Accordingly, it is customary to simply write

$$\ldots \to V \to W \to 0 \tag{12.8.2}$$

when the sequence truncates at a point.

A sequence is said to be *exact at a point* V_k iff $\mathrm{Im}(\phi_{k-1}) = \ker(\phi_k)$. A sequence is called an *exact sequence* if it is exact at all points.

In the beginning of a sequence

$$0 \longrightarrow U \xrightarrow{\phi} V \longrightarrow \cdots \tag{12.8.3}$$

it is exact at the point U iff the map ϕ is injective.

The image of the map $0 : 0 \to U$ is trivially $\mathrm{Im}(0) = \{\vec{0}\} \subset U$. Then exactness at U is equivalent to $\ker(\phi) = \{\vec{0}\}$, which is equivalent to require that ϕ is injective.

In the end of a sequence

$$\cdots \longrightarrow V \xrightarrow{\phi} W \longrightarrow 0 \tag{12.8.4}$$

it is exact at the point W iff the map ϕ is surjective.

The kernel of the map $0 : W \to 0$ is trivially $\ker(0) = W$. Then exactness at W is equivalent to $\mathrm{Im}(\phi) = W$, which is equivalent to require that ϕ is surjective.

An exact sequence of only two objects is called a *very short exact sequence*.

$$0 \longrightarrow V \xrightarrow{\phi} W \longrightarrow 0 \tag{12.8.5}$$

In view of the two comments above, exactness of this sequence is equivalent to require that the map ϕ is bijective. Writing a very short exact sequence is thence a way to state that the morphism ϕ is an isomorphism.

An exact sequence of only three objects is called a *short exact sequence*.

$$0 \longrightarrow U \xrightarrow{\phi} V \xrightarrow{\pi} W \longrightarrow 0 \tag{12.8.6}$$

One can also define a *long exact sequence* to be an exact sequence of four objects.

$$0 \longrightarrow V_1 \xrightarrow{\ \phi\ } V_2 \xrightarrow{\ \psi\ } V_3 \xrightarrow{\ \pi\ } V_4 \longrightarrow 0 \tag{12.8.7}$$

We shall not get involved into long exact sequences here.

Exactness of a short sequence amounts to require that ϕ is injective, π is surjective and $\mathrm{Im}(\phi) = \ker(\pi)$. It is impressive how much information is encoded in exactness and how much of it can be dug out also without knowing any detail about the objects and maps involved.

Let us first define a *splitting* of a short exact sequence to be a map $\alpha : W \to V$ such that $\pi \circ \alpha = \mathrm{id}_W$ We shall represent a splitting by the diagram

$$0 \longrightarrow U \xrightarrow{\ \phi\ } V \xrightarrow{\ \pi\ } W \longrightarrow 0 \tag{12.8.8}$$

Let us stress that this is not a commutative diagram. We are not assuming or requiring that $\alpha \circ \pi = \mathrm{id}_V$. If we did then $\alpha = \pi^{-1}$ and π would be bijective. Then, in view of exactness, $U = 0$ and the sequence would be very short.

Let us first show that given a splitting $\alpha : W \to V$, there exists a unique map $\beta : V \to U$ such that

$$\forall v \in V : \ v = \phi \circ \beta(v) + \alpha \circ \pi(v) \tag{12.8.9}$$

The map β has the property $\beta \circ \phi = \mathrm{id}_U$.

$$0 \longrightarrow U \xrightarrow{\ \phi\ } V \xrightarrow{\ \pi\ } W \longrightarrow 0 \tag{12.8.10}$$

For any $v \in V$, we want to first to define an element $\beta(v) =: u \in U$. Let us consider $\alpha \circ \pi(v) \in V$ and the element $v^\perp := v - \alpha \circ \pi(v) \in V$.
One can easily show that $\pi(v^\perp) = \pi(v) - \pi \circ \alpha \circ \pi(v) = \pi(v) - \pi(v) = 0$, i.e. that $v^\perp \in \ker(\pi) = \mathrm{Im}(\phi)$; then there exists an element $u \in U$ such that $\phi(u) = v^\perp$.
However, the map ϕ is injective, thus u is also the unique element in U such that $\phi(u) = v^\perp$.
Then we can define $\beta : V \to U : v \mapsto u$, where u is the only element such that $\phi(u) = v - \alpha \circ \pi(v)$, or, equivalently, $v = \phi \circ \beta(v) + \alpha \circ \pi(v)$.
On the other hand, the map β is unique. For, let us suppose that there is another map β' with the same property (12.8.9). Then one has $\phi \circ \beta'(v) + \alpha \circ \pi(v) = \phi \circ \beta(v) + \alpha \circ \pi(v)$ and $\phi(\beta(v) - \beta'(v)) = 0$. Then $\beta(v) - \beta'(v) \in \ker(\phi) = \{0\}$, and consequently $\beta' = \beta$.
Now that we showed that a map $\beta : V \to U$ exists (and it is unique), we can check that for any $v = \phi(u)$ one has

$$\beta \circ \phi(u) = u \tag{12.8.11}$$

In fact, one has $v^\perp = \phi(u) - \alpha \circ \pi(\phi(u)) = \phi(u) = v$ (notice that $\pi \circ \phi(u) = 0$, by exactness) and its (unique, since ϕ is injective) ϕ-preimage is u. Then $\beta \circ \phi(u) = u$, as claimed.

If one assumes a short exact sequence with a map $\beta : V \to U$ such that $\beta \circ \phi = \mathrm{id}_U$, then the sequence splits, i.e. one can define the split map $\alpha : W \to V$.

Let us consider $w \in W$ and we wish to identify an element $\alpha(w) \in V$ such that $\pi \circ \alpha(w) = w$. Let us first consider $v \in V$, which is a π-preimage of w, i.e. such that $\pi(v) = w$. The element v exists since π is surjective, though of course it is not unique. The elements in the form $\tilde{v} = v + \bar{v}$ with $\bar{v} \in \ker(\pi)$ are the ones and the only ones such that $\pi(\tilde{v}) = w$. These are the elements among which $\alpha(w)$ has to be searched.

Then we can define $v' = \phi \circ \beta(v) \in V$ and check that $\pi(v') = \pi \circ \phi \circ \beta(v) = 0$. Then $v' \in \ker(\pi)$ and one can define

$$\alpha : W \to V : w \mapsto v - v' \tag{12.8.12}$$

Let us suppose that in the beginning we chose another π-preimage of w, namely $\tilde{v} = v + \bar{v}$ with $\bar{v} \in \ker(\pi) = \text{Im}(\phi)$. Then we would set $\tilde{v}' = \phi \circ \beta(\tilde{v}) \in V$ which is also in $\ker(\pi)$. One can easily check that

$$\tilde{v} - \tilde{v}' = (v + \bar{v}) - \phi \circ \beta(\tilde{v}) = v + \bar{v} - \phi \circ \beta(v + \bar{v}) = v + \bar{v} - \phi \circ \beta(v) - \phi \circ \beta(\bar{v}) = v + \bar{v} - v' - \bar{v} = v - v' \tag{12.8.13}$$

where we used the fact that $\bar{v} \in \text{Im}(\phi)$, hence there exists a $u \in U$ such that $\bar{v} = \phi(u)$ and $\phi \circ \beta(\bar{v}) = \phi \circ \beta \circ \phi(u) = \phi(u) = \bar{v}$. Hence it does not really matter which preimage one initially selects, in the end the same element $v - v' \in V$ is finally picked up. Then we can set $\alpha(w) := v - v'$ and check that

$$\pi \circ \alpha(w) := \pi(v) - \pi(v') = w \qquad \Rightarrow \pi \circ \alpha = \text{id}_W \tag{12.8.14}$$

Thus the map α is a splitting of the sequence.

Then, as a matter of fact, one could define a splitting of a short sequence either by the map α or the map β.

We can also prove the following:

Theorem (12.8.15): *If a short exact sequence splits then $V \simeq \phi(U) \oplus \alpha(W)$.*

Proof: *Let us consider an element $v \in V$, set $u := \beta(v) \in U$ and $w := \pi(v) \in W$ and define $v^\parallel := \phi(u) = \phi \circ \beta(v)$ and $v^\perp := \alpha(w) = \alpha \circ \pi(v)$.*

We have first to show that any element $v \in V$ can be split as $V = \phi(U) + \alpha(W)$. For,

$$v^\parallel + v^\perp = \phi \circ \beta(v) + \alpha \circ \pi(v) = v \tag{12.8.16}$$

where we used the property ([12.8.9](#)) of the map β.

Then we have to show that the sum is direct. Let us suppose that $v = \alpha(w) = \phi(u)$ and apply π on the left to obtain

$$\pi(v) = \pi \circ \alpha(w) = \pi \circ \phi(u) = 0 \quad \Rightarrow \pi \circ \alpha(w) = w = 0 \tag{12.8.17}$$

If one applies β on the left, one has

$$\beta(v) = \beta \circ \alpha(w) = 0 = u \quad \Rightarrow u = 0 \tag{12.8.18}$$

Since one has $v = v^\parallel + v^\perp = 0$, then v is necessarily zero and the sum is direct. ∎

A short exact sequence splits iff one can find a complement $\tilde{W} \subset V$ to the subspace $\phi(U) \subset V$. Since one can *always* find complements to linear subspaces, then short exact sequences of vector spaces always splits.

Let us stress that given a subgroup $H \subset G$ one in general cannot find a complement subgroup $K \subset G$ such that $G = H \oplus K$. Thus short exact sequences of groups do not in general split.

Given a short exact sequence that splits we can consider the *split* sequence

$$0 \xleftarrow{} U \xleftarrow[\beta]{} V \xleftarrow[\alpha]{} W \xleftarrow{} 0 \qquad (12.8.19)$$

One can show that also the split sequence is exact.

First of all, let us show that $\beta \circ \alpha = 0$. Let us consider $w \in W$ and set $v := \alpha(w)$ and $u = \beta(v)$. Then one has

$$\phi(u) = v - \alpha \circ \pi(v) = \alpha(w) - \alpha \circ \pi \circ \alpha(w) = \alpha(w) - \alpha(w) = 0 \quad \Rightarrow u \in \ker(\phi) = \{0\} \quad \Rightarrow u = 0 \qquad (12.8.20)$$

Then we have $\beta \circ \alpha = 0$.

The map α has to be injective. We know that $\pi \circ \alpha = \mathrm{id}_W$. If one supposes that α is not injective, then the composition on the left with any map cannot be injective. Hence it cannot be $\pi \circ \alpha = \mathrm{id}_W$, since id_W is injective, which is a contradiction.

The map β has to be surjective. We know that $\beta \circ \phi = \mathrm{id}_U$. If one supposes that β is not surjective, then the composition on the right with any map cannot be surjective. Then it cannot be $\beta \circ \phi = \mathrm{id}_U$, since id_U is surjective, which is a contradiction.

Finally, one has to have $\ker(\beta) = \mathrm{Im}(\alpha)$. We already showed that $\mathrm{Im}(\alpha) \subset \ker(\beta)$ which is a consequence of $\beta \circ \alpha = 0$. Then we just have to show that $\ker(\beta) \subset \mathrm{Im}(\alpha)$. Let us consider $v \in \ker(\beta)$, $\beta(v) = 0$. However, $v = \alpha \circ \pi(v) + \phi \circ \beta(v) = v = \alpha \circ \pi(v)$. Then v is the α-image of the element $\pi(v) = w$.

Moreover, the split sequence does, in fact, split by the split map $\phi : U \to W$, since one has $\beta \circ \phi = \mathrm{id}_U$.

The theory of exact sequences (as well as category theory itself) is sometimes called *abstract nonsense*. In some sense, also topology can be called the same. These areas can be viewed as a collection of definitions and lemmas each of which relatively trivial to prove, once one is sufficiently familiar with the context, although sometimes the proof can be quite long. On the other hand they are so general in applications that one usually prove the lemmas *before* specifying to what context it will eventually be applied. We did not specified what the vector spaces contains, whether they are arrows in space or continuous functions. It does not matter since we are discussing about general properties which are valid for all sequences of vector spaces, *whatever*.

In other words, they are *nonsense* because we do not select a particular meaning to the objects, they are not *abstract* because it does not really matter. Still, these lemmas can be used in different contexts to prove concrete things (as, homology theorems which are used in classification efforts in different areas of mathematics). Often one can create a context in order to prove a theorem, as it sometimes happens in topology where one *defines* odd (unnatural) topologies on relatively familiar sets in order to prove properties.

Going abstract is not new in mathematics, it is often a reason for people to try to keep stuck to *real intuition* as particles in QFT (sarcastic mode: off) but it is also the reason why mathematics is so powerful. Mathematics talks about relations between things, not things. It is natural born abstract. Having some intuition about things is always a good thing but sometimes it goes in the opposite direction, the best thing would be to be able to work both with intuition and abstract.

It is also understood that it is not really worth studying all the proofs, since they can be reproduced when needed. If you want to practice further with diagram chasing and exact sequences watch this.

9. Lie algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space with an additional operation $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the *Lie bracket* (or *commutator*), with the additional properties

(a) $[\xi,\zeta] = -[\zeta,\xi]$ (antisymmetry)

(b) $[\lambda\xi + \mu\eta, \zeta] = \lambda[\xi,\zeta] + \mu[\eta,\zeta]$ (bilinearity)

and, of course, one also has $[\zeta, \lambda\xi + \mu\eta] = \lambda[\zeta,\xi] + \mu[\zeta,\eta]$

(c) $[\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]] = 0$ (Jacobi identity)

We shall see below some examples of Lie algebras; vector fields on a manifold form a Lie algebra. Observables in Hamiltonian mechanics with Poisson brackets form a Lie algebra. Matrices with the commutator $[A.B] := AB - BA$ form a Lie algebra as well. Vectors of $\mathbb{R}^3$ with the vector product form a Lie algebra.

Any (non-commutative) associative algebra A is also a Lie algebra by defining the Lie bracket as

$$[a,b] = ab - ba \qquad (12.9.1)$$

> Of course, we can do it also when A is commutative, although in that case the Lie bracket is always zero, which is pretty trivial, yet a Lie algebra.

A special Lie algebra is associated to all Lie groups, namely left invariant vector fields on a Lie group G form a sub Lie algebra of all vector fields, though finite dimensional, which is denoted by $\mathfrak{g}$.

A *Lie algebra representation* on a vector space V is a map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ which is a Lie algebra homomorphism. On $\mathrm{End}(V)$, we fix a canonical Lie algebra structure defined by commutators of linear maps.

A subspace $U \subset V$ is called *isotropic* iff $\forall \xi \in \mathfrak{g}, \forall u \in U : \rho(\xi)(u) = u$. It is called *invariant* iff $\forall \xi \in \mathfrak{g}, \forall u \in U : \rho(\xi)(u) \in U$. Clearly, an isotropic subspace is invariant, not necessarily the other way around.

A representation is called *irreducible* iff it admits only $\{0\}$ and V as invariant subspaces.

If we consider a map $\varphi : V \to W$ between two representations $\rho_1 : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho_2 : \mathfrak{g} \to \mathrm{End}(W)$ such that $\forall \xi \in \mathfrak{g} : \rho_2(\xi)(\varphi(v)) = \varphi(\rho_1(\xi)(v))$, the subspaces $\ker(\varphi) \subset V$ and $\mathrm{Im}(\varphi) \subset W$ are invariant subspaces.

In fact, we have

$$v \in \ker(\varphi) \iff \varphi(v) = 0 \iff \rho_1(\xi)(v) \in \ker(\varphi) \qquad (12.9.2)$$

and

$$w \in \mathrm{Im}(\varphi) \iff \exists v : \varphi(v) = w \iff w' := \rho_2(\xi)(w) = \varphi(\rho_1(\xi)(v)) \iff \exists v' = \rho_1(\xi)(v) : w' = \varphi(v') \iff w' = \rho_2(\xi)(w) \in \mathrm{Im}(\varphi) \qquad (12.9.3)$$

Accordingly, if ρ_1 and ρ_2 are irreducible, then either $\varphi = 0$ ($\mathrm{Im}(\varphi) = \{0\}$, hence $\ker(\varphi) = V$) or it is an isomorphism ($\ker(\varphi) = 0$ and $\mathrm{Im}(\varphi) = W$).

If $V = W$ and $\varphi : V \to V$ is irreducible, then $\varphi = \alpha\mathbb{I}$. In fact, $\varphi - \alpha\mathbb{I}$ is degenerate, its image cannot be V, it cannot be an isomorphism, therefore it has to be 0.   ∎

This is completely analogous to the proof of the Schur lemma for group representations, of which it is essentially the infinitesimal version.

Let us remark there is a complete classification of (semisimple) Lie algebras and their representations.

Universal enveloping algebra and Casimirs

A Lie algebra $\mathfrak{g}$, in particular, is a vector space. Hence we can define its contravariant tensor algebra $T_0^*(\mathfrak{g})$ containing all finite linear combinations of finite tensor products of elements of $\mathfrak{g}$.

Tensors $t \in T_0^p(\mathfrak{g})$ are defined as maps $t : \mathfrak{g}^p \to \mathbb{K}$. Then we have

$$T_0^*(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots \tag{12.9.4}$$

Consider the bilateral ideal I generated in $T_0^p(\mathfrak{g})$ by the elements

$$\xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta] \tag{12.9.5}$$

and consider the quotient $U(\mathfrak{g}) := T_0^p(\mathfrak{g})/I$, which is called the *universal enveloping algebra* of $\mathfrak{g}$.

The universal enveloping algebra $U(\mathfrak{g})$ is a a free associative algebra in which the Lie algebra structure is realized by the commutators $[t, r] := t \otimes r - r \otimes t$ which are compatible with the associative structure.

For an element $C \in U(\mathfrak{g})$ that commutes with all elements in $\mathfrak{g} \subset U(\mathfrak{g})$, and any representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ extends to $U(\rho) : U(\mathfrak{g}) \to \mathrm{End}(V)$, and we can define a map

$$\varphi_C : V \to V : v \mapsto U(\rho)(C)(v) =: C \cdot v \tag{12.9.6}$$

That is a map between representations since

$$C \cdot (\rho(\xi)(v)) = \rho(\xi)(C \cdot v) \iff [C, \xi](v) = 0 \tag{12.9.7}$$

Hence, by Schur lemma, $\varphi_C(v) = C \cdot v = \alpha v$.

Representations of $\mathfrak{sl}(2, \mathbb{C})$

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is made of traceless 2×2 complex matrices, which in fact form a complex three dimensional algebra. One can easily check that the commutator of two matrices in $\mathfrak{sl}(2, \mathbb{C})$ is still in $\mathfrak{sl}(2, \mathbb{C})$.

Of course, a linear combinations of traceless matrices is still traceless. Thus they form a vector space.

If we trace the commutator, any commutator not only of matrices in $\mathfrak{sl}(2, \mathbb{C})$, it is traceless by the cyclic property of the trace $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

A (complex) basis of such matrices are Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12.9.8}$$

and $S \in \mathfrak{sl}(2, \mathbb{C})$ iff it is a complex linear combination of $\tau_i = -\frac{i}{2}\sigma_i$, i.e. $S = \zeta^i \tau_i$, or a linear combination of $u_i = -i\sigma_i$, i.e. $S = z^i u_i$.

For future reference, let us remark that the products of Pauli matrices are

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ij}{}^k\sigma_k \qquad\qquad \Rightarrow \quad \tau_i\tau_j = -\tfrac{1}{4}\delta_{ij}\mathbb{I} + \tfrac{1}{2}\epsilon_{ij}{}^k\tau_k \qquad\qquad (u_iu_j = -\delta_{ij}\mathbb{I} + \epsilon_{ij}{}^k u_k) \qquad\qquad (12.9.9)$$

which are equivalent to the pair of conditions

$$\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbb{I} \qquad [\sigma_i,\sigma_j] = 2i\epsilon_{ij}{}^k\sigma_k \qquad \Rightarrow \quad \{\tau_i,\tau_j\} = -\tfrac{1}{2}\delta_{ij}\mathbb{I} \qquad [\tau_i,\tau_j] = \epsilon_{ij}{}^k\tau_k \qquad (\{u_i,u_j\} = -2\delta_{ij}\mathbb{I} \qquad [u_i,u_j] = 2\epsilon_{ij}{}^k u_k) \qquad (12.9.10)$$

Let us remark that one should not grow too fond of the representation of $\mathfrak{sl}(2,\mathbb{C})$ in terms of Pauli matrices or, for that matter, of any other specific representation. The algebra is concerned with linear combinations and commutators. As a matter of fact, when one fixes a specific representation, e.g. Pauli matrices, also other things have being fixed, e.g. the product rule (12.9.9) or the anticommutators (12.9.10).

These extra structures characterise that specific representation. However, they are not shared by other representations of the same algebra. Changes of algebra representations are defined to preserved linear combinations and commutators, they are not required to preserve products or anticommutators.

Keep that in mind, because often in proofs one takes roundabout arguments to use commutators only, instead of simply computing objects in the specific representation, just to show that the result is a property of the algebra, hence of all of its representations.

Now we want to develop tools to analyse representations, classify irreducible representations, especially *finite dimensional* irreducible representations of the algebra $\mathfrak{sl}(2,\mathbb{C})$, which has real dimension 6, 3 complex dimension, and it is explicitly spanned over $\mathbb{C}$ by the matrices σ_i (or, equivalently, by $\tau_i = -\tfrac{i}{2}\sigma_i$, or $u_i = -i\sigma_i$).

Let us consider an irreducible representation $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ over a (possibly infinite dimensional) complex vector space V. We first look for a maximal commuting Lie sub-algebra.

The sub-space $\mathfrak{h} := \mathrm{Span}(u_3) \subset \mathfrak{sl}(2,\mathbb{C})$ is of course a commuting sub-algebra. If we try to add something else to it we have

$$[a^i u_i, u_3] = 2a^i \epsilon_{i3}{}^k u_k = 2a^1\epsilon_{13}{}^2 u_2 + 2a^2\epsilon_{23}{}^1 u_1 = -2a^1 u_2 + 2a^2 u_1 \qquad (12.9.11)$$

Thus, $a^1 = a^2 = 0$, and any element $a = a^3 u_3$ commuting with u_3 is already in $\mathfrak{h}$, which is hence maximal.

The basis of $\mathfrak{sl}(2,\mathbb{C})$ can be adapted to the maximal commuting Lie sub-algebra, in the form which is called *ladder form*. In this case, we define $\tau_\pm := i\tau_1 \mp \tau_2 = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)$ and $h := i\tau_3 = \tfrac{1}{2}\sigma_3$ so that the commuting relations become

$$[\tau_\mp, h] = [i\tau_1 \pm \tau_2, i\tau_3] = [i\tau_1, i\tau_3] \pm [\tau_2, i\tau_3] = \tau_2 \pm i\tau_1 = \pm(i\tau_1 \pm \tau_2) = \pm\tau_\mp$$
$$[\tau_+, \tau_-] = [i\tau_1 - \tau_2, i\tau_1 + \tau_2] = 2i[\tau_1, \tau_2] = 2i\tau_3 = 2h \qquad\qquad (12.9.12)$$

If we consider any eigenvector of h, we have $h(v_k) = \alpha_k v_k$ and

$$\tau_- \circ h(v_k) - h \circ \tau_-(v_k) = \tau_-(v_k) \qquad \Rightarrow h \circ \tau_-(v_k) = (\alpha_k - 1)\tau_-(v_k) \qquad (12.9.13)$$

$$\tau_+ \circ h(v_k) - h \circ \tau_+(v_k) = -\tau_+(v_k) \qquad \Rightarrow h \circ \tau_+(v_k) = (\alpha_k + 1)\tau_+(v_k) \qquad (12.9.14)$$

$$\tau_+ \circ \tau_-(v_k) - \tau_- \circ \tau_+(v_k) = 2h(v_k) = 2\lambda_k v_k \qquad (12.9.15)$$

In view of these conditions, $\tau_-(v_k)$ is an eigenvector of h for the eigenvalue $(\alpha_k - 1)$, while $\tau_+(v_k)$ is an eigenvector of h for the eigenvalue $(\alpha_k + 1)$. Hence, starting from an eigenvector v_0, we can generate out of it other eigenvectors by applying $\tau_\pm$, iteratively. Let us set $v_1 := \tau_+(v_0)$, $v_2 := \tau_+(v_1)$, ..., $v_{k+1} := \tau_+(v_k)$ for all $k \geq 0$ (until we possibly get a zero vector). Similarly, let us set $v_{-1} := \tau_-(v_0)$, $v_{-2} := \tau_+(v_{-1})$, ..., $v_{k-1} := \tau_-(v_k)$ for all $k \leq 0$, again possibly until we hit a zero vector.

In this way, we have a family of eigenvectors v_k each with a different eigenvalue $\alpha_k = \alpha_0 + k$, where α_0 is the eigenvalue of v_0 and k is an integer. The family can be finite, hence bounded from above and below, in which case we can start from the least eigenvalue α_0 and $0 \leq k \leq n - 1$.

In mathematical literature often one starts from the maximal eigenvalue, instead. That is no difference except for notation, since one can restart the analysis using $-h$.

The family can be infinite but can have a minimal (or maximal) eigenvalue α_0 and then extend to $k \geq 0$ (or $k \leq 0$).

Or, finally, one can have an infinite family which extends unbounded in both direction, $k \in \mathbb{Z}$.

We could proceed by restricting to finite dimensional irreducible representations or by going a longer detour using a Casimir of the algebra. The second way is a bit longer, however, it teaches us something more general about infinite dimensional representations as well. Thus, by now, let us keep it general and notice that

$$\tau^2 := -\tfrac{1}{2}\left(\tau_+ \circ \tau_- + \tau_- \circ \tau_+\right) - h^2 = \tfrac{1}{2}\left((\tau_1)^2 + i\tau_2\tau_1 - i\tau_1\tau_2 + (\tau_2)^2 + (\tau_1)^2 + i\tau_1\tau_2 - i\tau_2\tau_1 + (\tau_2)^2\right) + (\tau_3)^2 = (\tau_1)^2 + (\tau_2)^2 + (\tau_3)^2 \qquad (12.9.16)$$

is in fact a Casimir (or, anyway, an element that commutes with all elements) of the algebra.

Let us compute

$$[\tau^2, \tau_1] = \tau_2[\tau_2, \tau_1] + [\tau_2, \tau_1]\tau_2 + \tau_3[\tau_3, \tau_1] + [\tau_3, \tau_1]\tau_3 = -\tau_2\tau_3 - \tau_3\tau_2 + \tau_3\tau_2 + \tau_2\tau_3 = 0 \qquad (12.9.17)$$

Similarly, for τ_2 and τ_3. ∎

Accordingly, we know that, for an irreducible representation (finite or infinite), τ^2 is proportional to the identity, say $\tau^2 = \alpha\mathbb{I}$. Hence, if we consider an eigenvector v_k of h, say $h(v_k) = (\alpha_0 + k)v_k$, we have

$$\{\tau_+, \tau_-\}(v_k) = -2\tau^2(v_k) - 2h^2(v_k) = -2(\alpha + (\alpha_0 + k)^2)v_k = -2(\alpha + \alpha_0^2 + k^2 + 2k\alpha_0)v_k \quad \propto v_k \qquad (12.9.18)$$

We also know the commutator (12.9.15), in particular

$$[\tau_+, \tau_-](v_k) = 2h(v_k) = 2(\alpha_0 + k)v_k \qquad (12.9.19)$$

By adding and subtracting, we obtain

$$\tau_+\left(\tau_-(v_k)\right) = -(\alpha + \alpha_0^2 + k^2 + 2k\alpha_0 - \alpha_0 - k)v_k \qquad \tau_-(v_{k+1}) = -(\alpha + \alpha_0^2 + k^2 + 2k\alpha_0 + \alpha_0 + k)v_k \qquad (12.9.20)$$

We can rewrite the second one as $\tau_-(v_k) = -(\alpha + \alpha_0^2 + 2k\alpha_0 - \alpha_0 + k^2 - k)v_{k-1}$. When we replace that into the first we obtain an identity and we are left with

$$\tau_-(v_k) = -(\alpha + \alpha_0^2 + 2k\alpha_0 - \alpha_0 + k^2 - k)v_{k-1} \qquad (12.9.21)$$

That tells us that if we start from an eigenvector v_k of h, we apply a sequence of h, τ_+ and τ_-, we still obtain one of the $v_{k'}$ or zero.

Accordingly, by applying (h, τ_+, τ_-) (or, equivalently, any element in $\mathfrak{sl}(2,\mathbb{C})$) to an eigenvector v_0, we stay in the subspace generated by a strip

$$F := (\ldots, (\tau_-)^k(v_0), \ldots, \tau_-(v_0), v_0, \tau_+(v_0), \ldots, (\tau_+)^k(v_0), \ldots) \tag{12.9.22}$$

which is called a *Fock strip*. In other words, the subspace $V_F := \mathrm{Span}(F)$ generated by a Fock strip is always an invariant subspace of V. Since we assumed V to be an irreducible representation (so to have $\tau^2 v = \alpha v$), then we have that $V = V_F$ is spanned by the Fock strip. In other words, any irreducible representation, being it finite or infinite, is generated by a Fock strip. In any irreducible representation, eigenspaces are 1 dimensional, the corresponding eigenvalues separated by 1, which is a pretty strict structure.

> Notice, as a take home remark, that we got Fock strip structure from a ladder basis *and* the Casimir. Had we restricted to finite dimensional irreducible representations from the beginning, we would have known from the beginning that eigenspaces are finite in number, even without considering the Casimir.

For a finite dimensional representation, the Fock strip is finite (and $\dim(V) = n$ elements long). There exists a minimal eigenvalue α_0, the others being $\alpha_k = \alpha_0 + k$ for $0 \le k \le n-1$. If we fix the eigenvector v_0 for the eigenvalue α_0, we can define $v_{k+1} := \tau_+(v_k)$ and we know $\tau_+(v_{n-1}) = 0$ as well as $\tau_-(v_0) = 0$. We know how h acts on the Fock strip

$$h(v_0) = \alpha_0 v_0 \qquad h(v_1) = (\alpha_0 + 1)v_1 \qquad \ldots \qquad h(v_k) = (\alpha_0 + k)v_k \qquad \ldots \qquad h(v_{n-1}) = (\alpha_0 + n - 1)v_{n-1} \tag{12.9.23}$$

how τ_+ acts

$$\tau_+(v_0) =: v_1 \qquad \tau_+(v_1) =: v_2 \qquad \ldots \qquad \tau_+(v_k) =: v_{k+1} \qquad \ldots \qquad \tau_+(v_{n-2}) =: v_{n-1} \qquad \tau_+(v_{n-1}) =: 0 \tag{12.9.24}$$

When we set $k = n$ into (12.9.21) we know the coefficient has to vanish, which gives $\alpha = -\alpha_0^2 - 2n\alpha_0 + \alpha_0 - n^2 + n$.

Also we know that α_0 is minimal, hence

$$\tau_-(v_1) = \tau_-(\tau_+(v_0)) = \tau_+(\tau_-(v_0)) + [\tau_-, \tau_+](v_0) = -2h(v_0) = -2\alpha_0 v_0 \tag{12.9.25}$$

Substituting this value of α back into (12.9.21) we obtain

$$\tau_-(v_k) = (2n\alpha_0 - 2k\alpha_0 + n^2 - n - k^2 + k)v_{k-1} \tag{12.9.26}$$

For $k = 1$ one has $\tau_-(v_1) = (2n\alpha_0 - 2\alpha_0 + n^2 - n)v_0$. Comparing with (12.9.25) we get

$$2n\alpha_0 - 2\alpha_0 + n^2 - n = -2\alpha_0 \quad \Rightarrow n\alpha_0 = -\frac{n(n-1)}{2} \quad \Rightarrow \alpha_0 = -\frac{n-1}{2} =: -j \tag{12.9.27}$$

Then, for any finite dimensional irreducible representation $\rho^{(j)}$, j is a semi-integer, called the *spin* of the representation.

> Let us stress that we can write everything in terms of the spin. The dimension is $n = 2j + 1$, the minimal eigenvalue $\alpha_0 = -j$, the Casimir $\tau^2 = -j(j+1)\mathbb{I}$. We have $h(v_k) = (-j + k)v_k$, $v_{k+1} = \tau_+(v_k)$, and $\tau_-(v_k) = k(2j - k + 1)v_{k-1}$.
> The constraints we used to compute α and α_0 come precisely from maximality of v_{n-1} and minimality of v_0, the point being that we should not be surprised that when we consider an infinite Fock strip with a minimal eigenvalue, we have one constraint less, so we are not able to compute one of the parameters and we are left with an extra parameter in the generic infinite dimensional reoresentations.

Let us compute the Casimir action to be, as expected:

$$\tau^2(v_k) = -\tfrac{1}{2}\left(\tau_+(\tau_-(v_k)) + \tau_-(v_{k+1})\right) - h^2 v_k =$$

$$= -\tfrac{1}{2}\left((2n\alpha_0 - 2k\alpha_0 + n^2 - n - k^2 + k)v_k + (2n\alpha_0 - 2k\alpha_0 - 2\alpha_0 + n^2 - n - k^2 - k)v_k)\right) - (\alpha_0 + k)^2 v_k =$$

$$= -\left(\left(2n\alpha_0 - 2k\alpha_0 - \alpha_0 + n^2 - n - k^2\right) + (\alpha_0^2 + k^2 + 2\alpha_0 k)\right)v_k = -\left(2n\alpha_0 - \alpha_0 + n^2 - n + \alpha_0^2\right)v_k = \alpha v_k \qquad (12.9.28)$$

which is in fact proportional to the identity on V.

To summarize, one writes everything in term of n, or $j := \frac{n-1}{2}$.

For $n = 1$ ($j = 0$), one has $V = \mathbb{C}$, one only eigenvalue $\alpha = 0$.

For $n = 2$ ($j = \tfrac{1}{2}$), one has $V = \mathbb{C}^2$, and two eigenvalues $\alpha_0 = -\tfrac{1}{2}$, $\alpha_1 = \tfrac{1}{2}$.

For $n = 3$ ($j = 1$), one has $V = \mathbb{C}^3$, and three eigenvalues $\alpha_0 = -1$, $\alpha_1 = 0$, $\alpha_2 = \tfrac{1}{2}$.

For $n = 4$ ($j = \tfrac{3}{2}$), one has $V = \mathbb{C}^4$, and four eigenvalues $\alpha_0 = -\tfrac{3}{2}$, $\alpha_1 = -\tfrac{1}{2}$, $\alpha_2 = \tfrac{1}{2}$, $\alpha_3 = \tfrac{3}{2}$. And so on.

The representations are graphically represented in the space of eigenvalues by noting the multiplicity of each eigenvalue.
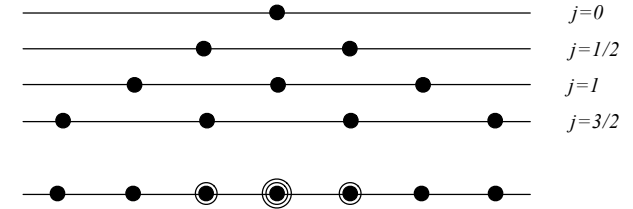
Fig. 12.2: Root graphs of representations of $\mathfrak{sl}(2, C)$

In irreducible representations the multiplicity is always one. More generally, we can represent a representation by noting the multiplicity of eigenvalues, which easily guide to the decomposition of the representations as a sum of irreducible representations. That is called a *root graph*. For example, in Fig. 12.2 we have the representation $\rho^{(3/2)} \oplus \rho^{(1)} \oplus \rho^{(1/2)} \oplus \rho^{(0)} \oplus \rho^{(0)}$.

Given an irreducible representation $\rho^{(j)}$, we know quite a lot about it. In particular, we can reconstruct the matrices to represent $S \in \mathfrak{sl}(2, \mathbb{C})$ in the basis of eigenvectors $(v_0, v_1, \ldots, v_{2j})$ of $\mathbb{C}^{2j+1}$.

Let us start quite trivially from $j = 0$. We know that $\rho^{(0)}(S) : \mathbb{C} \to \mathbb{C}$ with the basis v_0. We have $h(v_0) = 0$, $\tau_+(v_0) = 0$, $\tau_-(v_0) = 0$. Accordingly, $\rho^{(0)}(\xi) = 0$. That is called the *trivial representation* and any element $\xi \in \mathfrak{sl}(2, \mathbb{C})$ is mapped to the 0 (1×1)-matrix.

Next case is $j = \tfrac{1}{2}$ which is called the *fundamental representation* and denoted by $\rho^{(1/2)}$. At the level of algebra, we have $\xi = ib^i \sigma_i \in \mathfrak{sl}(2, \mathbb{C})$ and $\rho^{(1/2)}(\xi) : \mathbb{C}^2 \to \mathbb{C}^2$. In $\mathbb{C}^2$ we have two eigenvectors $v_\pm$ of the eigenvalue $\alpha_\pm = \pm\tfrac{1}{2}$. We have

$$\begin{array}{lll} h(v_-) = -\tfrac{1}{2}v_- & \tau_+(v_-) = v_+ & \tau_-(v_-) = 0 \\ h(v_+) = \tfrac{1}{2}v_+ & \tau_+(v_+) = 0 & \tau_-(v_+) = v_- \end{array} \qquad (12.9.29)$$

Then accordingly we have

$$\begin{array}{lll} \sigma_3(v_-) = -v_- & \sigma_1(v_-) = (\tau_+ + \tau_-)(v_-) = v_+ & \sigma_2(v_-) = -i(\tau_+ - \tau_-)(v_-) = -iv_+ \\ \sigma_3(v_+) = v_+ & \sigma_1(v_+) = (\tau_+ + \tau_-)(v_+) = v_- & \sigma_2(v_+) = -i(\tau_+ - \tau_-)(v_+) = iv_- \end{array} \qquad (12.9.30)$$

and in the basis (v_+, v_-) we have the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (12.9.31)$$

This is called *fundamental representation* since one has

$$\lambda^{(1/2)}(\xi) = i\vec{a} \cdot \vec{\sigma} = \xi \tag{12.9.32}$$

Let $m = 3$ (hence $j = 1$); we have eigenspaces E_{-1}, E_0 and E_1 and eigenvectors $e_- \in E_{-1}$, $e_0 \in E_0$ and $e_+ \in E_1$. We know that:

$$
\begin{aligned}
h(e_-) &= -e_- & h(e_0) &= 0 & h(e_+) &= e_+ \\
\tau_+(e_-) &= e_0 & \tau_+(e_0) &= e_+ & \tau_+(e_+) &= 0 \\
\tau_-(e_-) &= 0 & \tau_-(e_0) &= 2e_- & \tau_-(e_+) &= 2e_0
\end{aligned}
\tag{12.9.33}
$$

Thus we have (by ordering the basis as (e_+, e_0, e_-))

$$
h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad
\tau_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad
\tau_- = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}
\tag{12.9.34}
$$

Accordingly we obtain:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \qquad
\sigma_2 = i \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix} \qquad
\sigma_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\tag{12.9.35}
$$

One can check that those matrices obey the required commutation relations. An element in $\mathfrak{sl}(2, \mathbb{C})$ is any complex combination of these matrices. This representation $\rho^{(1)}$ is called the *adjoint representation*.

We can also use the information we learned about (finite dimensional) irreducible representations, the other way around. Let us denote by sym^k the following algebra representation

$$\mathrm{sym}^k(\xi)\psi^{A_1 A_2 \ldots A_k} = k!\xi_{B_1}^{(A_1}\psi^{B_1 A_2 \ldots A_k)} \tag{12.9.36}$$

which is the symmetrised tensor product of $(n-1)$ copies of the fundamental representation $\rho^{(1/2)}$, i.e. $A, B, \ldots = 1, 2$.

In particular, by setting $\xi = h = \frac{1}{2}\sigma_3$, we obtain in the basis of eigenvectors of h, setting $k = 2j$:

$$
\mathrm{sym}^{2j}(h) = \begin{pmatrix}
j & 0 & 0 & \ldots & 0 \\
0 & j-1 & 0 & \ldots & 0 \\
0 & 0 & j-2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & -j
\end{pmatrix}
\tag{12.9.37}
$$

Hence the representation sym^{2j} is of rank $n = 2j + 1 = k + 1$ and it has n eigenspaces of dimension 1. Consequently, the representation sym^k is irreducible and thence $\mathrm{sym}^{2j} \simeq \rho^{(j)}$.

If we consider an infinte dimensional irreducible representation with a minimal eigenvalue α_0, we have an infinite Fock strip $F = (v_0, v_1, \ldots, v_k, \ldots)$. We can go through the analysis made for finite dimensional representations, only we fail to have a maximal eigenvalue, from which there we determined the constraint $\alpha = -\alpha_0^2 - 2n\alpha_0 + \alpha_0 - n^2 + n$, while we still can argue from minimality of α_0 that

$$\tau_-(v_1) = -(\alpha + \alpha_0^2 + \alpha_0)v_0 = -2\alpha_0 v_0 \qquad \Rightarrow \alpha = -\alpha_0(\alpha_0 + 1) \tag{12.9.38}$$

and we can express everything in terms of the minimal eigenvalue $\alpha_0 \in \mathbb{R}$ (which is real since h is Hermitian).

We have:

$$
\begin{array}{lllll}
h(v_0) = \alpha_0 v_0 & h(v_1) = (\alpha_0 + 1)v_1 & \ldots & h(v_k) = (\alpha_0 + k)v_k & \ldots \\
\tau_+(v_0) =: v_1 & \tau_+(v_1) =: v_2 & \ldots & \tau_+(v_k) =: v_{k+1} & \ldots \\
\tau_-(v_0) = 0 & \tau_-(v_1) = -2\alpha_0(k-1)v_0 & \ldots & \tau_-(v_k) = -(k-1)(2\alpha_0 + k)v_{k-1} & \ldots
\end{array}
\tag{12.9.39}
$$

and $\tau^2 = -\alpha_0(\alpha_0 + 1)\mathbb{I}$.

Representations of $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$ is made of anti-Hermitian, traceless 2×2 complex matrices, which in fact forms a real three dimensional algebra.

One can easily check that the commutator of two matrices is $\mathfrak{su}(2)$ is still in $\mathfrak{su}(2)$.

Of course, a linear combinations of traceless and anti-Hermitian matrices is still traceless and anti-Hermitian. Thus they form a vector space.

If we consider complex conjugation of a commutator of matrices in $\mathfrak{su}(2)$ we have

$$[U_1, U_2]^\dagger = (U_1 U_2 - U_2 U_1)^\dagger = U_2^\dagger U_1^\dagger - U_1^\dagger U_2^\dagger = -(U_1 U_2 - U_2 U_1) = -[U_1, U_2] \tag{12.9.40}$$

which is still anti-Hermitian.

A (real) basis of such matrices are τ_i and $U \in \mathfrak{su}(2)$ iff it is a real linear combination of $\tau_i = -\frac{i}{2}\sigma_i$, i.e. $U = \theta^i \tau_i$.

A representation of $\mathfrak{su}(2)$ defines the matrices for the generators σ_i exactly as for $\mathfrak{sl}(2, \mathbb{C})$. A representation of $\mathfrak{su}(2)$ is just obtained by restricting to real linear combinations. Accordingly, irreducible representations for $\mathfrak{su}(2)$ are labelled by spin j as $\rho^{(j)} : \mathfrak{su}(2) \to \text{End}(V)$ with $V = \mathbb{C}^{2j+1}$.

10. Affine spaces

Affine spaces have a great importance in gravitational theories, mainly because connections are naturally elements of an affine space. Also, in general field theories, affine spaces are important since jet bundles $J^k C$ of order k are affine bundles over $J^{k-1} C$. This is the reason why one can define quasilinear equations on manifolds, essentially regardless the geometric character of fields.

An *affine space modelled on a vector space V* is a set $\mathbb{A}$ together with an operation

$$\mu : \mathbb{A} \times \mathbb{A} \to V : (P, O) \mapsto \mu(P, O) =: P - O \tag{12.10.1}$$

(1) $\forall P, R, Q \in \mathbb{A} : \mu(P, Q) = \mu(P, R) + \mu(R, Q)$ $(P - Q = P - R + R - Q)$

(2) $\forall Q \in \mathbb{A}, \forall v \in V : \exists! P \in A$ such that $P - Q = v$

In view of axiom (2) one can also define a map

$$\nu : V \times \mathbb{A} \to \mathbb{A} : (v, Q) \mapsto P := v + Q \tag{12.10.2}$$

where, $\forall Q \in \mathbb{A}$ and $v \in V$, $P = \nu(v, Q)$ is *the* point in $\mathbb{A}$ for which $v = P - Q$. Then one has $P = \nu(\mu(P, Q), Q)$ or, equivalently, $v = \mu(\nu(v, Q), Q)$ (or, shortly, $P = (P - Q) + Q$). This map $\nu : V \times \mathbb{A} \to \mathbb{A}$ turns out to be a (faithful and transitive) group action of V (as an additive group) on $\mathbb{A}$, i.e.

$$\nu(u + w, Q) = \nu(u, \nu(w, Q)) \qquad \nu(0, Q) = Q \tag{12.10.3}$$

> One can easily check it by setting $P = \nu(u + w, Q)$ and $R = \nu(w, Q)$ so that $u + w = \mu(P, Q)$, $w = \mu(R, Q)$, and $u = \mu(P, R)$ and (12.10.3) reads as axiom (1).
> For the second condition, one can simply set $R = Q$ in axiom (1) to obtain $\mu(Q, Q) = 0$, which, in turn, reads as $\nu(0, Q) = Q$.

Thus affine spaces naturally carry an action of an additive group V.

For any given point $O \in \mathbb{A}$, one can also define a map $i_O : \mathbb{A} \to V : P \mapsto P - O = \mu(P, O)$. This map is bijective and it induces a linear structure on $\mathbb{A}$. Unfortunately, the linear structure defined on $\mathbb{A}$ does depend on O. Let us denote by $\mathbb{A}_O$ the vector space structure defined on $\mathbb{A}$ in this way. Then $i_O : \mathbb{A}_O \to V$ becomes an isomorphism of vector spaces.

This is the origin of the naive description of affine spaces as *a vector space which forgot about its origin* which is sometimes given in the literature as an intuitive description of affine spaces. Often, affine spaces are regarded as vector spaces, i.e. one works on $\mathbb{A}_O$ instead of on $\mathbb{A}$, using the maps i_O and checking that results are independent of the choice of O. For example, this is how defines the dimension, a reference frame and Cartesian coordinates on affine spaces. However, this is, from our perspective, a mistake since affine spaces should be regarded as *manifolds*, not as vector spaces.

> Let us refer to Section 14.4 for the definition of tangent space at a point in a manifold. The space $A_O := \mathbb{A} \times \{O\}$ is isomorphic to $T_O\mathbb{A}$. On the other hand, A_O is clearly isomorphic to $\mathbb{A}_O \simeq V$, by the restriction of the map $\mu : \mathbb{A} \times \mathbb{A} \to V$.
> For any $(P, O) \in A_O$, we can define the curve $\gamma : \mathbb{R} \to \mathbb{A} : s \mapsto \gamma(s) = O + s(P - O)$. Then the map $i : A_O \mapsto T_O\mathbb{A} : (P, O) \mapsto [\gamma]$ turns out to be a linear isomorphism. For, let us consider a basis e_i in V, and Cartesian coordinates v^i defined on $\mathbb{A}$ (the point Q corresponds to coordinates v^i such that $Q - O = v^i e_i$). In those coordinates, any vector $v = [\gamma]$ has a linear representative $T_0^1\gamma : s \mapsto (x^i + v^i s)e_i$ (x^i being the coordinates of the origin O). Then just define $(P, O) \in A_O$ and $P = (T_0^1\gamma)(1) = O + v$ and show that $i(P, O) = v$.
> On this basis, eventually one can regard affine spaces as manifolds which allow a special global structure, namely the affine combinations which are defined hereafter.

There is also another operation that one can define on affine spaces, called *affine combinations*. Let us consider n points $A_i \in \mathbb{A}$ and n real numbers $s_i \in \mathbb{R}$ and an origin $O \in \mathbb{A}$; let us define the affine combination as the point $A = \nu(v, O) \in \mathbb{A}$ where

$$v = \frac{\sum_{i=1}^{n} s_i \mu(A_i, O)}{\sum_{i=1}^{n} s_i} \tag{12.10.4}$$

This is also called the *weighted average* of the points A_i with respect to the *weights* $t_i := s_i/s$, where we set $s := \sum_{i=1}^n s_i$. Notice that one has $\sum_{i=1}^n t_i = 1$ as the sum of weights. This affine combination is also denoted by

$$A = \sum_{i=1}^n t_i A_i = \left(\sum_{i=1}^n t_i(A_i - O) \right) + O \tag{12.10.5}$$

The main property of affine combinations is that they do not depend on the choice of the origin O.

Let in fact O' be another origin and

$$\left(\sum_{i=1}^n t_i(A_i - O') \right) + O' = \left(\sum_{i=1}^n t_i((A_i - O) + (O - O')) \right) + O' = \left(\sum_{i=1}^n t_i(A_i - O) \right) + (O - O') + O' = \left(\sum_{i=1}^n t_i(A_i - O) \right) + O = A \tag{12.10.6}$$

Accordingly, the affine combination is also denoted by

$$A = \sum_{i=1}^n t_i A_i \tag{12.10.7}$$

which is however just a short hand notation for (12.10.4).

In particular, for two points $A = A_1$ and $B = A_2$, one has the weights t and $1 - t$; the affine combination reads as

$$C = (t(A - O) + (1 - t)(B - O)) + O =: \boldsymbol{\mu}_t(A, B) \tag{12.10.8}$$

and it is independent of the choice of O.

The map $\boldsymbol{\mu}_t : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is also denoted by

$$\boldsymbol{\mu}_t : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : (A, B) \mapsto \boldsymbol{\mu}_t(A, B) =: tA + (1 - t)B \tag{12.10.9}$$

and it denotes a point on the line through A and B.

From a more fundamental point of view, one should define an *affine space* to be a set $\mathbb{A}$, together with an (one parameter) operation:

$$\boldsymbol{\mu}_t : \mathbb{A} \times \mathbb{A} \to \mathbb{A} : (P, Q) \mapsto \boldsymbol{\mu}_t(P, Q) \tag{12.10.10}$$

which satisfies the following properties $\forall A, B, C \in \mathbb{A}$ and $\forall r, s, t \in \mathbb{R}$

(1) $\boldsymbol{\mu}_0(A, B) = B$
(2) $\boldsymbol{\mu}_1(A, B) = A$
(3) $\boldsymbol{\mu}_{rt(1-t)}(\boldsymbol{\mu}_s(C, B), A) = \boldsymbol{\mu}_t(\boldsymbol{\mu}_{rs(1-t)}(C, A), \boldsymbol{\mu}_{rt(1-s)}(B, A))$.

This approach is more fundamental, since it does not assume the vector space V from the beginning and defines the affine structure on $\mathbb{A}$ alone. One can show that, for any origin $O \in \mathbb{A}$, one can endow $\mathbb{A}$ with a linear structure, denoted by $\mathbb{A}_O$, and that the vector spaces $\mathbb{A}_O$ are all isomorphic as the origin is changed. Then one defines V as an abstract copy of such vector spaces and one has isomorphisms $i_O : \mathbb{A}_O \to V$. Thus one can define the map $\mu : \mathbb{A} \times \mathbb{A} \to V$ as

$$\mu : \mathbb{A} \times \mathbb{A} \to V : (P, O) \mapsto \mu(P, O) := i_O(P) \tag{12.10.11}$$

and consequently

$$\nu : V \times \mathbb{A} \to \mathbb{A} : (v, P) \mapsto \nu(v, P) := (i_P)^{-1}(v) \tag{12.10.12}$$

However, this way is particularly hard in the beginning and quite long. It shows that, given an affine space, it can always be defined as an affine space modelled on a suitable vector space V. For this reason and for the sake of simplicity, we defined directly affine spaces modelled on a vector space.

Now that we defined affine spaces as objects, we have to define morphisms between affine spaces, namely *affine maps*. Let $\mathbb{A}$ and $\mathbb{B}$ be two affine spaces. A map $\Phi : \mathbb{A} \to \mathbb{B}$ is called an *affine map* iff it preserves affine combinations, i.e.

$$\Phi(tP + (1-t)Q) = t\Phi(P) + (1-t)\Phi(Q) \tag{12.10.13}$$

Affine automorphisms on $\mathbb{A}$ form a group with respect to composition, which is denoted by $\mathrm{GA}(\mathbb{A})$.

Affine spaces with affine maps form a category that we shall denote by $\mathfrak{Aff}$.

Let us fix an origin $O \in \mathbb{A}$; then all points in $\mathbb{A}$ can be represented by vectors in V, i.e. for any $P \in \mathbb{A}$ there is one and only one $v = P - O \in V$ and then $P = v + O$.

Let us now consider an affine map $\Phi : \mathbb{A} \to \mathbb{B} : A \mapsto B := \Phi(A)$. Let us set $O' = \Phi(O)$. Let us set $A = v + O$ and $B = w + O'$. Then the map Φ induces a map $\hat{\Phi} : V \to W : v \mapsto w = \hat{\Phi}(v)$ such that

$$\nu(\cdot, O) \begin{array}{c} \mathbb{A} \xrightarrow{\ \Phi\ } \mathbb{B} \\ \downarrow{\scriptstyle\mu(\cdot, O)} \quad \downarrow{\scriptstyle\mu(\cdot, O')} \\ V \xrightarrow{\ \hat{\Phi}\ } W \end{array} \qquad \hat{\Phi} := \mu(\cdot, O') \circ \Phi \circ \nu(\cdot, O) \tag{12.10.14}$$

We can easily show that the map $\hat{\Phi} : V \to W$ is linear.

Let us consider two vectors $v_1, v_2 \in V$, the corresponding points $A_1 = v_1 + O$ and $A_2 = v_2 + O$), and a scalar $\alpha \in \mathbb{R}$. If we wanted to define a linear structure on $\mathbb{A}$ we should define $A_1 + A_2$ and αA_1 as points in $\mathbb{A}$, i.e.

$$(v_1 + v_2) + O = (A_1 - O) + (A_2 - O) + O = A_1 + A_2 - O \qquad\qquad (\alpha v_1) + O = \alpha(A_1 - O) + O = \alpha A_1 + (1 - \alpha)O \tag{12.10.15}$$

respectively. Of course, both of these points do actually depend on the origin O and that is why the linear structure on $\mathbb{A}$ is not canonical.

Accordingly, let us denote by $B_1 = \Phi(A_1)$ and $B_2 = \Phi(A_2)$ the corresponding images. Then one has

$$B_1 = \Phi(A_1) = \Phi(v_1 + O) = \hat{\Phi}(v_1) + O' \tag{12.10.16}$$

and define accordingly $w_1 := \hat{\Phi}(v_1) = B_1 - O'$ and $w_2 := \hat{\Phi}(v_2) = B_2 - O'$.

Let us first show that the map $\hat{\Phi}$ does not depend on the choice of the origin. Let us consider a different origin $\hat{O} := \delta + O \in \mathbb{A}$ and, consequently, $\hat{O}' := \Phi(\hat{O}) \in \mathbb{B}$. Then, by using the new origins we would define a new map $\hat{\Phi}' : V \to W$. If one has $\hat{\Phi}' = \hat{\Phi}$, then it must be again $\hat{\Phi}'(v_1) = w_1$ iff

$$\hat{\Phi}(v_1) = \Phi(v_1 + \hat{O}) - \hat{O}' = \Phi(v_1 + \delta + O) - \hat{O}' = \hat{\Phi}(v_1 + \delta) + O' - \hat{O}' = \hat{\Phi}(v_1) + \hat{\Phi}(\delta) + O' - \hat{O}' = \hat{\Phi}(v_1) \tag{12.10.17}$$

which is satisfied since $\hat{\Phi}(\delta) = \hat{O}' - O' = \Phi(\hat{O}) - O' = \Phi(\delta + O) - O'$.

Then one has:

$$
\begin{aligned}
\hat{\Phi}(\alpha v_1 + \beta v_2) =&\Phi\left((\alpha v_1 + \beta v_2) + O\right) - O' = \Phi\left((\alpha v_1 + O) + (\beta v_2 + O) - O\right) - O' = \\
=&\Phi\left((\alpha A_1 + (1-\alpha)O) + (\beta A_2 + (1-\beta)O) - O\right) - O' = \Phi\left(\alpha A_1 + \beta A_2 + (1 - \alpha - \beta)O\right) - O' = \\
=&\alpha B_1 + \beta B_2 + (1 - \alpha - \beta)O' - O' = \alpha(\hat{\Phi}(v_1) + O') + \beta(\hat{\Phi}(v_2) + O') + (1 - \alpha - \beta)O' - O' = \\
=&\alpha\hat{\Phi}(v_1) + \beta\hat{\Phi}(v_2)
\end{aligned}
\tag{12.10.18}
$$

Thus we defined a covariant functor $\Lambda : \mathfrak{Aff} \to \mathfrak{Vect}$ which takes an affine space $\mathbb{A}$ and gives its model $V = \Lambda(\mathbb{A})$ and it takes an affine map Φ and gives its linear map $\Lambda(\Phi) := \hat{\Phi}$. None of these depend on the origin one may set.

Let us finally consider an affine endomorphism $\Phi : \mathbb{A} \to \mathbb{A}$. Since any point in $\mathbb{A}$ can be represented (uniquely) by a vector in V, we can describe the map Φ as a (non-linear) map $\hat{\Phi}$

$$
\hat{\Phi} : V \to V : v \mapsto \Lambda(\Phi)(v) + \delta
$$

where we set $\delta = \Phi(O) - O$. Then we see that an affine map is obtained as a composition of a linear map $\Lambda(\Phi)$ and a *translation* by an element $\delta \in V$.

Thus, in components the most general affine endomorphism reads as $\hat{\Phi} : V \to V : v^i \mapsto \Phi^i_j v^j + \delta^i$. The composition of two affine maps reads as

$$
\hat{\Psi} \circ \hat{\Phi} : V \to V : v^i \mapsto \Psi^i_j(\Phi^j_k v^k + \delta^j) + \delta'^i
\tag{12.10.19}
$$

Accordingly, the first map is expressed as a pair (Φ^j_i, δ^j), the second map is expressed as (Ψ^j_i, δ'^j) and the composition is $(\Psi^j_i \Phi^j_i, \Psi^i_j \delta^j + \delta'^i)$. This expresses the group $GA(\mathbb{A})$ as the semi-direct product $GA(\mathbb{A}) \simeq GL(V) \ltimes V$.

Let $(G, \bullet)$ and $(H, \cdot)$ be two groups and $\rho : G \times H \to G$ a right action of H on G such that the maps $\rho(\cdot, h)$ are group homomorphisms of G onto itself; the *semi-direct product* of G by H (on the right) is denoted by $G \rtimes H$ and it has as elements the pairs (g, h) and the product defined as

$$
(g_1, h_1) * (g_2, h_2) = (\rho(g_1, h_2) \bullet g_2, h_1 \cdot h_2) = ((g_1 \triangleleft h_2) \bullet g_2, h_1 \cdot h_2)
\tag{12.10.20}
$$

This product is associative

$$
\begin{aligned}
((g_1, h_1) * (g_2, h_2)) * (g_3, h_3) =&(h_1 \cdot h_2, (g_1 \triangleleft h_2) \bullet g_2, h_1 \cdot h_2) * (g_3, h_3) = (((g_1 \triangleleft h_2) \bullet g_2) \triangleleft h_3) \bullet g_3, (h_1 \cdot h_2) \cdot h_3) = \\
=&(((g_1 \triangleleft (h_2 \cdot h_3)) \bullet (g_2 \triangleleft h_3))) \bullet g_3, h_1 \cdot (h_2 \cdot h_3)) = (g_1, h_1) * ((g_2 \triangleleft h_3) \bullet g_3, h_2 \cdot h_3) = \\
=&(g_1, h_1) * ((g_2, h_2) * (g_3, h_3))
\end{aligned}
\tag{12.10.21}
$$

the element (e_G, e_H) is the neutral element, and the inverse is given by $(g, h)^{-1} = (g^{-1} \triangleleft h^{-1}, h^{-1}) = (g^{-1}, e_H) * (e_G, h^{-1})$ since one has

$$
\begin{aligned}
(g^{-1} \triangleleft h^{-1}, h^{-1}) * (g, h) =& ((g^{-1} \triangleleft h^{-1} \triangleleft h) \bullet g, h^{-1} \cdot h) = (e_G, e_H) \\
(g, h) * (g^{-1} \triangleleft h^{-1}, h^{-1}) =& ((g \triangleleft h^{-1}) \bullet (g^{-1} \triangleleft h^{-1}), h \cdot h^{-1}) = ((g \bullet g^{-1}) \triangleleft h^{-1}, e_H) = (e_G, e_H)
\end{aligned}
\tag{12.10.22}
$$

If H acts on the left on G by $\lambda : H \times G \to G$ so that the maps $\lambda(h, \cdot)$ are group homomorphisms, then one can define the semi-direct product (on the left) which is denoted by $H \ltimes G$ and it is formed by pairs (h, g) such that

$$
(h_1, g_1) * (h_2, g_2) = (h_1 \cdot h_2, \lambda(h_2, g_1) \bullet g_2) = (h_1 \cdot h_2, (h_1 \triangleright g_2) \bullet g_1)
\tag{12.10.23}
$$

This product is associative

$$((h_1, g_1) * (h_2, g_2)) * (h_3, g_3) = (h_1 \cdot h_2, (h_1 \triangleright g_2) \bullet g_1) * (h_3, g_3) = ((h_1 \cdot h_2) \cdot h_3, ((h_1 \cdot h_2) \triangleright g_3) \bullet ((h_1 \triangleright g_2) \bullet g_1)) =$$
$$= (h_1 \cdot (h_2 \cdot h_3), (h_1 \triangleright ((h_2 \triangleright g_3) \bullet g_2)) \bullet g_1) = (h_1, g_1) * (h_2 \cdot h_3, (h_2 \triangleright g_3) \bullet g_2) = (h_1, g_1) * ((h_2, g_2) * (h_3, g_3))$$

(12.10.24)

the element (e_H, e_G) is the neutral element, and the inverse is given by $(h, g)^{-1} = (h^{-1}, h^{-1} \triangleright g^{-1}) = (h^{-1}, e_G) * (e_H, g^{-1})$ since one has

$$(h, g) * (h^{-1}, h^{-1} \triangleright g^{-1}) = (h \cdot h^{-1}, (h \triangleright (h^{-1} \triangleright g^{-1})) \bullet g) = (e_H, e_G)$$
$$(h^{-1}, h^{-1} \triangleright g^{-1}) * (h, g) = (h^{-1} \cdot h, (h^{-1} \triangleright g) \bullet (h^{-1} \triangleright g^{-1})) = (e_H, e_G)$$

(12.10.25)

For example, a vector space is an additive group denoted by $(V, +)$ and $\mathrm{GL}(V)$ is a group which in fact acts on V by a left action. Because of linearity, the action is by group homeomorphisms. Then, one can define the semi-direct product $\mathrm{GL}(V) \ltimes V$ on the left for which the product is defined as

$$(\phi, v) * (\varphi, u) = (\phi \circ \varphi, \phi(u) + v)$$

(12.10.26)

References

On Affine spaces, Marcel Berger, Geometry I, Springer, Corrected Third Printing 2004, isbn: 3-540-11658-3

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Chapter 13. Topological spaces

1. Topological spaces

Let X be a set and τ be a collection of subsets of X. We say that τ is a *topology* of X if the following properties holds true:

(1) $\varnothing, X \in \tau$;

(2) for any (possibly infinite, even uncountably infinite) collection I of subsets $U_i \in \tau$ then the union $\cup_{i \in I} U_i \in \tau$;

(3) for any *finite* collection $I = \{1, 2, \ldots, n\}$ of subsets $U_i \in \tau$ then the intersection $\cap_{i=1}^n U_i \in \tau$.

> The main point of this definition is that it *is* a definition. The hard thing here is not rely on any *intuitive* notion of what *open* may mean. In $\mathbb{R}$ there is a topology in which the intervals (a, b) are *open*, but that is no better topology that any other we can define, provided it satisfies the axioms. The topology has to be *defined*, depending on the problem we are considering.

The sets $U \in \tau$ are called *opensets*. The *complement* $X - U = \{x \in X : x \notin U\}$ of an open set $U \in \tau$ is called a *closed set*. The collection of all closed sets is denoted by χ. One has that $\varnothing, X \in \chi$, arbitrary intersections of closed sets are closed, finite unions of closed sets are closed.

Given two topological spaces (X, τ_X) and (Y, τ_Y), a map $\phi : X \to Y$ is *continuous* iff the preimages of opensets in Y are open in X, namely for all $U \in \tau_Y : \phi^{-1}(U) \in \tau_X$.

> Notice we are defining continuous maps directly, not continuous maps *at a point*, as usual by using limits. Actually, we did not define limits, yet. This is more general and more fundamental.
> Notice that any map $\phi : X \to Y$ is continuous if we choose the trivial topology on Y or the discrete topology on X.

The identity $\mathrm{id} : X \to X$ is always continuous. The composition of continuous maps is continuous. Hence topological spaces with continuous maps form a category, denoted by $\mathfrak{Top}$. A continuous map, invertible with a continuous inverse, is called a *homeomorphism*. Homeomorphisms are categorical isomorphisms and one usually identifies homeomorphic topological spaces, since there is a one-to-one correspondence of opensets.

> One can also define *open maps* to be the maps $\phi : X \to Y$ such that the image of an openset is open, i.e. $\forall U \in \tau_X : \phi(U) \in \tau_Y$, as well as *closed maps* to be the maps $\phi : X \to Y$ such that the image of a closed set is closed, i.e. $\forall U \in \chi_X : \phi(U) \in \chi_Y$.
> However, open and closed maps are less relevant than continuous maps.

Examples of topologies

It is good to be familiar with many examples of topologies since they are usually used as examples and counter examples to properties. In other words they are used to fight against intuition. Let us start from some basic examples. More will soon follow.

- On any set X, the collection $\tau_T = \{\varnothing, X\}$ is a topology which is called the *trivial topology*. Also the set of all possible subsets of X is a topology $\tau_D = \mathcal{P}(X)$, which is called the *discrete topology*.

- A set X together with a topology τ is called a *topological space* and it is denoted by (X, τ).

A subset $Y \subset X$ of a topological space (X, τ_X) is a *topological subspace* itself when we define the *induced topology* on Y

$$\tau_Y = \{U \cap Y : U \in \tau_X\} \tag{13.1.1}$$

Then (Y, τ_Y) is a topological space on its own right.

If (X, τ_X) is a topological space and a subset $Y \subset X$ is endowed with the induced topology τ_Y, then the canonical embedding $i : Y \to X$ is continuous, since the preimage of an open set $U \in \tau_X$ is exactly $U \cap Y \in \tau_Y$.

- On $\mathbb{R}$ we can define the *standard topology* in which U is an openset iff $\forall x \in U \, \exists \delta > 0 : (x - \delta, x + \delta) \subset U$. This is a special case of topology induced by a distance; see below.

Let us consider a map $f : \mathbb{R} \to \mathbb{R}$ and let us endow $\mathbb{R}$ with its standard topology. The interval $B_r(x) = (x - r, x + r)$ is called a *ball* at x of radius $r > 0$. Let $U = B_\epsilon(f(x))$ be an open set of $\mathbb{R}$ and $f^{-1}(U)$ its preimage.

The map is continuous iff $f^{-1}(U)$ are opensets of $\mathbb{R}$, which in the standard topology means that it is a union of open intervals. Let us consider a point $x \in f^{-1}(U)$ then there must exist a ball $B_\delta(x) \subset f^{-1}(U)$, i.e.

$$\forall B_\epsilon(f(x)), \exists B_\delta(x) : f(B_\delta(x)) \subset B_\epsilon(f(x)) \tag{13.1.2}$$

i.e.

$$\forall \epsilon > 0, \exists \delta > 0 : (|x' - x| < \delta \Rightarrow |f(x') - f(x)| < \epsilon) \tag{13.1.3}$$

Thus the standard definition of continuity *at a point* is recovered.

Continuity for functions $f : \mathbb{R} \to \mathbb{R}$ with respect to the standard topology reduces to the usual definition of continuity, though the topological definition is simpler and a bit more general. For example, it provides a good control of continuity at isolated points in the domain, where continuity at a point fails.

For, let us consider $D = (0, 1) \cup \{2\} \subset \mathbb{R}$ and $C = (0, 1] \subset \mathbb{R}$ both endowed with the induced topology by the standard topology in $\mathbb{R}$. Then consider the map $f : D \to C$ defined as

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & x = 2 \end{cases} \tag{13.1.4}$$

One can check that the map is continouus, it is one-to-one, thus invertible. The inverse map is $\bar{f} : C \to D$ defined by

$$\bar{f}(y) = \begin{cases} y & 0 < y < 1 \\ 2 & y = 1 \end{cases} \tag{13.1.5}$$

That is the inverse, though it is not continuous, since $\{2\}$ is open in D and its preimage is $\bar{f}^{-1}(2) = \{1\}$ which in fact is not open in C. In fact, C and D are not homeomorphic, not by f anyway.

- Let us consider a space with 2 points, $X = \{a, b\}$. It has 4 subsets, $\mathcal{P}(X) = \{\varnothing, \{a\}, \{b\}, X\}$. A topology is a subset $\tau \subset \mathcal{P}(X)$.

  To fix a subset we only have to say, for each element of X, if it belongs or not to the subset. If we agree that we set 1 for *belong* and 0 for *not belong*, for example 10 means $\{a\}$, 01 means $\{b\}$, 00 means $\varnothing$ and 11 means X.
  That means for a set with n elements we have 2^n subsets and topologies have to be chosen among 2^{2^n} options. However, since we know $\varnothing$ and X are always there we just have to choose among 2^{2^n-2} candidates.
  For $n = 2$ we have $2^2 = 4$ candidates. We can list them as 00, 01, 10, 11, the first being the trivial topology, the last being the discrete topology. We have to decide whether the other two candidates meet the axioms to be topologies.

The 01 candidate is $\{\varnothing, \{b\}, X\}$. In this case we can check directly that it is closed by unions and intersections, hence it is a topology. The same for 10 candidate, namely $\{\varnothing, \{a\}, X\}$. Hence we have 4 topologies on X.

That is not the end of the story, though. We have 2^n maps $\phi : X \to X$, $n!$ of which are bijective maps, thus potentially homeomorphisms. Since X is finite the $n!$ bijective maps $\phi : X \to X$ are permutations. We already discuss how to denote permutations. For $n = 2$, $n! = 2$. The permutations are $\Pi_2 = \{\mathbb{I}, (1\,2)\}$. A permutation of X clearly extends to $\mathcal{P}(X)$. The identity always extends to the identity. The swap $(1\,2)$ swap $\{a\}$ with $\{b\}$ and leaves $\varnothing$ and X fixed.

In other words, the swap $(1\,2)$ maps the topology $\{\varnothing, \{a\}, X\}$ into $\{\varnothing, \{b\}, X\}$ and vice versa. These two topologies on X are homeomorphic and on X we have 4 topologies and 3 classes up to homeomorphisms.

- We can try this way for $X = \{a, b, c\}$. The subsets are 8, two of which are always included in any topology. We have to decide whether the other 6 subsets belong to the topology or not. Hence we have $2^6 = 64$ candidates.

We order the 6 subsets as $(\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\})$ and we denote a candidate by a string of 6 binary digits. For example, $011\,100$ corresponds to the candidate $\{\varnothing, \{b\}, \{c\}, \{a, b\}, X\}$, which, as one can easily check, does not obey the axioms (for example, $\{b\} \cup \{c\}$ is not an openset).

In this case, we see we have 29 topologies on X corresponding to

$$
\begin{array}{cccccc}
000\,000 & 000\,010 & 001\,100 & 100\,010 & 010\,101 & 110\,110 \\
100\,000 & 000\,001 & 010\,010 & 001\,010 & 001\,011 & 011\,101 \\
010\,000 & 110\,100 & 100\,001 & 010\,001 & 101\,110 & 011\,011 \\
001\,000 & 101\,010 & 010\,100 & 001\,001 & 110\,101 & 111\,111 \\
000\,100 & 011\,001 & 100\,100 & 100\,110 & 101\,011 &
\end{array}
\tag{13.1.6}
$$

while the other 35 candidates fail to obey the axioms.

If this is already quite tedious, the hard part is to write down the action of permutations Π_3 on topologies to determine the classes of homeomorphisms. For example the permutation (123) acts on subsets as

$$
\begin{array}{cccccc}
a & b & c & ab & ac & bc \\
b & c & a & bc & ab & ac
\end{array}
\tag{13.1.7}
$$

In other words, the permutation (123) of Π_3 acts in subsets as the permutation (123)(465) in Π_6. In this way, we lift the action of permutations from X to topologies. For example the homeomoprhism represented by the permutation (123) acts on the topology $101\,110$ to obtain the topology $110\,101$. These two topologies then belong to the same class of homeomorphism.

By repeating the argument for all the 6 permutations and all the 29 topologies we obtain 9 classes

$$
\begin{gathered}
000\,000 \\
100\,000 = 010\,000 = 001\,000 \\
000\,100 = 000\,010 = 000\,001 \\
110\,100 = 101\,010 = 011\,001 \\
001\,100 = 010\,010 = 100\,001 \\
010\,100 = 100\,100 = 100\,010 = 001\,010 = 010\,001 = 001\,001 \\
100\,110 = 010\,101 = 001\,011 \\
101\,110 = 110\,101 = 101\,011 = 110\,110 = 011\,101 = 011\,011 \\
111\,111
\end{gathered}
\tag{13.1.8}
$$

Of course, it is clear that this is not the way to go. For a set of 10 elements we have $2^{1022} \simeq 4.5 \cdot 10^{307}$ candidates, 8977053873043 topologies and 4717687 classes of topologies up to homeomorphisms. Not to mention, that usually one discusses topologies for infinite sets.

Order of topologies

All possible topologies on a set X are partially ordered by inclusion $\tau_1 \subset \tau_2$. In that case, we say that τ_1 is *coarser* (or *weaker*) topology than τ_2 or that τ_2 is *finer* (or *stronger*) than τ_1. Of course, there are topologies on X that cannot be compared. The trivial topology is the weakest topology of all, the discrete one the strongest.

If we consider a continuous map $\phi : (X, \tau_X) \to (Y, \tau_Y)$, then $\phi : (X, \tau_X) \to (Y, \tau_Y')$ is continuous if τ_Y' is weaker than τ_Y. Also $\phi : (X, \tau_X') \to (Y, \tau_Y)$ is continuous if τ_X' is stronger than τ_X.

We can generalize the induced topology on a subset. Consider a topological space (X, τ_X) and a set map $\phi : Y \to X$. Then we can define a topology on Y, i.e. $\tau_Y = \{\phi^{-1}(U) : U \in \tau_X\}$, which is called the *topology induced* by the map ϕ. The induced topology by ϕ makes the map ϕ continuous. As a matter of fact it is the weakest topology that makes the map ϕ continuous.

> If one removes any openset from τ_Y then the map ϕ becomes discontinuous.

The induced topology on a subset is the topology induced by the inclusion map $i : Y \to X$.

Analogously, if (X, τ_X) is a topological space and $\pi : X \to Y$ is a surjective map, one can define on Y a topology

$$\tau_Y = \{W \subset Y : \phi^{-1}(W) \in \tau_X\} \qquad (13.1.9)$$

which is called the *quotient topology*. Also in this case the map π turns out to be continuous. The quotient topology defined on Y is the strongest topology for which π is continuous.

> If we add any other openset to the quotient topology τ_Y then its preimage is not an openset of X and π is not continuous any longer.
>
> Let us remark that if we removed the request that π is surjective, all points in Y which are not in the image of π would bet a discrete topology.

Neighbourhoods of a point

For any point $x \in X$, an *neighbourhood* of x is a set ν_x such that there exists an open set $U_x \in \tau_X$ such that $x \in U_x \subset \nu_x$. A neighbourhood ν_x can be open, closed, or anything else. The set of all open neighbourhoods of a point x is denoted by N_x.

By definition, if $\nu \in N_x$ and $\nu \subset U \in \tau$, then $U \in N_x$. If $\nu, \nu' \in N_x$, then $\nu \cap \nu' \in N_x$.

A family of open neighbourhoods $B_x \subset N_x$ is called a *neighbourhood basis* for the point x iff for any open neighbourhood $U_x \in \tau_X$ of x there exists a neighbourhood $V_x \in B_x$ such that $V_x \subset U_x$.

Given a neighbourhood basis B_x at any point x, then open sets are defined as arbitrary unions of elements in $B = \cup_{x \in X} B_x$.

> Of course, arbitrary unions of elements in B are open sets, since $B \subset \tau_X$.
>
> On the other hand, let $U \in \tau_X$ and $x \in U$. Then U is a neighbourhood of x and there exists $V_x \in B_x$ such that $V_x \subset U$. Thus one can define $W = \cup_{x \in U} V_x$. Obviously, $W \subset U$ and $U \subset W$, hence $W = U$, i.e. $U = \cup_{x \in U} V_x$.

A subset $D \subset X$ is *dense* iff for all $x \in X$ and all $U \in N_x$, one has $U \cap D \neq \varnothing$.

For a subset $S \subset X$ consider the set $K(S) := \{C \in \chi : S \subset C\}$ of all closed sets containing S and define $\overline{S} = \cap_{C \in K(S)} C$. Then $\overline{S} \in \chi$ is a closed set, called the *closure* of S. By construction it is the smallest closed set containing S.

Also consider the set $O(S) := \{U \in \tau : U \subset S\}$ of all opensets contained into S and define $\underline{S} = \cup_{U \in O(S)} U$. Then $\underline{S} \in \tau$ is an openset, called the *interior* of S. By construction it is the largest openset contained into S.

Of course, we have $\underline{S} \subset S \subset \overline{S}$. We also define the *boundary* of S to be the closed set $\partial S =\subset \overline{S} \cap \overline{X - S}$.

A point $x \in \underline{S}$ is also called an *internal* point to S. A point $x \in \partial S$ is also called a *boundary* point to S, while a point in $X - \overline{S}$ is called an *external* point to S.

A *limit point* of S is $x \in X : x \in \overline{S - \{x\}}$, and an *isolated point* of S is $x \in S :\notin \overline{S - \{x\}}$.

The set of all limit points of S is denoted by $L(S)$.

One can easily show that a point of S is internal iff it is there contained with an open neighbourhood and that $x \in \overline{S} \iff \forall U \in N_x : S \cap U \neq \varnothing$.

> No, nobody cares if that sounds good, you need to prove because your intuition is based on the standard topology only!
>
> That is equivalent to its negation $x \notin \overline{S} \iff \exists U \in N_x : S \cap U = \varnothing$.

[$\Rightarrow$] Consider $x \notin \overline{S} \Rightarrow x \in X - \overline{S} \in \tau \Rightarrow \exists U_x \subset X - \overline{S}$. Thus $X - \overline{S}$ is an open neighbourhood of x.
$S \cap (X - \overline{S}) \subset \overline{S} \cap (X - \overline{S}) = \varnothing$. Thus there exists a neighnourhood $U = X - \overline{S} : S \cap U = \varnothing$.
[$\Leftarrow$] $S \cap U = \varnothing \Rightarrow S \subset X - U \in \chi \Rightarrow \overline{S} \subset X - U \in \chi \Rightarrow \overline{S} \cap U_x = \varnothing \Rightarrow x \notin \overline{S}$. ∎

Try and prove $\overline{S} = S \cup L(S)$ and $\overline{S} = S \cup \partial S$.

As a corollary we know that if $S \in \chi$ is closed then $S = \overline{S}$ iff $L(S) \subset S$.

$\quad S \in \chi \iff S = \overline{S} \iff S = S \cup L(S) \iff L(S) \subset S$. ∎
$\quad$ A subset $S \subset X$ is closed iff it contains all its limit points.

A *basis* for a topology τ on X is a collection of opensets $\mathcal{B} \subset \tau$ such that $\forall U \in \tau, \exists B \in \mathcal{B} : B \subset U$. Given a basis any openset can be written as the union of elements in the basis. Intervals are a basis for the standard topology of $\mathbb{R}$, balls are a basis for the standard topology in a space with a distance. The union $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$ of neighbourhood bases at a point is a basis.

Given two topologies (X, τ_1) and (X, τ_2) on the same set X and their respective neighbourhood bases $\mathcal{B}_x^1$ and $\mathcal{B}_x^2$ at any point x, then we have that $\tau_1 \subset \tau_2$ iff $\forall x \in X, \forall U \in \mathcal{B}_x^1, \exists V \in \mathcal{B}_x^2 : V \subset U$. Hence $\tau_1 = \tau_2$ iff $\forall x \in X, \forall U \in \mathcal{B}_x^1, \exists V \in \mathcal{B}_x^2 : V \subset U$ and $\forall x \in X, \forall V \in \mathcal{B}_x^2, \exists U \in \mathcal{B}_x^1 : U \subset V$.

A *sub-basis* is a family $\mathcal{S} \subset \tau$ such that the collection $\mathcal{B} = \{\cap_1^n S_i : S_i \in \mathcal{S}\}$ of all finite intersections of elements of the sub-basis form a basis. Given a sub-basis $\mathcal{S}$, opensets are defined as arbitrary unions of finite intersections of $S_\alpha \in \mathcal{S}$.

More examples of topologies

Given family of topological spaces (X_α, τ_α) with $\alpha \in A$ we can define their *topological sum* to be the union $X = \cup_A X_\alpha$ with the *sum topology* defined as

$$\tau = \{U \subset X : \forall \alpha \in A : U \cap X_\alpha \in \tau_\alpha\} \tag{13.1.10}$$

The topology τ is the strongest topology on X for which all inclusions $i_\alpha : X_\alpha \to X$ are continuous, which is a generalization of the induced topology.

We can also define the *topological product* $\Pi X := \Pi_A X_\alpha = \{(x_\alpha) : \forall \alpha \in A : x_\alpha \in X_\alpha\}$ equipped with the *product topology*, which is the weakest topology $\Pi \tau_\alpha$ on ΠX fo which all projections $p_\alpha : \Pi X \to X_\alpha : (x_\alpha) \mapsto x_\alpha$ are continuous. In the product topology opensets are arbitrary unions of basic neighbourhoods $B_\alpha = \{(x_\alpha) : x_\alpha \in U_\alpha \in \tau_\alpha$ and only finitely many $U_\alpha \neq X_\alpha\}$.

$\quad$ Indeed, for a finite product B_α are rectangular basis, namely the product of basic neighbourhoods in each factor. For infinite products, one allows non-trivial neighbourhoods only in a finite number of components, which in almost all of them we take $U_\alpha = X_\alpha$.

A *distance* on X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following properties

(1) $d(x, x) = 0$;
(2) if $x \neq y$ then $d(x, y) > 0$ $\qquad\qquad\qquad\qquad$ (positivity);
(3) $d(x, y) = d(y, x)$ $\qquad\qquad\qquad\qquad\qquad$ (symmetry);
(4) $d(x, z) \leq d(x, y) + d(y, z)$ $\qquad\qquad\qquad$ (triangular inequalities).

Sometimes a distance is called a *metric*. For us a metric will have an important different meaning thus we use *distance* instead.

The typical example of a space with a distance is a vector space V with a Euclidean inner product δ, which is a metric space by defining $d^2(v, w) = \delta(v - w, v - w)$.

This chain of canonical constructions (vector space, inner product out of a basis, distance out of the inner product, topology out of the distance) is a endless source of misunderstandings.

As a special case, we can fix an orthonormal basis so that $V \simeq \mathbb{R}^n$ and the inner product is in canonical form. As a further special case, we have $\mathbb{R}$ in which $d(v, w) = |v - w|$.

- A set X with a distance d is called a *metric space* and it is denoted by (X, d). In any metric space (X, d) we can define a *standard topology*, also called the *metric topology*, or the *topology of balls*, by starting from a neighbourhood basis $B_x = \{B_r(x)\}$ where

$$B_r(x) = \{y \in X : d(y, x) < r\} \tag{13.1.11}$$

is called a *ball* of radius r and center x. The topology is then defined by saying that an openset U is a union of balls or, equivalently, if for all $x \in U$ there exists a ball $B_r(x) \subset U$ (so that $U = \cup_{x \in U} B_r(x)$). Accordingly, a metric space is always a topological space with a neighbourhood basis made of balls, and that includes $\mathbb{R}^n$ with its standard topology.

A topological space homeomorphic to a space with distance is called a *metrizable space*.

- On $\mathbb{R}$ we can define a basis of neighbourhoods $N_x = \{B_r(x) := [x - r, x + r) \subset \mathbb{R} : r > 0\}$. The topology generated by these is called the *right half-open topology*. In this topology the intervals $[a, b)$ are open.

- Consider 2 copies of $\mathbb{R}$ and their disjoint union $\hat{X} = \mathbb{R} \coprod \mathbb{R}$. A point in $\hat{X}$ is a pair (α, x) with $\alpha = 1, 2$ and $x \in \mathbb{R}$.

On $\hat{X}$ we can identify

$$(\alpha, x) \sim (\beta, y) \iff \alpha \neq \beta \wedge x = y \neq 0 \tag{13.1.12}$$

In the quotient space $X = \hat{X}/\sim$ we have points $\mathbf{x} = \{(1, x), (2, x)\}$ for all $x \neq 0$, and $0_1 = (1, 0)$ as well as $0_2 = (2, 0)$ (which in fact are not identified).

Being a quotient we have a surjective map $\pi : \hat{X} \to X$, thus we can define on X the quotient topology τ_X. Then (X, τ_X) is a topological space and it is called the *line with 2 origins*.

On X, there are two types of open intervals. If $\mathbf{a}, \mathbf{b}$ are both positive or both negative, the interval $(\mathbf{a}, \mathbf{b})$ is an open interval. On the other hand, intervals that contain the origin may contain 0_1, or 0_2, or both $\{0_1, 0_2\}$. Indeed, these are open intervals in X, open sets are arbitrary unions of open intervals.

- On $\mathbb{R}$, we can define the relation $x \sim y \iff y - x = 2k\pi$ for some $k \in \mathbb{Z}$. This is an equivalence relation, the quotient set is denoted by $S_1 = \mathbb{R}/\sim$. It can be equipped with the quotient topology, so that it is a topological space called the *circle*.

One can show S_1 is homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ equipped with the induced topology, which is hence denoted by S_1 as well.

- Consider $\mathbb{R}^2$ with the following equivalence relation

$$(x, y) \sim (x', y') \iff \exists a, b \in \mathbb{Z} : x' = x + 2a\pi, y' = y + 2b\pi \tag{13.1.13}$$

The quotient set together with the quotient topology is called a *torus* and it is denoted by $\mathbb{T}_2$.

It is homeomorphic to the product $S_1 \times S_1$, to the subsets in $\mathbb{R}^4$ (with the induced topology)

$$\{(x, y, w, z) \in \mathbb{R}^4 : x^2 + y^2 = 1 \wedge w^2 + z^2 = 1\} \tag{13.1.14}$$

as well as with the subset of $\mathbb{R}^3$ (with the induced topology) which can be parameterized by two angles (α, β) as

$$x = \cos\alpha(2 + \cos\beta) \qquad y = \sin\alpha(2 + \cos\beta) \qquad z = \sin\beta \tag{13.1.15}$$

These 3 torus are homeomorphic, hence they represent the same topological space (up to homeomorphisms).

> We shall show they also are diffeomorphic, so that they will represent the same smooth manifold (up to diffeomorphisms). However, when one consider them as Riemannian manifolds (namely smooth manifold with a metric defined on them) they are not isometric, thus they define different Riemannian spaces. In particular, the first 2 are flat manifolds, while the 3rd one is curved.
> This will highlight how *being the same space* is not something only about the object itself. On the contrary it is a categorical concept in which transformations play a major role. Transformations decide if two objects are the same or not. Surprisingly enough, that has a lot to do with physics, especially in a relativistic theory.

• Consider $X = \mathbb{R} \times [0, 1)$. A point in X is in the form (x, α), where clearly $x \in \mathbb{R}$ and $\alpha = [0, 1)$. We have a total order on X defined as $(x, \alpha) < (y, \beta) \iff x < y \vee (x = y \wedge \alpha < \beta)$.

Because X is totally order we can define open intervals $(a, b) := \{x \in X : a < x < b\}$ and rays (which come in 2 types *right rays* and *left rays*, i.e. $(a >:= \{x \in X : a < x\}$ and $< a) := \{x \in X : x < a\})$. As usual these generate a topology, called the *order topology*, by saying that opensetes are arbitrary unions of these internals.

Then X with the order topology τ_X is a topological space..

> In the proof above about the topology defined on $\mathbb{R} \times [0, 1)$ all we used is the order in $\mathbb{R}$ and $[0, 1)$.
> We can consider $\{0, 1, \ldots, n\} \times [0, 1)$ and define on it the order topology as we did on $\mathbb{R} \times [0, 1)$. It turned out that with that topology $\{0, 1, \ldots, n\} \times [0, 1)$ is homeomorphic to $[0, n) \simeq [0, 1)$.
> Next, we can repeat the argument for $\omega_0 \times [0, 1)$ which turned out to be homeomorphic to $R^+ := \{x \in \mathbb{R} : x \geq 0\}$.
> Although we have very little insight about ω_1 the only clear things is that it comes with a well-order on it that makes all its subsets countable. That is more than enough for repeating the above argument and define a topology on $\omega_1 \times [0, 1)$.
> Unfortunately, that is the definition of a topological space which is called the *long line*. It is not homeomorphic to a line.
> As a side remark, $\mathbb{R} \times [0, 1)$ is much longer than the long line despite its simpler definition.

• Consider a rectangle $\hat{X} = [0, 1] \times [-1, 1] \subset \mathbb{R}^2$ on which we identify

$$(x, y) \sim (x', y') \Leftarrow x = x' > 0 \tag{13.1.16}$$

In the quotient space $X = \hat{X}/\sim$ there are two types of points: $\mathbf{x} = \{(x, y) : \forall y \in [-1, 1], x > 0\}$ for all $x \in (0, 1)$ and $\mathbf{0}_y = \{(0, y)\}$ for all $y \in [-1, 1]$.

We can define the quotient topology τ_X on X and the topological space (X, τ_X) is called the *Warsaw space*.

We have 2 types of neighbourhoods: $\forall 0 < a < b : (\mathbf{a}, \mathbf{b}) := \{\mathbf{x} \in X : 0 < a < x < b\}$ and $\forall y \in [-1, 1], 0 < a : [\mathbf{0}_y, \mathbf{a}) := \{\mathbf{0}_y\} \cup \{\mathbf{x} \in X : 0 < x < a\}$. As usual, opensets are arbitrary unions of neighbourhoods.

Countability and separability

A topological space (X, τ) is called *first-countable* iff, for each $\bar{x} \in X$, one has a countable neighbourhoods basis, namely a countable family $B_{\bar{x}} = \{U_i : i \in \mathbb{N}, \bar{x} \in U_i\}$ such that for any neighbourhood $U_{\bar{x}}$ there exists $U_i \in B_{\bar{x}} : U_i \subset U_{\bar{x}}$.

> Let $B_{\bar{x}} = \{B_i : i \in \mathbb{N}\}$ be a countable neighbourhoods basis. We can always consider a new countable neighbourhood basis $B'_{\bar{x}} = \{B'_i : i \in \mathbb{N}\}$ in which $B'_i \subset B'_j$ whenever $i \leq j$, e.g. just take $B'_i = \cap_{i \leq k} B_k$ and show it is again a neighbourhood basis.
>
> Thus there is no loss of generality in assuming that on first countable spaces one has a countable neighbourhood basis $B_{\bar{x}} = \{B_i : i \in \mathbb{N}, (i \leq j \Rightarrow B_i \subset B_j)\}$.

Balls in $\mathbb{R}^n$ are an example of a neighbourhood basis for the standard topology. The space $\mathbb{R}^n$ with the standard topology is also first-countable since one can use a basis with rational radius, which is countable.

A topological space (X, τ) is called T_0 (or *Kolmogorov space*) if, for any two distinct points $x, y, \in X$ ($x \neq y$), there exists an open set $U \in \tau$ such that either U contains x but not y or $y \in U$ but $x \notin U$.

Of course, a topological space with the trivial topology is not T_0 if it has more than one point. Also, the product of T_0 spaces is T_0.

A topological space (X, τ) is T_1 (or a *accessible space*, or *Fréchet space*) iff, for any two distinct points $x, y \in X$, there exist an open neighbourhood $x \in U_x \in \tau$ such that $y \notin U_x$.

> The space $X = \{a, b\}$ with the topology $\tau = \{\varnothing, \{a\}, X\}$ is T_0 but not T_1.

Any T_1 space is also T_0.

The product of T_1 spaces is T_1.

In a topological space (X, τ), X is T_1 iff any point singleton $\{x\}$ is a closed set.

> [$\Rightarrow$] Consider $X - \{x\} = \{y \in X : y \neq x\}$. Then for any $y \in X - \{x\}$ there exists a neighbourhood V_y such that $x \notin V_y$, hence $V_y \subset X - \{x\}$. Accordingly, the set $X - \{x\} = \cup_{y \in X - \{x\}} V_y$ is open. Hence $\{x\}$ is closed.
> [$\Leftarrow$] $\{x\} \in \chi$ and $y \neq x$: $X - \{x\} \in \tau$ is open, $y \in X - \{x\}$, and $x \neq X - \{x\}$ ∎

A topological space (X, τ) is T_2 (or *Hausdorff*, or *separated*) iff there exist two open neighbourhoods $U_x \in \tau$ and $U_y \in \tau$ such that $U_x \cap U_y = \varnothing$.

The standard topology on $\mathbb{R}^n$ makes it a T_2 topological space. The line with 2 origins is T_1 (for example, 0_1 has a neighbourhood $\{0_1\} \cup \{x \neq 0 - a < x < a\}$ that does not contain 0_2) but not T_2 (0_1 and 0_2 have no disjoint neighbourhodds).

The product of T_2 spaces is T_2.

Theorem: *if Y is T_2 and $f, g : X \to Y$ are continuous, then $\{x \in X : f(x) = g(x)\}$ is closed.*

Proof: *let us define $S = \{x \in X : f(x) \neq g(x)\}$ and consider $x \in S$. There are two disjoint open sets $V_{f(x)}$ and $V_{g(x)}$ to which correspond two opensets $U_x = f^{-1}(V_{f(x)})$ and $U'_x = g^{-1}(V_{g(x)})$ both containing x. Then we can consider a neighbourdood $U = U_x \cap U'_x$ such that $f(U) \subset V_{f(x)}$ and $g(U) \subset V_{g(x)}$. Then for any point $y \in U$ we have that $f(y) \neq g(y)$ (since $f(y) \in V_{f(x)}$ and $g(y) \in V_{g(x)}$ which are disjoint). Hence $y \in S$, hence $U \subset S$.*

Accordingly, $x \in S$ is there contained with a whole neighbourhood U, hence S is open, hence its complement $X - S = \{x \in X : f(x) = g(x)\}$ is closed. ∎

A space (X, τ) is *regular* iff $\forall C \in \chi$ and $x \notin C : \exists U_C, U_x \in \tau : C \subset U_C, x \in U_x$ and $U_C \cap U_x = \varnothing$.

A regular T_1 space is called T_3.

A T_3 space is also T_2. The product of T_3 spaces is T_3.

A space (X, τ) is *normal* iff $\forall A, B \in \chi : \exists U_A, U_B \in \tau : A \subset U_A, B \subset U_B$ and $U_A \cap U_B = \varnothing$.

A normal T_1 space is called T_4.

Beware that the product of T_4 spaces is not necessarily T_4. A T_4 space is T_3. A metrizable space is T_4.

A space (X, τ) is normal iff $\forall A \in \chi, U \in \tau : A \subset U$ there exists $V \in \tau : A \subset V \subset U$.

Limits

Let us consider a *sequence of points* $x : \mathbb{N} \to X : n \mapsto x_n$ in a topological space (X, τ).

We say that the sequence x_n *converges* to a point $\bar{x} \in X$ iff for all neighbourhood $U_{\bar{x}} \in \tau$, there exists a $n_0 \in \mathbb{N}$ such that one has

$$\forall n > n_0 : x_n \in U_{\bar{x}} \tag{13.1.17}$$

In that case we say that the sequence is *eventually in $U_{\bar{x}}$*. We also say that $\bar{x}$ is the limit point of the sequence and write $\lim_{n \to \infty} x_n = \bar{x}$ for it.

Notice we did not say that any sequence has a limit. The sequence

$$x_{2n} = y_1 \qquad x_{2n+1} = y_2 \tag{13.1.18}$$

does not converges if $y_1 \neq y_2$, at least if there exists a neighbourhood U_1 of y_1 which does not contain y_2, which is certainly true if the topology on X is at least T_0.
Of course, in the trivial topology any sequence converges to any point. In the discrete topology only the eventually constant sequences converge (*eventually constant* meaning there exists a $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : a_n = \bar{a}$ is constant).
Nor we said that when the limit exists, it is unique.
However, if the topology is T_2 then the limit is unique.
In fact, suppose there are two distinct limit points y_1 and y_2. Since the topology is T_2 there exist two disjoint neighbourhoods U_1 and U_2. By the definition of limit point, the sequence should be eventually in U_1 and in U_2 which, however, are disjoint. ∎

In the line with two origins Y consider the sequence $x_n = 1/n$. That sequence converges both to 0_1 and 0_2.

Theorem: *If we consider a subset $S \subset X$, if (X, τ) is first-countable, for all $x \in \bar{S}$ there is a sequence $x_n \in S$ such that $\lim_{n \to \infty} x_n = x$.*

Proof: $\forall x \in \bar{S}$ we know $\forall U_x : U_x \cap S \neq \varnothing$ We can consider a countable family B_n of neighbourhoods of x, such that $\ldots \subset B_n \ldots \subset B_2 \subset B_1 \subset B_0$.

Once can choose $x_n \in B_n(x) \cap S$ to be a sequence in S. Moreover, for all open neighbourhoods V_x of x, there exists $n_0 : B_{n_0}(x) \subset V_x \Rightarrow \forall n > n_0 : x_n \in V_x$. Thus we have $\lim_{n\to\infty} x_n = x$. ∎

Given a map $\phi : X \to Y$, we say that it *converges to $\bar{y}$, for $x \to \bar{x}$*, iff for any (eventually non-constant) sequence x_n converging to $\bar{x}$, the sequence $\phi_n = \phi(x_n)$ converges to $\bar{y}$. In that case, we write

$$\lim_{x \to \bar{x}} \phi(x) = \bar{y} \tag{13.1.19}$$

The map $\phi : X \to Y$ is *continuous at $\bar{x}$* iff $\lim_{x\to\bar{x}} \phi = \phi(\bar{x})$.

A map is continuous iff it is continuous at any point.

Theorem: *given a map $\phi : X \to Y$ and setting $\bar{y} := \phi(\bar{x})$; if $\forall U_{\bar{y}} \exists U_{\bar{x}} : (x \in U_{\bar{x}} \Rightarrow \phi(x) \in U_{\bar{y}})$ then ϕ is continuous at $\bar{x}$.*

Notice that it reproduces the usual definition of limits in $\mathbb{R}^n$ with the standard topology, where $x \in U_{\bar{x}}$ specifies to $|x - \bar{x}| < \delta$.

Proof: *for any sequence x_n converging to $\bar{x} \in U_{\bar{x}}$, there exists $n_0 \in \mathbb{N} : (n > n_0 \Rightarrow x_n \in U_{\bar{x}})$. Then, by hypothesis, $\phi_n = \phi(x_n) \in U_{\bar{y}}$, which is $\lim_{n\to\infty} \phi_n = \bar{y}$.* ∎

Lemma *(13.1.20): given a map $\phi : X \to Y$, a point $\bar{x} \in X$, and setting $\bar{y} = \phi(\bar{x})$;*

$(\forall U_{\bar{y}}, \exists U_{\bar{x}} : \phi(U_{\bar{x}}) \subset U_{\bar{y}})$ *iff for all subsets $S \subset X$ such that $\bar{x} \in \bar{S}$, one has $\phi(\bar{x}) \in \overline{\phi(S)}$.*

Proof: *[⇒] consider any subset $S \subset X$ such that $\bar{x} \in \bar{S}$ (so that $\forall U_{\bar{x}} : U_{\bar{x}} \cap S \neq \varnothing$) and a neighbourhood $U_{\bar{y}}$ of $\bar{y} = \phi(\bar{x})$.*

Then $\phi(U_{\bar{x}}) \cap \phi(S) \neq \varnothing$, which means that $U_{\bar{y}} \cap \phi(S) \neq \varnothing$. Thus $\bar{y} \in \overline{\phi(S)}$.

[⇐] for all $U_{\bar{y}}$, consider $S = \phi^{-1}(U_{\bar{y}})$. Then, by hypothesis, $\phi(\bar{x}) \in \overline{\phi(S)}$. Then, $\forall V_{\phi(\bar{x})} : V_{\phi(\bar{x})} \cap \phi(S) \neq \varnothing \Rightarrow \phi^{-1}(V_{\phi(\bar{x})}) \cap S \neq \varnothing$.

There exists $U_{\bar{x}} \subset \phi^{-1}(V_{\phi(\bar{x})}) \iff \phi(U_x) \subset V_{\phi(x)}$. ∎

Theorem: *if X is first-countable and $\lim_{x\to\bar{x}} \phi = \phi(\bar{x})$ then $\forall U_{\phi(\bar{x})} \exists U_{\bar{x}} : \phi(U_{\bar{x}}) \subset U_{\phi(\bar{x}))}$.*

Proof: *let us consider a subset $S \subset X : \bar{x} \in \bar{S}$. Then, since $\bar{x} \in S$, there exists a sequence $x_n \in S$ converging to $\bar{x}$ and we can set $y_n := \phi(x_n)$. Then, by hypothesis, $\lim_{n\to\infty} y_n = \phi(\bar{x})$. Since $y_n \in \phi(S)$, then $\bar{y} := \phi(\bar{x}) \in \overline{\phi(S)}$. For each $S \subset X : \bar{x} \in \bar{S} \Rightarrow \phi(\bar{x}) \in \overline{\phi(S)}$. The thesis follows by Lemma (13.1.20).* ∎

As a matter of fact, we have to competing definitions of what it means to be continuous at a point x, one based on sequences, one the usual definitions based on neighbourhoods of $\bar{x}$. The 2 definitions are equivalent in first-countable spaces.

In a more general topology, it is still true that the one based on neighbourhoods implies the one based on sequences, not the other way around. The one based on sequences is indeed stricter and we chose it to be more fundamental.

Connectiveness

A topological space (X, τ) is *connected* iff the only subsets which are both open and closed are $\varnothing$ and X. A subset is connected if it is connected in its induced topology. A maximal connected subset is called a *connected component*. The number of connected components in a topological space is clearly invariant under homeomorphism, so that a space with 2 connected components cannot be homeomorphic to a connected space.

Theorem: *let Y be a discrete space. (X, τ) is a connected space iff all continuous maps $\phi : X \to Y$ are constant on X.*

Proof: *Let us consider $x \in X$, $y := \phi(x)$. Since Y has the discrete topology, $\{y\}$ and $Y - \{y\}$ are open sets, hence, being ϕ continuous, their preimages $U = \phi^{-1}(y)$ and $X - U = \phi^{-1}(Y - \{y\})$ are opensets in X. Accordingly, U is both open and closed and it contains x.*

Hence X is connected iff $U = X$ iff $\phi(U) = \{y\}$ is constrant. ∎

Consequently, if X is the disjoint union of a number of connected components X_α, then any continuous map $\phi : X \to Y$ in a discrete space Y is constant on any connected component. These kind of maps are also called *locally constant maps*.

If X is the union of different connected components $X = \cup_i X_i$, then, by essentially the same argument, a map $\phi : X \to Y$ into a discrete space Y is constant on each connected component X_i. Such a maps are also called *locally constant maps*.

The product of connected spaces is connected.

The image of a connected space X along a continuous map $\phi : X \to Y$ is connected.

Suppose the image $\phi(X) \subset Y$ were not connected, then there exists a continuous map $\delta : \phi(X) \to D$ into a discrete D is not constant, iff $\delta \circ \phi : X \to D$ is not constant, iff X is not connected.

Notice that in $\mathbb{R}$ with the standard topology, a subset $I \subset \mathbb{R}$ is connected iff it is an internal.

[$\Rightarrow$] If S is connected and not an interval. Then there exists $a, b \in S$ and $c \in \mathbb{R} - S$ such that $a < c < b$. Define $U_a := \{x \in S : x < c\}$ and $U_b = \{x \in S : x > c\}$. We see that $a \in U_a \neq \varnothing$, $b \in U_b \neq \varnothing$, $U_a \cup U_b = S$, and $U_a \cap U_b = \varnothing$. Then S is not connected.

[$\Leftarrow$] if I is an interval and not connected, then there exists a continuous function $\delta : I \to \mathbb{Z}_2$ which is locally constant. Define $U_0 = \delta^{-1}(0)$ and $U_1 = \delta^{-1}(1)$. Take $a \in U_0$ and $b \in U_1$ and assume, with no loss of generality, that $a < b$.

Let us set $U = \{x \in I : x < b, \delta(x) = 0\}$. Then $U \subset \mathbb{R}$ is bounded from above by b and consequently, it has a lower upper bound $\alpha = \sup(U)$, hence there exists a sequence $a : \mathbb{N} \to I$ converging to α. On this sequence δ is constantly 0, then $\delta(\alpha) = 0$.

However, if there exists $x \in (\alpha, b] : \delta(x) = 0$ then α would not be the lower upper bound. Then $\delta((\alpha, b]) = \{1\}$. For any sequence $b : \mathbb{N} \to (\alpha, \beta]$ converging to α, one has $\delta(b_n) \to 1$, hence $\delta(\alpha) = 1$. That is a contradiction is δ is continuous. ∎

Theorem: *if the map $\phi : [a, b] \to \mathbb{R}$ is continuous then $[\phi(a), \phi(b)] \subset \phi([a, b])$*

Proof: *$[a, b]$ is an interval, $\phi([a, b])$ is an interval. If $\phi(a) < \phi(b)$ then $[\phi(a), \phi(b)] \subset \phi([a, b])$.* ∎

Corollary: *$\phi : X \to \mathbb{R}$ is continuous, X is connected, and $\phi(a) < \phi(b)$, then $\forall y : \phi(a) \leq y \leq \phi(b) : y \in \mathrm{Im}\phi$.* ∎

Corollary: *$\phi : [a, b] \to [a, b]$ is continuous, $\exists x \in [a, b] : \phi(x) = x$.* ∎

Proof: *Let us consider $g : [a, b] \to \mathbb{R} : x \mapsto g(x) = \phi(x) - x$. Since $\phi : [a, b] \to [a, b]$, $a \leq \phi(a) \leq b$ and $a \leq \phi(b) \leq b$, then $g(a) = \phi(a) - a \geq 0$ and $g(b) = \phi(b) - b \leq 0$.*

Then there exists $x \subset n[a, b] : g(x) = 0$, iff $\phi(x) = x$ ∎

This is the first instance of fixed point theorems.

We call a *path* a continuous map $\gamma : [0, 1] \to X$. the point $x = \gamma(0)$ is called the *origin* of the path, $y = \gamma(1)$ is called the *target*. A path for which $\gamma(0) = \gamma(1)$ is called a *loop*.

Any continuous map $\gamma' : [a, b] \to X$ can be reduced to a path by a suitable reparameterization. An *arc* is a path $\gamma : [0, 1] \to X$ which is a homeomorphisms on the image.

A space (X, τ) is called *path-connected* if for any 2 points x, $y \in X$ there exists a path from $\gamma(0) = x$ to $\gamma(1) = y$.

Theorem: *If (X, τ) is path-connected, then it is connected*

Proof: *Suppose X is path-connected and not connected. There are two non-empty, disjoint opensets U_1, U_2 that cover X. Let $x \in U_1$ and $y \in U_2$, since X is path connected there exists a path $\gamma : [0,1] \to X$ from x to y. Consider $0 \in I_1 = \gamma^{-1}(U_1)$ and $1 \in I_2 = \gamma^{-1}(U_2)$. The sets I_1 and I_2 are 2 non-empty, disjoint, open sets that cover $[0,1]$. That is a contradiction, since $[0,1]$ is connected being an interval.* ∎

The product of path-connected spaces is path-connected. If the map $\phi : X \to Y$ is continuous and X is path-connected, then $\phi(X)$ is path-connected.

We have a counter example to the conjecture that connected implies path-connected.

Consider $X = \{(x,y) \in \mathbb{R}^2 : y = \sin(\frac{1}{x}) : x \in (0,1]\} \cup \{(0,0)\}$ with the topology induced by $\mathbb{R}^2$. One can show X is connected. Let us show here it is not path-connected, in particular, there is no continuous map $\gamma : [0,1] \to X$ from $(0,0)$ to $(1, \sin(1)) \in X$. Let us remark that for $x(s) > 0$ one has $y(s) = \sin\left(\frac{1}{x(s)}\right)$.

If such a path existed, then $\lim_{s \to 0} \gamma(s) = (0,0)$: consider the ball $B_{\frac{1}{2}}((0,0))$ and let us show that there is a subsequence to $(0,0)$ eventually out of the ball. It is enough to consider $x = \frac{2}{(4k+1)\pi}$ for $k \in \mathbb{Z}$ on which $y = 1$ is constant, and outside the ball. ∎

Paths are also important to define the *fundamental group* of a space. Being it an topological invariant, it is important to prove that spaces are not homeomorphic (when they have different fundamental groups). The fundamental group deserve a detailed treatment on their own.

Compactness

Given a topological space (X, τ), an *open cover* of a subset $S \subset X$ is a family of opensets $\{U_\alpha \in \tau : \alpha \in I\}$ such that $S \subset \cup_{\alpha \in I} U_\alpha$.

An open cover $\{V_j : j \in J\}$ is a *subcover* of $\{U_i : I \in I\}$ iff there is a map $\iota : J \to I$ such that $\forall j \in J : V_j = U_{\iota(j)}$. The subset $S \subset X$ in a space (X, τ) is *compact* iff for any open cover $\{U_i : I \in I\}$ there is a locally finite subcover $\{V_j : j \in J \to I\}$ defined by an (injective) map $\iota : J \to I$.

Theorem: *the continuous image of a compact subset is compact.*

Proof: *let $\phi : X \to Y$ be a continuous map, X compact and $\{U_\alpha : \alpha \in I\}$ an open cover of the image $\phi(X) \subset Y$. Consider $V_\alpha = \phi^{-1}(U_\alpha)$; the family $\{V_\alpha : \alpha \in I\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{V_\alpha : \alpha \in J\}$ given by $\iota : J \to I$ of X.*

Then $\{U_\alpha \in J\}$ is a finite subcover of $\{U_\alpha : \alpha \in I\}$ and it still covers the image $\phi(X)$. Therefore $\phi(X)$ is compact ∎

Definition: *given a map $\phi : X \to Y$ such that any preimage $\phi^{-1}(K)$ of a compact set $K \subset Y$ is compact, then the map ϕ is called a *proper map*.*

If $C \subset X$ is closed and X is compact then C is compact.

If $C \in \chi$ is closed and $\{U_\alpha : \alpha \in I\}$ is an open cover of C, then $\{U_\alpha : \alpha \in I'\}$ is an open cover of X obtained by assuming $0 \notin I$, setting $I' = I \cup \{0\}$ and $U_0 = X - C$. Since X is compact, there exists a finite subcover $\{U_\alpha : \alpha \in J'\}$, which in particular still covers C. Then C is compact. ∎

Theorem: *$K \subset \mathbb{R}$ is compact iff K is closed and bounded.*

Proof: *let us split the proof into 3 steps.*

[*K* is compact, then *K* is bounded]: *consider $x \in K$ and the balls $B_n(x)$ around it of radius $n \in \mathbb{N}$. Then $K \subset X \subset \cup_{n \in \mathbb{N}} B_n(x)$ then there is a finite subcover $J \subset \mathbb{N}$ that still cover K. Since $n_1 < n_2 \Rightarrow B_{n_1}(x) \subset B_{n_2}(x)$, there exists $n_0 : K \subset B_{n_0}(x)$, which is pretty much what we mean by bounded.*

[*K* is compact, then *K* is closed]: *consider $x \in X - K$. Then $\forall y \in K$ we have that $x \neq y$, hence, since $\mathbb{R}$ is T_2, there exists 2 disjoint neighbourhoods $x \in V_y, y \in U_y$. Thus $\{U_y : y \in K\}$ is an open cover of K and there exists a finite subcover $\{U_y : y \in \{y_1, y_2, \ldots, y_n\}\}$.*

On the other hand, consider $U_x = \cap_{i=1}^{n} V_{y_i}$; that is openset and it contains x. If it contained a point $y \in K$, then y would not belong to any of the corresponding U_{y_i} and $\{U_y : y \in \{y_1, y_2, \ldots, y_n\}\}$ would not cover K. Thus U_x is an open neighbourhood of x contained in $X - K$. Then $X - K = \cup_{x \in X - K} U_x$ is openset, then K is closed.

[Let us remark there are shorter proofs on $\mathbb{R}$ which take advantage of $\mathbb{R}$ having further properties than being T_2. This proof, however, extend to any T_2 space.]

[*K* is a closed and bounded interval in $\mathbb{R}$, then it is compact] *[we prove this only for intervals.]*

If $a = b$ the interval $[a, b] = \{a\}$ is clearly compact. Then assume $a < b$ and, seeking for a contradiction, that $[a, b]$ is not compact. Let $\{U_\alpha : \alpha \in I\}$ and open cover of $[a, b]$ which has no finite subcover and define $S = \{x \in [a, b] : [a, x] \text{ has a finite subcover}\} \ni a$ which is non-empty and bounded from above by b.

The, as for any non-empty bounded subset in $\mathbb{R}$, we can define $\alpha = \sup(S) \in [a, b]$. Consider $\delta > 0 : a + \delta < \alpha$, hence $a < \alpha \leq b$.

Now take $V \in \{U_\alpha : \alpha \in I\} : \alpha \in V$ and fix $\epsilon > 0 : a < \alpha - \epsilon < \alpha \leq b, [\alpha - \epsilon, \alpha + \epsilon] \subset V$. Let us pick $x_0 : \alpha - \epsilon < x_0 < \alpha$; since $x_0 < \alpha$, $[a, x_0]$ can be covered by a finite subcover $\{U_j : j \in J \subset I\}$. Then $\{U_j : j \in J \subset I\} \cup \{V\}$ is a finite subcover of $[a, \alpha + \epsilon]$ and α cannot be the upper bound of $[a, \alpha]$ that are covered by some finite subcover. ∎

Here $\mathbb{R}$ is nothing special except that the order topology coincides with the standard topology, which makes the proof easier. Indeed, the result is true in any metrizable space:

Proposition: *if X is a metrizable space and $S \subset X$. K is compact iff K is closed and bounded.*

If X is T_2 (in particular, metrizable) and $K \subset X$ is compact, then K is closed. Also, the product of compact spaces is compact.

If X is compact and the map $\phi : X \to \mathbb{R}$ is continuous, then ϕ assumes in X its max and min.

X is compact, then $\phi(X) \subset \mathbb{R}$ is compact, then $\phi(X)$ is closed and bounded. Set $b = \sup(\phi(X))$ (which exists because the image is bounded). However, $b \in \phi(X)$ (since the image is closed, hence it contains its limit points). Therefore there exists $x_b \in X$ such that $\phi(x_b) = b$ and ϕ attend its maximum. Analogously, for the minimum. ∎

Theorem: *If $K \subset X$ is compact and $E \subset S$ is infinite, then E has a limit point in K.*

Proof: *suppose E has no limit point in K.*

$\forall x \in K, \exists U_x : U_x \cap E \subset \{x\}$.

$\{U_x : x \in K\}$ is an open cover of K, which is compact, then there exists a finite subcover $\{U_\alpha : \alpha \in I\}$ still covering K (as well as $E \subset K$). However, each U_α contains at most one point of E, hence $\cup_{\alpha=1}^{n} U_\alpha$ contains at most n points of E and it still covers it. Thus E has to be finite, which is a contradiction. ∎

Paracompact spaces and partitions of unity

An open cover is *locally finite* if $\forall x \in S, \exists U_x \in \tau : U_x \cap U_\alpha \neq \varnothing$ for a finite number of opensets U_α of the open cover.

Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of X. A *refinement* of the cover $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_\beta : \beta \in J\}$ together with a map $\iota : I \to J$ such that for any $\beta \in J$ such that $V_\beta \subset U_{\iota(\beta)}$.

That is not a subcover as in the definition of compact set. Here we discard some elements of original cover as we did there. However, here we can also shrink the remaining ones.

Let us also remark that if the map $\iota : J \to I$ were not injective, it does not really make any difference for subcovers. In fact, if $\iota(\beta_1) = \iota(\beta_2) = \alpha$ it means that one can pick U_α multiple times to be included into the subcover, even though the subcover, being a set, cannot contain U_α more than once.

In refinements, the situation is different since V_{β_1}, V_{β_2} may be different subsets of U_α and potentially be both inserted in the refinement.

When the map $\iota : I \to J$ is not injective, we can define an equivalence relation $\beta_1 \sim \beta_2 \iff \iota(\beta_1) = \iota(\beta_2)$ (so that the map ι is, by construction, compatible with the quotient and it induces a map $\iota' : J' \to I$ in the quotient $J' = J/\sim$). Then we can reduce to a new refinement $\mathcal{W} = \{W_\alpha : \alpha \in J'\}$ by setting $W_\alpha = \cup_{\beta \sim \alpha} V_\beta$. Also if $\mathcal{V}$ is locally finite, then $\mathcal{W}$ is locally finite as well. That is still a refinement of $\mathcal{U}$, just for the map $\iota' : J' \to I$, which is now injective. It is called a *reduced refinement*. We can easily restrict to reduce refinements and understand the specification of being *reduced*.

A topological space (X, τ) is *paracompact* iff any open cover has a refinement which is locally finite.

The long line is Hausdorff (all order topologies are), connected, even T_4 though not paracompact. Also, the product of paracompact spaces does not need to be paracompact. Every metrizable space is paracompact.

For a continuous function $\varphi : X \to \mathbb{R}$ the *support* of φ is the subset $\mathrm{supp}(\varphi) := \bar{S} \subset X$, where the set $S \subset X$ is defined as

$$S = \{x \in X : \varphi(x) \neq 0\} \tag{13.1.21}$$

A *partition of unity* subordinate to a locally finite open cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is a collection of (continuous) functions $\varphi_\alpha : X \to [0,1] \subset \mathbb{R}$ ($\alpha \in I$) such that

(1) $\mathrm{supp}(\varphi_\alpha) \subset U_\alpha$ (hence, at any point $x \in X$, there are only a finite number of functions φ_α which are non-zero);

(2) at any $x \in X$, $\sum_{\alpha \in I} \varphi_\alpha(x) = 1$

Notice that, since the cover is locally finite, the sum in (2) is a finite sum at any x.

One can show that a space X is normal iff every locally finite open cover has a partition of unity subordinate to it.

Also any Hausdorff paracompact space is regular and any regular paracompact space is normal. Then we have the following theorem:

Theorem: *Let (X, τ) be a T_2 topological space; X is paracompact iff for any open cover $\mathcal{U}$ there exists a partition of unity subordinate to $\mathcal{U}$.*

Proof: *For any open cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of X there exists a locally finite subcover $\mathcal{V} = \{V_\alpha : \alpha \in J\}$ given by the injective map $\iota : J \to I$. Since a T_2 paracompact space is regular and then normal there exists a partition of unit subordinate to $\mathcal{V}$ which is locally finite. However, any partition of unity subordinate to $\mathcal{V}$ is also subordnate to $\mathcal{U}$.* ∎

We shall often use the theorem to have the existence of a partition of unity on manifolds. Usually we shall use it for a *smooth* partition of unity on a smooth manifold.

The manifolds $\mathbb{R}^n$ are T_2, paracompact and hence, for any open cover, there exists a partition of unity (both a continuous one and a smooth one) subordinate to the open cover.

The partition of unity is often used to build global objects on (topological and differential) manifolds. In those cases, usually one starts from local objects $g_{(\alpha)}$ defined on a coordinate patch, and then considers an affine combination as

$$g = \sum_{\alpha \in I} \varphi_\alpha g_{(\alpha)} \tag{13.1.22}$$

using the partition of unity $(U_\alpha, \varphi_\alpha)$ as weights.

The object g defined in this way is continuous (or differentiable) since the weights φ_α go to zero before the corresponding $g_{(\alpha)}$ become discontinuous. In this construction we shall use pretty much all we said until now: we need the cover defining coordinates to be locally finite, which can be done on manifolds, which for this reason will be required to be paracompact, we need a smooth partition of the unity (which can be achieved by a series of theorems which show how one can smooth continuous objects to become differentiable), we need fields g which are sections of suitable bundles, with local representations which can be glued together to obtain a global section.

Using these techniques, we shall show, for example, that any manifold allows a global Euclidean metric on it.

2. Topological manifolds

Two topological spaces (X, τ_X) and (Y, τ_Y) are *locally homeomorphic* if for any point $x \in X$ there exists an open neighbourhood $U_x \subset X$ and a homeomorphism $\phi : U_x \to V \subset Y$. Of course, V must be an open set in Y and a neighbourhood of $\phi(x)$.

If (Y, τ_Y) is T_2 and (X, τ_X) and (Y, τ_Y) are locally homeomorphic, one cannot conclude that (X, τ_X) is T_2.

Consider Y the line with two origins. Of course, $Y_1 := Y - \{0_2\}$ and $Y_2 := Y - \{0_1\}$ are opensets in Y and they are both homeomorphic to $\mathbb{R}$.

On the other hand, we already mentioned that 0_1 and 0_2 cannot be separated, any neighbourhood of 0_1 is not disjoint from any neighbourhood of 0_2. Accordingly, although $\mathbb{R}$ is T_2 and locally homeomorphic to Y, Y is not T_2.

For two spaces to be locally homeomorphic, one does not refer to any global map from one space to the other. The notion of being locally homeomorphic should not be confused with a *local homeomorphism*.

Given two spaces X and Y, a local homeomorphism $\phi : X \to Y$ is a global map such that at any point $x \in X$ and $y = \phi(x) \in Y$ one can restrict ϕ to suitable opensets $U_x \subset X$ and $V_y \subset Y$, namely $\phi_x : U_x \to V_y$, so that ϕ_x is a homeomorphism. A local homeomorphism is a global map, which may fail to be globally homeomorphism, but can be restricted to a homeomorphism around any point.

Consider $\mathbb{R}$ with the equivalence relation $x \sim y$ iff $\frac{1}{2\pi}(x - y) \in \mathbb{Z}$. The quotient space $\mathbb{R}/\sim$ is called a *circle* and it is denoted by S_1.

In particular, the projection $\pi : \mathbb{R} \to S_1$ is not a homeomorphisms since $\pi(x) = \pi(x + 2k\pi)$ for any $k \in \mathbb{Z}$. On the other hand, for any $x \in \mathbb{R}$, one can find a neighbourhood U_x *small enough* so that the restriction $\pi_x : U_x \to V_x$ is in fact a homeomorphism, where we set $V_x = \pi(U_x)$.

Definition: *a topological space* (M, τ_M) *is called a topological locally Euclidean space iff it is locally homeomorphic to* $\mathbb{R}^m$ *(with the standard topology).*

The number m is fixed for M to be a locally Euclidean space. It is called the *dimension* of M and it is clearly a topological invariant.

That means that one can cover M with open sets U_α ($\alpha \in I$) and for each such open set U_α there exists a homeomorphism $\varphi^\alpha : U_\alpha \to V_\alpha := \varphi^\alpha(U_\alpha) \subset \mathbb{R}^m$. The pair $(U_\alpha, \varphi^\alpha)$ is called a *chart*. The inverse $\varphi_\alpha := (\varphi^\alpha)^{-1} : V_\alpha \to U_\alpha$ of a chart is called a (local) *parameterisation*.

A collection $\mathcal{A} := \{(U_\alpha, \varphi^\alpha) : \alpha \in I\}$ of (compatible) charts which covers the whole M is called an *atlas*. A locally Euclidean spaces, by definition, can be covered by an atlas.

Given a chart $(U_\alpha, \varphi^\alpha)$, a point $p \in U_\alpha \subset M$ corresponds to a point $\varphi^\alpha(p) \in V_\alpha \subset \mathbb{R}^m$, i.e. an m-tuple x^μ of real numbers. The numbers x^μ (as well as the point $(x^\mu) \in \mathbb{R}^m$) are called the *coordinates* of the point p.

In a topological setting, charts are always *compatible*. Given 2 charts $(U_\alpha, \varphi^\alpha)$ and (U_β, φ^β) defined on 2 overlapping regions, i.e. $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, one can define *transition maps*

$$\varphi^\alpha{}_\beta := \varphi^\alpha \circ \varphi_\beta : V_{\alpha\beta} := \varphi^\alpha(U_{\alpha\beta}) \to V_{\beta\alpha} := \varphi^\beta(U_{\alpha\beta}) \tag{13.2.1}$$

which is a map from $V_{\alpha\beta} \subset \mathbb{R}^m$ into $V_{\beta\alpha} \subset \mathbb{R}^m$. In other words, transition maps are not on the M space at all, they are local maps on $\mathbb{R}^m$. Anyway, transition maps are trivially continuous, being compositions of continuous maps.

> Later on, we shall require more regularity to transition maps, in particular being smooth (and we know what smooth maps are on $\mathbb{R}^m$), or analytical, to define smooth, or analytical, locally Euclidean spaces. In topology they are continuous by construction.
>
> Let us stress that we have some ado with intersections, e.g. $U_{\alpha\beta}$, which we could avoid since there exists an (obviously unique) map $\varnothing : \varnothing \to \varnothing$ which would fill in the gaps nicely, anyway.

Generally, a point $p \in M$ can belong to more than one chart. Let us suppose that $p \in U_\alpha \cap U_\beta$ so that it can be represented by two different coordinate sets, namely $x^\mu = \varphi^\alpha(p)$ and $x'^\mu = \varphi^\beta(p)$. Of course, one has

$$x'^\mu = \varphi^\beta(p) = \varphi^\beta \circ \varphi_\alpha(\varphi^\alpha(p)) = \varphi^\beta \circ \varphi_\alpha(x^\mu) = \varphi^\beta{}_\alpha(x^\mu) \tag{13.2.2}$$

Accordingly, transition maps $\varphi^\beta{}_\alpha$ transform coordinates of a point p in a chart α to the coordinates of the same point in a different chart β. Again that has nothing to do with the space M, it is entirely related to $\mathbb{R}^m$ and coordinates.

f did not notice until now, mind the order. The 2 maps $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$ are one the inverse to each other. More generally, by construction, transition maps and they have the following properties

(1) $\varphi^\alpha{}_\alpha = \mathrm{id}$ $\qquad$ (on V_α)

(2) $\varphi^\alpha{}_\beta \circ \varphi^\beta{}_\alpha = \mathrm{id}$ $\qquad$ (on $V_{\alpha\beta} = \varphi^\alpha(U_{\alpha\beta})$)

(3) $\varphi^\alpha{}_\beta \circ \varphi^\beta{}_\gamma \circ \varphi^\gamma{}_\alpha = \mathrm{id}$ $\qquad$ (on $V_{\alpha\beta\gamma} = \varphi^\alpha(U_{\alpha\beta\gamma})$)

which are called *cocycle identities*.

Definition: *a topological space* (M, τ_M) *is called a topological manifold iff M is T_2, paracompact and, locally Euclidean space.*

Locally Euclidean spaces with continuous map form a category $\mathfrak{Euc}_0$. One can regard this category as a more structured version of $\mathfrak{Top}$, which is quite convenient to have since topological spaces are too general to be used in practice in (our) applications. Similarly, topological manifolds with continuous maps form a further structured category denoted by $\mathfrak{Man}_0$.

> Let us briefly discuss here some of the semantic about locally Euclidean spaces and manifolds. Manifolds are definitely more popular in the mathematical literature. On the other hand, most of the constructions on manifold can be adapted to a general locally Euclidean space.
>
> The category of manifolds have beed criticised because it is supposed to have too many morphisms. For example, set maps between 2 fixed sets $f : A \to B$ is a set, linear maps between 2 fixed vector spaces $f : A \to B$ is a vector space. Unfortunately, that fails to be true for manifolds since maps between 2 manifolds $\phi : M \to N$ do not form a manifold. Thus mathematicians often are searching for something better to replace them.
>
> On the other hand, in physics observers will describe physical entities locally and one wants to abstract from them something which does not depend on the observer. That is similar to how locally Euclidean spaces are assembled from coordinates to become abstract objects.

Let us also remark how a locally Euclidean space is constructed with an atlas on the top, although one can check in the definition that the atlas is not part of the definition. One can have two equivalent atlases on a single locally Euclidean space, it still is the same locally Euclidean space. That is very annoying if one wants to classify locally Euclidean spaces. For that reason mathematicians define the *maximal* atlas which is in one-to-one correspondence with abstract locally Euclidean spaces and go out for a classification.

From a physical perspective, a maximal atlas is quite unnatural. One wishes to have *enough* observers to describe everything, not necessarily *all possible* observers. In physical applications, for example in mechanics, one wants to consider spaces covered with coordinates, not necessarily all possible coordinates. At most we want to maintain our freedom to change coordinates at will although hardly ever one is required to exhibit all possible Lagrangian coordinates for a system before starting studying it.

Finally, let us remark that usually locally Euclidean spaces are manifolds. Of course, there are some examples in which it fails (non-Hausdorff, non-paracompact, ...) but it is far from clear (to me at least) if these cases can be excluded from physics, or even how they can be excluded from mathematics. For example, one can require some further properties on the covering of charts to enforce Hausdorff and or paracompactness.

In this situation one wise attitude is to keep an open mind and study which structure is relevant in which context, postponing decisions for later times in a long term strategy. In this book we shall try to keep locally Euclidean spaces distinct from manifolds and check how we can survive on locally Euclidean spaces.

Now suppose one has a family of *patches* $V_\alpha \in \mathbb{R}^m$ (with $\alpha \in I$), i.e. open sets, ideally homeomorphic to balls. For any pair of patches V_α and V_β, one can have two subsets $V_{\beta\alpha} \subset V_\beta$ and $V_{\alpha\beta} \subset V_\alpha$ and a homeomorphism $\varphi^\alpha{}_\beta : V_{\beta\alpha} \to V_{\alpha\beta}$. If one has no such homeomorphism, one can set $V_{\alpha\beta} = V_{\beta\alpha} = \varnothing$. Let us also suppose that the family of homeomorphisms $\varphi^\alpha{}_\beta$ obeys cocycle identities.

Thus we can build a locally Euclidean space M, which is unique modulo homeomorphisms. Let us start by defining the disjoint union of the patches V_α, i.e.

$$X = \coprod_{\alpha \in I} V_\alpha \tag{13.2.3}$$

A point of X is in the form (α, x) with $\alpha \in I$ and $x \in V_\alpha$. We can define an equivalence relation $\sim$ on X by setting

$$(\alpha, x) \sim (\beta, y) \iff y = \varphi^\beta{}_\alpha(x) \tag{13.2.4}$$

This is (can be extended to) an equivalence relation and we can define the quotient space $M = X/\sim$. A point in M is an equivalence class $[(\alpha, x)]$ which, besides the representative (α, x), contains all the other equivalent representatives (β, y) with $y = \varphi^\beta{}_\alpha(x)$, if any.

Let us show that M is a locally Euclidean space.

Let us set $U_\alpha = \{([(\alpha, x)] : \alpha \in I, x \in V_\alpha\}$ and define the map $\varphi^\alpha : U_\alpha \to V_\alpha : [(\alpha, x)] \mapsto x$. The sets U_α are an open cover of M (in the quotient topology induced by the projection $\pi : X \to M$) and the charts φ_α are homeomorphisms. Thus M isa locally Euclidean space.

Transition maps between the charts φ^α and φ^β are by construction the maps $\varphi^\beta{}_\alpha$.

Let us now suppose there is another topological manifold M' with an atlas (W_α, ψ^α) with transition maps $\psi^\alpha{}_\beta = \psi^\alpha \circ \psi_\beta \equiv \varphi^\alpha{}_\beta$ (then $\psi^\alpha \circ \psi_\beta \circ \varphi^\beta \circ \varphi_\alpha = \text{id}$) and let us show that M' is homeomorphic to M.

One can define the maps $\psi_\alpha \circ \varphi^\alpha : U_\alpha \to W_\alpha : p \mapsto p'$. They do not depend on the chart since in a nearby chart β one has the map

$$\psi_\beta \circ \varphi^\beta : U_\beta \to W_\beta \tag{13.2.5}$$

which glue together since

$$\psi_\beta \circ \varphi^\beta = \psi_\alpha \circ \left(\psi^\alpha \circ \psi_\beta \circ \varphi^\beta \circ \varphi_\alpha\right) \circ \varphi^\alpha = \psi_\alpha \circ \text{id} \circ \varphi^\alpha = \psi_\alpha \circ \varphi^\alpha \tag{13.2.6}$$

and thus they define a global homeomorphism $\phi : M \to M'$.

Then one can define topological locally Euclidean spaces by glueing together patches (*any* topological locally Euclidean space can be constructed in this way) modulo homeomorphisms. Notice that the topological locally Euclidean space is an intrinsic object defined independently of charts as its points are defined as equivalence classes of coordinate representations. This provides an equivalent definition of locally Euclidean space which is particularly insightful for physics. Then if the locally Euclidean space is T_2 and paracompact it is a manifold.

Local expressions for maps

Objects on a locally Euclidean space can be systematically defined locally and then glued together to define a global and intrinsic object. Let us here sketch the standard procedure, which then differ by details depending on the object one is considering, with maps $\phi : M \to N$. We go down from an intrinsic object to local representations depending on the chart and up again from the local representations to the global object.

Let us consider a point $p \in M$ and let us set $q := \phi(p) \in N$. Then choose 2 charts $(U_\alpha, \varphi^\alpha) : x \in U_\alpha$ and $(V_i, \psi^i) : y \in V_i$ and define the *local expression* of the map ϕ in the two charts as

$$\phi^i{}_\alpha := \psi^i \circ \phi \circ \varphi_\alpha : \varphi^\alpha(U_\alpha) \subset \mathbb{R}^m \to \psi^i(V_i) \subset \mathbb{R}^n : x^\mu \mapsto y^i := \phi^i{}_\alpha(x) \tag{13.2.7}$$

where x^μ are local coordinates on U_α of the point $p \in U_\alpha$ and y^i are local coordinates on V_i of the point $q \in V_i$.

> I have an habit, which is an abuse of notation, about local expressions of objects as functions of the coordinates. The local map $\phi^i{}_\alpha(x)$ is meant to depend on all the coordinates x^μ and it should be written as $\phi^i{}_\alpha(x^\mu)$ or as $\phi^i{}_\alpha(x^1, x^2, \ldots, x^\mu)$.
>
> However, by doing that this will conflict with Einstein convention, we shall use in linear algebra (and we will introduce below in due time). In that notation, $\phi^i{}_\alpha(x^\mu)$ can be confused with a collection of numbers, something like $(\phi^i{}_\alpha(x^1), \phi^i{}_\alpha(x^2), \ldots, \phi^i{}_\alpha(x^m))$ which is not what we mean. For that reason
> > *we systematically understand coordinate indices when they are arguments in a local map.*
>
> Also, if you are not familiar with Einstein notations, you may find weird to put indices up. We'll discuss this later, by know all indices up, down, before, or after, namely ${}^i{}_{\alpha\beta}t^{ab}{}_\mu$, are always indices (unless we say otherwise explicitly). Powers are indicated as $\left({}^i{}_{\alpha\beta}t^{ab}{}_\mu\right)^{-\frac{1}{2}}$.
>
> Since indicating all the charts by indices is often cumbersome, we shall often understand the indices for charts making clear in the context what we mean. Sorry for that, get used to it. If you feel it is just notation, go and explain the long division to an ancient roman without using positional notation.

If we change charts, the corresponding local expression changes as

$$\phi^i{}_\alpha = \psi^i \circ \phi \circ \varphi_\alpha = \psi^i \circ \psi_j \circ \psi^j \circ \phi \circ \varphi_\beta \circ \varphi^\beta \circ \varphi_\alpha = \psi^i{}_j \circ (\psi^j \circ \phi \circ \varphi_\beta) \circ \varphi^\beta{}_\alpha = \psi^i{}_j \circ \phi^j{}_\beta \circ \varphi^\beta{}_\alpha \tag{13.2.8}$$

Hence transition maps not only changes coordinates from one cart to another. They also change local expressions of a map from a chart to another. The transformation (13.2.8) of local expressions are also called *transformation laws* for (local expressions of) maps.

To summarize, we take a global map ϕ and we describe it by a collection of local expressions $\phi^i{}_\alpha$ that depend on the charts, obeying transformation laws (13.2.8). All local expressions are written as functions in $\mathbb{R}^n$, so that we can use, with some extra care, what we learned about calculus in $\mathbb{R}^n$.

On the other hand, given the family of local expressions $\phi^i{}_\alpha : \varphi^\alpha(U_\alpha) \subset \mathbb{R}^m \to \psi^i(V_i) \subset \mathbb{R}^n$ which obeys maps transformation laws (13.2.8), there is a unique global map $\phi : M \to N$ having those as local expressions.

We have to find a way to rebuild $\phi(p) = q$ out of local expressions. Of course, we have $\phi(p) = \psi_i \circ \phi^i{}_\alpha \circ \varphi^\alpha(p)$, however, that is not the global map ϕ, just its restriction to U_α; so we can write

$$\phi|_{U_\alpha} = \psi_i \circ \phi^i{}_\alpha \circ \varphi^\alpha \tag{13.2.9}$$

If we change charts, we find similarly $\phi|_{U_\beta} = \psi_j \circ \phi^j{}_\beta \circ \varphi^\beta$. If we restrict to a non-empty intersection $p \in U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \varnothing$, the two expression agree

$$\phi|_{U_\alpha}(p) = \psi_i \circ \phi^i{}_\alpha \circ \varphi^\alpha(p) = \psi_i \circ \psi^i{}_j \circ \phi^j{}_\beta \circ \varphi^\beta{}_\alpha \circ \varphi^\alpha(p) = (\psi_i \circ \psi^i) \circ \psi_j \circ \phi^j{}_\beta \circ \varphi^\beta \circ (\varphi_\alpha \circ \varphi^\alpha)(p) = \psi_j \circ \phi^j{}_\beta \circ \varphi^\beta(p) = \phi|_{U_\beta}(p) \tag{13.2.10}$$

and, of course, because of that, there is a unique map $\phi|_{U_\alpha \cup U_\beta}$ which restricts to $\phi|_{U_\alpha}$ and $\phi|_{U_\beta}$.

Thus we can extend the map ϕ to the union of the domains of all charts, i.e. to the whole M, and get the global $\phi : M \to N$.

By the way, this idea is the core business of sheaf theory. So it is time to sketch it even though, as it happens for category theory, for us it is mainly a way to highlight the important things in a context not much a computational or theoretical tool.

Sheaves

Let us consider a topological space (X, τ), even though we shall mainly use topological (and smooth) manifolds with sheaves.

Let us first observe that, to any topological space X, we can associate a category $\mathfrak{O}(X)$ whose objects are opensets of X and arrows $U \to V$ are defined when $U \subset V$, which is a preorder.

Given another category $\mathfrak{G}$, e.g. for example the category of groups or any subcategory of $\mathfrak{Set}$, a $\mathfrak{G}$-*valued presheaf* is a controvariant functor $F : \mathfrak{O}(X) \to \mathfrak{G}$

Then for any $U \in \tau$ one has an object $F(U)$, sometimes denoted by $\Gamma(U; F)$. An element $s \in F(U)$ is called a *local section* on U. An element $\sigma \in F(X)$ is called a *global section.*

Then, for any morphism $i : V \to U$, when $V \subset U$, one has a map $\rho_V^U : F(U) \to F(V)$, called the *restriction map* from U to V. If $s \in F(U)$ is a local section, $\rho_V^U(s) \in F(V)$ is a local section on V, it is denoted for short by $s|_V$, and it is called the *restriction of s to V*. The restriction map has the functorial properties, namely the restriction from U to U itself is the identity and if we have $S \subset V \subset U$ the corresponding restriction maps obey the property

$$\rho_S^U = \rho_S^V \circ \rho_V^U \tag{13.2.11}$$

Of course, these properties are automatically satisfied since we defined presheaves as controvariant functors.

The maps $C(M, N)$ defines a presheaf $C(U, N)$, as the functions $F(U) : U \to \mathbb{R}$. We shall also consider below an example of presheaf $(\cdot) : U \to G$ in which we associate to an openset an element of a commutative group G.

A presheaf F is a *sheaf* if is has the following 2 properties for an openset U and an open cover $\{U_i \subset U : i \in I\}$ of U and

(1) for 2 sections $s, t \in F(U)$: if $\forall i \in I : s|_{U_i} = t|_{U_i}$, then $s = t$ (locality)

(2) $s_i \in F(U_i)$ and $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, then $\exists! s \in F(U) : s|_{U_i} = s_i$ (glueing)

The section s mentioned in axiom 2 is called the *glueing* of the local section s_i. It exists for axiom (2) and it is unique by axiom (1).

The examples of presheaves listed above (maps and functions) are also examples of sheaves. Below we shall add the sheaf of sections of any fiber bundle and physical fields *are* sections of fiber bundles.

Again, this is a framework, it does not depend on what we are dealing with case by case. As a matter of fact, this will be our definition of geometric object or physical field, and we shall borrow from sheaf theory the mechanisms of *localizing* global objects (on charts of a manifold) and *glueing* together local expression to define global objects. Of course, in any specific context, these mechanisms are there before and without sheaves, just as maps between topological spaces are there even without category theory. However, sheaf theory, as category theory, provides a unifying view in otherwise unrelated contexts and allows to discuss general tools forgetting about unessential details.

> Let us remark, as a side note, that there is another reason for sheaf theory, which is among the original motivations to develop the theory in the first place.
> Originally, Grothendieck, I think talking about algebraic manifolds, complained about the structure of the corresponding category. He noticed that in $\mathfrak{Set}$, $\mathfrak{Vect}$, ..., given two objects A, B and considering maps $\mathrm{Hom}(A, B)$ that is again a set, a vector space, ..., i.e. it is again an object in the category itself. This allows to apply to $\mathrm{Hom}(A, B)$ tools developed for vector spaces, as well as considering $\mathrm{Hom}(\mathrm{Hom}(\cdot, \cdot), \cdot)$ (which as we shall see is the starting idea to tensor algebra).
> However, maps between two manifolds (in all different regularities), say $\mathcal{C}^k(M, N)$, are not a manifold.
> In many areas of mathematics (and mathematical physics), is important keep under control the group of isomorphisms on the manifold itself, if not for other reasons than often one wants to quotient those out and dealing with infinite dimensional, vaguely known beasts as homeomorphisms, diffeomorphisms, ... is always uneasy.
> That is, anyway, (one of) the boundaries of the territory we fully master, so we are better to keep eyes wide open. Even if we do not really know yet what is relevant and what is not.

3. Topologies on maps

Let M, N be 2 locally Euclidean spaces (or topological — or, later, $\mathcal{C}^k$, smooth, or analytical — manifolds). Let us denote by $\mathcal{C}(M, N)$ the set of continuous maps $\phi : M \to N$.

Here we chose to work with locally Euclidean spaces. However, some of the results extend to a general topological spaces, some only to *locally compact* topological spaces, some to T_2 topologic spaces. Topological manifolds are, of course, T_2 and locally compact and all our applications will be about manifolds. We wish to keep an eye open on T_2 and eventually sunder the results for locally Euclidean spaces and manifolds.

Here the first real mess is how to define *locally compact spaces*. The literature has no accepted definition for it, some assuming T_2 to force all definitions to be equivalent, others restricting to topological vector spaces. We shall use the following general definition:

Definition: *a topological space* (X, τ) *is locally compact at* $x \in X$ *iff, for any (open) neighbourhood* $x \in U_x \in \tau$, *there exists a compact neighbourhood* $K \subset U_x$ *of* x. *As a remind, we remark that a compact neighbourhood contains* x *together with an open neighbourhood* V_x. *Altogether, we have that* $\forall U_x, \exists K, V_x$, *with* U_x, V_x *opensets and* K *compact, such that*

$$x \in V_x \subset K \subset U_x \tag{13.3.1}$$

The space X *is locally compact iff it is locally compact at any point* $x \in X$.

A locally Euclidean space M is clearly locally compact, since it is enough to check locally compactness on small neighbourhoods and $\mathbb{R}^m$ is locally compact.

The space of maps $\mathcal{C}(M, N)$ can be turned into a topological space by defining a topology on $\mathcal{C}(M, N)$. Usually, we have two options for such a topology. They are called the *weak* (or *compact-open*) and *strong topology*.

The 2 topologies on compact manifolds reduce to the same one. On non-compact spaces, the weak topology is coarser than the strong topology which better controls maps behaviour out of a compact. However, since we assume N to be at least a locally Euclidean space its topology is automatically metrizable (in fact it is generated by balls in $\mathbb{R}^n$), the weak topology on $\mathcal{C}(M, N)$ is produced by a distance as well, at least for a compact M. It, in fact, relates to the topology generated by the distance on $\mathcal{C}(M, N)$ as

$$D(f, g) = \sup_{x \in M} d(f(x), g(x)) \tag{13.3.2}$$

although the mechanism needs to be clarified with some care.

The weak topology is also equivalent to a topology defined on local expression on maps, controlling the bounds on local expressions, which is often used in analysis.

Also the strong topology is pretty closed to a metrizable topology, it is in fact a Baire topology which is quite closed to the standard topology on a complete metric space. We will not enter in the complete study of properties, we just start accumulating some aspects which will be relevant for us.

Weak topology

Let M and N be 2 locally Euclidean spaces, $K \subset M$ be a compact, $V \subset N$ be an openset. Let us define the subset of maps $B(K, V) = \{\phi \in \mathcal{C}(M, N) : \phi(K) \subset V\}$. These form a sub-basis of the *weak* (or *compact-open*) topology. The consequent topological space is denoted by $\mathcal{C}_W(M, N)$.

Since $B(K, V)$ are a sub-basis, we can infer many properties of the topology in terms of the properties of the sets $B(K, V)$. First of all, let us remark that we have

$$K_1 \subset K_2 \Rightarrow B(K_2, V) \subset B(K_1, V) \qquad V_1 \subset V_2 \Rightarrow B(K, V_1) \subset B(K, V_2) \tag{13.3.3}$$

as well as

$$B(K_1, V) \cap B(K_2, V) = B(K_1 \cup K_2, V)) \qquad B(K, V_1 \cap V_2) = B(K, V_1) \cap B(K, V_2) \tag{13.3.4}$$

These can be easily showed by double inclusions.
They also imply $\cap_{i,j} B(K_i, V_j) \subset B(K_1 \cup K_2, V_1 \cup V_2)$ simply by

$$\cap_{i,j} B(K_i, V_j) = B(K_1, V_1) \cap B(K_2, V_1) \cap B(K_1, V_2) \cap B(K_2, V_2) = B(K_1, V_1 \cap V_2) \cap B(K_2, V_1 \cap V_2) = B(K_1 \cup K_2, V_1 \cap V_2) \subset B(K_1 \cup K_2, V_1 \cup V_2) \tag{13.3.5}$$

A good way to list the properties is grouping them in terms of their categorical meaning. In order to do so, let us first consider the category $(\mathfrak{Euc}^0)^* \times \mathfrak{Euc}^0$, objects being pairs of locally Euclidean spaces (M, N), morphisms being denoted by $(f, g) : (M, N) \to (M', N')$ and corresponding to pairs of morphisms, $f : M \to M'$ in $(\mathfrak{Euc}^0)^*$ and $g : N \to N'$ in $\mathfrak{Euc}^0$. Then we know $g : N \to N'$ is simply a continuous map. The category $(\mathfrak{Euc}^0)^*$ is

called the *opposite category*, it is obtained by $\mathfrak{Euc}^0$ by keeping the same objects and simply flipping the arrows, i.e. a morphism $f : M \to M'$ in $(\mathfrak{Euc}^0)^*$ corresponds to a continuous map $f : M' \to M$.

> This may seem somehow unnatural. On the other hand, if you are (or when you will be) familiar with tensors, it resembles pretty much what happens there with the dual which, of course, is not defined on a general locally Euclidean space.

We want to regard maps as a functor $\mathcal{C}_W(\cdot, \cdot) : (\mathfrak{Euc}^0)^* \times \mathfrak{Euc}^0 \to \mathfrak{Top}$. The action on objects is clearly $\mathcal{C}_W : (M, N) \mapsto \mathcal{C}_W(M, N)$, on morphisms we set $\mathcal{C}_W(f, g) : \mathcal{C}_W(M, N) \to \mathcal{C}_W(M', N') : h \mapsto g \circ h \circ f$.

> As $h \in \mathcal{C}(M, N)$, $f \in \mathcal{C}(M', M)$, and $g \in \mathcal{C}(N, N')$ are continuous maps, then $g \circ h \circ f : M' \to N'$ is a continuous map, i.e. $g \circ h \circ f \in \mathcal{C}(M', N')$. That is a sort of being well-defined at a point h.
>
> It has nothing to do with the map $h \mapsto g \circ h \circ f$ and its properties, in particular, with its continuity which has to do with the topology we gave to $\mathcal{C}_W(M, N)$ (and not much directly with the topologies on M and N).
>
> Clearly, if $\mathcal{C}_W(f, g)$ has to be a morphism : $\mathcal{C}_W(M, N) \to \mathcal{C}_W(M', N')$ in the category $\mathfrak{Top}$, it has to be continuous with respect to the weak topology we fixed on objects. And that needs to be proven, since it does not descend from continuity of $g \circ h \circ f$ which just say that $\mathcal{C}_W(f, g)(h) \in \mathcal{C}_W(M', N')$.

Let us denote $(f, g)_* := \mathcal{C}_W(f, g)$ for short, and we need to prove that the map $(f, g)_* : \mathcal{C}_W(M, N) \to \mathcal{C}_W(M', N') : h \mapsto g \circ h \circ f$ is continuous in the weak topology.

> For, notice that by setting $g_*(h) := g \circ h$ and $f^*(h) := h \circ f$, we get

$$(g_*)^{-1}(B(K, g(V))) = B(K, V) \qquad\qquad (f^*)^{-1}(B(K, V)) = B(f(K), V)$$
$$\Rightarrow \qquad ((f, g)_*)^{-1}(B(K, g(V))) = (g_*)^{-1} B(f(K), g(V)) = B(f(K), V) \tag{13.3.6}$$

> That means that if we consider an openset in $\mathcal{C}_W(M', N')$, by definition, the openset is union of finite intersections of basic sets $B(K, V)$. There is not much to worry about intersections, since any finite intersection is again a basic set by (13.3.5). Then we fix an openset $W' \subset N'$, a compact $K' \subset M'$ and the corresponding basic set $B(K', W') \subset \mathcal{C}_W(M', N')$. That defines a compact $f(K') = K \subset M$ and an openset $W = g^{-1}(W') \subset N$, as well as the corresponding basic set $B(K, W) \subset \mathcal{C}_W(M, N)$. By (13.3.6), for $h \in B(K, W)$, we have

$$(f, g)_*(h)(K') = g \circ h \circ f(K') = g \circ h(K) \subset g(W) = W' \qquad \Rightarrow (f, g)_*(h) \in B(K', W') \qquad \Rightarrow (f, g)_*(B(K, W)) = B(K', W') \tag{13.3.7}$$

> In other words, the preimage of a basic set is another basic set, which proves continuity.
>
> Let us stress that, in order to prove continuity in the weak topology, we essentially found a way to fit the preimage of basic sets in terms of basic sets. Also, in view of locally compactness, we can show that given X, Y, Z we have that the composiztion

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \tag{13.3.8}$$

> is continuous.
>
> For, consider a basic set $B(K, W)$ with a compact set $K \subset X$ and an openset $W \subset Z$ and consider $(f, g) \in \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y)$ such that $f \circ g \in B(K, W) \subset \mathcal{C}(X, Z)$. Since we know $g(K) \subset Y$ is compact, for any $y \in g(K)$ there exists an openset U_y and a compact K_y so that $U_y \subset K_y$. On the other hand $\cup_{y \in g(K)} U_y$ covers $g(K)$ which is comnpact. Hence we can extract a finite subcover $\{U_i : i = 1..n\}$ which still covers $g(K)$. Let us set $U := \cup_{i=1}^n U_i$ which is an openset and the corresponding $H := \cup_{i=1}^n K_i$ which is a compact and $U \subset H$. Then we have $f \in B(H, W)$ and $g \in B(K, U)$, which shows $\circ$ is a continuous map.

Now we defined $\circ : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ we can define the continuous embedding

$$c : Y \to \mathcal{C}(X,Y) : y \mapsto c_y \tag{13.3.9}$$

where $c_y \in \mathcal{C}(X,Y)$ is the constant map $c_y(x) = y$.

Continuity follows from observing that $c^{-1}(B(K,U)) = U$. The embedding comes from the fact that, for any $x \in X$, $c(U) = B(\{x\},U) \cap c(Y)$.

The role of X is quite unessential; for any (non-empty) space X we can define a copy of Y as $c : Y \to \mathcal{C}(X,Y)$. They altogether are denoted as $c : Y \to \mathcal{C}(*,Y)$.

Then we have a morphism $\circ : \mathcal{C}(X,Y) \times \mathcal{C}(*,X) \to \mathcal{C}(*,Y)$. The spaces $\mathcal{C}(*,X) \simeq X$ and $\mathcal{C}(*,Y) \simeq Y$ are copies of X and Y. Then we define the *evaluation map* as

$$\text{ev} : \mathcal{C}(X,Y) \times X \to Y : (h,x) \mapsto h(x) \tag{13.3.10}$$

which is continuous in the weak topology.

If we have a continuous map $f : X \times Y \to Z$ we define its *right-adjoint* to be $f^\sharp : X \to \mathcal{C}(Y,Z)$ as

$$f^\sharp(x) : Y \to Z : y \mapsto f(x,y) \tag{13.3.11}$$

On the other hand, if we have $g : X \to \mathcal{C}(Y,Z)$ we define its *left-adjoint* to be

$$g^\flat : X \times Y \to Z : (x,y) \mapsto g(x)(y) \tag{13.3.12}$$

Let us finally add that if (X,Y) are T_0 (T_1, T_2, or T_3) then $\mathcal{C}(X,Y)$ is T_0 (T_1, T_2, or T_3).

We can also give an equivalent definition of compact-open topology by using local charts. Let us fix a map $\phi \in \mathcal{C}(M,N)$, 2 charts (U_α,φ) and (V_i,ψ) on N and M, $K \subset U_\alpha$ a compact, and $\epsilon > 0$. Any map, including ϕ, has a local expression $f := \psi \circ \phi \circ \varphi^{-1}$ in the relevant chars, as well as $g := \psi \circ \lambda \circ \varphi^{-1}$ for the local expression of $\lambda : M \to N$. Using that, we can transform maps λ in $\mathcal{C}(M,N)$ to functions $G : K \to \mathbb{R}$ by associating to λ, and its local expression g, a function

$$G : K \to \mathbb{R} : x \mapsto |g(x) - f(x)| \tag{13.3.13}$$

Then we can define

$$C(\alpha,i,\epsilon,K,\phi) := \{\lambda \in \mathcal{C}(M,N) : \lambda(K) \subset V_i, \forall x \in K : G(x) < \epsilon\} \tag{13.3.14}$$

That is an hybrid definition with local expressions, however, one can find that the topology generated by the sub-basis $C(\alpha,i,\epsilon,K,\phi)$ is in fact equivalent to the compact-open topology defined above intrinsically.

Given 2 topologies on a set X, namely τ_1 and τ_2, and given two sub-bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for them, we have two following criterion to compare the topologies.

Theorem: $\forall S' \in \mathcal{B}_2, \forall x \in S' : \exists S_i \in \mathcal{B}_1 : i = 1..n : x \in \cap_{i=1}^n S_i \subset S'$ iff $\tau_2 \subset \tau_1$

Strong topology

Let M and N be 2 topological manifolds. For a map $\phi : M \to N$ let $\Gamma_\phi \subset M \times N$ denote the graph of the map. Let W be an openset of $M \times N$ such that $\Gamma_\phi \subset W \subset M \times N$

Let us define the sets

$$B_S(\phi, W) := \{\psi \in \mathcal{C}(M, N) : \Gamma_\psi \subset W\} \tag{13.3.15}$$

which form a basis of the strong topology on $\mathcal{C}(M, N)$.

The topological space defined by the strong topology is denoted by $\mathcal{C}_S(M, N)$.

If M is compact $\mathcal{C}_W(M, N) = \mathcal{C}_S(M, N)$.

4. Čech cohomology

Let G be an Abelian group and X a topological space with a locally finite open cover $\cup_{i \in I} U_i = X$. Of course, on topological manifolds one always has locally finite open coverings. The Abelian group can be written in additional $(G, +)$ or multiplicative $(G, \cdot)$ notation.

Given a locally finite open covering of X let us set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$ whenever such sets are non-empty. Being the covering locally finite sooner or later these intersections will become all empty.

Let us define a *0-cochain* a function associating at each U_i an element of G, denoted by $(g) : U_i \mapsto (g)_i$.

A *1-cochain* a function associating at each intersection U_{ij} an element of G, denoted by $(g) : U_{ij} \mapsto (g)_{ij}$.

A *2-cochain* a function associating at each intersection U_{ijk} an element of G, denoted by $(g) : U_{ijk} \mapsto (g)_{ijk}$ and so on for k-cochains.

The set $C^k(\mathcal{U}, G)$ of k-cochains is a group with the product

$$(g + h)_{i_0 i_1 \dots i_k} = (g)_{i_0 i_1 \dots i_k} + (h)_{i_0 i_1 \dots i_k} \qquad (g \cdot h)_{i_0 i_1 \dots i_k} = (g)_{i_0 i_1 \dots i_k} \cdot (h)_{i_0 i_1 \dots i_k} \tag{13.4.1}$$

> Here everything depends on the cover one chooses in the beginning. For that reason one should consider finer and finer coverings and compute some sort of limit with respect to the cover. One can show, that at some point, the objects we are going to define will become constant with respect to further refinements, and they become associated to the space X more than to its cover $\mathcal{U}$.

We can define the *coboundary* operator δ taking k-cochains and returning $(k + 1)$-cochains. For $k = 1$, one sets

$$(\delta g)_{ijk} = (g)_{ij} - (g)_{ik} + (g)_{jk} \qquad (\delta g)_{ijk} = (g)_{jk} \cdot (g)_{ik}^{-1} \cdot (g)_{ij} \tag{13.4.2}$$

For $k = 0$, one sets

$$(\delta g)_{ij} = (g)_i - (g)_j \qquad (\delta g)_{ij} = (g)_i \cdot (g)_j^{-1} \tag{13.4.3}$$

One can easily show that $\delta\delta(g) = e$; for a 0-cochain (g), one has $(\delta g)_{ij} = g_i + g_j$ (or $(\delta g)_{ij} = g_i \cdot g_j^{-1}$) and

$$
\begin{aligned}
(\delta\delta g)_{ijk} =& (\delta g)_{jk} - (\delta g)_{ik} + (\delta g)_{ij} = g_j - g_k - g_i - g_k + g_i - g_j = g_j - g_k + g_k - g_i + g_i - g_j = 0 \\
(\delta\delta g)_{ijk} =& (\delta g)_{jk} \cdot (\delta g)_{ik}^{-1} \cdot (\delta g)_{ij} = g_j \cdot g_k^{-1} \cdot (g_i \cdot g_k^{-1})^{-1} \cdot g_i \cdot g_j^{-1} = g_j \cdot g_k^{-1} \cdot g_k \cdot g_i^{-1} \cdot g_i \cdot g_j^{-1} = 1
\end{aligned}
\tag{13.4.4}
$$

We say that a cochain (g) is a *cocycle* iff $(\delta g) = 1$. It is a *coboundary* iff there exists a chain (h) such that $(\delta h) = (g)$. In view of the identity $\delta^2 = 1$, any coboundary is also a cocycle, though not the other way around.

Let us denote by $Z^k(\mathcal{U}, G)$ the set of k-cocycles and by $B^k(\mathcal{U}, G)$ the set of k-coboundaries, which inherit a group structure from cochains, since one can show that $\delta(g + h) = \delta g + \delta h$. Two k-cocycles are said to be *cohomologically equivalent* $(g) \sim (h)$ iff $(g^{-1} \cdot h)$ is a coboundary.

This is an equivalence relation on $Z^k(\mathcal{U}, G)$ and we define the quotient $H^k(\mathcal{U}, G) = Z^k(\mathcal{U}, G)/\sim = Z^k(\mathcal{U}, G)/B^k(\mathcal{U}, G)$ which inherits a group structure from $Z^k(\mathcal{U}, G)$. The group $H^k(\mathcal{U}, G)$ is called the *k-cohomology Čech group*.

One can extract some information about a topological space X from these cohomology groups $H^k(\mathcal{U}, G)$ (once one suitably gets rid of the dependence of the open covering).

This structure is quite flexible, even though one should apply extra care when allowing more general stuff. One can repeat the discussion for the group $\mathcal{G} = \{\gamma : U \to G\}$ of point dependent group elements, with the pointwise sum inheredit from G. One can choose a different group for any open set (e.g. homotopy groups based at a point). Some of the discussion above can be even adapted to non-commutative groups, though in that case a lot of the structure is eventually lost due to non-commutativity. However, the framework in that case can be applied to manifolds to classify inequivalent transition functions.

Classification of manifolds

The whole construction of Čech cohomology relies on the fact that the group G is commutative. Let us keep the terminology also for the case in which the group is non-commutative, a case in which things are much more difficult to be treated in this generality. We shall consider some simple cases which do not misbehave too wildly taking the (pseudo)group $\mathrm{Hom}_{loc}(X)$ of homeomorphisms defined locally on X.

The transition maps $(\varphi) : U_{ij} \mapsto (\varphi)_{ij}$ can be regarded as a 1-cochain. The coboundary of such a cochain is

$$
(\delta\varphi)_{ijk} = (\varphi)_{jk} \circ (\varphi)_{ik}^{-1} \circ (\varphi)_{ij} = \mathrm{id}
\tag{13.4.5}
$$

Thus (φ) is in fact a cocycle in view of cocycle identities (which are called in that way because of it).

Let us consider a homeomorphism $\phi : X \to Y$ between two topological manifolds and consider two coverings U_i of X and $V_i = \phi(U_i)$ of Y. Let $(\varphi)_{ij}$ be the cocycle of transition maps on X and $(\psi)_{ij}$ be the cocycle of transition maps on Y. The homeomorphism $\phi : X \to Y$ restricts to local homeomorphisms $\phi_i = \psi_i \circ \phi \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_i(V_i)$ and hence defines a 0-cochain $(\phi)_i = \phi_i$.

One has

$$
\phi_j = \psi_j \circ \phi \circ \varphi_j^{-1} = \psi_{ji} \circ \phi_i \circ \varphi_{ij} \qquad \Rightarrow \psi_{ij} \circ \phi_j = \phi_i \circ \varphi_{ij}
\tag{13.4.6}
$$

We say that the 1-cocycles $(\varphi)_{ij}$ and $(\psi)_{ij}$ are *cohomologically equivalent* iff there exists a 0-cochain $(\phi)_i$ such that $\psi_{ij} \circ \phi_j = \phi_i \circ \varphi_{ij}$ (notice that for a commutative group this reduces to the notion of coboundaries, namely in additive notation to $(\psi) - (\varphi) = \delta(\phi)$, in standard Čech setting). However, this defines an equivalence relation between 1-cocycles and it defines a cohomology class $H^1 = Z^1/B^1$. One immediately has that topological manifolds modulo homeomorphism are in one-to-one correspondence with H^1 defined in this way.

Notice also that X is homeomorphic to $\mathbb{R}^n$ iff its transition cocycle is a coboundary (as one obtains just setting $\psi_{ij} = \text{id}$).

The same consideration can be done to classify isomorphic bundles. We shall do it in due time.

References

Hirsch

Chapter 14. Smooth Manifolds

1. The category of smooth manifolds

We defined the categories of locally Euclidean topological spaces $\mathfrak{Euc}^0$ and of topological manifolds $\mathfrak{C}^0(\mathfrak{M})$.

It may happen that a space M can be covered with an atlas such that its transition maps $\varphi^i{}_j : \mathbb{R}^m \to \mathbb{R}^m$ are not only homeomorphisms, but they turn out to be more regular.

> To be precise, transition maps are defined on $\varphi_j(U_{ij})$ and into $\varphi_i(U_{ij})$ and *there* they are homeomorphisms. Writing that $\varphi^i{}_j : \mathbb{R}^m \to \mathbb{R}^m$ is a, rather usual, abuse of language.

For example, if transition maps between 2 charts are of class $\mathcal{C}^k$ then we say that the 2 charts are $\mathcal{C}^k$-compatible. Given an atlas of $\mathcal{C}^k$-compatible charts, then M is a locally Euclidean space of class $\mathcal{C}^k$. When we have 2 $\mathcal{C}^k$ atlases, we say the are $\mathcal{C}^k$-*compatible* iff their union is still a $\mathcal{C}^k$ atlas. Two $\mathcal{C}^k$ atlases define the same locally Euclidean space or manifold.

On a $\mathcal{C}^k$-manifold, one can define $\mathcal{C}^k$-maps. In fact, let us consider two $\mathcal{C}^k$-manifolds, namely M and N, and a map $\Phi : M \to N$. Let us consider a pair of charts, one $(U_\alpha, \varphi^\alpha)$ around $x \in M$ and another (V_i, ϕ^i) around $y = \Phi(x) \in N$. The local expression of the map Φ is given by local maps as

$$\Phi^i{}_\alpha = \phi^i \circ \Phi \circ \varphi_\alpha : \mathbb{R}^m \to \mathbb{R}^n \tag{14.1.1}$$

In another pair of charts (U_β, φ^β) and (V_j, ϕ^j), one has another local expression for the same map, namely

$$\Phi^j{}_\beta = \phi^j \circ \Phi \circ \varphi_\beta = \phi^j \circ \phi_i \circ \phi^i \circ \Phi \circ \phi_\alpha \circ \phi^\alpha \circ \varphi_\beta = \phi^j{}_i \circ \Phi^i{}_\alpha \circ \varphi^\alpha{}_\beta \tag{14.1.2}$$

If this local expression is of class $\mathcal{C}^k$ in a pair of charts, it has the same regularity in any pair of charts. Being of class $\mathcal{C}^k$ thus does not depend on the charts and it is a feature of Φ.

> Here, of course, we take for granted the notion of a $\mathcal{C}^k$-map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and we define a $\mathcal{C}^k$-map Φ between manifolds as a map with local expressions $\Phi^i{}_\alpha$ given by (14.1.1) which are $\mathcal{C}^k$-map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Notice that, on $\mathcal{C}^3$ manifolds, one cannot define $\mathcal{C}^4$ maps, since a map which is $\mathcal{C}^4$ in a pair of charts is not necessarily $\mathcal{C}^4$ in other charts, since transition functions are $\mathcal{C}^3$ only.

On the other hand, one can define $\mathcal{C}^h$-maps on $\mathcal{C}^k$ manifolds when $h \leq k$.

The category of locally Euclidean $\mathcal{C}^k$-spaces (or $\mathcal{C}^k$-manifolds) with $\mathcal{C}^k$-maps is denoted by $\mathfrak{Euc}^k$ (or $\mathfrak{C}^k(\mathfrak{M})$). Analogously, we can define the category of $\mathcal{C}^\infty$-manifolds with $\mathcal{C}^\infty$-maps which is denoted by $\mathfrak{C}^\infty(\mathfrak{M})=\mathfrak{Man}$, and analytic manifolds with analytic maps which is denoted by $\mathfrak{C}^\omega(\mathfrak{M})$.

Diffeomorphisms

On a smooth manifold M, one can define smooth maps $\Phi : M \to N$, endomorphisms $\Phi : M \to M$, and isomorphisms $\Phi : M \to M$, which are also called *diffeomorphisms*.

A $\mathcal{C}^\infty$-map can be one-to-one, so having an inverse map, and still the inverse function fail to be continuous. In these cases of course the morphism is not invertible as such. It has an inverse in the category of sets, not in the category of smooth manifolds.

A $\mathcal{C}^\infty$-map can be one-to-one, so having a continuous inverse map, and still failing to be invertible in the category of smooth manifolds. That happens when the inverse map exists, it is continuous (thus it is invertible in the category of topological manifold), though the inverse in not smooth.

The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$ is one-to-one and it has a continuous inverse $f^{-1} : \mathbb{R} \to \mathbb{R} : y \mapsto \sqrt[3]{y}$. However, the inverse is not smooth, since its derivative is

$$(f^{-1})'(y) = \tfrac{1}{3}y^{-2/3} = \frac{1}{3\sqrt[3]{y^2}} \tag{14.1.3}$$

which fails to be defined at $y = 0$.

A global endomorphism is a diffeomorphism it has a Jaconian which is anywhere non-degenerate on M. The set of maps which are isomorphisms in the category of smooth manifolds, form a group by composition, the identity being the neutral element, and it is denoted by $\mathrm{Diff}(M)$. That is a pretty bad group. It is infinite dimensional and very little is known in general about it. Unfortunately, it is a fundamental structure of spacetimes in relativistic theories.

Let us also mention a bit of abuse of notation which is sometimes common about smooth manifolds. A *manifold* refers to a particular object of the category of smooth manifolds, or to the equivalence class of objects modulo diffeomorphisms. Two spheres of different radius are two different manifolds, and one, for example, can map one into the other by a diffeomorphism, so that they correspond to one single class of manifolds up to diffeomorphisms. When we shall say that spacetime is a manifold, actually we usually mean that spacetime is a particular manifold, or any manifold which is diffeomorphic to it. In other words, we often identify diffeomorphic manifolds. This is due to the covariance principle and the associated hole arguments which is discussed in Chapter 2.

Extended diffeomorphisms

Since the devil is in the details, we can also define a slightly different set of maps, called *extended diffeomorphisms* and denoted by $\mathrm{Diff}_*(M)$. They can be defined on locally Euclidean smooth spaces as well as on smooth manifolds. An *extended diffeomorphism* is a global map $\phi : M \to M'$ which is a homeomophism, and it is smooth with a smooth inverse everywhere up to a finite number of points.

We can give an example on $\mathbb{R}^2$. Consider the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, x^2 y + y^3)$ in Cartesian coordinates. One can prove that Φ is a homeomorphism on the whole plane and a diffeomorphism out of the origin but not at the origin.

> In fact, we can define the inverse $\bar{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2 : (x', y') \mapsto (x', y(x', y'))$, where $y = y(x', y')$ is a (the unique) solution of $y' = x'^2 y + y^3$. This always exists since the function $f : \mathbb{R} \to \mathbb{R} : y \mapsto y^3 + x'^2 y$ is everywhere monotone, being strictly growing (its derivative being $f'(y; x') = 3y^2 + x'^2$) except when $(x, y) = (0, 0)$. Hence it has a continuous inverse $y = y(x', y')$ which defines an everywhere continuous inverse map $\bar{\Phi}$.

Moreover, the Jacobian of the transfromation Φ is

$$J = \begin{pmatrix} 1 & 0 \\ 2xy & x^2 + 3y^2 \end{pmatrix} \tag{14.1.4}$$

which is singular only at $(x, y) = (0, 0)$. At the origin, the Jacobian exists although it is degenerate, hence Φ is everywhere differentiable but not a diffeomorphism at the origin.

In fact, at the point (x, y) the Jacobian maps the vector $\vec{v} = v^1 e_1 + v^2 e_2$ to $J(\vec{v}) = v^1 e_1 + (2xyv^1 + (x^2 + 3y^2)v^2)e_2$. This maps independent vectors to independent vectors except at the origin, where all vectors are mapped into $\mathrm{Span}(e_1)$. Hence we see that extended diffemorphisms do not preserve tangency of curves.

Consider the curves $\gamma_\pm : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (x = \pm s, y = s)$ which have the tangent vectors $\vec{v}_\pm = \pm e_1 + e_2$ at the origin. The transformation Φ maps these curves to $\Phi \circ \gamma_\pm : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (s, 2s^3)$ which both have tangent vector contained into $\mathrm{Span}(e_1)$ at the origin.

Therefore the transformation Φ maps two transverse curves $\gamma_\pm$ to two tangent curves $\tilde{\gamma}_\pm = \Phi \circ \gamma_\pm$. Accordingly, the transformation $\bar{\Phi}$ maps two curves $\tilde{\gamma}_\pm : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (s, 2s^3)$ which are tangent at the origin, to two curves $\gamma_\pm$ transverse at the origin.

Functions and curves on a manifold

Since $\mathbb{R}$ (and $\mathbb{C}$) is a smooth manifold itself, we have smooth maps $F : M \to \mathbb{R}$ ($F : M \to \mathbb{C}$) which are called *(real) functions* (or *complex functions*) on M. The set of (real or complex) functions on M form an algebra (and a ring) denoted by $\mathcal{F}(M)$ (or $\mathcal{F}(M; \mathbb{C})$).

Even if $\mathbb{R}$ and $\mathbb{C}$ are fields, $\mathcal{F}(M)$ and $\mathcal{F}(M; \mathbb{C})$ are not, since there exist functions with zeros for which $1/F$ is not a function, since it is not defined on the zeros of F.

Also maps $\gamma : \mathbb{R} \to M$ are provided with a special name. They are called *(parameterised) curves* on M. The set of all parameterised curves is denoted by $\Gamma(M)$, they are also sections of a special bundle $\pi : \mathbb{R} \times M \to \mathbb{R}$.

> As for maps, when we write $\gamma : \mathbb{R} \to M$ we do not really mean that the map is defined on the whole $\mathbb{R}$, we just what it to be defined on some interval I, being it open or closed depending in the context, usually at least such that $0 \in I$.

Functions and curves are rather special maps, since one can regard $\mathbb{R}$ (or $\mathbb{C}$) as manifolds with a canonical global chart fixed on them. In other words, when we specialize from maps to functions or curves we *can* keep the coordinate on $\mathbb{R}$ fixed once and for all.

For example, functions can be describe in terms of their local expressions. Let $F : M \to \mathbb{R}$ be a function and $(U_\alpha, \varphi^\alpha)$ a chart around $p \in M$. The local expression is defined as

$$F_\alpha := F \circ \varphi_\alpha : \varphi^\alpha(U_\alpha) \to \mathbb{R} \tag{14.1.5}$$

which we usually denote simply by $F_\alpha : \mathbb{R}^m \to \mathbb{R}$ by the usual abuse of notation.

If we change the chart (U_β, φ^β) on M we get transformation laws for functions on M in the form

$$F_\alpha = F \circ \varphi_\alpha = F \circ \varphi_\beta \circ \varphi^\beta \circ \varphi_\alpha = F_\beta \circ \varphi^\beta{}_\alpha \tag{14.1.6}$$

Functions form a sheaf and they can be localized as well as one can define global functions by gluwing together local expressions which agree on the overlaps of an open cover.

A similar discussion can be done for parameterized curves $\gamma : \mathbb{R} \to M$ which can be described by local expressions in a chart

$$\gamma^\alpha := \varphi^\alpha \circ \gamma : \mathbb{R} \to \mathbb{R}^m : s \mapsto \gamma^\mu(s) \tag{14.1.7}$$

which glue together by their transformation laws

$$\gamma^\alpha = \varphi^\alpha \circ \gamma = \varphi^\alpha \circ \varphi_\beta \circ \varphi^\beta \circ \gamma = \varphi^\alpha{}_\beta \circ \gamma^\beta \tag{14.1.8}$$

2. Examples of smooth manifolds

Let us give some examples of locally Euclidean smooth spaces.

Generally speaking, to prove that something is a locally Euclidean smooth space, we shall define a cover by charts and check that transition functions are smooth. This is a bit awkward with respect to the mathematical definition of manifolds on books, which usually requires to start from quite a special topological space (T_2, paracompact) and then to define on it a canonical maximal atlas.

The two definitions turn out to be almost equivalent since many constructions extends from locally Euclidean smooth space to manifolds and any finite atlas defines a maximal atlas (essentially by adding all compatible charts even if in this case one does not really need it).

For us, a locally Euclidean smooth space is a set which *allows* an atlas, so a set with two atlases may be regarded as the same manifold if the two atlases are $\mathcal{C}^k$-compatible, i.e. if the union of the two atlases is still an atlas of the same regularity class.

Vector spaces

Any vector spaces V of dimension n is a manifold. If one fixes a basis e_i of it then one can define a global chart

$$c : V \to \mathbb{R}^n : v \mapsto v^i \qquad v = v^i e_i \tag{14.2.1}$$

Being global, this chart forms an atlas on its own.

Such coordinates are called *Cartesian coordinates*. Given two systems of Cartesian coordinates transition map takes the form $w^i = \Lambda^i_j v^j$, where Λ^i_j is the transition matrix between the two bases. If one forms an atlas with all possible Cartesian coordinates on V then transition maps take values in $GL(V)$, which of course is a much smaller group than $\text{Diff}(\mathbb{R}^n)$.

Of course, on the manifold $\mathbb{R}^2$ we can use Cartesian coordinates (x, y) as well as any other coordinate system which is compatible with it. For example, *polar coordinates* (r, θ) are $\mathcal{C}^\infty$-compatible since transition maps

$$\varphi : (r, \theta) \mapsto (x = r\cos\theta, y = r\sin\theta) \tag{14.2.2}$$

are $\mathcal{C}^\infty$ maps defined on the plane except a semi line.

Although polar coordinates are perfectly fine on $\mathbb{R}^2$ as a manifold, they do not preserves linear structure (transition maps are in $\text{Diff}(\mathbb{R}^2)$, not in $GL(\mathbb{R}^2)$). In some sense, Cartesian coordinates are characteristic of vector spaces.

Affine spaces

Cartesian coordinates can be extended on affine spaces as well. Let $\mathbb{A}$ be an affine space of dimension n and consider a *reference frame*, i.e. an origin and a basis (O, e_i) in the model V. A global chart is associated to the reference frame as

$$c : \mathbb{A} \to \mathbb{R}^n : A \mapsto v^i \qquad A = v^i e_i + O \tag{14.2.3}$$

Again this chart is global and it forms an atlas on its own. Given two systems of Cartesian coordinates transition map takes the form

$$w^i = \Lambda^i_j v^j + \delta^i \tag{14.2.4}$$

where Λ^i_j is the transition matrix between the two bases and δ^i are the components of the vector $O - O'$. If one forms an atlas with all possible Cartesian coordinates on $\mathbb{A}$ then transition maps take values in $GA(\mathbb{A})$.

Also in this case, since affine spaces are also manifolds, one can consider more general coordinates on them. However, on affine spaces one can always restrict to special atlases which have transition maps valued in $GA(\mathbb{A})$, only.

Projective spaces

Let us here consider (real) projective spaces $\mathbf{P}_n = \mathbf{P}(\mathbb{R}^{n+1})$. Points in $\mathbf{P}_n$ denote directions in $\mathbb{R}^{n+1}$.

For, let us consider $\mathbb{R}^{n+1} - \{0\}$ and define the equivalence relation

$$x \sim y \iff \exists \lambda \in \mathbb{R} - \{0\} : y = \lambda x \tag{14.2.5}$$

The quotient space $\mathbf{P}_n := (\mathbb{R}^{n+1} - \{0\})/\sim$ is called the n-th (real) projective space. A point $p \in \mathbf{P}_n$ is a class $p = [x^0, x^1, \ldots x^n] = \{(\lambda x^0, \lambda x^1, \ldots \lambda x^n) : \lambda \in \mathbb{R} - \{0\}\}$, where x^μ cannot be all simultaneously zero. We shall here show that the projective space $\mathbf{P}_n$ is a manifold and build an atlas for it.

Let us consider the points $U_{(0)} = \{p = [x^0, x^1, \ldots, x^n] : x^0 \neq 0\} \subset \mathbf{P}_n$ and define the map

$$\varphi_{(0)} : U_{(0)} \to \mathbb{R}^n : p = [x^0, x^1, \ldots, x^n] \mapsto (\tfrac{x^1}{x^0}, \ldots, \tfrac{x^n}{x^0}) \tag{14.2.6}$$

The map $\varphi_{(0)}$ is obviously one-to-one on $U_{(0)}$. Since x^μ cannot be all simultaneously zero, we can cover the whole $\mathbf{P}_n$ with $n+1$ charts $(U_{(\mu)}, \varphi_{(\mu)})$ defined as above. The charts induce a topology on $\mathbf{P}_n$ which makes $\mathbf{P}_n$ a (topological) manifold.

Let us compute the transition map between $\varphi_{(0)}$ and $\varphi_{(1)}$. The point $(x^1, \ldots, x^n) \in \mathbb{R}^n$ is mapped by $\varphi_{(0)}^{-1}$ to $[1, x^1, \ldots, x^n] = p$. Let us now suppose that $p \in U_{(1)}$ (i.e. $x^1 \neq 0$) so that $\varphi_{(1)}(p) = (\tfrac{1}{x^1}, \ldots, \tfrac{x^n}{x^1})$. Thus the transition map $\varphi_{(10)}$ is defined over $V_{(i)} = \{(x^1, \ldots, x^n) : x^i \neq 0\} \subset \mathbb{R}^n$ and it is

$$\varphi_{(10)} : V_{(1)} \to V_{(1)} : (x^1, x^2, \ldots, x^n) \mapsto (\tfrac{1}{x^1}, \tfrac{x^2}{x^1}, \ldots, \tfrac{x^n}{x^1}) = (\tfrac{1}{x^1}, \tfrac{x^2-1}{x^1} + \tfrac{1}{x^1}, \ldots, \tfrac{x^n-1}{x^1} + \tfrac{1}{x^1}) \tag{14.2.7}$$

which is a $\mathcal{C}^\infty$-map on $V_{(1)}$. Accordingly, the projective space $\mathbf{P}_n$ is a $\mathcal{C}^\infty$-manifold. It is covered by $(n+1)$-charts $(U_{(\mu)}, \varphi_{(\mu)})$ which are called the *affine charts*.
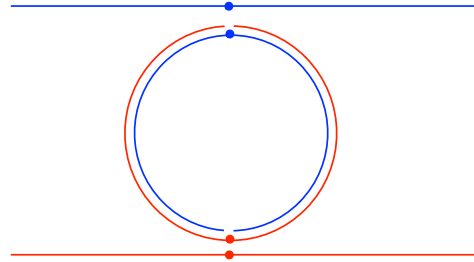
Fig. 14.1: The projective space $\mathbf{P}_1$

Let us consider $\mathbf{P}_1$ as the set of directions in the plane $\mathbb{R}^2$. A point $p \in \mathbf{P}_1$ is in the form $p = [x, y]$ with $(x, y) \neq (0, 0)$. We can cover it with two charts: $\varphi_{(0)}$ is defined on $U_{(0)} = \{[x, y] : x \neq 0\}$ which corresponds to the whole $\mathbf{P}^1$ except a point $P_0 = [0, y]$ with $y \neq 0$ so that $\{P_0\} = \{[0, 1]\} \subset \mathbf{P}_1$. The other chart is $\varphi_{(1)}$ which is defined on $U_{(1)} = \{[x, y] : y \neq 0\}$ which corresponds to the whole $\mathbf{P}_1$ except the point $P_1 = [x, 0]$ with $y \neq 0$ so that $\{P_1\} = \{[1, 0]\} \subset \mathbf{P}_1$.

Since both $U_{(0)}$ and $U_{(1)}$ are homeomorphic to $\mathbb{R}$ we already discovered that $\mathbf{P}_1$ is in a sense made by adding a point to $\mathbb{R}$. As one gets near the point P_1 in the chart $\varphi_{(0)}$ the image point goes to the infinity of $\mathbb{R}$. The point P_1 on the other hand is mapped by $\varphi_{(1)}$ to the point $0 \in \mathbb{R}$.

Thus $\mathbf{P}_1$ is obtained by glueing together two copies of $\mathbb{R}$ along the map $\varphi_{(0,1)} : \mathbb{R} - \{0\} \to \mathbb{R} - \{(0)\} : x \mapsto 1/x$ as shown in the Figure. We shall see hereafter that the result is a circle S_1.

Let us then consider $\mathbf{P}_2$ as the set of directions in the space $\mathbb{R}^3$. A point $p \in \mathbf{P}_2$ is in the form $p = [x, y, z]$ with $(x, y, z) \neq (0, 0, 0)$. We can cover it with three charts: $\varphi_{(0)}$ is defined on $U_{(0)} = \{[x, y, z] : x \neq 0\}$ which corresponds to the whole $\mathbf{P}_2$ except the line $L_0 = \{[0, y, z] : (y, z) \neq (0, 0)\} \subset \mathbf{P}_2$. Since y and z cannot be zero at the same time, the subset L_0 is diffeomorphic to a copy of $\mathbf{P}_1$. The chart $\varphi_{(1)}$ is defined on $U_{(1)} = \{[x, y, z] : y \neq 0\}$ which corresponds to the whole $\mathbf{P}_2$ except the line $L_1 = \{[x, 0, z] : (y, z) \neq (0, 0)\} \subset \mathbf{P}_2$ and it is again diffeomorphic to a copy of $\mathbf{P}_1$. The last chart

$\varphi_{(2)}$ is defined on $U_{(2)} = \{[x, y, z] : z \neq 0\}$ which corresponds to the whole $\mathbf{P}_2$ except the line $L_2 = \{[x, y, 0] : (x, y) \neq (0, 0)\} \subset \mathbf{P}_2$ and it is again diffeomorphic to a copy of $\mathbf{P}_1$.

Since $U_{(0)}$, $U_{(1)}$ and $U_{(2)}$ are homeomorphic to $\mathbb{R}^2$ we already discovered that $\mathbf{P}_2$ is in a sense made by glueing together a copy of $\mathbf{P}_1$ to $\mathbb{R}^2$. As one gets near a point on L_1 in the chart $\varphi_{(0)}$ the image point goes to the infinity of $\mathbb{R}^2$ in a specific direction (meaning that approaching L_1 on the other side corresponds to go to infinity on the opposite direction).

Thus $\mathbf{P}_2$ is obtained by glueing together three copies of $\mathbb{R}^2$ along the map $\varphi_{(0,1)} : \mathbb{R}^2 - \{(0, z)\} \to \mathbb{R}^2 - \{(0, z)\} : (x, z) \mapsto (1/x, z/x)$. However, this is not a sphere S_2 (in fact $\mathbf{P}_2$ is not orientable).

The sphere S_n

Let us consider a sphere S_n of dimension n sitting in $\mathbb{R}^{n+1}$ by the embedding $i : S_n \to \mathbb{R}^{n+1}$ as the unit sphere, namely $i(p) = (x^0, x^1, \ldots, x^n)$ which satisfy the equation $(x^0)^2 + (x^1)^2 + \ldots + (x^n)^2 = 1$.

We shall define on S_n an atlas made of two charts, which exhibits the manifold structure of S_n. Let us consider two points $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0)$ on S_n, called the *north pole* and the *south pole*, respectively. Let us set $(S_n)_N := S_n - \{N\}$ and $(S_n)_S := S_n - \{S\}$ and define the two maps

$$\varphi_N : (S_n)_N \to \mathbb{R}^n : (x^0, x^1, \ldots, x^n) \mapsto x_N^i := \frac{x^i}{1 - x^0} \qquad \varphi_S : (S_n)_S \to \mathbb{R}^n : (x^0, x^1, \ldots, x^n) \mapsto x_S^i := \frac{x^i}{1 + x^0} \qquad (14.2.8)$$

The maps given above correspond to project a point P on the sphere from one pole to the equatorial plane (and of course they fail to be defined when the point P coincides with the pole from which one is projecting). Reasoning in $\mathbb{R}^{n+1}$, one has the line through the pole (e.g. N) and $P = (x^0, x^1, \ldots, x^n) \in (S_n)_N$ (thence with $x^0 \neq 1$) as

$$\gamma : \mathbb{R} \to \mathbb{R}^{n+1} : s \mapsto \begin{pmatrix} s(x^0 - 1) + 1 \\ sx^1 \\ \vdots \\ sx^n \end{pmatrix} \qquad (14.2.9)$$

which intersects the equatorial plane $(0, x^1, \ldots, x^n)$ at the point $(0, \frac{x^i}{1-x^0})$ when $s = \frac{1}{1-x^0}$. The map φ_N follows by identifying the equatorial plane with $\mathbb{R}^n$.

We can compute the transition map $\varphi_{(NS)} : \mathbb{R}^n - \{(0)\} \to \mathbb{R}^n - \{(0)\}$ by solving for (x^0, x^i) the system

$$\begin{cases} x_S^i = \dfrac{x^i}{1 + x^0} \\ (x^0)^2 + (x^1)^2 + \ldots + (x^n)^2 = 1 \end{cases} \qquad \begin{cases} x^0 = \dfrac{1 - |x_S|^2}{1 + |x_S|^2} \\ x^i = \dfrac{2x_S^i}{1 + |x_S|^2} \end{cases} \qquad (14.2.10)$$

which gives the inverse chart

$$\varphi_S^{-1} : \mathbb{R}^n \to (S_n)_S : x_S^i \mapsto \begin{pmatrix} \frac{1 - |x_S|^2}{1 + |x_S|^2} \\ \frac{2x_S^i}{1 + |x_S|^2} \end{pmatrix} \qquad (14.2.11)$$

where we set $|x_S|^2 = (x_S^1)^2 + \ldots + (x_S^n)^2$. The map φ_S^{-1} is also known as a *parameterisation* of the sphere.

Then transition maps are

$$\varphi_{(NS)} = \varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\} : x_S^i \mapsto x_N^i := \frac{x_S^i}{|x_S|^2} \qquad (14.2.12)$$

which is a diffeomorphism where it is defined. Accordingly, S_n is a $\mathcal{C}^\infty$-manifold.

The transition map for $n = 1$ reads as

$$\varphi_{(NS)} : \mathbb{R}^1 - \{0\} \to \mathbb{R}^1 - \{0\} : x_S \mapsto x_N := \tfrac{1}{x_S} \qquad (14.2.13)$$

which in fact coincides with the transition map of projective space $\mathbf{P}_1$.

Thus, both S_1 and $\mathbf{P}_1$ are obtained by glueing together two patches isomorphic to $\mathbb{R}$. On S_1 one has two patches with coordinates x_S and x_N, respectively. On $\mathbf{P}_1$ one has two patches with coordinates x and y, respectively.

We can define two local maps between S_1 and $\mathbf{P}_1$ defined by their local expressions as

$$\Phi_1 : (S_1)_S \to U_{(1)} : x_S \mapsto x \qquad\qquad \Phi_2 : (S_1)_N \to U_{(2)} : x_N \mapsto y \qquad (14.2.14)$$

These two local diffeomorphisms glue together because they are compatible with transition maps

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\ \varphi_{SN}\ } & S_1 \\
\Phi_1 \downarrow & & \downarrow \Phi_2 \\
\mathbf{P}_1 & \xrightarrow{\ \varphi_{(0,1)}\ } & \mathbf{P}_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
x_S & \xmapsto{\ \varphi_{SN}\ } & x_N = \tfrac{1}{x_S} \\
\Phi_1 \uparrow & & \downarrow \Phi_2 \\
x & \xmapsto{\ \varphi_{(0,1)}\ } & y = \tfrac{1}{x}
\end{array}
\qquad (14.2.15)
$$

and they define a global diffeomorphism $\Phi : S_1 \to \mathbf{P}_1$.

3. Manifolds with boundary

Let us consider $H_+^n = \{x^\mu \in \mathbb{R}^n : x^0 \geq 0\}$ the (closed) *half-plane* in $\mathbb{R}^n$. The half-plane H_+^n is a subset of $\mathbb{R}^n$ and it gets a topology induced by the standard topology in $\mathbb{R}^n$. Accordingly, it is a topological space.

Open sets in H_+^n are of two types. If you consider a ball U in $\mathbb{R}^n$ which does not intersect the hyperplane $x^0 = 0$, then $U \cap H_+^n$ is an open set in H_+^n. These are either the empty set $\varnothing$ or they coincide with U and, in that case, they are called *ball-shaped open sets*.

If the ball U intersects the hyperplane $x^0 = 0$, then $U \cap H_+^n$ is called a *half-ball-shaped* open set in H_+^n. An half-ball-shaped open set in H_+^n is made by its interior which is a ball of $\mathbb{R}^n$, together with a ball in $\mathbb{R}^{n-1}$.

However, H_+^n is not a topological locally Euclidean space.

In order to be a locally Euclidean space there should be an atlas and, consequently, any point would be locally homeomorphic to an open set in $\mathbb{R}^n$.

On the other hand, if one considers x^μ with $x^0 = 0$ any of its neighbourhood is not homeomorphic to a ball in $\mathbb{R}^n$ since they all are in fact half-balls.

In H^n_+ one can define two types of points: a point $x \in H^n_+$ is *interior* if it has a neighbourhood which is homeomorphic to a ball of $\mathbb{R}^n$. A point $x \in H^n_+$ is *boundary* if any neighbourhood of it is homeomorphic to a half-ball of $\mathbb{R}^n$.

> Mind the asymmetry in quantificators.

One can define two subsets: $\text{Int}(H^n_+) = \{x^\mu \in \mathbb{R}^n : x^0 > 0\}$ is made of all interior points and $\partial H^n_+ = \{x^\mu \in \mathbb{R}^n : x^0 = 0\}$ is made of all boundary points. Of course, one has that $H^n_+ = \text{Int}(H^n_+) \cup \partial H^n_+$ as well as $\text{Int}(H^n_+) \cap \partial H^n_+ = \varnothing$.

Both the *interior part* $\text{Int}(H^n_+)$ and the *boundary* ∂H^n_+ of the half plane are sub-manifolds. While the interior part is a manifold of dimension n covered by the restriction of a global chart in $\mathbb{R}^n$, the boundary $\partial H^n_+ \simeq \mathbb{R}^{n-1}$ is a sub-manifold of dimension $n-1$.

A *local $\mathcal{C}^k$-map* on H^n_+ is a map $\phi : U \to V$ for two open sets U and V in H^n_+ (thus possibly half-balls of $\mathbb{R}^n$) such that there exist two open sets U' and V' in $\mathbb{R}^n$ such that $U \subset U'$ and $V \subset V'$ and a $\mathcal{C}^k$-map $\hat\phi : U' \to V'$ which restricts to ϕ on $U \subset U'$. The map $\hat\phi$ is called an *extension* of ϕ.

A *local $\mathcal{C}^k$-transformation* on H^n_+ is a $\mathcal{C}^k$-map $\phi : H^n_+ \to H^n_+$ which is invertible and the inverse map $\phi^{-1} : H^n_+ \to H^n_+$ is a local $\mathcal{C}^k$-transformation on H^n_+, too.

> A local $\mathcal{C}^k$-transformation $\phi : H^n_+ \to H^n_+$ maps boundary points to boundary points and internal points to internal points.
>
> Let us consider an internal point $x \in \text{Int}(H^n_+)$, which means that one can find a ball $D \subset H^n_+$ which is a neighbourhood of x. Then $\phi(D) \subset H^n_+$ is a ball, since it coincides with $\hat\phi(D)$ and the extension $\hat\phi$ maps balls into balls. Then $\phi(x)$ has a ball-shaped neighbourhood $\phi(D)$, hence $\phi(x)$ is internal.
>
> On the other hand, let us consider a boundary point $x \in \partial H^n_+$ and suppose, for the sake of argument, that $\phi(x)$ is internal. Then it would have a ball-shaped neighbourhhod $D \subset H^n_+$ and $\hat\phi^{-1}(D) \subset H^n_+$ would be a ball-shaped neighbourhood of x, which then would be internal, contradicting the hypothesis. Then $\phi(x)$ is boundary as well.

A *manifold with boundary* (of dimension m) is a topological space M which allows an atlas $(U_\alpha, \varphi^\alpha)$ with charts $\varphi^\alpha : U_\alpha \to H^m_+$ which are $\mathcal{C}^k$-transformations defined on the half plane. Transition maps $\varphi^\alpha{}_\beta = \varphi^\alpha \circ \varphi_\beta$ are in fact local $\mathcal{C}^k$-transformations on the half plane H^m_+.

> Equivalently, one can define manifolds with boundary as a collection of open patches $V_\alpha \subset \mathbb{H}^m_+$ together with a family of glueing local $\mathcal{C}^k$-transformations $\varphi^\alpha{}_\beta : V_\beta \to V_\alpha$ obeying cocycle identities. The construction closely follows the construction for manifolds.
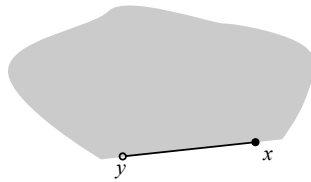
Fig. 14.2: Example of a set which has a boundary of a boundary which is not a manifold with boundary

One can define *interior points* of M as the points $x \in M$ which are mapped to interior points by a chart (and then by any chart). The set of interior points is denoted by $\text{Int}(M) \subset M$ and it is a manifold (just using the induced atlas).

Boundary points of M are the points $x \in M$ which are mapped to boundary points by a chart (and then by any chart). The set of boundary points is denoted by $\partial M \subset M$ and it is called the *boundary* of M. It is a sub-manifold of dimension $(m-1)$ (just using the induced atlas).

> Notice that the boundary ∂M of a manifold with boundary M is a manifold, not a manifold with boundary.

Since manifolds could be defined as manifolds with boundaries with an empty boundary (i.e. with $\partial M = \varnothing$), then one has $\partial\partial M = \varnothing$.

in order to have boundaries of boundaries one should define more general objects, see for example Figure 14.1, which are not manifolds with boundaries since the point x (nor point y) has neighbourhoods which are not balls or half-balls.

4. Tangent vectors

Tangent vectors are defined to be equivalence classes of curves on a manifold M of dimension $\dim(M) = m$.

Let us consider the set $\Gamma_p(M)$ of all (parameterised) curves in M based at $p \in M$. We define the relation

$$\gamma \sim \gamma' \iff \forall F : M \to \mathbb{R}, T_0^1(F \circ \gamma) = T_0^1(F \circ \gamma') \tag{14.4.1}$$

where $F \circ \gamma, F \circ \gamma' : \mathbb{R} \to \mathbb{R}$ and T_0^1 denotes the first Taylor polynomial at $s = 0$.

Although, the equivalence relation is obviously independent of the chart, let us consider its expression in a fixed chart (U, x^μ). The local expression of the curves in coordinates are

$$\gamma : \mathbb{R} \to M : s \mapsto x^\mu = \gamma^\mu(s) \qquad \gamma' : \mathbb{R} \to M : s \mapsto x^\mu = \gamma'^\mu(s) \tag{14.4.2}$$

while the local expression of the function F is given by

$$F : M \to \mathbb{R} : x^\mu \mapsto F(x) \tag{14.4.3}$$

Thus the functions $F \circ \gamma, F \circ \gamma' : \mathbb{R} \to \mathbb{R}$ are expressed as

$$F \circ \gamma : \mathbb{R} \to \mathbb{R} : s \mapsto F(\gamma(s)) \qquad F \circ \gamma' : \mathbb{R} \to \mathbb{R} : s \mapsto F(\gamma'(s)) \tag{14.4.4}$$

so that the Taylor expansion reads as

$$T_0^1(F \circ \gamma) = F(p) + \partial_\mu F(p)\dot\gamma^\mu(0)s \qquad T_0^1(F \circ \gamma') = F(p) + \partial_\mu F(p)\dot\gamma'^\mu(0)s \tag{14.4.5}$$

If the equality $T_0^1(F \circ \gamma) = T_0^1(F \circ \gamma')$ must hold for any function F, that means it must be $\dot\gamma^\mu(0) = \dot\gamma'^\mu(0)$.

Thus two curves γ and γ' are equivalent iff

$$\begin{cases} \gamma(0) = \gamma'(0) =: p \\ \dot\gamma^\mu(0) = \dot\gamma'^\mu(0) \end{cases} \tag{14.4.6}$$

We can check directly that conditions (14.4.6) do not depend on the chart, i.e. if two curves are equivalent in one chart, they are equivalent in any compatible chart.

Let us consider a new chart $x'^\mu = x'^\mu(x)$. The new local expression of the two curves are

$$\gamma : \mathbb{R} \to M : s \mapsto x'^\mu = x'^\mu(\gamma(s)) \qquad \gamma' : \mathbb{R} \to M : s \mapsto x'^\mu = x'^\mu(\gamma'(s)) \tag{14.4.7}$$

and the conditions for them to be compatible reads as

$$\begin{cases} x'^{\mu}\left(\gamma(0)\right) = x'^{\mu}\left(\gamma'(0)\right) \\ J_{\alpha}^{\mu}\,\dot{\gamma}^{\alpha}(0) = J_{\alpha}^{\mu}\,\dot{\gamma}'^{\alpha}(0) \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \gamma(0) = \gamma'(0) =: p \\ \dot{\gamma}^{\mu}(0) = \dot{\gamma}'^{\mu}(0) \end{cases} \tag{14.4.8}$$

since transition functions $x'^{\mu} = x'^{\mu}(x)$ are invertible (and thus the Jacobians are invertible as well).

Of course, the independence of the chart of (14.4.6) follows more directly for the fact that they come as local expression of an intrinsic condition. Still the direct proof is somehow interesting on itself since it does not rely on the notion of *intrinsic* and eventually it contributes to *define* what intrinsic means in differential geometry.

Let us define the *tangent space*

$$T_p M = \Gamma_p(M)/\sim \qquad\qquad TM = \coprod_{p \in M} T_p M \tag{14.4.9}$$

and a point $[\gamma] \in TM$ is called a *tangent vector* to M at the point $p = \gamma(0) \in M$.

This definition works for a locally Euclidean space. On a manifold we can *also* define the equivalence relation on $\Gamma(M)$ and define directly $TM = \Gamma(M)/\sim$. Since equation (14.4.5) have to hold true for any function F, when the space M is Hausdorff that condition implies both (14.4.6) as if $\gamma(0) \neq \gamma'(0)$ there exists a function F such that $F(\gamma(0)) \neq F(\gamma'(0))$. This fails on non-Hausdorff spaces, for example on the line with 2 origins, consequently in general we need to define $T_x M$ first.

The idea is extremely simple and fundamental. The velocity of a motion in a manifold is certainly tangent to the manifold. Thus the motions which share the same velocity do *define* a tangent vector.

This reminds the fact that tangent vectors are defined using derivatives and derivatives have been introduced to model velocities.

[That is not exactly historically accurate: derivatives have been introduced as inverse to the problem of area, i.e. integrals. However, one could say that they *could* have been introduced to model velocities.]

Let us denote by $T_x M \subset TM$ the subset of all tangent vectors at a point $x \in M$.

Let us define on $T_{x_0} M$ the operations

$$[\gamma] + [\gamma'] = [x_0^{\mu} + v^{\mu}s] + [x_0^{\mu} + w^{\mu}s] = [x_0 + (v^{\mu} + w^{\mu})s] \qquad \lambda[\gamma] = \lambda[x_0^{\mu} + v^{\mu}s] = [x_0 + \lambda v^{\mu}s] \tag{14.4.10}$$

One should show that these definitions do not depend on the representative chosen for the curves.

Then, endowed with these two operations, the set $T_{x_0} M$ is a vector space.

Notice how, given a chart, a tangent vector $[\gamma]$ is identified by the $2m$ numbers $(\gamma^{\mu}(0), \dot{\gamma}^{\mu}(0))$ which are in fact shared by all representatives of the class $[\gamma]$.

We aim hereafter to show what the numbers $(\gamma^{\mu}(0), \dot{\gamma}^{\mu}(0))$ represent and how they depend on the chart.

For the first item let us consider a chart and the m-families of curves (locally defined within the domain U of the chart)

$$c_{\alpha} : \mathbb{R} \to U : s \mapsto x^{\mu} = x_0^{\mu} + \delta_{\alpha}^{\mu} s \tag{14.4.11}$$

The curve c_{α} is called the αth *coordinate curve* through the point x_0.

The corresponding tangent vectors $[c_{\alpha}]$ are also denoted by $\partial_{\alpha} = [c_{\alpha}]$ and they form a basis of the vector space $T_{x_0} M$.

In fact, a general tangent vector at x_0 is in the form

$$[\gamma] = [x_0^\mu + v^\mu s] = [x_0^\mu + v^\alpha \delta_\alpha^\mu s] = v^\alpha [c_\alpha] = v^\alpha \partial_\alpha \qquad (14.4.12)$$

so that ∂_α are generators of $T_{x_0} M$.

The zero vector $\vec{0} \in T_{x_0} M$ is the class of the constant motion $\vec{0} = [x_0^\mu]$. Thus a linear combination $v^\alpha \partial_\alpha$ equals the zero vector iff

$$\vec{0} = v^\alpha \partial_\alpha = [x_0^\mu + v^\mu s] = [x_0^\mu] \qquad \Longleftrightarrow \quad v^\mu = 0 \qquad (14.4.13)$$

Consequently, the vectors ∂_α are independent. Being independent generators they form a basis.

Since $T_{x_0} M$ has a basis of m elements one has $\dim(T_{x_0} M) = m$.

The basis ∂_α of $T_{x_0} M$ is called the *natural basis* induced by the chart.

Thus if a tangent vector $[\gamma]$ is identified by $2m$ numbers, namely $(x_0^\mu, v^\mu) := (\gamma^\mu(0), \dot\gamma^\mu(0))$, the first m numbers $x_0^\mu := \gamma^\mu(0)$ are the coordinates of the application point x_0, while the second set of m numbers $v^\mu := \dot\gamma^\mu(0)$ are the components of the vector $\vec{v} = [\gamma]$ in the basis ∂_α, namely one has $\vec{v} = v^\mu \partial_\mu$. As far as the second issue is concerned, if we consider a new chart (U, x'^μ) and the transition functions $x'^\mu = x'^\mu(x)$ are known, then the coordinate curve c_α in the old chart, defines a tangent vector ∂_α which can be expanded in the new natural basis ∂'_α, in fact

$$\partial_\alpha = [c_\alpha] = [x'^\mu(x_0^\lambda + \delta_\alpha^\lambda s)] = [x_0'^\mu + J_\lambda^\mu \delta_\alpha^\lambda s] = [x_0'^\mu + J_\alpha^\lambda \delta_\lambda^\mu s] = J_\alpha^\lambda \partial'_\lambda \qquad (14.4.14)$$

where we applied to this case the equation (14.4.12). Thus a tangent vector $\vec{v} = v^\alpha \partial_\alpha = v^\alpha J_\alpha^\mu \partial'_\mu = v'^\mu \partial'_\mu$ is expressed by the components v^α in the old chart and by the components

$$v'^\mu = J_\alpha^\mu v^\alpha \qquad (14.4.15)$$

in the new chart.

We can define an atlas for TM made of natural coordinates (x^μ, v^μ) depending on an atlas of M such that transition functions on TM are given by

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^\mu = J_\alpha^\mu v^\alpha \end{cases} \qquad (14.4.16)$$

If charts on M are compatible of class $\mathcal{C}^k$, then the corresponding natural charts on TM are of class $\mathcal{C}^{k-1}$ (due to the Jacobian which reduces regularity by one order). Accordingly, if M is a smooth manifold, its tangent TM is a smooth manifold dimension $\dim(TM) = 2m$. Of course, once TM is endowed with the structure of smooth manifold by the natural atlas, then the atlas can be enriched by adding all $\mathcal{C}^\infty$-compatible charts.

We shall endow TM with a richer structure of (vector) fiber bundle in Section 18.2. In any event, one has a special map $\pi : TM \to M : (x, \vec{v}) \mapsto x$ which associates to any tangent vector its application point. This is a surjective smooth map called the *projection* and we shall show it is of maximal rank, namely $\mathrm{rank}(\pi) = m$.

Higher order tangent spaces

Similarly to tangent vectors, we can define *higher order jet prolongations* by considering equivalence relations which use higher order Taylor expansion. For example, we can define T^2M as the space of equivalence classes $[\gamma]^2$ of curves such that

$$
\begin{cases}
\gamma(0) = \gamma'(0) \\
\dot{\gamma}^\mu(0) = \dot{\gamma}'^\mu(0) \\
\ddot{\gamma}^\mu(0) = \ddot{\gamma}'^\mu(0)
\end{cases}
\tag{14.4.17}
$$

Points in T^2M are identified by $3m$ numbers (x^μ, v^μ, a^μ). By changing chart on M this induces a new natural chart on T^2M which is related to the old natural chart by

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
v'^\mu = J^\mu_\alpha v^\alpha \\
a'^\mu = J^\mu_\alpha a^\alpha + J^\mu_{\alpha\beta} v^\alpha v^\beta
\end{cases}
\tag{14.4.18}
$$

Notice that while v^α is a pointwise linear object (i.e. at any point $x \in M$ the transformation of v^μ is linear), this is not the same for a^μ (which is pointwise affine in a and quadratic in v). If we regarded a as an object on TM (not on M) then a can be regarded as an affine object since its transformation laws are in the form

$$
a'^\mu = A^\mu_\alpha(x) a^\alpha + B^\mu(x, v)
\tag{14.4.19}
$$

This remark will be stated more precisely in terms of fiber bundle structure in Section 18.2.

This construction can be easily extended to any order k; if M is a smooth manifold, then T^kM is a smooth manifold dimension $\dim(T^kM) = (k+1)m$.

One can also consider the tangent space to the manifold TM, denoted by TTM. It is a manifold of dimension $\dim(TTM) = 4m$ with natural coordinates $(x^\mu, v^\mu, u^\mu, a^\mu)$ whose transformation laws read as

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
v'^\mu = J^\mu_\alpha v^\alpha \\
u'^\mu = J^\mu_\alpha u^\alpha \\
a'^\mu = J^\mu_\alpha a^\alpha + J^\mu_{\alpha\beta} v^\alpha u^\beta
\end{cases}
\tag{14.4.20}
$$

Accordingly, the map $\Phi : T^2M \to TTM : (x, v, a) \mapsto (x, v, v, a)$ is a global map and it shows that T^2M is a sub-manifold of TTM identified by the constraint $u^\mu = v^\mu$.

Directions vs. tangent vectors

In some sense, tangent vectors are somehow fundamentally awkward as objects of differential geometry. They are associated to curves, *parameterised* curves, and they do depend on the parameterisation while differential geometry should study properties of the *trajectories*, i.e. the properties of sub-manifolds which are independent of parameterisation.

By changing parameterisation of the curve $\gamma'(s') = \gamma(s(s'))$ by the reparameterisation $\phi(s') = s$, the tangent vector transforms as

$$v'^{\mu} = \frac{d\gamma'^{\mu}}{ds'}(s') = \dot{\phi}(s')\frac{d\gamma^{\mu}}{ds}(s(s')) = \dot{\phi}v^{\mu} \tag{14.4.21}$$

Thus the tangent vector changes by reparameterisation from the vector $\vec{v} = v^{\mu}\partial_{\mu}$ to a vector $\vec{v}\,'$ parallel to it. Accordingly, the direction of the tangent vector depends on the trajectory, while the length of the tangent vector depends on the parameterisation.

Let us consider equivalent tangent vectors which are applied at the same point x_0 and are parallel to each other. Let us denote by $\mathbf{P}(M)$ the quotient space. A point $[(x, \vec{v})] \in \mathbf{P}(M)$ contains all tangent vectors at $x \in M$ which are parallel to $\vec{v}$, thence representing a *tangent direction* to M.

Another way to discuss the issue is identifying parameterised curves $\gamma : \mathbb{R} \to M$ with their graph in $\mathbb{R} \times M$. In this way, parameterised curves on M can be regarded as trajectories in $\mathbb{R} \times M$. Then we can consider a parameterisation of $\hat{\gamma} : \mathbb{R} \to \mathbb{R} \times M$ of the graph of γ and consider geometric properties of $\hat{\gamma}$ as the properties which do not depend on the parameterisation fixed on $\hat{\gamma}$.

In this way, we can regard features and properties of the parameterised curve γ (e.g. its tangent vector) as geometric properties of its graph $\hat{\gamma}$. This is due to the fact that the trajectory identified by $\hat{\gamma}$ does contain a complete information about the parameterisation of γ. On the other hand the parameterisation of $\hat{\gamma}$ is completely uncorrelated to the parameterisation of γ. Hence the tangent vector $(x, \dot{\gamma}) \in TM$ is represented by the direction $[(x, \dot{\gamma})] \in \mathbf{P}(\mathbb{R} \times M)$ which of course, does not depend on the parameterisation chosen on $\hat{\gamma}$.

This shows that the initial motivation is somehow false; tangent vectors can be seen as geometric objects in differential geometry, provided that we regard parameterised curves in M as trajectories on $\mathbb{R} \times M$. This is also what one does in homogeneous formalism by describing motions in space as worldlines in spacetime.

5. Vector fields

A *vector field* is a map $X : M \to TM$ associating a tangent vector at any point such that $\pi \circ X = \mathrm{id}_M$. Accordingly, a vector field map is locally in the form

$$X : M \to TM : x^{\mu} \mapsto (x^{\mu}, X^{\mu}(x)) \tag{14.5.1}$$

which corresponds to associate to the point $x \in M$ the vector $X = X^{\mu}(x)\partial_{\mu} \in T_x M$.

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$. Of course, the sum of two vector fields is again a vector field and the product of a vector field by a scalar $\lambda \in \mathbb{R}$ is again a vector field. Accordingly, $\mathfrak{X}(M)$ is an (infinite dimensional) real vector space.

A vector field X can be multiplied by a function $F \in \mathcal{F}(M)$ to obtain a vector field

$$\cdot : \mathcal{F}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (F, X) \mapsto (F \cdot X) \tag{14.5.2}$$

By this operation, vector fields also form a $\mathcal{F}(M)$-module.

For any function $F : M \to \mathbb{R}$ on M, one can define a new function

$$X(F) : M \to \mathbb{R} : x \mapsto X^\mu(x) \frac{\partial F}{\partial x^\mu}(x) \tag{14.5.3}$$

The function $X(F)$ does not depend on the chart used on M. Let $x'^\mu = x'^\mu(x)$ be new coordinates. In view of (14.4.14) one has the vector field

$$X = X^\mu(x)\partial_\mu = X^\mu(x)J^\alpha_\mu \partial'_\alpha = X'^\alpha(x)\partial'_\alpha \qquad X'^\alpha(x) = X^\mu(x)J^\alpha_\mu \tag{14.5.4}$$

The function F has two local expressions in the two charts which are related by transformation laws

$$F'(x'(x)) = F(x) \tag{14.5.5}$$

so that partial derivatives behave as

$$J^\alpha_\mu \partial'_\alpha F' = \partial_\mu F \tag{14.5.6}$$

Thus the function $X(F)$ reads as

$$X(F) = X^\mu(x)\frac{\partial F}{\partial x^\mu} = X^\mu(x)J^\alpha_\mu \frac{\partial F'}{\partial x'^\alpha} = X'^\alpha(x)\frac{\partial F'}{\partial x'^\alpha} \tag{14.5.7}$$

and the local expressions do not depend on the chart.

To define a global vector field X on a manifold M, one can provide component functions $X^\mu(x)$ in each coordinate patch. In the intersection of two coordinate patches, the components must obey the transformation laws

$$X'^\mu(x') = J^\mu_\alpha(x)X^\alpha(x) \tag{14.5.8}$$

If they do, the two local expressions glue together to define a global vector field on the union of the two patches. By iteration over all charts in an atlas, one defines a global vector field.

> In general we shall see that transformation laws of the components of an object always contain all information about its global structure. That is particularly satisfactory from a physical viewpoint when charts are identified with observers. That tells us that one observer can describe locally an object by means of its components. In order to achieve a complete global description, one has to know the subjective descriptions of a number of observers and add knowledge about how to compare observations of different observers.
>
> At that point one obtains an intrinsic, objective, absolute description of the object which becomes independent of the observer. In other words, strange as it may sound, one can obtain an *absolute* description of the reality exactly starting from a collection of relative descriptions.
>
> Theory of relativity claims that the description of physical reality can and should be done in this way. Accordingly, it should be called theory of the absolute, not theory of relativity.

The operation defined by vector fields has the following properties

(a) Linear: $X(\lambda F + \mu G) = \lambda X(F) + \mu X(G)$

(b) Leibniz rule: $X(F \cdot G) = X(F) \cdot G + F \cdot X(G)$

Then a vector field is called a *derivative* of the algebra of functions on M.

We can also define an operation between vector fields called the *commutator* defined by

$$[X, Y](F) = X(Y(F)) - Y(X(F)) \tag{14.5.9}$$

By definition, the commutator of two vector fields takes a function F to return a function $[X, Y](F) = X(Y(F)) - Y(X(F))$. It is hence an operator on the algebra of functions.

One can easily show it is in fact again a vector field.

One has simply:

$$
\begin{aligned}
[X, Y](F) &= X(Y(F)) - Y(X(F)) = X^\mu \partial_\mu \left(Y^\alpha \partial_\alpha F\right) - Y^\mu \partial_\mu \left(X^\alpha \partial_\alpha F\right) = \\
&= X^\mu \partial_\mu Y^\alpha \partial_\alpha F + {\color{red}X^\mu Y^\alpha \partial_{\mu\alpha} F} - Y^\mu \partial_\mu X^\alpha \partial_\alpha F - {\color{red}Y^\mu X^\alpha \partial_{\mu\alpha} F} = \\
&= \left(X^\mu \partial_\mu Y^\alpha - Y^\mu \partial_\mu X^\alpha\right) \partial_\alpha F
\end{aligned}
\tag{14.5.10}
$$

which corresponds to the vector field

$$[X, Y] = \left(X^\mu \partial_\mu Y^\alpha - Y^\mu \partial_\mu X^\alpha\right) \partial_\alpha \tag{14.5.11}$$

Thus the commutator defines an operation $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which has the following properties

(a) antisymmetry: $[X, Y] = -[Y, X]$

(b) bilinearity: $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$ (as well as, of course, $[Z, \lambda X + \mu Y] = \lambda[Z, X] + \mu[Z, Y]$)

(c) Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

The Jacobi identity can be proven by noticing that

$$[X, [Y, Z]](F) = XYZ(F) - XZY(F) - YZX(F) + ZYX(F) \tag{14.5.12}$$

so that

$$
\begin{aligned}
[X, [Y, Z]](F) + [Y, [Z, X]](F) + [Z, [X, Y]](F) = {}&{\color{red}XYZ(F)} - {\color{green}XZY(F)} - {\color{blue}YZX(F)} + \underline{\color{red}ZYX(F)} + {\color{blue}YZX(F)} - \underline{\color{green}YXZ(F)} + \\
&- \underline{\color{blue}ZXY(F)} + {\color{green}XZY(F)} + \underline{\color{blue}ZXY(F)} - \underline{\color{red}ZYX(F)} - {\color{red}XYZ(F)} + \underline{\color{green}YXZ(F)} = 0
\end{aligned}
\tag{14.5.13}
$$

Thus $\mathfrak{X}(M)$ with the commutator is endowed by a structure of Lie algebra.

Flows and their infinitesimal generators

A *flow* on a manifold is a 1-parameter subgroup of diffeomorphisms $\Phi : \mathbb{R} \to \mathrm{Diff}(M) : s \mapsto \Phi_s$. Being a subgroup, we mean that the family $\Phi_s : M \to M$ obeys the following properties

(a) $\Phi_0 = \mathrm{id}_M$

(b) $\Phi_r \circ \Phi_s = \Phi_{r+s}$

Equivalently, we can require that the map $\Phi : \mathbb{R} \to \mathrm{Diff}(M)$ is a group homomorphism and Φ is an action of the group $(\mathbb{R}, +)$ on M.

For any point $x \in M$, one can define a curve

$$\gamma_x : \mathbb{R} \to M : s \mapsto \gamma_x(s) = \Phi_s(x) \tag{14.5.14}$$

which is called the *integral curve* based at x.

Trajectories of integral curves are a foliation of M. Any point $x \in M$ is on some integral trajectory (x is on γ_x in the first place) and integral trajectories are disjoint.

Let us suppose $y \in M$ belongs to two different integral trajectories, e.g. $y = \gamma_x(s) = \gamma_{x'}(s')$, and then we can show that γ_x and $\gamma_{x'}$ define the same integral trajectory. In fact

$$y = \gamma_x(s) = \Phi_s(x) = \Phi_{s'}(x') \quad \Rightarrow x' = \Phi_{s'}^{-1} \circ \Phi_s(x) = \Phi_{s-s'}(x) \tag{14.5.15}$$

Then any point z which belongs to the integral curve γ_x, namely $z = \gamma_x(r)$, does belong to the integral curve $\gamma_{x'}$ as well

$$z = \gamma_x(r) = \Phi_r(x) = \Phi_{r-s+s'}(x') = \gamma_{r-s+s'}(x') \tag{14.5.16}$$

Of course, one can easily show the opposite inclusion, so that the two integral trajectories are the same.

Another way of stating the same result, is that one can define an equivalence relation among points in M which says two points are equivalent if they belong to the same integral curve. That is an equivalence relation.

Let us remark that, in general, we cannot say that the quotient space $M/\sim$ is a manifold.

Let us consider on $\mathbb{R}^2$ the flow $\Phi_\theta : (x, y) \mapsto (x' = \cos\theta x + \sin\theta y, y' = -\sin\theta x + \cos\theta y)$. One can easily show that this is a flow on $M = \mathbb{R}^2$.

Integral trajectories are circles centered at the origin (plus the origin on its own which can be considered as a degenerate circle). Then points are equivalent iff they stay at equal distance from the origin. Integral trajectories are labeled for example by the radius r of the circle, with $r \geq 0$ ($r = 0$ being the orbit of the origin)

The quotient space $\mathbb{R}^2/\sim$ is the semi-line $Q = \{r \in \mathbb{R} : r \geq 0\}$, which is not a manifold.

Since any point $x \in M$ belongs to one and only one integral curve γ_x, we can compute the tangent vector $\dot\gamma_x$ at x, i.e.

$$\dot\gamma_x = \frac{d\Phi_s^\mu}{ds}(x)\Big|_{s=0} \partial_\mu \tag{14.5.17}$$

That is a vector field and it is called the *infinitesimal generator* of the flow.

The flow of a vector field

Let $X = X^\mu(x)\partial_\mu$ be a vector field on a manifold M. An *integral curve* of X based at x is a solution of the Cauchy problem

$$\begin{cases} \dot\gamma_x(s) = X^\mu(\gamma_x(s)) \\ \gamma_x^\mu(0) = x^\mu \end{cases} \tag{14.5.18}$$

that is a curve whose tangent vector is always agreeing with the vector field X. Since we are always considering very regular vector fields the components X^μ are smooth and Cauchy theorem guarantees that solutions always exist and depend smoothly on the initial condition x.

Thus there are integral curves passing through any point x and integral trajectories are disjoint in view of the uniqueness part of Cauchy theorem. Let us suppose that all integral curves are defined for all values of the parameter $s \in \mathbb{R}$; in this case we say that the vector field X is *complete*.

> There are general results such that for example if M is compact then any vector field is complete. More generally, if X is somehow uniformly bounded (as it trivially happens on a compact space) then it is complete.

Thus, for any complete X, we can define a family of diffeomorphisms

$$\Phi_s : M \to M : x \mapsto \gamma_x(s) \tag{14.5.19}$$

which are trivially a flow. The flow Φ_s so defined is called the flow of the vector field X and one can check that the infinitesimal generator of the flow Φ_s coincides with the original vector field X.

6. Regular Approximations

Many important results about manifolds can be expressed in terms of the topology we gave to maps.

Let us first notice that $r \leq k \Rightarrow \mathcal{C}^k(M, N) \subset \mathcal{C}^r(M, N)$, i.e. regular more maps are a subset of less regular ones. Then we have a family of embeddings

$$\ldots \to \mathcal{C}^k(M, N) \to \ldots \to \mathcal{C}^2(M, N) \to \mathcal{C}^1(M, N) \to \mathcal{C}^0(M, N) \tag{14.6.1}$$

which form an inverse family. We can define on the inverse limit $\mathcal{C}^\infty(M, N)$ the finest topology which makes all inclusions $i^k : \mathcal{C}^\infty(M, N) \to \mathcal{C}^k(M, N)$ continuous, both starting from the weak and strong topology on $\mathcal{C}^k(M, N)$. That defines the weak and the strong topologies on $\mathcal{C}^\infty(M, N)$, which contains exactly all $U \cap \mathcal{C}^\infty(M, N)$, for U openset on some $\mathcal{C}^k(M, N)$. As for finite k, we denote the corresponding topological spaces as $\mathcal{C}^\infty_W(M, N)$ and $\mathcal{C}^\infty_S(M, N)$.

Let M and N be two $\mathcal{C}^r$-manifolds with $1 \leq r \leq \infty$ and $k \in \mathbb{N} : 1 \leq k < r$. Then we have the following results:

(i) the subset of $\mathcal{C}^r$-maps $\mathcal{C}^r(M, N) \subset \mathcal{C}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(ii) the subset of $\mathcal{C}^r$-diffeomorphisms $\mathrm{Diff}^r(M, N) \subset \mathrm{Diff}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(iii) the subset of $\mathcal{C}^r$-embeddings $\mathrm{Emb}^r(M, N) \subset \mathrm{Emb}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(iv) the subset of closed $\mathcal{C}^r$-embeddings $\mathrm{cEmb}^r(M, N) \subset \mathrm{cEmb}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(v) the subset of $\mathcal{C}^r$-immersions $\mathrm{Imm}^r(M, N) \subset \mathrm{Imm}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(v) the subset of $\mathcal{C}^r$-submesions $\mathrm{Subm}^r(M, N) \subset \mathrm{Subm}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology;

(vi) the subset of proper $\mathcal{C}^r$-maps $\mathrm{Pro}^r(M, N) \subset \mathrm{Pro}^k(M, N)$ is dense in the strong $\mathcal{C}^k$-topology.

We shall not prove these results (see Hirsch). All items follow from item *(i)* together with the lemmas that the set of $\mathrm{Imm}^r(M, N)$ (as well as $\mathrm{Subm}^r(M, N)$, $\mathrm{Emb}^r(M, N)$, $\mathrm{cEmb}^r(M, N)$, $\mathrm{Pro}^r(M, N)$) is open in $\mathcal{C}^r_S(M, N)$ for all $1 \leq r \leq \infty$.

As a corollary, there exists a diffeomorphism in $\mathrm{Diff}^r(M, N)$ iff there exists a diffeomorphism in $\mathrm{Diff}^k(M, N)$ for any $1 \leq k < r \leq \infty$.

Being dense subsets means when we have a $\mathcal{C}^k$-diffeomorphism (embedding, ...), we can approximate it, as closed as we wish, by a more regular $\mathcal{C}^r$-diffeomorphism (embedding, ...), for any $1 \leq k < r \leq \infty$. In simpler words, we do not need to worry too much about regularity.

Also if we have a $\mathcal{C}^r$-differential structure on M, then we can define a $\mathcal{C}^k$-differential structure on it, for any $r \leq k$. In other words, usually we can assume we are on a smooth manifold.

7. Sub-manifolds

As in any category, a manifold S is said to be a *sub-manifold* of a manifold M iff there exists a monomorphism $i : S \to M$ which is called the *canonical*
embedding. The canonical embedding must be an isomorphism on its image $i(S) \subset M$.

In particular, the image $i(S)$ (with the topology induced by M) must be homeomorphic to S. For example, the ones shown below are not sub-manifolds.

Fig. 14.3: Two examples of injectve images of $\mathbb{R}$ which are not sub-manifolds.

The reader should show an example of open sets in $\mathbb{R}$ and an openset in $\mathbb{R}^2$ which do not correspond each other as they should if the inclusion were a homeomorphism on its image.

If we choose coordinates k^a on S and coordinates x^μ on M then the canonical embedding reads as

$$i : S \to M : k^a \mapsto x^\mu(k) \tag{14.7.1}$$

and we have the Jacobians $J_a^\mu := \partial_a x^\mu$ for free. Since the canonical embedding must be maximal rank then this Jacobian must be maximal rank, namely $\mathrm{rank}(J_a^\mu) = \dim(S)$.

If we consider a function $F : M \to \mathbb{R}$ the *level set* of F is defined to be $F^{-1}(a) = \{x \in M : F(x) = a\} \subset M$. Of course, this can be empty if the value $a \in \mathbb{R}$ is never attained by F or it can be the whole M if the function F is constant and $\forall x \in M : F(x) = a$.

Except for these situations, the preimage $F^{-1}(a)$ is computed by fetching a local expression of F in a chart and solving the equation $F(x^\mu) = a$. If $dF := \partial_\mu F dx^\mu \neq 0$, for example $\partial_1 F \neq 0$, then we can solve $x^1 = \phi(x^2, \ldots, x^m)$ by the implicit function theorem. Thus $F^{-1}(a)$ is generically a hypersurface in M which is locally parameterised by $(x^2, \ldots, x^m)$ and whose coordinate expression is $x^1 = \phi(x^2, \ldots, x^m)$.

Then $F^{-1}(a)$ is a sub-manifold of M and the canonical embedding is given by

$$i : F^{-1}(a) \to M : (x^2, \ldots, x^m) \mapsto (\phi(x^2, \ldots, x^m), x^2, \ldots, x^m) \tag{14.7.2}$$

If one has two functions $F, G : M \to \mathbb{R}$, one can define a vector value $\vec{F} : M \to \mathbb{R}^2 : x \mapsto (F(x), G(x))$.

The level set is defined as $\vec{F}^{-1}(a, b) = \{x \in M : \vec{F}(x) = (a, b)\}$ and generically one can solve, for example,

$$\begin{cases} x^1 = \phi^1(x^3, \ldots, x^m) \\ x^2 = \phi^2(x^3, \ldots, x^m) \end{cases} \tag{14.7.3}$$

and the level set $\vec{F}^{-1}(a, b)$ is a sub-manifold of dimension $m - 2$ (codimension 2) with the canonical immersion

$$i : \vec{F}^{-1}(a, b) \to M : (x^3, \ldots, x^m) \mapsto (\phi^1(x^3, \ldots, x^m), \phi^2(x^3, \ldots, x^m), x^3, \ldots, x^m) \tag{14.7.4}$$

This can be done wherever

$$\det \begin{pmatrix} \frac{\partial F}{\partial x^1} & \frac{\partial F}{\partial x^2} \\[2mm] \frac{\partial G}{\partial x^1} & \frac{\partial G}{\partial x^2} \end{pmatrix} \neq 0 \tag{14.7.5}$$

Where this fails, one can decide to solve for other coordinates. This can be done provided that the matrix

$$\begin{pmatrix} \frac{\partial F}{\partial x^1} & \frac{\partial F}{\partial x^2} & \cdots & \frac{\partial F}{\partial x^m} \\[2mm] \frac{\partial G}{\partial x^1} & \frac{\partial G}{\partial x^2} & \cdots & \frac{\partial G}{\partial x^m} \end{pmatrix} \tag{14.7.6}$$

is maximal rank, i.e. it is of rank 2.

More generally, k functions F_i (with $k \leq m$) define a sub-manifold of codimension k provided that

$$\left(\frac{\partial F_i}{\partial x^\mu} \right) = \begin{pmatrix} \frac{\partial F_1}{\partial x^1} & \frac{\partial F_1}{\partial x^2} & \cdots & \frac{\partial F_1}{\partial x^m} \\[2mm] \frac{\partial F_2}{\partial x^1} & \frac{\partial F_2}{\partial x^2} & \cdots & \frac{\partial F_2}{\partial x^m} \\[2mm] \cdots & & \cdots & \\[2mm] \frac{\partial F_k}{\partial x^1} & \frac{\partial F_k}{\partial x^2} & \cdots & \frac{\partial F_k}{\partial x^m} \end{pmatrix} \tag{14.7.7}$$

is maximal rank, i.e. it is of rank k.

If the canonical embedding can be written as $i : S \to M : k^a \mapsto (0, \ldots, 0, k^1, \ldots, k^n)$ then the coordinates on M are called *adapted* to S.

If the canonical embedding can be written as $i : S \to M : k^a \mapsto (\phi^1(k), \ldots, \phi^{m-n}(k), k)$ then the coordinates on M are called *concordant* to S.

Given a function $F : M \to \mathbb{R}$, one can consider a whole family of sub-manifolds $S_a := F^{-1}(a)$. Since each point of M is on a level set and two level sets cannot intersect, the family S_a defines an equivalence relation on M (two points are equialent iff they belong to the same level set).

If S_a are all of the same dimension k, they are said to form a *foliation* of rank k. Since vectors tangent to S_a are also particular tangent vectors to M, a foliation defines subspaces

$$\Delta(x) := TS_a \subset TM \tag{14.7.8}$$

where we set $F(x) = a$. A family of subspaces $\Delta : M \to TM : x \mapsto \Delta(x)$ with $\Delta(x)$ being a subspace of T_xM is called a *distribution* (of subspaces). If all subspaces $\Delta(x)$ has the same dimension, then the distribution is said to be of *constant rank.*

What we showed above is that any foliation of constant rank defines a distribution of constant rank. These are special distributions in the form $\Delta(x) := TS_a$ which are called *integrable* distributions. In the next Subsection, we shall deal with the problem of determining if a distribution is integrable (since there are distributions which are not, i.e. distributions which are not associated to a foliation).

Frobenius theorem

Let us consider a distribution Δ of constant rank k. As we did for integral curves, we can search for a sub-manifold S of dimension k such that for any $z \in S$ one has $T_zS = \Delta(z)$.

As a matter of fact a (never vanishing) vector field X defines a distribution of rank 1

$$\Delta(x) = \{(x, v) : v = \lambda X(x)\} \subset TM \tag{14.7.9}$$

We shall show that rank 1 distributions are always integrable and that is why integral curves always exist.

Unlike what happens for integral curves, higher rank distributions are not always integrable. To see it, let us start by defining a *generator* of the distribution Δ to be a vector field $X \in \mathfrak{X}(M)$ such that $X(x) \in \Delta(x)$ at all points.

If the distribution is integrable, then X is a vector field which happens to be tangent to the foliation sub-manifolds everywhere. We say that the distribution Δ is involutive iff for any two generators X, Y of the distribution, then the commutator $[X, Y]$ is again a generator.

If the distribution is integrable, the generators are tangent to the foliation. The commutator of two vector fields tangent to a manifold S is again tangent to it (as a consequence of naturality of the commutator). Accordingly, an integrable distribution is also involutive. In other words, being involutive is a necessary condition for a distribution to be integrable.

Frobenius theorem states that, conversely, all involutive distributions are also integrable.

Thus being involutive is not only necessary but also sufficient condition to be integrable. Under many viewpoints, Frobenius theorem plays the same role of (and it generalises) Cauchy theorem for existence and uniqueness of integral curves.

Of course, if a distribution is of rank 1 and X is a generator, then any other generator is proportional to X. Since the commutator of two parallel vectors is still parallel to them, rank 1 distributions are always involutive and hence always integrable.

One has:

$$[X, f(x) \cdot X] = X(f) \cdot X \tag{14.7.10}$$

which is parallel to X and hence still a generator.

Let n be a vector field on a Riemannian manifold (M, g). Besides defining the distribution of all vectors parallel to n, one can define also the distribution of rank $m - 1$ of hyperplanes orthogonal to n, namely

$$\Delta = \{v : g(v, n) = 0\} \tag{14.7.11}$$

Let us check whether it is involutive. Let us consider two generators $X = X^\mu \partial_\mu$ (namely $X^\mu n_\mu = 0$) and $Y = Y^\nu \partial_\nu$. The commutator is

$$g([X, Y], n) = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) n_\nu = -X^\mu Y^\nu (\nabla_\mu n_\nu - \nabla_\nu n_\mu) \tag{14.7.12}$$

for any torsionless connection. Then the distribution is involutive (and hence integrable) if $\nabla_{[\mu} n_{\nu]} = 0$.

Any vector field n which does not obey this condition provides an example of a distribution of rank $m - 1$ which is not integrable.

Foliations can be quite weird. On a torus $\mathbb{T}_2$ one can consider coordinates (α, β) and the distribution defined by $X = a \frac{\partial}{\partial \alpha} + b \frac{\partial}{\partial \beta}$ with $a, b \in \mathbb{R}$. That is a rank 1 distribution, hence it is involutive, hence it is integrable. The integral surface of this distribution are integral curves of X. If $a/b \in \mathbb{Q}$ (i.e. if a and b are *rationally dependent*) the corresponding integral curves are closed, they are circles S_1 and sub-manifolds of $\mathbb{T}_2$. On the other hand, if a and b are *rationally independent*, the curve cannot close and they are copies of $\mathbb{R}$ densely immersed in $\mathbb{T}_2$. Being densely immersed, they are embedded, the topology of the manifold $\mathbb{R}$ does not coincide with the topology induced by the immersion.

More regular distributions correspond to *regular foliations* in which leaves of the foliation are embedded as sub-manifolds. If we declare equivalent points in M is the belong to the same leaf sub-manifold, one can consider a quotient space $S = M/\sim$ in which each point is a leaf of the foliation. If this set is a manifold itself, we have a projection $\pi : M \to S$ for which $[M \to S]$ is a *fiber bundle*. A bundle's total space is always foliated into its fibers and fibres are sub-manfiolds.

In view of this, we shall often deal with regular foliations as fiber bundles.

References

add Lee, Hirsch, ...

Chapter 15. Tensor fields and differential forms

1. Covectors and 1-forms

A vector space V of dimension n is isomorphic to $\mathbb{R}^n$ and it gets standard topology from it.

We can consider the set V^* of linear maps $\alpha : V \to \mathbb{R}$. When V is finite dimensional linearity implies continuity (α is a homogeneous first degree polynomial in the components of the vector $v \in V$, hence continuous).

One can define a linear stucture on V^* by setting

$$(\alpha + \beta) : V \to \mathbb{R} : v \mapsto \alpha(v) + \beta(v) \qquad\qquad (\lambda\alpha) : V \to \mathbb{R} : v \mapsto \lambda\alpha(v) \tag{15.1.1}$$

> Notice the overloading of symbols, e.g. in $(\alpha + \beta) : V \to \mathbb{R} : v \mapsto \alpha(v) + \beta(v)$. The first $+$ symbol in $(\alpha + \beta)$ is the sum to be defined on V^*. The last $+$ symbol in $\alpha(v) + \beta(v)$ is the sum in $\mathbb{R}$.
> Similarly for the understood product in $(\lambda\alpha)$ and in $\lambda\alpha(v)$.

The maps $(\alpha + \beta)$ and $\lambda\alpha$ so defined are linear and hence they are elements of V^*.

> The maps α, β are covectors and hence they are linear. Then one has

$$\begin{aligned}(\alpha + \beta)(\lambda v + \mu w) =&\alpha(\lambda v + \mu w) + \beta(\lambda v + \mu w) = \lambda\alpha(v) + \mu\alpha(w) + \lambda\beta(v) + \mu\beta(w) = \lambda(\alpha(v) + \beta(v)) + \mu(\alpha(w) + \beta(w)) = \\ =&\lambda(\alpha + \beta)(v) + \mu(\alpha + \beta)(w)\end{aligned} \tag{15.1.2}$$

> and

$$(\lambda\alpha)(\mu v + \nu w) = \lambda\alpha(\mu v + \nu w) = \lambda\mu\alpha(v) + \lambda\nu\alpha(w) = \mu(\lambda\alpha)(v) + \nu(\lambda\alpha)(w) \tag{15.1.3}$$

The vector space V^* so defined is called the *dual* of V. Elements of V^* are called *covectors*. Let us stress that being covectors defined as maps, a covector is known once its value on any vector v is known. This has already be used to define $(\alpha+\beta)$ as the only map such that $(\alpha+\beta)(v) = \alpha(v)+\beta(v)$.

> In the infinite dimensional case linearity does not imply continuity and one has two notions of dual. A broader definition of *algebraic dual* made of linear maps and a stricter definition of *topological dual* made of linear continuous maps. In the finite dimensional cases, the two notions coincide.

Einstein convention

We are using everywhere, except where explicitly said, *Einstein index convention*. This is done by a set of notations which has become universal in GR and proven to be useful in differential geometry.

Einstein convention in a nutshell is the following rules:

(a) indices have a *family* which is represented by the alphabet used for them. In each family, indices range among fixed limits. For example, we usually define a family denoted by lowercase Greek indices (like in x^μ). These are called *world indices* and they refer to quantities defined on spacetime M. If spacetime is of dimension $\dim(M) = m = 4$ they usually range between 0 and 3.

(b) indices have a *position* which can be *up* or *down* like in x^μ or α_μ. Powers are denoted by $(x^\mu)^2$ in order not to conflict with the notation for up indices.

(c) Objects denoted by the same letter and indices different by order, family or position are different objects. For example, α^μ is different from α_μ, α_μ is different from α_i, and $F_{\mu\nu}$ is, in general, different from $F_{\nu\mu}$.

(d) We shall often consider linear combinations of terms made by products of objects with indices. In each term, one index can appear zero, one or two times. If it appears zero times it means it does not appear, if it appears once, it is called a *free index*, if it appears twice it is called a *dumd index*. When indices appear twice in a term, they appear once up and once down. All terms in a linear combination must have the same free indices (possibly in different order). They may have different dumb indices.

(e) On any pair of dumb indices in a term, a sum over the range of the family of the indices is understood. For example $v^{\mu i}\alpha_\mu$ has one free index i and a pair of dumb indices μ and, assuming from the context that μ is a world index ranging from 0 to 3, the sum understood is

$$v^{\mu i}\alpha_\mu = \sum_{\mu=0}^{3} v^{\mu i}\alpha_\mu \tag{15.1.4}$$

(f) being dumb indices summed over, one can replace any pair of dumb indices by any other index in the same family, provided that we do not break rule (d). In a linear combination, one can replace a free index with another index in the same family, provided that the operation is done in each term and one does not break rule (d).

(g) Of course, equalities between linear combinations are to be understood as a single linear combination equalized to 0. Terms with no free indices are (real) numbers, sometimes called *scalars*, and then, if they are non-zero, one can divide by them.

We probably should mention that in fact there are *two* Einstein convention, one called the *abstract Einstein convention* and one called the *concrete Einstein convention*. Here we use the abstract Einstein convention. In the concrete Einstein convention, an index only refers to the collection for its values; for example x^μ only stands for the collection (x^0, x^1, x^2, x^3), while in the abstract notation an index also remembers the family (which in turn refer to the space on which the object is defined). For example, one can have two manifolds M and N with $\dim(M) = m = 4$ and $\dim(N) = n = 2$ and then denoting coordinates x^μ on M and x^i on N where, besides usual world coordinates ranging from 0 to 3, we defined a new (mid-Latin lowercase) family ranging from 1 to 2.

One can determine which coordinate has been using by the index family (and alphabet). On the contrary, the concrete Einstein convention would be ambiguous since x^1 could refer to the second coordinate on M as well as the first coordinate on N and one would need to use another name for

coordinates on N, for example y^i. Although we would avoid unnecessary overloading (and we would often use a different name for coordinates in the different spaces) we like to keep possibility to give the same name to similar objects on different spaces. For example, one usually denotes the natural basis for tangent vectors by ∂_α and can distinguish the space by the index family instead changing the symbol for it.

If we consider, for example, an expression like

$$A^i_j x^j = B^i \tag{15.1.5}$$

we are using a index family, say ranging from 1 to n. The expression (15.1.5) is syntactically correct since each term has only one free index i, while the first term (the one on the left hand side) also has a dumb index j appearing two times once up and once down. The expression has a sum understood

$$A^i_j x^j = B^i \qquad \Longleftrightarrow \qquad \sum_{j=1}^n A^i_j x^j = B^i \qquad \Longleftrightarrow \qquad \begin{cases} A^1_1 x^1 + A^1_2 x^2 + \ldots + A^1_n x^n = B^1 \\ A^2_1 x^1 + A^2_2 x^2 + \ldots + A^2_n x^n = B^2 \\ \ldots \\ A^n_1 x^1 + A^n_2 x^2 + \ldots + A^n_n x^n = B^n \end{cases} \tag{15.1.6}$$

Since the expression has a free index it can be expanded as a set of n equality, i.e. a (linear) system. Thus, at the very least, Einstein convention provides a very compact way to write complex structures like systems of equations. The framework can be seen to extend matrix formalism (which is as compact as Einstein convention) since it can deal with objects with more that one or two indices. As we shall see below, it will also provide a framework to keep trace of transformation rules of the objects which is particularly relevant in GR, since GR consists exactly in keeping track of the dependence of objects and formulae from the observer, i.e. dependence on the chart. See here for extending the Einstein convention with symmetrisation and antisymmetrisation of indices.

Changing the basis

Let us consider a basis ∂_μ fixed on V. Then any vector of $v \in V$ can be written as a linear combination of elements of the basis, the coefficients of the expansion being denoted by v^μ and called the *components* of the vector with respect to the basis. Precisely, for any vector $v \in V$ there exists a set of components v^μ such that

$$v = v^\mu \partial_\mu \tag{15.1.7}$$

Let us stress here an obvious fact. We learnt to compute components by using an inner product $< \cdot, \cdot >$ on V. For example, for an orthonormal basis ∂_i, one has

$$v^1 = < v, \partial_1 > \tag{15.1.8}$$

However, the components are more fundamental and they come before the definition of the scalar product. For example, in $\mathbb{R}^2$ one can consider a vector $v = (2, 3)$ and a basis $(\partial_1 = (1, 1), \partial_2 = (1, -1))$. The components has to satisfy the system

$$v^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + v^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{15.1.9}$$

This can be solved to find the unique solution $v^1 = \frac{5}{2}$ and $v^2 = -\frac{1}{2}$ without resorting to any notion of inner product.

:Index: :AIndex: :Symbols: :Notation:

If we consider a new basis ∂'_μ in V, each of the new vectors can be expanded with respect to the old basis. Namely, we know there exists a transformation matrix $\bar{J}^\mu_\alpha$ such that

$$\partial'_\mu = \bar{J}^\alpha_\mu \partial_\alpha \qquad (15.1.10)$$

Here the bar denotes the inverse matrix and notation has been chosen for later convenience.

The same vector v can be expanded along the new basis, namely $v = v'^\mu \partial'_\mu$ and one has

$$v = v'^\mu \partial'_\mu = v'^\mu \bar{J}^\alpha_\mu \partial_\alpha = v^\alpha \partial_\alpha \qquad (15.1.11)$$

Being the expansion along a basis unique, one has

$$v^\alpha = v'^\mu \bar{J}^\alpha_\mu \qquad \Rightarrow v'^\mu = J^\mu_\alpha v^\alpha \qquad (15.1.12)$$

Here we used the fact that $\bar{J}^\mu_\alpha$ is the inverse matrix to J^α_μ, i.e.

$$\bar{J}^\mu_\alpha J^\alpha_\nu = \delta^\mu_\nu \qquad J^\alpha_\nu \bar{J}^\nu_\beta = \delta^\alpha_\beta \qquad (15.1.13)$$

and δ^α_β is called the Kronecker delta and it is defined to be 1 when $\alpha = \beta$ and 0 when $\alpha \neq \beta$.
We also used the trivial fact that

$$\delta^\alpha_\beta v^\beta = \sum_{\beta=1}^m \delta^\alpha_\beta v^\beta = v^\alpha \qquad (15.1.14)$$

The effect of the Kronecker delta when it is contracted (i.e. summed over) with an index of another object is simply to replace the contracted index with the other free index of the delta.

Here a pattern is emerging. When the basis is changed and new objects are compared with the old ones, up indices get multiplied by J^μ_α (like in (15.1.12)) while down indices get multiplied by the inverse matrix $\bar{J}^\alpha_\mu$ (like in (15.1.10)). This pattern will be confirmed over and over hereafter. Of course, this is only due to the fact that we are setting indices up and down exactly in view of this property. We already knew that components transform with the matrix J and that it why we denoted them with an up index.

The convention about the position of indices may sound awkward in the beginning. It contains a number of conventional issues (which is called transition matrix and which is called the inverse; which is called the old basis and which is called the new one; we are considering new objects expansion in the old basis or old objects expansions in the new basis).
Of course, all this notation is conventional and one could produce an equally well-defined framework in which down indices transform with a matrix P and up indices transform with the inverse $\bar{P}$. This would be a matter of convention. However, what is not a matter of convention is that indices with different position transform with matrices which are one the inverse of the other. That is the only non-conventional fact.

Now we are ready to consider the issue of bases in V^*. Being V^* a vector space we can look for a basis for it. We shall hereafter show that once a basis ∂_α of V has been fixed, then a basis ∂^α of V^* is defined and it is called the *dual basis*.

Let us start by defining n objects in V^*, as the covectors which act on $v = v^\alpha \partial_\alpha$ to give

$$\partial^\alpha(v) = v^\alpha \qquad \Longleftrightarrow \quad \partial^\alpha(\partial_\beta) = \delta^\alpha_\beta \qquad (15.1.15)$$

The maps ∂^{α} so defined are in fact maps $\partial^{\alpha} : V \to \mathbb{R}$ and they are linear; in fact

$$\partial^{\alpha}(\lambda v + \mu w) = \lambda v^{\alpha} + \mu w^{\alpha} = \lambda \partial^{\alpha}(v) + \mu \partial^{\alpha}(w) \tag{15.1.16}$$

Accordingly, the maps ∂^{α} are elements of V^{*}.

  Second step is to show that ∂^{α} are generators of V^{*}.

    Let us consider $\alpha \in V^{*}$ a general covector. Since we know α, we know in particular the numbers $\alpha_{\mu} := \alpha(\partial_{\mu})$.

    We shall show that $\alpha = \alpha_{\alpha}\partial^{\alpha}$, where the equality has to be understood as equality of covectors, i.e. equality of maps. In other words, equality can be proven by proving that the two maps act equally on any $v \in V$.

    Let us start computing

$$\alpha(v) = \alpha(v^{\mu}\partial_{\mu}) = v^{\mu}\alpha(\partial_{\mu}) = v^{\mu}\alpha_{\mu} \tag{15.1.17}$$

    on the left hand side. Analogously, on the right hand side one has

$$\alpha_{\alpha}\partial^{\alpha}(v) = \alpha_{\alpha}v^{\alpha} = v^{\mu}\alpha_{\mu} = \alpha(v) \tag{15.1.18}$$

    which proves that $\alpha = \alpha_{\alpha}\partial^{\alpha}$, i.e. that any covector α can be expanded as a linear combination of the covectors ∂^{α}, i.e. that the covectors ∂^{α} are generators of V^{*}.

  Third step is to show that ∂^{α} are independent.

    Let us consider a linear combination $\lambda_{\alpha}\partial^{\alpha} = 0$. Here the 0 on the right hand side has to be understood as the zero covector, i.e. the (linear) map which maps any vector v to $0 \in \mathbb{R}$.

    Thus being the linear combination zero, means that for any vector $v \in V$ one has

$$\lambda_{\alpha}\partial^{\alpha}(v) = 0 \qquad \Longleftrightarrow \quad \lambda_{\alpha}v^{\alpha} = 0 \tag{15.1.19}$$

    Since this have to be true for all v, in particular we can set $v = \partial_{1}$ to obtain $\lambda_{1} = 0$. Then we can set $v = \partial_{2}$ to obtain $\lambda_{2} = 0$. and so on until we prove that necessarily one has $\lambda_{\alpha} = 0$.

    Since the only linear combination which is zero is the one with all coefficients zero, the covectors ∂^{α} are independent.

  Then ∂^{α} are independent generators and they form a basis in V^{*}. Of course, they depend on the choice of the basis ∂_{μ} in V. The basis ∂^{α} is hence called the *dual basis* of the basis in V. Unless we explicitly say otherwise, when a basis is chosen in V the dual basis is always used in V^{*}.

    Let us also remark the trivial fact that we did not used an inner product to define the dual basis.

  If we change the basis in V defining a new basis $\partial'_{\mu} = \bar{J}^{\alpha}_{\mu}\partial_{\alpha}$, then the new basis ∂'_{μ} defines a new dual basis ∂'^{α}. We can easily compute that the new dual basis can be expanded in terms of the old dual basis as

$$\partial'^{\mu} = J^{\mu}_{\alpha}\partial^{\alpha} \tag{15.1.20}$$

    Let us apply the equation (15.1.20) to an arbitrary vector $v = v^{\mu}\partial_{\mu} = v'^{\mu}\partial'_{\mu} \in V$. On the left hand side one gets

$$\partial'^{\mu}(v) = v'^{\mu} \tag{15.1.21}$$

while on the right hand side one gets

$$J^\mu_\alpha \partial^\alpha(v) = J^\mu_\alpha v^\alpha = v'^\mu = \partial'^\mu(v) \tag{15.1.22}$$

This shows the thesis.

Then, for a covector $\alpha = \alpha_\mu \partial^\mu = \alpha'_\mu \partial'^\mu$, one has $\alpha_\nu \partial^\nu = \alpha'_\mu J^\mu_\nu \partial^\nu$ which implies, being the expansion along a basis unique, that the components of the covector transform as

$$\alpha'_\mu J^\mu_\nu = \alpha_\nu \qquad \Longleftrightarrow \qquad \alpha'_\mu = \alpha_\nu \bar{J}^\nu_\mu \tag{15.1.23}$$

Here again the pattern; the dual basis has an up index and transforms with J, while the components of the covector have a down index and transform with $\bar{J}$.

Let us consider the scalar $\alpha(v)$ in the old and new basis.

$$\alpha'_\mu v'^\mu = \alpha_\nu \bar{J}^\nu_\mu J^\mu_\sigma v^\sigma = \alpha_\nu \delta^\nu_\sigma v^\sigma = \alpha_\nu v^\nu \tag{15.1.24}$$

Accordingly, the scalar $\alpha(v)$ is expressed as $\alpha_\nu v^\nu$ in all bases.
On the contrary, if we allowed an expression like $\alpha'_\mu \beta'_\mu$

$$\alpha'_\mu \beta'_\mu = \alpha_\nu \bar{J}^\nu_\mu \beta_\sigma \bar{J}^\sigma_\mu \tag{15.1.25}$$

which is definitely different from $\alpha_\mu \beta_\mu$.
That means that if we defined a quantity to be $\alpha'_\mu \beta'_\mu$ we should specify in which basis it reads in that forms since in all other bases the expression would not be simply $\alpha_\mu \beta_\mu$. Since choosing a basis is something we do, we expect intrinsic properties to be independent of that choice. Accordingly, we are betting that we are not interested in quantities like $\alpha_\mu \beta_\mu$ and that such quantities will not enter physical laws. That is why rule (d) of Einstein convention forbids such expressions.

Covectors on a complex vector space

When we consider a complex vector space V, we immediately face a problem: we have two duals, one of linear maps $\alpha : V \to \mathbb{C}$ and one of anti-linear maps $\hat{\alpha} : V \to \mathbb{C}$. These are denoted by $\mathbb{V}^*$ and $V^\dagger$, respectively.

Moreover, we can iterate the process and define $(V^*)^*$, $(V^*)^\dagger$, $(V^\dagger)^*$, and $(V^\dagger)^\dagger$, and so on. For future convenience let us set $\bar{V} := (V^\dagger)^*$.

If ∂_α is a basis of V, then we can define

$$\partial^\alpha : V \to \mathbb{C} : v \mapsto v^\alpha \qquad \hat{\partial}^\alpha : V \to \mathbb{C} : v \mapsto (v^\alpha)^\dagger \qquad \bar{\partial}_\alpha : V^\dagger \to \mathbb{C} : \hat{\alpha} \mapsto \hat{\alpha}_\alpha \tag{15.1.26}$$

where $v = v^\alpha \partial_\alpha$ is any vector in V and $\hat{\alpha} = \hat{\alpha}_\alpha \hat{\partial}^\alpha$ is an element in $V^\dagger$. While ∂^α and $\bar{\partial}_\alpha$ are linear maps, $\hat{\partial}^\alpha$ is anti-linear. Hence $\partial^\alpha \in V^*$, $\hat{\partial}^\alpha \in V^\dagger$, and $\bar{\partial}_\alpha \in (V^\dagger)^*$.

The elements $\hat{\partial}^\alpha \in V^\dagger$ form a basis. For any $\hat{\alpha} \in V^\dagger$, we have $\hat{\alpha}(\partial_\mu) =: \hat{\alpha}_\mu$ and we can show that $\hat{\alpha} = \hat{\alpha}_\mu \hat{\partial}^\mu$. In fact, one has

$$\hat{\alpha}(v) = \hat{\alpha}(v^\alpha \partial_\alpha) = (v^\alpha)^\dagger \hat{\alpha}(\partial_\alpha) = (v^\alpha)^\dagger \hat{\alpha}_\alpha = \hat{\alpha}_\mu (v^\mu)^\dagger = \hat{\alpha}_\mu \hat{\partial}^\mu(v) \tag{15.1.27}$$

Hence $\hat{\partial}^\alpha$ are generators of $V^\dagger$.

They are also independent. In fact, if one has $\lambda_\alpha \hat{\partial}^\alpha = 0$, then

$$0 = \lambda_\alpha \hat{\partial}^\alpha(\partial_\mu) = \lambda_\alpha \delta^\alpha_\mu = \lambda_\mu \tag{15.1.28}$$

Then, in the end, $\hat{\partial}^\alpha$ are a basis of $V^\dagger$.

Then ∂^α is the *dual basis* of V^* for an argument which is identical to the argument discussed above for real vector spaces, $\hat{\partial}^\alpha$ is the *dual conjugate basis* of $V^\dagger$ as we just showed, $\bar{\partial}_\alpha$ is the *conjugate basis* of $\bar{V}$ since it is the dual basis to the dual conjugate basis $\hat{\partial}^\alpha$. Let us remark that both $V^\dagger$ and $\bar{V}$ can be defined just because we have complex conjugation on $\mathbb{C}$.

If we consider a new basis $\partial'_\alpha = \bar{J}^\mu_\alpha \partial_\mu$ in V, of course we define the new dual, conjugate dual, and conjugate bases. Of course, for the dual basis we have $\partial'^\alpha = J^\alpha_\mu \partial^\mu$, for the components we have $\alpha'_\mu = \bar{J}^\nu_\mu \alpha_\nu$, as in the real case, and so that $\alpha'_\alpha \partial'^\alpha = \alpha_\alpha \partial^\alpha$.

For the dual conjugate basis, we have $\hat{\partial}'^\alpha = (\bar{J}^\alpha_\mu)^\dagger \hat{\partial}^\mu$, in fact, we have

$$\hat{\partial}'^\alpha(v) = (v'^\mu)^\dagger \delta^\alpha_\mu = (v^\nu J^\mu_\nu)^\dagger \delta^\alpha_\mu = (J^\alpha_\mu)^\dagger (v^\nu)^\dagger \delta^\mu_\nu = (J^\alpha_\mu)^\dagger \hat{\partial}^\mu(v) \tag{15.1.29}$$

for all $v \in V$.

> Let us stress that, by now, $(J^\alpha_\mu)^\dagger$ means: *take the element (α, μ) of the Jacobian and then take the complex conjugation of it as a complex number.*
>
> If we wrote $(J^\dagger)^\alpha_\mu$ that would have meant: *take the Jacobian, conjugate it as a matrix, (i.e. transpose and complex conjugate) and then take the element (α, μ).*
>
> The two operations are different and because of that they are denoted differently.

Let us stress that this is an exception to the position of indices as we formulated above. Since $\hat{\partial}^\alpha$ has an up index, it is (wrongly) expected to transform as $\hat{\partial}'^\alpha = J^\alpha_\mu \hat{\partial}^\mu$, not as it correctly transforms, i.e. $\hat{\partial}'^\alpha = (J^\alpha_\mu)^\dagger \hat{\partial}^\mu$.

We can also notice other cracks in the Einstein notation: for example, one $\partial'^\alpha(\partial_\mu) = \delta^\alpha_\mu$ and there is nothing highlighting that the map ∂'^α is, in fact, anti-linear. That means, the matrix representing the objects forgets of whether it comes from a linear or an anti-linear map, which is annoying.

Because of these reasons, we introduce a modified family of indices, denoted by $\hat{\alpha}, \hat{\beta}, \ldots$ and called *hat-Greek indices*, for objects in $V^\dagger$. Or equivalently, one can regard to the hat as a further abstract property of the index, besides its family and position.

According to this extension of Einstein notation, the dual conjugate basis is denoted by $\hat{\partial}^{\hat{\alpha}}$ and it is defined by $\hat{\partial}^{\hat{\alpha}}(v) = (v^{\hat{\alpha}})^\dagger$. The hat-index is informing us that the corresponding map is anti-linear in that argument, as well as that it transforms as $\hat{\partial}'^{\hat{\alpha}} = (J^{\hat{\alpha}}_{\hat{\mu}})^\dagger \hat{\partial}^{\hat{\mu}}$.

The action of $\hat{\partial}^{\hat{\alpha}}$ on the elements of the basis is then $\hat{\partial}^{\hat{\alpha}}(\partial_\mu) = \delta^{\hat{\alpha}}_\mu$, so that we can rewrite the action more verbosely as

$$\hat{\partial}^{\hat{\alpha}}(v) = (v^\mu)^\dagger \hat{\partial}^{\hat{\alpha}}(\partial_\mu) = (v^\mu)^\dagger \delta^{\hat{\alpha}}_\mu =: (v^\dagger)^{\hat{\alpha}} \tag{15.1.30}$$

> What we mean here is that we *define* the conjugate component $(v^\dagger)^{\hat{\alpha}}$ of a vector to be $(v^\mu)^\dagger \delta^{\hat{\alpha}}_\mu$.
>
> By doing that and by not understanding $\delta^{\hat{\alpha}}_\mu$, the hats are preserved at each step and they correctly account for the complex conjugations.

The components of an element $\hat{\alpha} \in V^\dagger$ are accordingly

$$\hat{\alpha}(\partial_\mu) = \hat{\alpha}_{\hat{\alpha}} \delta^{\hat{\alpha}}_\mu =: \hat{\alpha}_\mu \tag{15.1.31}$$

and we have $\hat{\alpha} = \hat{\alpha}_{\hat{\alpha}} \hat{\partial}^{\hat{\alpha}} = \hat{\alpha}_\mu \delta^\mu_{\hat{\alpha}} \hat{\partial}^{\hat{\alpha}}$. We shall be back to the extended Einstein notation in a while when discussing canonical isomorphisms and anti-isomorphisms.

For components in $V^\dagger$, we have transformation rules $(\hat\alpha')_{\hat\alpha} = (\bar{J}^\dagger)^{\hat\beta}_{\hat\alpha} \hat\alpha_{\hat\beta}$, in fact:

$$(\hat\alpha')_{\hat\alpha} = \hat\alpha(\partial'_\mu)\delta^\mu_{\hat\alpha} = \hat\alpha(\bar{J}^\nu_\mu \partial_\nu)\delta^\mu_{\hat\alpha} = (\bar{J}^\nu_\mu)^\dagger \delta^\mu_{\hat\alpha} \hat\alpha(\partial_\nu) = \delta^{\hat\beta}_\nu (\bar{J}^\nu_\mu)^\dagger \delta^\mu_{\hat\alpha} \hat\alpha_\sigma \delta^\sigma_{\hat\beta} = (\bar{J}^\dagger)^{\hat\beta}_{\hat\alpha} \hat\alpha_{\hat\beta} \tag{15.1.32}$$

where we set $(\bar{J}^\dagger)^{\hat\beta}_{\hat\alpha} := \delta^{\hat\beta}_\nu (\bar{J}^\nu_\mu)^\dagger \delta^\mu_{\hat\alpha}$ and, in fact, $(\hat\alpha')_{\hat\alpha}(\hat\partial')^{\hat\alpha} = (\hat\alpha)_{\hat\alpha}\hat\partial^{\hat\alpha}$.

As far as the conjugate basis $\bar\partial_{\hat\mu}$ is concerned, it is defined by

$$\bar\partial_{\hat\mu}(\hat\alpha) = \bar\partial_{\hat\mu}(\hat\alpha_{\hat\alpha}\hat\partial^{\hat\alpha}) = \hat\alpha_{\hat\alpha}\bar\partial_{\hat\mu}(\hat\partial^{\hat\alpha}) = \hat\alpha_{\hat\alpha}\delta^{\hat\alpha}_{\hat\mu} = \hat\alpha_{\hat\mu} \qquad\qquad \bar\partial_{\hat\mu}(\hat\partial^{\hat\alpha}) = \delta^{\hat\alpha}_{\hat\mu} \tag{15.1.33}$$

An element $\bar{v} \in \bar{V}$ is expanded as $\bar{v} = \bar{v}^{\hat\mu}\bar\partial_{\hat\mu}$. The components are defined as $\bar{v}^{\hat\mu} := \bar{v}(\partial^{\hat\mu})$ and, then we have

$$\bar{v}(\hat\alpha) = \bar{v}(\hat\alpha_{\hat\alpha}\hat\partial^{\hat\alpha}) = \hat\alpha_{\hat\alpha}\bar{v}(\hat\partial^{\hat\alpha}) = \hat\alpha_{\hat\alpha}\bar{v}^{\hat\alpha} = \bar{v}^{\hat\alpha}\hat\alpha_{\hat\beta}\delta^{\hat\beta}_{\hat\alpha} = \bar{v}^{\hat\alpha}\hat\alpha_{\hat\beta}\bar\partial_{\hat\alpha}(\hat\partial^{\hat\beta}) = \bar{v}^{\hat\alpha}\bar\partial_{\hat\alpha}(\hat\alpha_{\hat\beta}\hat\partial^{\hat\beta}) = \bar{v}^{\hat\alpha}\bar\partial_{\hat\alpha}(\hat\alpha) \tag{15.1.34}$$

When we change the basis, the components change as

$$(\bar{v}')^{\hat\mu} = \bar{v}\left((\hat\partial')^{\hat\mu}\right) = \bar{v}((J^{\hat\mu}_{\hat\nu})^\dagger \hat\partial^{\hat\nu}) = (J^{\hat\mu}_{\hat\nu})^\dagger \bar{v}(\hat\partial^{\hat\nu}) = (J^{\hat\mu}_{\hat\nu})^\dagger \bar{v}^{\hat\nu} \tag{15.1.35}$$

and the conjugate basis changes as $(J^\dagger)^{\hat\nu}_{\hat\mu}\bar\partial'_{\hat\nu} = \bar\partial_{\hat\mu}$, since one has

$$(J^\dagger)^{\hat\nu}_{\hat\mu}\bar\partial'_{\hat\nu}(\hat\alpha) = (J^\dagger)^{\hat\nu}_{\hat\mu}r\bar\partial'_{\hat\nu}(\hat\alpha'_{\hat\alpha}\hat\partial'^{\hat\alpha}) = (J^\dagger)^{\hat\nu}_{\hat\mu}\hat\alpha'_{\hat\alpha}\bar\partial'_{\hat\nu}((\hat\partial')^{\hat\alpha}) = (J^\dagger)^{\hat\nu}_{\hat\mu}\hat\alpha'_{\hat\alpha}\delta^{\hat\alpha}_{\hat\nu} = (J^\dagger)^{\hat\nu}_{\hat\mu}\hat\alpha'_{\hat\nu} = \hat\alpha_{\hat\mu} = \hat\alpha_{\hat\alpha}\delta^{\hat\alpha}_{\hat\mu} = \hat\alpha_{\hat\alpha}\bar\partial_{\hat\mu}(\hat\partial^{\hat\alpha}) = \bar\partial_{\hat\mu}(\hat\alpha_{\hat\alpha}\hat\partial^{\hat\alpha}) = \bar\partial_{\hat\mu}(\hat\alpha) \tag{15.1.36}$$

There is still a bit of mismatch in transformation rules due to the fact that in general, $(\bar{J})^\dagger \neq (J^\dagger)^{-1}$. This will be solved in restricting to orthonormal basis with respect to a Hermitian inner product. In this case, the transformation matrix J^μ_ν is unitary, and in fact $(J)^\dagger = \bar{J}$.

Covectors on a manifold

We discussed covectors of a general vector space V. If we consider a point $x \in M$, we have a vector space $V = T_xM$ of tangent vectors and we can consider its dual $V^* = T^*_xM$. A covector $\alpha \in T^*_xM$ acts as a linear map $\alpha : T_xM \to \mathbb{R}$ on vectors at x.

If we fix coordinates x^μ on M, they induce a natural basis ∂_μ of T_xM and this induces a dual basis ∂^μ of T^*_xM. Traditionally, the dual basis ∂^μ is denoted by $dx^\mu := \partial^\mu$.

We already know (see equations (14.4.14) and (14.4.15)) that, if we change local coordinates $x'^\mu = x'^\mu(x)$, then the natural basis and vector components change as

$$\partial_\alpha = J^\mu_\alpha \partial'_\mu \qquad\qquad v'^\mu = J^\mu_\alpha v^\alpha \tag{15.1.37}$$

This transformation rules perfectly agree with transformation rules assumed on V, namely (15.1.10) and (15.1.12), that is why on V we chose that notation. Accordingly, the whole discussion for transformation rules on V^* follows unchanged, just now the matrix J has to be understood as the Jacobian of the change of coordinates. Then on T^*_xM one has transformation rules for the dual basis and covector components as

$$dx'^\mu = J^\mu_\nu \, dx^\nu \qquad\qquad \alpha'_\nu = \bar{J}^\mu_\nu \alpha_\mu \tag{15.1.38}$$

which still follows the rule for indices positioning.

As for tangent vectors, we also defined the manifold TM which accounts for all tangent vectors at any point of M, here we can define

$$T^*M = \coprod_{x \in M} T_x^*M \qquad (15.1.39)$$

which is called the *cotangent* of M.

A point in T^*M (as any point in a disjoint union) is a pair (x, α) with $\alpha \in T_x^*M$. If we fix a chart (U, φ) with coordinates x^μ in M then the application point x can be represented in terms of its coordinates x^μ while the covector $\alpha = \alpha_\mu \, dx^\mu$ can be represented by its components α_μ. Hence we have a natural coordinate system (x^μ, α_μ) on T_x^*M which is induced by the chart chosen on M. This is well defined over all covectors applied to a point in the domain U of the chart chosen on M. Accordingly, the whole T^*M is covered by natural charts associated to an atlas in M, which exhibits an atlas of T^*M which is a smooth manifold of dimension $2m$.

Transition functions on T^*M are given by transformation rules of covectors, namely by

$$\begin{cases} x'^\mu = x'^\mu(x) \\ \alpha'_\nu = \bar{J}_\nu^\mu \alpha_\mu \end{cases} \qquad (15.1.40)$$

which are in fact smooth if $x'^\mu = x'^\mu(x)$ are.

We can also define a map $\pi : T^*M \to M : (x, \alpha) \mapsto x$, which is called the *projection*, which takes a covector and gives its application point. In coordinates the map reads as $\pi(x^\mu, \alpha_\mu) = x^\mu$ and it is surjective and maximal rank.

Everybody knows at least one example of covector. If we consider a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$, we know there exists a linear map dF_x such that

$$F(x + h) = F(x) + dF_x(h) + o(h) \qquad (15.1.41)$$

which is called the *differential* of F at the point x. The linear map takes an increment h, which is in fact a vector of $\mathbb{R}^2$ applied at x (though here one mixes the linear, affine and manifold structures of $\mathbb{R}^2$), and returns a number (the linear estimate for the increment of F). The map is linear by construction and hence it is a covector of (the manifold) $\mathbb{R}^2$ at the point x.

Notice also that in that notation $dx^\mu(h) = h^\mu$, i.e. dx^μ is precisely the dual basis.

Fields of covectors or *1*-forms

As we defined the Lie algebra $\mathfrak{X}(M)$ of vector fields over M, we can define fields of covectors, which are called 1-*forms*. A 1-form is a map $\omega : M \to T^*M$ such that $\pi \circ \omega = \mathrm{id}_M$. Then $\omega(x)$ is a covector at x and a 1-form is assigning a covector at any point of M.

Locally, a 1-form is represented by

$$\omega = \omega_\mu(x) \, dx^\mu \qquad (15.1.42)$$

and $\omega_\mu(x)$ are called the *component functions*.

The set of all 1-forms on M will be denoted by $\Omega^1(M)$. It is obviously a (real) vector space since 1-forms can be added and multiplied by a number. While we defined an action of vector fields on functions, 1-forms do not act on function. They instead act on vector fields, since one can in fact define

$$i_X\omega := \omega(X) = \omega_\mu X^\mu \tag{15.1.43}$$

the result being a function on M. This is called the *contraction* of ω along X and it is also denoted by $X\lrcorner\omega := i_X\omega$.

Contractions have the following properties

$$i_X\left(f\cdot\omega + g\cdot\theta\right) = f\cdot i_X\omega + g\cdot i_X\theta \qquad i_{f\cdot X+g\cdot Y}\omega = f\cdot i_X\omega + g\cdot i_Y\omega \tag{15.1.44}$$

and we say that contraction is $\mathcal{F}(M)$-bilinear.

Functions or *0*-forms

Smooth functions on a smooth manifold M form an algebra $\mathcal{F}(M)$; functions can be added, multiplied by a number and multiplied

$$(f\cdot g)(x) = f(x)g(x) \tag{15.1.45}$$

Functions on M are also called 0-*forms* and the set of 0-forms is also denoted by $\Omega^0(M)$.

The contraction of a 1-form ω along a vector field X gives a 0-form $i_X\omega \in \Omega^0(M)$. Accordingly, the contraction along X can be regarded as a map $i_X : \Omega^1(M) \to \Omega^0(M)$.

We can also generalise the differential of a function defined in $\mathbb{R}^n$ to a general manifold M. Let $F : M \to \mathbb{R}$ be a function and define locally

$$dF = \frac{\partial F}{\partial x^\mu}\,dx^\mu \tag{15.1.46}$$

If we change coordinates $x'^\mu = x'^\mu(x)$ the function has a different local expression $F'(x')$ though the value of th function at a point does not depend on the chart, i.e.

$$F'(x'(x)) = F(x) \tag{15.1.47}$$

Taking the derivative ∂_μ of both sides

$$J_\mu^\nu\partial'_\nu F' = \partial_\mu F \tag{15.1.48}$$

provides the transformation rules of the partial derivatives. Then one has that the two local expressions of the differentials agree on the overlap of the charts

$$dF' = \frac{\partial F'}{\partial x'^\mu}\,dx'^\mu = \frac{\partial F'}{\partial x'^\mu}J_\nu^\mu\,dx^\nu = \frac{\partial F}{\partial x^\nu}\,dx^\nu = dF \tag{15.1.49}$$

and shows that the differential does not depend on the chart and is globally defined.

The differential is then defined as an operator $d : \Omega^0(M) \to \Omega^1(M)$ with the following properties

$$d(\lambda F + \mu G) = \lambda dF + \mu dG \qquad d(F\cdot G) = dF\cdot G + F\cdot dG \tag{15.1.50}$$

Of course, constants can be regarded as special functions, by the map $i : \mathbb{R} \to \Omega^0(M)$, which is an injective linear map. The image of the map i coincides with the kernel $\ker d$, since the only functions with vanishing differential are the constants.

We then have the sequence

$$0 \to \mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \tag{15.1.51}$$

which is the beginning of the so-called *augmented de Rham complex*. We shall complete it below, in due time.

Canonical and non-canonical isomorphisms

Before proceeding let us discuss the iteration of the dual procedure. Being V^* a vector space itself, one can consider its dual $V^{**} = (V^*)^*$. Points in V^{**} are linear maps $\hat{w} : V^* \to \mathbb{R}$.

We can define a linear map $\Phi : V \to V^{**} : v \mapsto \Phi(v)$ where $\Phi(v) \in V^{**}$ is that elements such that $\forall \alpha \in V^*$

$$\Phi(v)(\alpha) = \alpha(v) \tag{15.1.52}$$

The element $\Phi(v) \in V^{**}$ is well identified since we have its action on any covector α.

Moreover, its action is linear, in fact

$$\Phi(v)(\lambda\alpha + \mu\beta) = (\lambda\alpha + \mu\beta)(v) = \lambda\alpha(v) + \mu\beta(v) = \lambda\Phi(v)(\alpha) + \mu\Phi(v)(\beta) \tag{15.1.53}$$

so that it really defines an element in V^{**}.

We now want to show that Φ is in fact an isomorphism. For, we shall define another $\Psi : V^{**} \to V$ and show that $\Psi = \Phi^{-1}$.

Thus first step, is defining $\Psi : V^{**} \to V : \hat{w} \mapsto \Psi(\hat{w})$ to be the only element $v \in V$ such that $\forall \alpha \in V^*$

$$\alpha(v) = \hat{w}(\alpha) \tag{15.1.54}$$

On V^{**}, we can use the dual-dual basis and relative components. Once $\hat{w}$ is given we have $\forall \alpha \in V^*$

$$\alpha(v) = \alpha_\nu v^\nu \qquad \hat{w}(\alpha) = \alpha_\nu \hat{w}^\nu \tag{15.1.55}$$

from which $v^\nu = \hat{w}^\nu$ follows. Then v is uniquely determined by $\hat{w}$.

As a second step one can check that $\Psi \circ \Phi = \mathrm{id}_V$ and $\Phi \circ \Psi = \mathrm{id}_{V^{**}}$, which proves that $\Psi = \Phi^{-1}$.

Just notice that we used for both Φ and Ψ essentially the same property, (15.1.52) and (15.1.54).

Thus, since Φ has an inverse it is an isomorphism. Then V and V^{**} are isomorphic as vector spaces.

Show that both the maps Φ and Ψ are linear, i.e. for example that

$$\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(w) \tag{15.1.56}$$

Notice that the isomorphism Φ here defined does not rely on any additional structure on V, in particular it does not depend on the basis or on an inner product. For this reason, we shall call it a *canonical isomorphism*.

To be honest, one should have noticed that V and V^{**} are isomorphic just because they have the same dimension and any two vector spaces V and W with the same dimension are isomorphic. In fact, one can fix two basis e_i and f_i in V and W, respectively, and define the only linear map $\Phi : V \to W$ such that

$$\Phi(e_i) = f_i \tag{15.1.57}$$

That is an isomorphism (it maps a basis into a basis).

However, this isomorphism may depend on the choice of the bases (and thus is not canonical). The same argument could be applied to V and V^* which in fact have the same dimension. Let us fix a basis ∂_μ in V and the dual basis ∂^μ in V^*. Ignoring for a while Einstein conventions, we can define a linear map by extending

$$\Phi(\partial_\mu) = \partial^\mu \tag{15.1.58}$$

Unfortunately, by changing the basis $\partial'_\mu = \bar{J}^\nu_\mu \partial_\nu$ one has

$$\Phi(\partial'_\mu) = \Phi(\bar{J}^\nu_\mu \partial_\nu) = \bar{J}^\nu_\mu \Phi(\partial_\nu) = \bar{J}^\nu_\mu \partial^\nu \neq \partial'^\mu = J^\mu_\nu \partial^\nu \tag{15.1.59}$$

Thus to define the map Φ, the equation (15.1.58) is not enough since one should also specify in which basis equation (15.1.58) is given, since in general it holds true in some bases but not in others.

In principle, there are infinitely many isomorphisms between V and V^*; one can define one isomorphism for every basis chosen on V. But if then one wants to write down the isomorphic image of a vector v, one does not know how to do it if not after fixing a basis.

We describe this situation by saying that V and V^* are isomorphic but not canonically. In the case of the isomorfism between V and V^{**} we did not simply show that isomorphisms exist but we single out one particular isomorphism which allows to map elements of V and V^{**} back and forth and eventually identify V and V^{**}.

When a canonical isomorphism exists between two vector spaces, as we showed for V and V^{**}, the vector spaces and their elements are identified by an abuse of notation. Unfortunately, sometimes when one starts to abuse notation and to identify different things, then notation becomes obscure unless one is well aware of the abuse of language.

That is more natural than it seems at first; we learnt to identify real numbers with zero-degree polynomials, even though polynomials are functions (or algebraic objects, depending on the definition and context), not numbers. We identified rational numbers with their embedding in real numbers even when real numbers are defined by Dedekin classes of rational numbers and then they are not rational numbers on their own. We identified integers with their embedding in rational numbers, even though rational numbers are classes of pairs of integers. And so on.

We can define an isomorphism between V and V^* also using an inner product $< \cdot, \cdot >: V \times V \to \mathbb{R}$.

In particular, we can define the linear maps $\flat : V \to V^*$ and $\sharp : V^* \to V$ as

$$v^\flat(u) = < v, u > \qquad\qquad < \alpha^\sharp, u > = \alpha(u) \tag{15.1.60}$$

One should show that they define good maps, that they are linear and that they are the inverse one of the other. This can be done intrinsically, though it is much easier to do it using a basis. Let us fix a basis ∂_μ in V and the corresponding dual basis in V^*. The inner product is represented by a matrix $g_{\mu\nu} = < \partial_\mu, \partial_\nu >$ which is invertible in view of the fact that inner product is non-degenerate. Let us denote the inverse matrix by $g^{\mu\nu}$ (which is an inner product on V^*, but that is another story to tell).

The two maps are defined as

$$\flat : V \to V^* : v^\mu \partial_\mu \mapsto v^\mu g_{\mu\nu} \partial^\nu \qquad\qquad \sharp : V^* \to V : \alpha_\mu \partial^\mu \mapsto \alpha_\mu g^{\mu\nu} \partial_\nu \tag{15.1.61}$$

These are obtained just specifying equations (15.1.60) in the bases we fixed. They are manifestly good linear maps and the inverse to each other.

Of course, these isomorphisms are defined regardless the choice of a basis, but they depend on an inner product. Thus these isomorphisms are not canonical in the category of vector spaces (while they are in the category of Euclidean vector spaces).

> By the way, choosing a basis is like choosing an inner product since there is a unique (definite positive) inner product for which the basis is orthonormal.

In any event, we shall identify objects which are canonically isomorphic (like V and V^{**}) and we shall not identify objects which are non-canonically isomorphic (like V and V^*). Accordingly, we identify $V^{**} \simeq V$, so that $V^* \simeq V^{***}$.

Since $V^{**} \simeq V$, we can regard vectors are linear maps on covectors defining their action as

$$v(\alpha) = \alpha(v) \tag{15.1.62}$$

Canonical isomorphisms and anti-isomorphisms

Before going to discuss tensors, let us go back briefly to complex vector spaces and discuss their canonical isomorphisms.

For a finite dimensional complex vector space V, we discussed the dual V^*, the dual conjugate $V^\dagger$ as well as the conjugate $\bar{V}$ spaces, They are all a sort of dual, although, for example, we did not discuss $(V^*)^\dagger$, $(\bar{V})^*$, $(\bar{V})^\dagger$, $(V^\dagger)^\dagger$, and so on. The reason for this, is that there are a number of canonical isomorophisms and anti-isomorphisms among these objects, which eventually will show that, up to identifications induced by canonical isomorphisms, the only spaces left are V, V^*, $\bar{V}$, and $\bar{V}^* \simeq V^\dagger$, that $V \approx \bar{V}$ and $V^* \approx \bar{V}^*$ are anti-isomorphic (and consequently we shall not identify them with each other) and, moreover, inner products further define non-canonical isomorphisms. For example, an Hermitian inner product will define isomorphisms $V \sim \bar{V}^*$ and $\bar{V} \sim V^*$. Since the situation is quite intricate, it is a good exercise to get used to notation.

> Given two vector spaces V and W, we write $V \simeq W$ iff the are canonically isomorphic to each other. We write $V \approx W$ iff they are canonically anti-isomorphic and $V \sim W$ iff they are non-canonically isomorphic.

As usual, following the real cases, one already knows that $V \simeq V^{**} \simeq \ldots$, $V^* \simeq V^{***} \simeq \ldots$, $\bar{V} \simeq \bar{V}^{**} \simeq \ldots$, and $\bar{V}^* \simeq \bar{V}^{***} \simeq \ldots$ are canonically isomorphic.

In order to keep it simple, since we already discussed the ideas in details, let us simply lists maps and show that they are (anti-)isomorphisms. Let us start by all canonical isomorphisms.

Let us consider the map

$$j : V \to (V^\dagger)^\dagger : v \mapsto j(v) \qquad \forall \hat\alpha \in V^\dagger : j(v)(\hat\alpha) = (\hat\alpha(v))^\dagger \tag{15.1.63}$$

For,

$$j(v)(\lambda\hat\alpha + \mu\hat\beta) = (\lambda\hat\alpha(v) + \mu\hat\beta(v))^\dagger = \lambda^\dagger j(v)(\hat\alpha) + \mu^\dagger j(v)(\hat\beta) \qquad (j(v) \in (V^\dagger)^\dagger)$$

$$j(\lambda v + \mu u)(\hat\alpha) = (\hat\alpha(\lambda v + \mu u))^\dagger = \lambda\hat\alpha(v)^\dagger + \mu\hat\alpha(u)^\dagger = \lambda j(v)(\hat\alpha) + \mu j(u)(\hat\alpha) \qquad (j \text{ is linear}) \tag{15.1.64}$$

In view of these identifications, we have $V \simeq (V^*)^* \simeq (V^\dagger)^\dagger$ and, consequently, we can define the action of $v \in V$ on V^* and $V^\dagger$.

For $v = v^\alpha \partial_\alpha$, we have

$$j(v)(\hat{\alpha}) := (\hat{\alpha}(v))^\dagger = (\hat{\alpha}_{\hat{\alpha}})^\dagger \delta^{\hat{\alpha}}_{\hat{\beta}} v^\beta = (\hat{\alpha}^\dagger)_\beta v^\beta \tag{15.1.65}$$

which clearly is vanishing for all $\hat{\alpha}$ iff $v = 0$. That means that the map $j : V \to (V^\dagger)^\dagger$ is injective. Since it is a map between spaces of the same dimension, it is also surjective, i.e. an isomorphism.

As long as canonical anti-isomorphisms are concerned, we have a canonical anti-isomorphism $\hat{k} : V^* \to V^\dagger : \alpha \mapsto \hat{k}(\alpha) = \hat{\alpha}$ defined as

$$\hat{k}(\alpha)(v) = \hat{\alpha}(v) = (\alpha(v))^\dagger \tag{15.1.66}$$

The image $\hat{\alpha}$ is in $V^\dagger$ because it is anti-linear in v, i.e.

$$\hat{\alpha}(\lambda v + \mu u) = (\alpha(\lambda v + \mu u))^\dagger = (\lambda\alpha(v) + \mu\alpha(u))^\dagger = \lambda^\dagger \alpha(v)^\dagger + \mu^\dagger \alpha(u)^\dagger = \lambda^\dagger \hat{\alpha}(v) + \mu^\dagger \hat{\alpha}(u) \tag{15.1.67}$$

Also the map $\hat{k}$ is anti-linear since

$$\hat{k}(\lambda\alpha + \mu\beta)(v) = (\lambda\alpha(v) + \mu\beta(v))^\dagger = \lambda^\dagger \hat{\alpha}(v) + \mu^\dagger \hat{\beta}(v) \tag{15.1.68}$$

Finally, it is invertible since we can also define the inverse anti-isomorphism $\hat{k}^{-1} : V^\dagger \to V^* : \hat{\alpha} \mapsto \hat{\hat{\alpha}}$ defined by $\hat{\hat{\alpha}}(v) = (\hat{\alpha}(v))^\dagger$. If $\hat{\alpha} \in V^\dagger$, the $\hat{\hat{\alpha}} \in V^*$ is linear.

In a basis ∂_a of V, the induced dual basis ∂^a and the dual conjugate basis $\partial^{\hat{\alpha}}$, the map is expressed as

$$\hat{k} : V^* \to V^\dagger : \alpha = \alpha_\alpha \partial^\alpha \mapsto \hat{\alpha} = (\alpha_\alpha)^\dagger \delta^\alpha_{\hat{\beta}} \partial^{\hat{\beta}} =: (\alpha^\dagger)_{\hat{\alpha}} \partial^{\hat{\alpha}} \tag{15.1.69}$$

Also, we have two canonical anti-isomorphisms $\hat{\imath} : V \to (V^\dagger)^* : v \mapsto \hat{\imath}(v)$ and $\hat{\jmath} : V \to (V^*)^\dagger : v \mapsto \hat{\jmath}(v)$ defined by

$$\hat{\imath}(v)(\hat{\alpha}) = \hat{\alpha}(v) \qquad\qquad \hat{\jmath}(v)(\alpha) = (\alpha(v))^\dagger \tag{15.1.70}$$

For the first

$$\begin{aligned}
\hat{\imath}(v)(\lambda\hat{\alpha} + \mu\hat{\beta}) &= \lambda\hat{\alpha}(v) + \mu\hat{\beta}(v) = \lambda\hat{\imath}(v)(\hat{\alpha}) + \mu\hat{\imath}(v)(\hat{\beta}) & (\hat{\imath}(v) \in (V^\dagger)^*) \\
\hat{\imath}(\lambda v + \mu u)(\hat{\alpha}) &= \hat{\alpha}(\lambda v + \mu u) = \lambda^\dagger \hat{\alpha}(v) + \mu^\dagger \hat{\alpha}(u) = \lambda^\dagger \hat{\imath}(v)(\hat{\alpha}) + \mu^\dagger \hat{\imath}(u)(\hat{\alpha}) & (\hat{\imath} \text{ is anti-linear})
\end{aligned} \tag{15.1.71}$$

For the second

$$\begin{aligned}
\hat{\jmath}(v)(\lambda\alpha + \mu\beta) &= \lambda^\dagger \alpha(v)^\dagger + \mu^\dagger \beta(v)^\dagger = \lambda^\dagger \hat{\jmath}(v)(\alpha) + \mu^\dagger \hat{\jmath}(v)(\beta) & (\hat{\jmath}(v) \in (V^*)^\dagger) \\
\hat{\jmath}(\lambda v + \mu u)(\alpha) &= \alpha(\lambda v + \mu u)^\dagger = \lambda^\dagger \alpha(v)^\dagger + \mu^\dagger \alpha(u)^\dagger = \lambda^\dagger \hat{\jmath}(v)(\alpha) + \mu^\dagger \hat{\jmath}(u)(\alpha) & (\hat{\jmath} \text{ is anti-linear})
\end{aligned} \tag{15.1.72}$$

Accordingly, we have a canonical isomorphism $(V^\dagger)^* \simeq (V^*)^\dagger =: \bar{V}$, which is canonically anti-isomorphic to $V \simeq (V^*)^* \simeq (V^\dagger)^\dagger$.

Notice that since $\bar{V} = (V^\dagger)^*$, then $\bar{V}^* = V^\dagger$. Accordingly, from now on we shall denote the relevant spaces by $\bar{V}$ and $\bar{V}^*$.

Before considering the maps induced by inner products, we can summarise the situation about the whole family of spaces which can be obtained from V by repeated duality and conjugation, modulo canonical isomorphisms. We are left with only four spaces:

$$v = v^\alpha \partial_\alpha \in V \simeq (V^*)^* \simeq (V^\dagger)^\dagger \qquad \alpha = \alpha_\alpha \partial^\alpha \in V^* \simeq \bar{V}^\dagger$$
$$\hat{v} = \hat{v}^{\hat{\alpha}} \bar{\partial}_{\hat{\alpha}} \in \bar{V} \simeq (V^\dagger)^* \simeq (V^*)^\dagger \qquad \hat{\alpha} = \hat{\alpha}_{\hat{\alpha}} \hat{\partial}^{\hat{\alpha}} \in \bar{V}^* \simeq V^\dagger$$

(15.1.73)

with canonical anti-isomorphisms along the columns.

A vector $v \in V$ acts both as a linear form on $\alpha \in V^*$ and an anti-linear form on $\hat{\alpha} \in \bar{V}^*$, namely

$$v(\alpha) := \alpha(v) = \alpha_\alpha v^\alpha \qquad v(\hat{\alpha}) := (\hat{\alpha}(v))^\dagger = (\hat{\alpha}_{\hat{\alpha}})^\dagger \delta^{\hat{\alpha}}_\beta v^\beta = (\hat{\alpha}^\dagger)_\beta v^\beta$$

(15.1.74)

A conjugate vector $\hat{v} \in \bar{V}$ acts both as a linear form on $\hat{\alpha} \in \bar{V}^*$ and an anti-linear form on $\alpha \in V^*$, namely

$$\hat{v}(\hat{\alpha}) := \hat{\alpha}_{\hat{\alpha}} \hat{v}^{\hat{\alpha}} \qquad \hat{v}(\alpha) := (\alpha(\hat{v}))^\dagger = (\alpha_\alpha)^\dagger \delta^\alpha_{\hat{\beta}} \hat{v}^{\hat{\beta}} = (\alpha^\dagger)_{\hat{\beta}} \hat{v}^{\hat{\beta}}$$

(15.1.75)

A covector $\alpha \in V^*$ acts both as a linear form on $v \in V$ and an anti-linear form on $\hat{v} \in \bar{V}$, namely

$$\alpha(v) = \alpha_\alpha v^\alpha \qquad \alpha(\hat{v}) := \alpha(\hat{v}) = \alpha_\alpha \delta^\alpha_{\hat{\beta}} (\hat{v}^{\hat{\beta}})^\dagger = \alpha_\alpha (\hat{v}^\dagger)^\alpha$$

(15.1.76)

A conjugate covector $\hat{\alpha} \in \bar{V}^*$ acts both as a linear form on $\hat{v} \in \bar{V}$ and an anti-linear form on $v \in V$, namely

$$\hat{\alpha}(\hat{v}) := \hat{v}(\hat{\alpha}) := \hat{\alpha}_{\hat{\alpha}} \hat{v}^{\hat{\alpha}} \qquad \hat{\alpha}(v) := \hat{\alpha}_{\hat{\alpha}} \delta^{\hat{\alpha}}_\beta (v^\beta)^\dagger = \hat{\alpha}_{\hat{\alpha}} (v^\dagger)^{\hat{\alpha}}$$

(15.1.77)

More generally, a linear map $\Phi : \bar{V} \to W$, in view of the canonical anti-isomorphism $V \approx \bar{V}$, uniquely induces an anti-linear map $\hat{\Phi} : V \to W$, and vice versa. Accordingly, from now on there is no much need of considering anti-linear maps on V, since they can be identified with linear maps on $\bar{V}$.

The space of linear maps $\Phi : V \to W$ is denoted by $L(V; W)$, the space of anti-linear maps $\hat{\Phi} : V \to W$ is denoted by $\hat{L}(V; W)$. They are vector spaces with obvious operations. We have canonical isomorphisms $\hat{L}(V; W) \simeq L(\bar{V}; W)$.

In fact, if $\hat{\Phi} : V \to W$ is an anti-linear map on V, we can associate to it a linear map $i(\hat{\Phi}) : \bar{V} \to W$ defined by

$$i(\hat{\Phi})(\hat{i}(v)) = \hat{\Phi}(v) \in W$$

(15.1.78)

In a basis ∂_α of V and a basis $\partial_i \in W$, the map $\hat{\Phi} : V \to W$ reads as $\hat{\Phi}(v^\alpha \partial_\alpha) = (v^\dagger)^{\hat{\alpha}} \hat{\Phi}^i_{\hat{\alpha}} \partial_i$. Then the corresponding linear map $i(\hat{\Phi})$ reads as $[i(\hat{\Phi})]^i_{\hat{\alpha}} = \hat{\Phi}^i_{\hat{\alpha}}$.

Accordingly, we can drop anti-linear maps (and forms) on V and V^* and replace them with linear maps (and forms) on $\bar{V}$ and $\bar{V}^*$. Of course, also anti-linear maps on $\bar{V}$ and $\bar{V}^*$ are replaced with linear maps on $\overline{\bar{V}} \simeq V$ and $\overline{(\bar{V}^*)} \simeq V^*$.

If we have two vector spaces V and U and a linear map $\Phi : V \to U$, then we can define the *dual operator* $\Phi^* : U^* \to V^*$ and the *dual conjugate operator* $\bar{\Phi}^* : \bar{U}^* \to \bar{V}^*$ so that

$$\Phi^*(\alpha)(v) = \alpha(\Phi(v)) \qquad \bar{\Phi}^*(\hat{\alpha})(v) = \hat{\alpha}(\Phi(v)) \qquad \forall \alpha \in U^*, \quad \hat{\alpha} \in \bar{U}^*, \quad v \in V$$

(15.1.79)

as well as the *conjugate operator* $\bar{\Phi} : \bar{V} \to \bar{U}$, ($\bar{U}$ as anti-linear maps on U^*)

$$\bar{\Phi}(\hat{v})(\alpha) = \hat{v}(\Phi^*(\alpha)) \qquad \forall \alpha \in U^*, \quad \hat{v} \in \bar{V} \tag{15.1.80}$$

Notice that all these operators are linear.

In a basis ∂_a on V and a basis ∂_i on U, we have $\Phi(\partial_a) = \Phi^i_a \partial_i$ ($\Rightarrow \Phi(v) = v^a \Phi^i_a \partial_i$) and consequently

$$\alpha = \alpha_i \partial^i, v = v^a \partial_a : \qquad \Phi^*(\partial^i) = \Phi^i_a \partial^a \qquad \Rightarrow \Phi^*(\alpha)(v) = \alpha_i \Phi^i_a v^a = \alpha(v^a \Phi^i_a \partial_i) \tag{15.1.81}$$

$$\hat{\alpha} = \hat{\alpha}_{\hat{\imath}} \hat{\partial}^{\hat{\imath}}, v = v^a \partial_a : \qquad \bar{\Phi}^*(\hat{\partial}^{\hat{\imath}}) = (\Phi^\dagger)^{\hat{\imath}}_{\hat{a}} \hat{\partial}^{\hat{a}} \qquad \Rightarrow \bar{\Phi}^*(\hat{\alpha})(v) = \hat{\alpha}_{\hat{\imath}} (\Phi^\dagger)^{\hat{\imath}}_{\hat{a}} \delta^{\hat{a}}_a (v^a)^\dagger = \hat{\alpha}(v^a \Phi^i_a \partial_i) \tag{15.1.82}$$

$$\alpha = \alpha_i \partial^i, \hat{v} = v^{\hat{a}} \hat{\partial}_{\hat{a}} : \qquad \bar{\Phi}(\hat{\partial}_{\hat{a}}) = (\Phi^\dagger)^{\hat{\imath}}_{\hat{a}} \hat{\partial}_{\hat{\imath}} \qquad \Rightarrow \bar{\Phi}(\hat{v})(\alpha) = \hat{v}^{\hat{a}} \delta^i_{\hat{\imath}} (\Phi^\dagger)^{\hat{\imath}}_{\hat{a}} (\alpha_i)^\dagger = \hat{v}(\alpha_i \Phi^i_a \partial^a) \tag{15.1.83}$$

Clearly enough, the correspondence $\Phi \mapsto \Phi^*$ (as well as the correspondence $\bar{\Phi} \mapsto \bar{\Phi}^*$) is linear, while the correspondences $\Phi \mapsto \bar{\Phi}^*$ and $\Phi \mapsto \bar{\Phi}$ are anti-linear.

Let us then consider a Hermitian inner product η on V. That establishes a non-canonical isomorphism $\flat : V \to \bar{V}^*$ (as well as $\flat : \bar{V} \to V^*$), the inverse being denoted by $\sharp : \bar{V}^* \to V$ (and $\sharp : V^* \to \bar{V}$). Since this is the case, in view of the canonical anti-isomorphisms, that induces also non-canonical anti-isomorphisms between V and V^* as well as between $\bar{V}$ and $\bar{V}^*$.

Just identify $\bar{V}^*$ with anti-linear forms on V and define

$$v^\flat(u) = \eta(v, u)^\dagger = \eta(u, v) \tag{15.1.84}$$

which is anti-linear in u (so that $v^\flat \in \bar{V}^*$) and linear in v (so that $\flat$ is an isomorphism).

We also define

$$\hat{v}^\flat(\hat{u}) = \eta(\hat{u}, \hat{v}) \tag{15.1.85}$$

which is anti-linear in $\hat{u}$ (so that $\hat{v}^\flat \in V^*$, which is also the space of anti-linear maps on $\bar{V}$) and linear in $\hat{v}$ (so that $\flat$ is an isomorphism). Consequently, by composition with the canonical anti-isomorphisms, one also has non-canonical anti-isomorphisms $\flat : V \to V^*$ and $\flat : V \to \bar{V}^*$.

In a basis ∂_a of V, the non-canonical isomorphisms read as

$$v^\flat = \eta_{\hat{a}b} v^b \hat{\partial}^{\hat{a}} \qquad\qquad \hat{v}^\flat = \eta_{a\hat{b}} \hat{v}^{\hat{b}} \partial^a \tag{15.1.86}$$

For the non-canonical anti-isomorphisms one has

$$v^\flat = \eta_{\hat{a}b} v^b \delta^{\hat{a}}_a \partial^a \qquad\qquad \hat{v}^\flat = \eta_{a\hat{b}} \hat{v}^{\hat{b}} \delta^a_{\hat{a}} \hat{\partial}^{\hat{a}} \tag{15.1.87}$$

Let us stress that had we a symmetric inner product, that would induce non-canonical isomorphisms $\flat : V \to V^*$ and $\flat : \bar{V} \to \bar{V}^*$, instead.

If we have two vector spaces, each with a hermitian inner product, (V, η) and (W, η'), for any linear map $\Phi : V \to W$, we can define the *adjoint map* $\Phi^\dagger : W \to V$ given by

$$\eta(v, \Phi^\dagger(w)) = \eta'(\Phi(v), w) \tag{15.1.88}$$

In an orthonormal basis ∂_a on V and an orthonormal basis ∂_i on W, the map Φ reads as $\Phi(\partial_a) = \Phi^i_a \partial_i$ and we have

$$(v^a)^\dagger \eta_{ad} w^i (\Phi^\dagger)^d_i = (v^a \Phi^i_a)^\dagger \eta_{ij} w^j \quad \Rightarrow (\Phi^\dagger)^d_i := \eta^{ad} (\Phi^k_a)^\dagger \eta_{ki} \tag{15.1.89}$$

Sheaf definition of vectors and covectors

As we announced, geometric objects can be introduced on locally Euclidean spaces in a sheaf perspective. A locally Euclidean space M of dimension m is covered with charts and it can be obtained (up to diffeomorphisms) by identifying patches $\hat{U}_\alpha := \varphi^\alpha(U_\alpha) \subset \mathbb{R}^m$ along an equivalence relation defined by transition maps $\varphi^\beta{}_\alpha : \varphi^\alpha(U_{\alpha\beta}) \to \varphi^\beta(U_{\alpha\beta})$.

For any openset U, we can consider vector fields defined on U, so we attach a Lie algebra $\mathfrak{X}(U)$ to any openset $U \subset M$. Given two intersecting opensets U and U', we should be able to identify vectors in $\mathfrak{X}(U)$ with vectors in $\mathfrak{X}(U')$ when they represent the same tangent vector.

Fix a point $p \in U \cap U'$ and 2 charts around p one $(U_\alpha, \varphi^\alpha)$ for U and one (U_β, φ^β) for U'. Then we already discussed that an element of $\mathfrak{X}(U)$ is described by $X^\mu(x)\partial_\mu$, the element of $\mathfrak{X}(U')$ by $X'^\mu(x')\partial'_\mu$.

The relation is obtained from the transition map $x'^\mu = x'^\mu(x)$ between the charts. We know that $\partial'_\mu = \bar{J}^\lambda_\mu \partial_\lambda$, thus $X'^\mu(x') = J^\mu_\lambda X^\lambda(x)$. Accordingly, we can represent elements in $\mathfrak{X}(U)$ by component local functions $X^\mu(x)$ and we define the projection maps to $U_{\alpha\beta}$ as

$$X'^\mu(x') = J^\mu_\lambda X^\lambda(x) \tag{15.1.90}$$

which are transformation laws. One can regard a vector field X as an intrinsic object, or as an equivalence class of local descriptions depending on charts. The transformation laws define the sheaf projection so that local descriptions glue together iff they come from a single intrinsic vector field, so that the two descriptions are in fact equivalent.

The same discussion can be done for function by using the transformation laws

$$F'(x') = F(x) \tag{15.1.91}$$

or for 1-forms, which again we can describe by their local components $\alpha_\mu(x)$, but using the transformation laws

$$\alpha'_\mu(x') = \bar{J}^\lambda_\mu \alpha_\lambda(x) \tag{15.1.92}$$

We see that transformation laws are identified with glueing conditions of local descriptions. Different transformations laws correspond to different geometric objects. When we know local descriptions and transformations laws we are able to rebuild intrinsic objects. This general idea will come back in multiple instances and contexts always highlighting that one needs local descriptions and transformation laws. Once this (that we shall name the *golden rule*) is accepted the difference between intrinsic formalism and the local one is recognized to be just that, a difference between two different languages and we can switch freely back and forth depending on the problem under consideration.

2. Tensors and tensor fields

Now that we defined vectors and covectors we can define general tensors. A *tensor* of rank (p, q) on a real vector space V is a multilinear map

$$t : (V)^p \times (V^*)^q \to \mathbb{R} \tag{15.2.1}$$

The set of all tensors of rank (p, q) on V will be denoted by $T_q^p(V)$.

For $(p, q) = (0, 1)$ we have a linear map like $\hat{w} : V^* \to \mathbb{R}$, i.e. an element of $V^{**} \simeq V$. Thus we identify $T_1^0(V) = V$.
For $(p, q) = (1, 0)$ we have a linear map like $\alpha : V \to \mathbb{R}$, i.e. an element of V^*. Thus we identify $T_0^1(V) = V^*$.
We extend the definition setting $T_0^0(V) = \mathbb{R}$.

The sets $T_q^p(V)$ have an obvious structure of vector spaces induced by the operations

$$(r + t)(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) = r(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) + t(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q)$$
$$(\lambda t)(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) = \lambda t(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) \tag{15.2.2}$$

One can also define an operation $\otimes : T_q^p(V) \times T_{q'}^{p'}(V) \to T_{q+q'}^{p+p'}(V)$

$$(t \otimes r)(v_1, \ldots, v_p, w_1, \ldots, w_{p'}, \alpha^1, \ldots, \alpha^q, \beta^1, \ldots, \beta^{q'}) = t(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) \cdot r(w_1, \ldots, w_{p'}, \beta^1, \ldots, \beta^{q'}) \tag{15.2.3}$$

which is called *tensor product*.

Let us now define the set $T(V) = \oplus_{(p,q)} T_q^p(V)$, which is made of finite linear combinations of tensors of different rank. The set $T(V)$ inherits a structure of (infinite dimensional) vector space. The tensor product is extended by linearity to $T(V)$ and $T(V)$ becomes an algebra, called the *tensor algebra*.

Everybody already encountered examples of tensors. An inner product on V is a bilinear form $< \cdot, \cdot >: V \times V \to \mathbb{R}$ (which is symmetric, non-degenerate, definite positive) and hence it is tensor in $T_0^2(V)$. Kinetic energy is a quadratic form associated to a (symmetric, non-degenerate, definite positive) bilinear form which is again a tensor in $T_0^2(V)$.

Complex tensors

Since tensors are linear maps, if we consider a complex vector space V we should consider linear and anti-linear maps. However, we already discussed that anti-linear maps on V induce linear maps on $\bar{V}$. Accordingly, to define tensors on a complex vector space V is enough to consider multilinear maps on all possible duals.

We define a tensor as a multilinear, complex valued, map

$$T_{q,\bar{q}}^{p,\bar{p}}(V) = \{t : V^p \times (V^*)^q \times \bar{V}^{\bar{p}} \times (\bar{V}^*)^{\bar{q}} \to \mathbb{C}\} \tag{15.2.4}$$

Those include anti-linear maps on V, which are represented as linear maps on $\bar{V}$.

More generally, given two vector spaces V and U, we can define linear maps $t : V^* \times U^* \to \mathbb{C}$. The space of such maps is denoted by $V \otimes U$ and it is called the *tensor product* of the two vector spaces. It is clearly a vector space, with a basis $e_a \otimes e_i$ which are defined as

$$(e_a \otimes e_i)(\alpha, \beta) = \alpha_a \beta_i \tag{15.2.5}$$

where e_a and e_i are bases in V and U, respectively. That defines *tensors* with indices in different families.

In particular, one can see that

$$T^{p,\bar{p}}_{q,\bar{q}}(V) = (V^*)^{\otimes p} \otimes (V)^{\otimes q} \otimes (\bar{V}^*)^{\otimes \bar{p}} \otimes (\bar{V})^{\otimes \bar{q}} \tag{15.2.6}$$

Then one can use tensor spaces to represent (up to canonical isomorphisms) all linear maps for some tensor product space to some other.

Bases of tensors

Let us hereafter define a basis for the vector spaces $T^p_q(V)$. We shall thus show that all spaces $T^p_q(V)$ are finite dimensional and accordingly the tensor algebra will be infinite dimensional, a basis of the tensor algebra being defined by the union of bases of each $T^p_q(V)$.

The procedure is quite similar to what we did for the dual space, which is in fact a special case. We shall first define a collection of tensors in $T^p_q(V)$, then we shall show that they generate $T^p_q(V)$, finally we shall prove that they are independent.

First step: let us fix a basis ∂_α of V and the corresponding dual basis ∂^α of V^* and consider $\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q} \in T^p_q(V)$, so that one has

$$\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) = v_1^{\mu_1} \ldots v_p^{\mu_p} \alpha^1_{\nu_1} \ldots \alpha^q_{\nu_q} \tag{15.2.7}$$

Second step: let us prove that these elements generate $T^p_q(V)$. Let us consider a generic element $t \in T^p_q(V)$ and the collection of components

$$t^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} = t(\partial_{\mu_1}, \ldots, \partial_{\mu_p}, \partial^{\nu_1}, \ldots, \partial^{\nu_q}) \tag{15.2.8}$$

We can now check that $t = t^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}$ by applying both sides to the same set of arguments.

One has

$$\begin{aligned} t(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) &= v_1^{\mu_1} \ldots v_p^{\mu_p} \alpha^1_{\nu_1} \ldots \alpha^q_{\nu_q} t(\partial_{\mu_1}, \ldots, \partial_{\mu_p}, \partial^{\nu_1}, \ldots, \partial^{\nu_q}) = \\ &= t^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}(v_1, \ldots, v_p, \alpha^1, \ldots, \alpha^q) \end{aligned} \tag{15.2.9}$$

Then $\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}$ are generators of $T^p_q(V)$.

Third step: $\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}$ are independent.

Consider a linear combination

$$\lambda^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q} = 0 \tag{15.2.10}$$

and apply to $(\partial_{\rho_1}, \ldots, \partial_{\rho_p}, \partial^{\sigma_1}, \ldots, \partial^{\sigma_q})$. It follows directly that $\lambda^{\sigma_1 \ldots \sigma_q}_{\rho_1 \ldots \rho_p} = 0$ and independence of $\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}$.

Then $\partial^{\mu_1} \otimes \ldots \otimes \partial^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}$ form a basis for $T^p_q(V)$.

Accordingly, $T^p_q(V)$ is finite dimensional and of dimension $\dim(T^p_q(V)) = \dim(V)^{p+q}$.

Universal property of tensor product

Let us define the tensor product of two vector spaces V and U by the universal property. A vector space $V \otimes U$ is called the *tensor product* of V and U if there exists a linear map $\otimes : V \times U \to V \otimes U$ with the following property:

for any bilinear map $f : V \times U \to Z$ in another vector space Z there exists a unique map $\bar{f} : V \otimes U \to Z$ such that the following diagram is commutative

$$
\begin{array}{ccc}
V \times U & \xrightarrow{\otimes} & V \otimes U \\
& {\scriptstyle f}\searrow & \downarrow {\scriptstyle \bar{f}} \\
& & Z
\end{array}
\tag{15.2.11}
$$

i.e. such that $\bar{f} \circ \otimes = f$.

If the tensor product exists, it is unique (up to isomorphisms).

In fact, let us suppose there are two candidates as a tensor product, namely $(V \otimes U, \otimes)$ and (W, φ). Then let us fix $Z = W$ and $f = \varphi$; by the universal property above, there exists a unique map $\bar{\varphi} : V \otimes U \to W$. However, also (W, φ) has the universal property, thus fixing $Z = V \otimes U$ and $f = \otimes$ then the map $\bar{\bar{\otimes}} : W \to V \otimes U$ and it is unique.

These two maps satisfy the commutation relations

$$
\bar{\varphi} \circ \otimes = \varphi \qquad \bar{\bar{\otimes}} \circ \varphi = \otimes \qquad \Rightarrow \bar{\varphi} \circ \bar{\bar{\otimes}} \circ \varphi = \varphi \qquad \Rightarrow \bar{\bar{\otimes}} \circ \bar{\varphi} \circ \otimes = \otimes
\tag{15.2.12}
$$

Accordingly, these maps make the following diagrams commutative:

$$
\begin{array}{ccc}
V \times U \xrightarrow{\otimes} V \otimes U && V \times U \xrightarrow{\varphi} W \\
{\scriptstyle \otimes}\searrow \quad \downarrow {\scriptstyle \bar{\bar{\otimes}} \circ \bar{\varphi}} && {\scriptstyle \varphi}\searrow \quad \downarrow {\scriptstyle \bar{\varphi} \circ \bar{\bar{\otimes}}} \\
V \otimes U && W
\end{array}
\tag{15.2.13}
$$

as the identity maps $\mathrm{id}_{V \otimes U}$ and id_W do. However, in view of the universality property, the maps making these diagram commutative are unique, then we have

$$
\bar{\bar{\otimes}} \circ \bar{\varphi} = \mathrm{id}_{V \otimes U} \qquad \bar{\varphi} \circ \bar{\bar{\otimes}} = \mathrm{id}_W
\tag{15.2.14}
$$

and the two maps $\bar{\bar{\otimes}}$ and φ are one the inverse of the other. Thus $V \otimes U \simeq W$.

Let us now show that

$$
T^p_q(V) \otimes T^{p'}_{q'}(V) \simeq T^{p+p'}_{q+q'}(V) \qquad \otimes : T^p_q(V) \times T^{p'}_{q'}(V) \to T^{p+p'}_{q+q'}(V) : (t, r) \mapsto t \otimes r
\tag{15.2.15}
$$

Let us consider a vector space Z with a basis e_i and the basis

$$\left(dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}, \, dx^{\rho_1} \otimes \ldots \otimes dx^{\rho_{p'}} \otimes \partial_{\sigma_1} \otimes \ldots \otimes \partial_{\sigma_{q'}} \right) \tag{15.2.16}$$

of $T_q^p(V) \times T_{q'}^{p'}(V)$. Any map $f : T_q^p(V) \times T_{q'}^{p'}(V) \to Z$ is given by setting

$$f\left(dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q}, \, dx^{\rho_1} \otimes \ldots \otimes dx^{\rho_{p'}} \otimes \partial_{\sigma_1} \otimes \ldots \otimes \partial_{\sigma_{q'}} \right) = f_{\nu_1 \ldots \nu_q \sigma_1 \ldots \sigma_{q'}}^{\mu_1 \ldots \mu_p \rho_1 \ldots \rho_{p'} \, i} e_i \tag{15.2.17}$$

Then we can define the map $\bar{f} : T_q^p(V) \otimes T_{q'}^{p'}(V) \to Z$ by

$$\bar{f}\left(dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \otimes dx^{\rho_1} \otimes \ldots \otimes dx^{\rho_{p'}} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q} \otimes \partial_{\sigma_1} \otimes \ldots \otimes \partial_{\sigma_{q'}} \right) = f_{\nu_1 \ldots \nu_q \sigma_1 \ldots \sigma_{q'}}^{\mu_1 \ldots \mu_p \rho_1 \ldots \rho_{p'} \, i} e_i \tag{15.2.18}$$

which in fact has the property $\bar{f} \circ \otimes = f$ and such a map is unique. Thus we show that $T_q^p(V) \otimes T_{q'}^{p'}(V) \simeq T_{q+q'}^{p+p'}(V)$.

In particular, we have, for example, that

$$T_1^1(V) = T_0^1(V) \otimes T_1^0(V) = V^* \otimes V \tag{15.2.19}$$

Changing the basis

If we change the basis in V by $\partial'_\mu = \bar{J}_\mu^\nu \partial_\nu$ the corresponding dual basis changes as $\partial'^\mu = J_\nu^\mu \partial^\nu$. Then the elements of the basis change as

$$\partial'^{\mu_1} \otimes \ldots \otimes \partial'^{\mu_p} \otimes \partial'_{\nu_1} \otimes \ldots \otimes \partial'_{\nu_q} = J_{\rho_1}^{\mu_1} \ldots J_{\rho_p}^{\mu_p} \bar{J}_{\nu_1}^{\sigma_1} \ldots \bar{J}_{\nu_q}^{\sigma_q} \partial^{\rho_1} \otimes \ldots \otimes \partial^{\rho_p} \otimes \partial_{\sigma_1} \otimes \ldots \otimes \partial_{\sigma_q} \tag{15.2.20}$$

and the components of the tensor change as

$$t'^{\sigma_1 \ldots \sigma_q}_{\rho_1 \ldots \rho_p} = \bar{J}_{\rho_1}^{\mu_1} \ldots \bar{J}_{\rho_p}^{\mu_p} J_{\nu_1}^{\sigma_1} \ldots J_{\nu_q}^{\sigma_q} t^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \tag{15.2.21}$$

Notice as these transformation rules follow the prescription of Jacobians for up indices and anti Jacobians for down indices.

Let us stress that if the tensor $t = 0$ then in one basis the components $t^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} = 0$ vanish. Then the components $t'^{\sigma_1 \ldots \sigma_q}_{\rho_1 \ldots \rho_p}$ vanish in any basis. This means that setting a tensor to zero does not depend on the basis (and when charts are indentifyed with observers the equation $t = 0$ has an absolute meaning, i.e. it is independent of the basis).

Tensors on a manifold

As for covectors, notation is ready to define tensors at a point x on a manifold M by just taking $V = T_x M$ and, consequently, by interpreting the matrix J as Jacobians of coordinate changes.

The tensor spaces $T_q^p(T_x M)$ are also denoted by $T_q^p(M)_x$ and the corresponding tensor algebra is denoted by $T_*^*(M)_x$. Then we can define the tensor spaces $T_q^p(M)$ and $T_*^*(M)$ by a disjoint union over to point on M

$$T_q^p(M) = \coprod_{x \in M} T_q^p(M)_x \qquad\qquad T_*^*(M) = \coprod_{x \in M} T_*^*(M)_x \tag{15.2.22}$$

The sets $T_q^p(M)$ with natural coordinates $(x^\mu, t_{\mu_1...\mu_p}^{\nu_1...\nu_q})$ and transformations rules

$$\begin{cases} x'^\mu = x'^\mu(x) \\ t'^{\sigma_1...\sigma_q}_{\rho_1...\rho_p} = \bar{J}^{\mu_1}_{\rho_1} \ldots \bar{J}^{\mu_p}_{\rho_p} J^{\sigma_1}_{\nu_1} \ldots J^{\sigma_q}_{\nu_q} t^{\nu_1...\nu_q}_{\mu_1...\mu_p} \end{cases} \tag{15.2.23}$$

are endowed with the structure of smooth manifolds whenever M is smooth.

As for covectors and vectors we can define the *projection map*

$$\pi : T_q^p(M) \to M : (x, t) \mapsto x \tag{15.2.24}$$

Tensor fields

As we defined the Lie algebra $\mathfrak{X}(M)$ of vector fields over M and 1-forms $\Omega^1(M)$, we can define *tensor fields*. A tensor field of rank (p, q) is a map $T : M \to T_q^p(M)$ such that $\pi \circ T = \mathrm{id}_M$. Then $T(x)$ is a tensor of rank (p, q) at x and a tensor field is assigning a tensor at any point of M.

Locally, a tensor field is represented by

$$T = T^{\nu_1...\nu_q}_{\mu_1...\mu_p}(x) dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q} \tag{15.2.25}$$

and $T^{\nu_1...\nu_q}_{\mu_1...\mu_p}(x)$ are called the *component functions*.

The set of all tensor fields of rank (p, q) on M will be denoted by $\mathcal{T}_q^p(M)$. It is obviously a (real) vector space since tensor fields can be added and multiplied by a number.

Let us consider a tensor field $T \in \mathcal{T}_1^2(M)$ locally expressed as

$$T = T^\rho_{\mu\nu}(x) dx^\mu \otimes dx^\nu \otimes \partial_\rho \tag{15.2.26}$$

We can associate to T an $\mathcal{F}(M)$-multilinear map $\hat{T} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Omega^1(M) \to \mathcal{F}(M)$ defined by

$$\hat{T}(X, Y, \omega) = T^\rho_{\mu\nu} X^\mu Y^\nu \omega_\rho \tag{15.2.27}$$

More generally, tensor fields are often defined also as $\mathcal{F}(M)$-multilinear map $\hat{T} : (\mathfrak{X}(M))^p \times (\Omega^1(M))^q \to \mathcal{F}(M)$.

The spaces of tensor fields and $\mathcal{F}(M)$-multilinear maps are canonically isomorphic and they will be identified. Accordingly, we shall drop the hat sign.

We can use transformation laws to define projections of a sheaf, which uses components to locally represent tensor in a chart doamin. As we already discussed for vectors and covectors, the sheaf formulation is equivalent to the intrinsic formulation

Let us finally provide an example of tensor field. One can assign an inner product in T_xM at each point $x \in M$. This is given by a symmetric, non-degenerate, definite positive, tensor field or rank $(2,0)$

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu \tag{15.2.28}$$

which is called a (strictly Riemannian) *metric* on M. Being the matrix $g_{\mu\nu}$ non-degenerate, it allows a pointwise inverse (denoted by $g^{\mu\nu}(x)$) which defines a tensor field of rank $(0,2)$

$$\bar{g} = g^{\mu\nu}(x)\, \partial_\mu \otimes \partial_\nu \tag{15.2.29}$$

which, by the way, pointwise defines an inner product on covectors.

The non-canonical isomorphisms between V and V^* discussed above, can be viewed in this context as tensor fields. We can define $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ and $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ by

$$X^\flat = g(X,\cdot) = X^\mu g_{\mu\nu} dx^\nu \qquad \omega^\sharp = \bar{g}(\omega,\cdot) = \omega_\mu g^{\mu\nu} \partial_\nu \tag{15.2.30}$$

Canonical isomorphisms

Let us consider the set of linear maps $\phi : V \to V$, which is denoted by $\mathrm{End}(V)$. We can canonically associate a $(1,1)$-tensor $\hat{\phi}$ to the endomorphism ϕ by

$$\hat{\phi}(v,\alpha) = \alpha(\phi(v)) \tag{15.2.31}$$

The map $\hat{\phi}$ is bilinear

$$\begin{aligned}
\hat{\phi}(\lambda v + \mu w, \alpha) &= \alpha(\phi(\lambda v + \mu w)) = \alpha(\lambda\phi(v)) + \mu\phi(w)) = \lambda\alpha(\phi(v)) + \mu\alpha(\phi(w)) = \lambda\hat{\phi}(v,\alpha) + \mu\hat{\phi}(w,\alpha) \\
\hat{\phi}(v, \lambda\alpha + \mu\beta) &= (\lambda\alpha + \mu\beta)(\phi(v)) = \lambda\alpha(\phi(v)) + \mu\beta(\phi(v)) = \lambda\hat{\phi}(v,\alpha) + \mu\hat{\phi}(v,\beta)
\end{aligned} \tag{15.2.32}$$

The inverse map associates an endomorphism ϕ to a $(1,1)$-tensor $\hat{\phi}$

$$\alpha(\phi(v)) = \hat{\phi}(v,\alpha) \tag{15.2.33}$$

One should prove that there is one and only one endomorphism ϕ obeying this property $\forall \alpha \in V^*$.

The two maps are inverse to each other and hence both one-to-one. They are also linear maps.

In fact

$$(\lambda\phi + \mu\psi)\widehat{}(v,\alpha) = \alpha((\lambda\phi + \mu\psi)(v)) = \lambda\alpha(\phi(v)) + \mu\alpha(\psi(v)) = \lambda\hat{\phi}(v,\alpha) + \mu\hat{\psi}(v,\alpha) \tag{15.2.34}$$

Accordingly, $T^1_1(V)$ and $\mathrm{End}(V)$ are canonically isomorphic.

Similarly, one has canonical isomorphisms between

$$
\begin{array}{ccc}
\{\phi : V \to V^*\} & \longleftrightarrow & \{\phi : V \times V \to \mathbb{R}\} \\
\{\phi : V^* \to V\} & \longleftrightarrow & \{\phi : V^* \times V^* \to \mathbb{R}\} \\
\{\phi : V^* \to V^*\} & \longleftrightarrow & \{\phi : V^* \times V \to \mathbb{R}\} \\
& \cdots & \\
\{\phi : V \times V \to V \times V^*\} & \longleftrightarrow & \{\phi : V \times V \times V \times V^* \to \mathbb{R}\}
\end{array}
\tag{15.2.35}
$$

Accordingly, in view of these canonical isomorphisms tensors do not only represent multilinear forms but also any linear map among products of V and V^*. Similarly, tensor fields represent $\mathcal{F}(M)$-linear maps among products of vector fields and 1-forms.

3. Tensor densities

Tensors are not the only objects we shall use. The components of a tensor transform under change of coordinates as

$$
t'^{\sigma_1\ldots\sigma_q}_{\rho_1\ldots\rho_p} = \bar{J}^{\mu_1}_{\rho_1}\ldots\bar{J}^{\mu_p}_{\rho_p} J^{\sigma_1}_{\nu_1}\ldots J^{\sigma_q}_{\nu_q} t^{\nu_1\ldots\nu_q}_{\mu_1\ldots\mu_p}
\tag{15.3.1}
$$

Let us define a *tensor density* of rank (p, q) and weight $w \in \mathbb{R}$ as an object with components with p indices down and q indices up which transform under change of coordinates as

$$
t'^{\sigma_1\ldots\sigma_q}_{\rho_1\ldots\rho_p} = (\det J)^{-w} \bar{J}^{\mu_1}_{\rho_1}\ldots\bar{J}^{\mu_p}_{\rho_p} J^{\sigma_1}_{\nu_1}\ldots J^{\sigma_q}_{\nu_q} t^{\nu_1\ldots\nu_q}_{\mu_1\ldots\mu_p}
\tag{15.3.2}
$$

Notice that also setting a tensor density to zero is something which is independent of the chart.

Tensor densities naturally arise from tensors. If $g = g_{\mu\nu}\, dx^\mu \otimes dx^\mu$ is a bilinear form (or, in particular, a metric) the determinant $\det(g)$ transforms as

$$
\det(g') = (\det(J))^{-2}\det(g)
\tag{15.3.3}
$$

so it is a scalar density of weight 2. On an orientable manifold, one can restrict to oriented atlases so that $\det(J) > 0$. In that case, we set $\sqrt{g}$ for the square root of the absolute value of the determinant of the matrix $g_{\mu\nu}$ which transforms as

$$
\sqrt{g'} = (\det(J))^{-1}\sqrt{g}
\tag{15.3.4}
$$

which is a scalar density of weight 1.

Of course, on a manifold with a metric g, one can convert any tensor density of weight w to a tensor by multiplication by $(\sqrt{g})^{-w}$ and vice versa.

Once again, we can define a sheaf which locally represents tensor densities as we did for tensors, but with a different restriction which accounts for the different glueing conditions.

Levi Civita tensor densities

On any manifold M of dimension m, one can define two canonical tensor densities $\epsilon^{\mu_1\ldots\mu_m}$ and $\epsilon_{\mu_1\ldots\mu_m}$ defined to be the sign $(-1)^{\sharp(\sigma)}$ of the permutation $\sigma \in \Pi_m$ which brings $(\mu_1 \ldots \mu_m)$ to $(1 \ldots m)$. They are called *Levi Civita tensor densities* (we shall see that they have different weights).

> It is to be understood that Levi Civita tensor densities have the same values in all charts, otherwise they would select a preferred class of observers for which their value is ± 1.

Let $P = P^\mu_\rho \, dx^\rho \otimes \partial_\mu$ be a tensor in $T^1_1(V)$ (let it be $\dim(V) = n$). This can be associated with a matrix

$$P^\mu_\rho \longrightarrow \begin{pmatrix} P^1_1 & P^1_2 & \ldots & P^1_n \\ P^2_1 & P^2_2 & \ldots & P^2_n \\ \vdots & \vdots & & \vdots \\ P^n_1 & P^n_2 & \ldots & P^n_n \end{pmatrix} =: P \tag{15.3.5}$$

Then the *determinant* of P is defined to be

$$\det(P) = \epsilon_{\mu_1\ldots\mu_n} P^{\mu_1}_1 \ldots P^{\mu_n}_n = \tfrac{1}{n!}\epsilon_{\mu_1\ldots\mu_n} P^{\mu_1}_{\nu_1} \ldots P^{\mu_n}_{\nu_n} \epsilon^{\nu_1\ldots\nu_n} \tag{15.3.6}$$

Notice that the determinant of a $(1,1)$-tensor is a scalar and its value does not depend on the chart.

> Let us start with $n = 2$. In dimension $n = 2$ we have

$$\det(P) = \epsilon_{\mu\nu} P^\mu_1 P^\nu_2 = P^1_1 P^2_2 - P^2_1 P^1_2 = \det\begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix} \tag{15.3.7}$$

> In dimension $n = 3$ we have

$$\det(P) = \epsilon_{\mu\nu\rho} P^\mu_1 P^\nu_2 P^\rho_3 = \epsilon_{1\nu\rho} P^1_1 P^\nu_2 P^\rho_3 + \epsilon_{2\nu\rho} P^2_1 P^\nu_2 P^\rho_3 + \epsilon_{3\nu\rho} P^3_1 P^\nu_2 P^\rho_3 =$$
$$= \epsilon_{1\nu\rho} P^1_1 P^\nu_2 P^\rho_3 - \epsilon_{\nu 2\rho} P^2_1 P^\nu_2 P^\rho_3 + \epsilon_{\nu\rho 3} P^3_1 P^\nu_2 P^\rho_3 = P^\mu_1 \Delta^1_\mu \tag{15.3.8}$$

> where we set

$$\Delta^1_1 := \epsilon_{1\nu\rho} P^\nu_2 P^\rho_3 \qquad \Delta^1_2 := -\epsilon_{\nu 2\rho} P^\nu_2 P^\rho_3 \qquad \Delta^1_3 := \epsilon_{\nu\rho 3} P^\nu_2 P^\rho_3 \tag{15.3.9}$$

> for the *algebraic complements*. This provides the Laplace rule for the expansion of the determinant with respect to the first column.
> In higher dimension, the proof is similar when one defines the algebraic complement of the element $(1,k)$ as

$$\Delta^1_k = \epsilon_{k\mu_1\ldots\mu_{k-1}\mu_{k+1}\ldots\mu_n} P^{\mu_1}_2 \ldots P^{\mu_{k-1}}_k P^{\mu_{k+1}}_{k+1} \ldots P^{\mu_n}_n \tag{15.3.10}$$

Since the determinant is associated to a matrix and a matrix can be also associated to tensors $T^2_0(V)$ (as well as to tensors in $T^0_2(V)$) the determinant is extended to these cases. If $A = A_{\mu\rho} \, dx^\mu \otimes dx^\rho$ ($B = B^{\mu\rho}\partial_\mu \otimes \partial_\rho$) is a tensor of rank $(2,0)$ (or $(0,2)$, respectively) then it is associated to a matrix

$$A_{\mu\rho} \longrightarrow \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} =: A \qquad B^{\mu\rho} \longrightarrow \begin{pmatrix} B^{11} & B^{12} & \ldots & B^{1n} \\ B^{21} & B^{22} & \ldots & B^{2n} \\ \vdots & \vdots & & \vdots \\ B^{n1} & B^{n2} & \ldots & B^{nn} \end{pmatrix} =: B \tag{15.3.11}$$

and the determinant is defined to be

$$\det(A) = \epsilon^{\mu_1 \dots \mu_n} A_{\mu_1 1} \dots A_{\mu_n n} \qquad\qquad \det(B) = \epsilon_{\mu_1 \dots \mu_n} B^{\mu_1 1} \dots B^{\mu_n n} \tag{15.3.12}$$

Let us remark that $\det(A)$ is a scalar density of weight 1, while $\det(B)$ is a scalar density of weight -1.

> Let us stress that the convention (15.3.5) to put components of a tensor of rank $(1,1)$ in a matrix and the conventions (15.3.11) to put components of a tensor of rank $(2,0)$ and $(0,2)$ in a matrix are independent notation.
> We chose the notation so that the matrix products read as
>
> $$PQ = P^\mu_\lambda Q^\lambda_\rho \quad AP = A_{\mu\lambda}P^\lambda_\rho \quad PB = P^\mu_\lambda B^{\lambda\rho} \quad AB = A_{\mu\lambda}B^{\lambda\rho} \quad BA = B^{\mu\lambda}A_{\lambda\rho} \tag{15.3.13}$$
>
> and
>
> $$^t PA = P^\lambda_\mu A_{\lambda\rho} \quad B\,^t P = B^{\mu\lambda}P^\rho_\lambda \tag{15.3.14}$$
>
> This is not that important since we are aiming to replace matrix formalism with index formalism. However, sometimes we shall like to compare index formula with well-known formula in matrix notation. For example, we would like to write transformation rules of a bilinear form
>
> $$A'_{\mu\nu} = \bar{J}^\rho_\mu A_{\rho\sigma} \bar{J}^\sigma_\nu \qquad \Longleftrightarrow \quad A' = {}^t\bar{J} A \bar{J} \tag{15.3.15}$$
>
> to recognize its well-known matrix form. [Just remember that usually in linear algebra the transition matrix P is defined to be $\bar{J}$ in our notation.]
> Let us finally remark that we did not even provide a notation for the matrix product AA or BB since they would imply a summation along two indices in the same position.

Let us now consider the transformation rules of Levi Civita tensor densities. We have

$$\begin{aligned}
\epsilon'_{\mu_1 \dots \mu_m} &= \det(J)\bar{J}^{\nu_1}_{\mu_1} \dots \bar{J}^{\nu_m}_{\mu_m}\epsilon_{\nu_1 \dots \nu_m} = \det(J)\det(\bar{J})\epsilon_{\mu_1 \dots \mu_m} = \epsilon_{\mu_1 \dots \mu_m} \\
\epsilon'^{\mu_1 \dots \mu_m} &= \det(\bar{J})J^{\mu_1}_{\nu_1} \dots J^{\mu_m}_{\nu_m}\epsilon^{\nu_1 \dots \nu_m} = \det(\bar{J})\det(J)\epsilon^{\mu_1 \dots \mu_m} = \epsilon^{\mu_1 \dots \mu_m}
\end{aligned} \tag{15.3.16}$$

which shows that only by assuming they are densities of the correct weight, then their value does not depend on the chart and one can say that $\epsilon_{1\dots m} = 1$ in any chart.

There are a lot of other important identities on contractions of the Levi Civita tensor densities. We shall here present a systematic way of proving them. Let us start from dimension 3. We shall denote by ϵ_{ijk} the Levi Civita tensor densities in dimension 3. One can also define the controvariant version ϵ^{ijk}. Indices are raised and lowered by a metric (call it g_{ij}). If one raises all indices of ϵ_{ijk}, what is obtained is not ϵ^{hlm} but it is denoted by $\epsilon^{hl\,m}_{\cdot\cdot\cdot}$. There is a simple relation between ϵ^{hlm} and $\epsilon^{hl\,m}_{\cdot\cdot\cdot}$, that is

$$\epsilon^{hl\,m}_{\cdot\cdot\cdot} = g^{hi}g^{lj}g^{mk}\epsilon_{ijk} = \det{}^{-1}(g)\epsilon^{hlm} \tag{15.3.17}$$

Hence, unless $\det(g) = 1$, raising and lowering need to be explicitly traced. Analogously, one has

$$\epsilon^{\cdot\cdot\cdot}_{hl\,m} = g_{hi}g_{lj}g_{mk}\epsilon^{ijk} = \det(g)\epsilon_{hlm} \tag{15.3.18}$$

The next fundamental identity is

$$\epsilon^{ijk}\epsilon_{lmn} = 3!\delta^{[i}_l \, \delta^j_m \, \delta^{k]}_n \tag{15.3.19}$$

This can be proven by simply fixing the free indices to some value (the right hand side expands to a sum of products of deltas with permutation sign; if an index is repeated then no term survive due to antisymmetry; if no repetition is present, just one term survive on the left; by evaluating the deltas, one gets exactly the sign of the permutation between (ijk) and (lmn)). There the square braces denotes complete anti-symmetrisation (who is not familiar with it can see below).

Form this fundamental identity, one can obtain by contraction the other relevant formulae.

$$\epsilon^{ijk}\epsilon_{imn} = 3!\tfrac{1}{3}(\delta^i_i \, \delta^{[j}_m \, \delta^{k]}_n + \delta^j_i \, \delta^{[k}_m \, \delta^{i]}_n + \delta^k_i \, \delta^{[i}_m \, \delta^{j]}_n) = 2!(3\delta^{[j}_m \, \delta^{k]}_n + \delta^{[k}_m \, \delta^{j]}_n + \delta^{[k}_m \, \delta^{j]}_n) = 2!\delta^{[j}_m \, \delta^{k]}_n \tag{15.3.20}$$

$$\epsilon^{ijk}\epsilon_{ijn} = 2!\tfrac{1}{2}(\delta^j_j \, \delta^k_n - \delta^k_j \, \delta^j_n) = 3\delta^k_n - \delta^k_n = 2\delta^k_n \tag{15.3.21}$$

$$\epsilon^{ijk}\epsilon_{ijk} = 2\delta^k_k = 3! \tag{15.3.22}$$

We shall denote by $\epsilon_{\alpha\beta\gamma\delta}$ the Levi Civita tensor densities in dimension four. One can also define the contravariant version $\epsilon^{\alpha\beta\gamma\delta}$.

Indices are raised and lowered by a metric (call it $g_{\alpha\beta}$). If one raises all indices of $\epsilon_{\alpha\beta\gamma\delta}$, what is obtained is not $\epsilon^{\mu\nu\rho\sigma}$ but it is denoted by $\epsilon^{\mu\nu\rho\sigma}_{\cdots\cdots}$. There is a simple relation between $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}_{\cdots\cdots}$, that is

$$\epsilon^{\mu\nu\rho\sigma}_{\cdots\cdots} = g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\epsilon_{\alpha\beta\gamma\delta} = \det{}^{-1}(g)\epsilon^{\mu\nu\rho\sigma} \tag{15.3.23}$$

Hence unless $\det(g) = 1$ (e.g. we are working in orthonormal coordinates) raising and lowering need to be explicitly traced. Analogously, one has

$$\epsilon^{\cdots\cdots}_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\epsilon^{\alpha\beta\gamma\delta} = \det(g)\epsilon_{\mu\nu\rho\sigma} \tag{15.3.24}$$

The next fundamental identity is

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = 4!\delta^{[\mu}_\alpha \, \delta^\nu_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta \tag{15.3.25}$$

This can be proven by simply fixing the free indices to some value (the right hand side expands to a sum of products of deltas with permutation sign; if an index is repeated then no term survive due to antisymmetry; if no repetition is present, just one term survive on the left; by evaluating the deltas, one gets exactly the sign of the permutation between $(\mu\nu\rho\sigma)$ and $(\alpha\beta\gamma\delta)$).

Form this fundamental identity, one can obtain by contraction the other relevant formulae.

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta} = 4!\tfrac{1}{4}(\delta^\mu_\mu \, \delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta - \delta^\nu_\mu \, \delta^{[\mu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta - \delta^\rho_\mu \, \delta^{[\nu}_\beta \, \delta^\mu_\gamma \, \delta^{\sigma]}_\delta - \delta^\sigma_\mu \, \delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\mu]}_\delta) = 3!(4\delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta - \delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta - \delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta - \delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta) = 3!\delta^{[\nu}_\beta \, \delta^\rho_\gamma \, \delta^{\sigma]}_\delta \tag{15.3.26}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\gamma\delta} = 3!\tfrac{1}{3}(\delta^\nu_\nu \, \delta^{[\rho}_\gamma \, \delta^{\sigma]}_\delta - \delta^\rho_\nu \, \delta^{[\nu}_\gamma \, \delta^{\sigma]}_\delta - \delta^\sigma_\nu \, \delta^{[\rho}_\gamma \, \delta^{\nu]}_\delta = 2!(4\delta^{[\rho}_\gamma \, \delta^{\sigma]}_\delta - \delta^{[\rho}_\gamma \, \delta^{\sigma]}_\delta - \delta^{[\rho}_\gamma \, \delta^{\sigma]}_\delta = 2!2!\delta^{[\rho}_\gamma \, \delta^{\sigma]}_\delta \tag{15.3.27}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\delta} = 2!2!\delta^{[\rho}_\rho \, \delta^{\sigma]}_\delta = 2!(\delta^\rho_\rho \, \delta^\sigma_\delta - \delta^\sigma_\rho \, \delta^\rho_\delta) = 2!(4\delta^\sigma_\delta - \delta^\sigma_\delta) = 3!\delta^\sigma_\delta \tag{15.3.28}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = 3!\delta^\sigma_\sigma = 4! \tag{15.3.29}$$

Thus, in arbitrary dimension m, one has $\epsilon^{\mu_1 \ldots \mu_m}$ and $\epsilon_{\nu_1 \ldots \nu_m}$ which can be contracted on k pair of indices to obtain:

$$\epsilon^{\mu_1 \ldots \mu_{m-k} \lambda_1 \ldots \lambda_k} \epsilon_{\nu_1 \ldots \nu_{m-k} \lambda_1 \ldots \lambda_k} = (m-k)! k! \delta^{\mu_1}_{[\nu_1} \ldots \delta^{\mu_{m-k}}_{\nu_{m-k}]} \tag{15.3.30}$$

We can also obtain the formula for taking the derivative of a determinant. Let us suppose we have an (invertible) matrix $A_{\mu\nu}(x)$ and a derivative d (it does not matter what exactly the derivative is since we shall only use Leibniz rule).

$$d(\det(A)) = d\left(\epsilon^{\mu_1 \ldots \mu_m} A_{\mu_1 1} \ldots A_{\mu_m m}\right) = \epsilon^{\mu_1 \ldots \mu_m} dA_{\mu_1 1} \ldots A_{\mu_m m} + \ldots + \epsilon^{\mu_1 \ldots \mu_m} A_{\mu_1 1} \ldots dA_{\mu_m m} \tag{15.3.31}$$

Thus we obtain m determinants in each of which one column has beed derived. Each determinant can be expanded with Laplace rule with respect to the column which has been derived

$$d(\det(A)) = dA_{\mu 1} \Delta^{\mu 1} + \ldots + \Delta^{\mu m} dA_{\mu m} = \Delta^{\mu\nu} dA_{\mu\nu} \tag{15.3.32}$$

where $\Delta^{\mu\nu}$ denotes the algebraic complement to the element $(\mu\nu)$. We know that algebraic complement enters the formula for the inverse matrix

$$\bar{A}^{\mu\nu} = \frac{1}{\det(A)} \Delta^{\mu\nu} \qquad \Rightarrow \Delta^{\mu\nu} = \det(A) \bar{A}^{\mu\nu} \tag{15.3.33}$$

so that one has

$$d(\det(A)) = \det(A) \bar{A}^{\mu\nu} dA_{\mu\nu} \tag{15.3.34}$$

Thus the derivative of the determinant is computed algebraically (as expected since the determinant is a polynomial) essentially (i.e. modulo the numeric factor $\det A$)) by taking the trace of the product of the inverse matrix $\bar{A}$ by the original matrix where all the entries have been derived dA.

One can repeat the proof for the derivative of a $(1, 1)$-tensor, to obtain

$$d(\det(J)) = \det(J) \bar{J}^{\mu}_{\nu} dJ^{\nu}_{\mu} \tag{15.3.35}$$

4. Pull-back and push-forward

Tensor fields can be dragged along smooth maps. In general we call *push-forward* when a tensor field is dragged in the same direction of the map, while we call *pull-back* when the dragging is in the opposite direction. Usually, we denote by Φ^* the pull-back along a map Φ, while we denote by Φ_* the push-forward. As a general result, all tensor fields can be both pulled back and pushed forward along diffeomorphisms. Depending on the rank, tensor fields are more or less friendly to push-forward or pull-back when the map is not a diffeomorphism.

Pulling back or pushing forward along the identity does not change the tensor field. If one has the composition of two maps the push-forward preserves the composition while the pull-back inverts the composition.

Functions, *0*-forms

Let us consider two manifolds M and N with a smooth map $\Phi : M \to N$.

Given a function $F : N \to \mathbb{R} \in \mathcal{F}(N)$, one can define a function $\Phi^*F := F \circ \Phi : M \to \mathbb{R}$ which is called the *pull-back* of F along the map Φ.

$$
\begin{array}{ccc}
 & & \mathbb{R} \\
 & \nearrow \quad \nearrow & \\
\Phi^*F \dashrightarrow & F & \\
M \xrightarrow{\quad \Phi \quad} N & &
\end{array}
\tag{15.4.1}
$$

Let x^μ be local coordinates on M around a point x, and y^i coordinates around $y = \Phi(x)$. The local expression of the map Φ reads as $\Phi : x^\mu \mapsto y^i(x)$.

> To be precise, let us assume that we restricted the domain U of coordinate x^μ and the domain V of coordinate y^i so that one has $\Phi(U) = V$.
> Also notice that formally the local expression is similar to a set of transition functions $x'^\mu = x'^\mu(x)$, just with the new coordinates called y^i. Just the meaning of symbols are different; here x^μ and y^i are coordinates of different points in different spaces, while in transition functions x'^μ and x^μ are different names of the same point.

The local expression of F is $F(y)$ and the pull-back is locally given by

$$
\Phi^*F : M \to \mathbb{R} : x \mapsto F(y(x))
\tag{15.4.2}
$$

Accordingly, pulling back a function amounts to just compute it $\Phi^*F(x) = F(y(x)) = F \circ \Phi$ along the new coordinates.

> Except for understanding the pull-back, this is quite similar to the relations between two local expressions in two different charts $F(x') = F(x'(x))$. Again, same formula with two different meanings.

Let us stress that *any* function can be pulled back along *any* map. On the contrary, this is not the case for push-forward. Given a function on M there is no general way to define a function on N. For pushing forward functions, one has to either restrict functions or restrict maps.

> Pushing forward have to overcome two problems. If the map Φ is not surjective, then one has no clue on the value of a candidate push-forward function on points which are not in the image of Φ.
> Second issue is when the map is not injective; if there are different points with the same image $\Phi(x_1) = \Phi(x_2) = y$ then in general $F(x_1) \neq F(x_2)$ and one cannot choose which is the value to be assigned at the point y.

It may happen that a function F on M is constant on level sets of the (surjective, possibly not injective) map Φ. In that case, the function F is called Φ-*projectable* and one can define the push-forward. Of course, only some functions are Φ-projectable.

Another strategy is restricting the map Φ so that one can push forward any function on M. For the map Φ must be a diffeomorphism and pushing forward a function on M along Φ is the same of pulling it back along the inverse map, i.e., for any $F : M \to \mathbb{R}$, one has $\Phi_*F := (\Phi^{-1})^*F$

Vector fields

Vector fields are operators on functions.

Let us consider $X \in \mathfrak{X}(M)$. The push-forward of X along the map $\Phi : M \to N$ is defined as the vector field $\Phi_*X \in \mathfrak{X}(N)$ which acts on a general function $F : N \to \mathbb{R}$ as

$$\Phi_*X(F) \circ \Phi = X(\Phi^*F) \qquad \Phi^* (\Phi_*X(F)) = X(\Phi^*F) \tag{15.4.3}$$

Locally, one has $X = X^\mu(x)\partial_\mu$ and $F(y)$, its pull-back being $\Phi^*F(x) = F(y(x))$. Then one has

$$X(\Phi^*F) = X^\mu(x)J^i_\mu(x)\, \partial_i F(y(x)) \tag{15.4.4}$$

where J^i_μ is the Jacobian of the map Φ. This could be seen as a vector field over N, if there exist functions $Y^i(y)$ such that $(\forall F \in \mathcal{F}(N))$

$$Y^i(y(x))\partial_i F(y(x)) = X^\mu(x)J^i_\mu(x)\, \partial_i F(y(x)) \qquad Y^i(y(x)) = X^\mu(x)J^i_\mu(x) \tag{15.4.5}$$

Unfortunately, if the map Φ is not surjective there is no clue about the value of Y^i on the points which are not in the image of Φ. We have to require Φ to be surjective.

Then, as for pulling back functions, we have two issues; if the map Φ is not injective, e.g. there are two points x_1 and x_2 such that $\Phi(x_1) = \Phi(x_2) = y$ then one, in general, cannot determine which value is to be assigned to $Y^i(y)$ unless the local component functions $X^\mu(x)J^i_\mu(x)$ are constant along the level sets of Φ. In that case, one can unambiguously define the components $Y^i(y)$ and the vector field X is called Φ-*projectable*.

Alternatively, the map Φ is injective, the level sets are made of only one point, and all vector fields are Φ-projectable. In this case, Φ is a diffeomorphism and one also has the inverse map $x(y)$ (by using Φ^{-1}) and

$$\Phi_*X = X^\mu(x(y))J^i_\mu(x(y))\, \partial_i \tag{15.4.6}$$

is a good vector field on N. Since Φ is invertible, one can also pull back a vector field $Y = Y^i(y)\partial_i \in \mathfrak{X}(N)$ as

$$\Phi^*Y = Y^i(y(x))\bar{J}^\mu_i(y(x))\partial_\mu \tag{15.4.7}$$

where $\bar{J}^\mu_i$ is the anti-Jacobian of the map Φ, i.e. the Jacobian of the inverse map $\Phi^{-1} : N \to M$.

One can show that the push-forward of the commutator of two vector fields is the commutator of their push-forward, i.e.

$$\Phi_*[X, Y] = [\Phi_*X, \Phi_*Y] \tag{15.4.8}$$

Let us fix a function F on N so that we can evaluate

$$\begin{aligned} \Phi^* (\Phi_*[X,Y](F)) =&[X,Y](\Phi^*F) = X(Y(\Phi^*F)) - Y(X(\Phi^*F)) = X (\Phi^* (\Phi_*Y(F))) - Y (\Phi^* (\Phi_*X(F))) = \\ =&\Phi^* (\Phi_*X (\Phi_*Y(F)) - \Phi_*Y (\Phi_*X(F))) = \Phi^* ([\Phi_*X, \Phi_*Y](F)) \end{aligned} \tag{15.4.9}$$

1-forms

We shall follow the same strategy for 1-forms, which are operators on vector fields. If $\omega \in \Omega^1(N)$ is a 1-form on N, we define the pull-back as the 1-form $\Phi^*\omega \in \Omega^1(M)$ such that for any vector field $X \in \mathfrak{X}(M)$ one has

$$\Phi^*\omega(X) = \omega(\Phi_*X) \tag{15.4.10}$$

Locally, we have

$$\omega(\Phi_*X) = \omega_i(y)X^\mu(x(y))J^i_\mu(x(y)) \qquad \Rightarrow \Phi^*\omega = \omega_i(y(x))J^i_\mu(x)\ dx^\mu \tag{15.4.11}$$

Notice that to define (15.4.11), one does not need the inverse of Φ. Accordingly, one can pull back any 1-form along any map. Of course, if Φ is a diffeomorphism then 1-forms can be pushed forward by pulling back along the inverse map.

Tensor fields

A tensor field of rank $(p,0)$ is called *controvariant*, while a tensor of rank $(0,q)$ is called *covariant*.

Notice that covariant (controvariant, respectively) tensors $T^*_0(V) = \oplus_p T^p_0(V)$ $(T^0_*(V) = \oplus_q T^0_q(V)$, respectively) forms a subalgebra of the tensor algebra. In fact, tensor products of covariant (controvariant, respectively) tensors are still covariant (controvariant, respectively).

We can define pull-back of a covariant tensor T on N as

$$\Phi^*T(X_1,\ldots,X_p) = \Phi^*\left[T(\Phi_*X_1,\ldots,\Phi_*X_p)\right] \tag{15.4.12}$$

Locally, one has

$$\Phi^*T = T_{i_1\ldots i_p}(y(x))J^{i_1}_{\mu_1}(x)\ldots J^{i_p}_{\mu_p}(x)\ dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \tag{15.4.13}$$

Notice that any covariant tensor can be pulled back along any map Φ. Of course, covariant tensors can be pushed forward along diffeomorphisms.

We can define push-forward of a controvariant tensor T on M as

$$\Phi^*\left[\Phi_*T(\omega_1,\ldots,\omega_q)\right] = T(\Phi^*\omega_1,\ldots,\Phi^*\omega_q) \tag{15.4.14}$$

Locally, one has

$$\Phi_*T = T^{\mu_1\ldots\mu_q}(x(y))J^{i_1}_{\mu_1}(x(y))\ldots J^{i_q}_{\mu_q}(x(y))\ \partial_{i_1} \otimes \ldots \otimes \partial_{i_q} \tag{15.4.15}$$

Notice that controvariant tensors can be pushed forward along diffeomorphisms Φ. Of course, they can be also pulled back.

For mixed tensors, we restrict to diffeomorpims.

The push-forward of a tensor field T of rank (p,q) on M is given by

$$\Phi^*\left[\Phi_*T(X_1,\ldots,X_p,\omega_1,\ldots,\omega_q)\right] = T(\Phi^*X_1,\ldots,\Phi^*X_p,\Phi^*\omega_1,\ldots,\Phi^*\omega_q) \tag{15.4.16}$$

while pull-back of a tensor field T of rank (p, q) on N is defined as

$$\Phi^* T(X_1, \ldots, X_p, \omega_1, \ldots, \omega_q) = \Phi^* \left[T(\Phi_* X_1, \ldots, \Phi_* X_p, \Phi_* \omega_1, \ldots, \Phi_* \omega_q) \right] \tag{15.4.17}$$

Locally, one has

$$\Phi_* T = T^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_p}(x(y)) J^{i_1}_{\mu_1}(x(y)) \ldots J^{i_q}_{\mu_q}(x(y)) \bar{J}^{\nu_1}_{j_1}(y) \ldots \bar{J}^{\nu_p}_{j_p}(y) \; dy^{j_1} \otimes \ldots \otimes dy^{j_p} \otimes \partial_{i_1} \otimes \ldots \otimes \partial_{i_q} \tag{15.4.18}$$

for push-forward, while one has

$$\Phi^* T = T^{j_1 \ldots j_q}_{i_1 \ldots i_p}(y(x)) J^{i_1}_{\mu_1}(x) \ldots J^{i_p}_{\mu_p}(x) \bar{J}^{\nu_1}_{j_1}(y(x)) \ldots \bar{J}^{\nu_q}_{j_q}(y(x)) \; dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_q} \tag{15.4.19}$$

for pull-back.

Usually, computing Jacobians and anti-Jacobians is quite tedious. For that reason, one usually computes push-forward and pull-back of tensor fields by following the following procedure:

- compute the components as functions of x or y as appropriate by using the expression of Φ or Φ^{-1}.

- act on the basis by transforming each vector or covector basis element. The action of maps of vectors are obtained by taking the partial derivatives of functions $\Phi_* F(x) = F(y(x))$ while the action of maps on covectors is found by differentiating both sides of the map expression $y^i = \Phi^i(x)$.

Tangent and cotangent maps

We can also try to use a map to induce single vectors instead of a whole vector field on new manifolds.

Let M and N be two manifolds, $\Phi : M \to N$ a map and $(x, v) \in TM$ a tangent vector to M. Since $v = [\gamma]$ is an equivalence class of curves based at x we can define a family of curves $\Phi \circ \gamma$ based at $\Phi(x) = y \in N$. Such a map on curves is compatible with the equivalence relation (if $\gamma \sim \gamma'$ then $\Phi \circ \gamma \sim \Phi \circ \gamma'$ since the map is smooth and preserved equality between derivatives) and it induces a map in the quotient

$$T\Phi : TM \to TN : (x, v = [\gamma]) \mapsto (\Phi(x), [\Phi \circ \gamma]) \tag{15.4.20}$$

which is called the *tangent map* of Φ. It is often useful to restrict the tangent map to vectors at x to obtain a map between tangent spaces $T_x \Phi : T_x M \to T_{\Phi(x)} N : v = [\gamma] \mapsto [\Phi \circ \gamma]$

In coordinates, the map reads as $\Phi : M \to N : x^\mu \mapsto y^i(x)$, the curve reads as $\gamma : s \mapsto \gamma(s)$. The tangent map then is given by

$$T_x \Phi : T_x M \to T_{\Phi(x)} N : v^\mu \partial_\mu \mapsto v^\mu J^i_\mu \partial_i \tag{15.4.21}$$

as one obtains by taking the tangent vector of the induced curve $\Phi \circ \gamma : s \mapsto y(\gamma(s))$.

Let us stress that the push-forward of a vector field when it is defined (i.e. when Φ is a diffeomorpism) is given by

$$(\Phi_* X)(y) = (T_x \Phi)(X(x)) \tag{15.4.22}$$

as one can check by comparing with (15.4.6). However, since we here are interested in mapping single vectors and not whole vector fields, the tangent map is defined along any smooth map.

If $\Phi : M \to M$ is the identity map then J_μ^i is the identity matrix and $T_x\Phi : T_xM \to T_xM$ is the identity. If we consider two maps $\Phi : M \to N$ and $\Psi : N \to Q$, then

$$T_x(\Psi \circ \Phi) = T_{\Phi(x)}\Psi \circ T_x\Phi \tag{15.4.23}$$

In fact, let z^A be coordinetes on Q, then one has

$$T_x(\Psi \circ \Phi)(v) = v^\mu J_\mu^A \partial_A = v^\mu J_i^A J_\mu^i \partial_A = T_{\Phi(x)}\Psi(v^\mu J_\mu^i \partial_i) = T_{\Phi(x)}\Psi \circ T_x\Phi(v) \tag{15.4.24}$$

as it follows from the fact that the Jacobian of a composition is the composition of Jacobians.

Similarly, we can define the *cotangent map* (or *transpose map*) as

$$T^*\Phi : T^*N \to T^*M : (\Phi(x), \alpha) \mapsto (x, T_{\Phi(x)}^*\Phi(\alpha)) \qquad \forall v \in TM : T^*\Phi(\alpha)(v) = \alpha(T_x\Phi(v)) \tag{15.4.25}$$

The map Φ has to be at least surjective, if one wants the map $T^*\Phi$ to be defined everywhere in its domain. It is often useful to restrict the cotangent map to covectors at $\Phi(x)$ to obtain a map between cotangent spaces

$$T_{\Phi(x)}^*\Phi : T_{\Phi(x)}^*N \to T_x^*M : \alpha \mapsto T^*\Phi(\alpha) \tag{15.4.26}$$

In coordinates, the map reads as $\Phi : M \to N : x^\mu \mapsto y^i(x)$. The cotangent map then is given by

$$T_{\Phi(x)}^*\Phi : T_{\Phi(x)}^*N \to T_x^*M : \alpha_i \, dy^i \mapsto \alpha_i J_\mu^i \, dx^\mu \tag{15.4.27}$$

as one obtains by considering the identity

$$\alpha(T_x\Phi(v)) = \alpha_i J_\mu^i v^\mu \tag{15.4.28}$$

Let us stress that the pull-back of a 1-form when it is defined (e.g. when Φ is a diffeomorpism) is given by

$$(\Phi^*\omega)(x) = \left(T_{\Phi(x)}^*\Phi\right)(\omega(\Phi(x))) \tag{15.4.29}$$

as one can check by comparing with (15.4.11).

If $\Phi : M \to M$ is the identity map then J_μ^i is the identity matrix and $T_x^*\Phi : T_x^*M \to T_x^*M$ is the identity. If we consider two maps $\Phi : M \to N$ and $\Psi : N \to Q$, then

$$T_{\Psi\circ\Phi(x)}^*(\Psi \circ \Phi) = T_{\Phi(x)}^*\Phi \circ T_{\Psi\circ\Phi(x)}^*\Psi \tag{15.4.30}$$

In fact, let z^A be coordinetes on Q, then one has

$$T_{\Psi\circ\Phi(x)}^*(\Psi \circ \Phi)(\alpha) = \alpha_A J_\mu^A \, dx^\mu = \alpha_A J_i^A J_\mu^i \, dx^\mu = T_{\Phi(x)}^*\Phi(\alpha_A J_i^A \, dy^i) = T_{\Phi(x)}^*\Phi \circ T_{\Psi\circ\Phi(x)}^*\Psi(\alpha) \tag{15.4.31}$$

as it follows from the fact that the Jacobian of a composition is the composition of Jacobians.

It is quite surprising how most of the properties involving (co)tangent maps can be proven just resorting to the behaviour of (co)tangent maps with respect to compositions.

The relation between the tangent map and the push-forward (as well as between the cotangent map and the pull-back) can be summarised by the following two commutative diagrams

$$\Phi_* X = T\Phi \circ X \circ \Phi^{-1} \qquad \Phi^* \boldsymbol{\omega} = T^*\Phi \circ \boldsymbol{\omega} \circ \Phi \tag{15.4.32}$$

5. Symmetrisation and antisymmetrisation

Within the subalgebra $T_*^0(V)$ of covariant tensors, one can define two important sets: *symmetric* and *antisymmetric* tensors.

> The same procedure can be used starting from controvariant tensors.
> Let us stress that (anti)symmetric tensors are not a subalgebra since tensor product does not preserve symmetry properties of tensors. As we shall see hereafter, one can define a new product which makes them algebras on their own, though not subalgebras of the tensor algebra.

Let us denote by Π_n the group of permutations of n elements and denote by $\sigma \in \Pi_n$ a permutation and by $(-1)^{\sharp(\sigma)}$ the sign of the permutation σ. A covariant tensor $t : V^p \to \mathbb{R}$ is *symmetric* if its value does not depend on the order of its argument, i.e. if for any $\sigma \in \Pi_p$ one has

$$t(v_1, \ldots, v_p) = t(\sigma(v_1, \ldots, v_p)) \tag{15.5.1}$$

> Any tensor of rank 1 (i.e. any covector) is symmetric (Π_1 contains only the identity).
> A tensor of rank 2 is symmetric if $t(v, w) = t(w, v)$, for all vectors v and w. This means that

$$t_{\mu\nu} v^\mu w^\nu = t_{\mu\nu} w^\mu v^\nu \quad \Rightarrow t_{\mu\nu} = t_{\nu\mu} \tag{15.5.2}$$

> and t is symmetric iff its components are symmetric.
> A tensor of rank 3 is symmetric if $t(v, u, w) = t(\sigma(v, u, w))$, i.e. if $\forall \sigma \in \Pi_3$

$$t_{\mu\nu\rho} = t_{\sigma(\mu\nu\rho)} \tag{15.5.3}$$

> and the components of t are symmetric.

A covariant tensor $t : V^p \to \mathbb{R}$ is *antisymmetric* if its value does not depend on the order of its argument modulo the sign of permutation, i.e. if for any $\sigma \in \Pi_p$ one has

$$t(v_1, \ldots, v_p) = (-1)^{\sharp(\sigma)} t(\sigma(v_1, \ldots, v_p)) \tag{15.5.4}$$

Any tensor of rank 1 (i.e. any covector) is antisymmetric (Π_1 contains only the identity).

A tensor of rank 2 is antisymmetric if $t(v, w) = -t(w, v)$, for all vectors v and w. This means that

$$t_{\mu\nu}v^{\mu}w^{\nu} = -t_{\mu\nu}w^{\mu}v^{\nu} \quad \Rightarrow t_{\mu\nu} = -t_{\nu\mu} \tag{15.5.5}$$

and t is antisymmetric iff its components are antisymmetric.

A tensor of rank 3 is antisymmetric if $t(v, u, w) = (-1)^{\sharp(\sigma)}t(\sigma(v, u, w))$, i.e. if $\forall \sigma \in \Pi_3$

$$t_{\mu\nu\rho} = (-1)^{\sharp(\sigma)}t_{\sigma(\mu\nu\rho)} \tag{15.5.6}$$

and the components of t are antisymmetric.

The set of all (covariant) symmettric tensors of rank p will be denoted by $S^p(V) \subset T_0^p(V)$, while the set of all (covariant) antisymmettric tensors of rank p will be denoted by $A^p(V) \subset T_0^p(V)$. Of course, both $S^p(V)$ and $A^p(V)$ are linear subspaces.

We can define two projectors

$$S : T_0^p(V) \to S^p(V) \qquad A : T_0^p(V) \to A^p(V) \tag{15.5.7}$$

For any (covariant) tensor t of rank p we set

$$S(t)(v_1, \ldots, v_p) = \frac{1}{p!}\sum_{\sigma \in \Pi_p} t(\sigma(v_1, \ldots, v_p)) \qquad A(t)(v_1, \ldots, v_p) = \frac{1}{p!}\sum_{\sigma \in \Pi_p} (-1)^{\sharp(\sigma)}t(\sigma(v_1, \ldots, v_p)) \tag{15.5.8}$$

They are called projectors since $S^2 = S$ and $A^2 = A$. In fact, when we apply S to a tensor which is already symmetric it leaves the tensor unchanged. Analogously, when we apply A to a tensor which is already antisymmetric it leaves the tensor unchanged.

Then one can define $S^p(V)$ to be the image of the projector S and $A^p(V)$ to be the image of the projector A.

As we said, any covector is symmetric and antisymmetric so that the action of the projectors of covectors is trivial. Let $t = t_{\mu\nu}\ dx^{\mu} \otimes\ dx^{\nu}$ be an arbitrary rank 2 tensor; the action of the projectors is defined by

$$S(t)(v, w) = \tfrac{1}{2}\left(t(v, w) + t(w, v)\right) = \tfrac{1}{2}\left(t_{\mu\nu} + t_{\nu\mu}\right)v^{\mu}w^{\nu} \qquad A(t)(v, w) = \tfrac{1}{2}\left(t(v, w) - t(w, v)\right) = \tfrac{1}{2}\left(t_{\mu\nu} - t_{\nu\mu}\right)v^{\mu}w^{\nu} \tag{15.5.9}$$

from which one obtains

$$S(t) = \tfrac{1}{2}\left(t_{\mu\nu} + t_{\nu\mu}\right)\ dx^{\mu} \otimes\ dx^{\nu} =: t_{(\mu\nu)}\, dx^{\mu} \otimes\ dx^{\nu} \qquad A(t)(v, w) = \tfrac{1}{2}\left(t_{\mu\nu} - t_{\nu\mu}\right)\ dx^{\mu} \otimes\ dx^{\nu} =: t_{[\mu\nu]}\, dx^{\mu} \otimes\ dx^{\nu} \tag{15.5.10}$$

Let us here extend Einstein conventions by adding a rule which prescribes that indices included in parentheses $(\ldots)$ are completely symmeterised

$$t_{(\mu_1\ldots\mu_k)} = \frac{1}{k!}\sum_{\sigma \in \Pi_k} t_{\sigma(\mu_1\ldots\mu_k)} \tag{15.5.11}$$

while indices included in square parentheses [...] are completely antisymmeterized

$$t_{[\mu_1 \dots \mu_k]} = \frac{1}{k!} \sum_{\sigma \in \Pi_k} (-1)^{\sharp(\sigma)} t_{\sigma(\mu_1 \dots \mu_k)} \tag{15.5.12}$$

If one wants to avoid (anti)symmetrisation of an index which happens to be in parentheses, then the corresponding index will be underlined, i.e. underlined indices do not participate to (anti)symmetrisation. For example,

$$t_{\alpha[\beta\underline{\mu\nu}\gamma]} = \tfrac{1}{2} \left(t_{\alpha\beta\mu\nu\gamma} - t_{\alpha\gamma\mu\nu\beta} \right) \tag{15.5.13}$$

In general, the action of A on a tensor t corresponds to a complete antisymmetrisation of the components of t, while the action of S corresponds to a complete symmetrisation. For example, for a rank three tensor $t = t_{\mu\nu\lambda} \ dx^\mu \otimes dx^\nu \otimes dx^\lambda$ one has

$$S(t) = t_{(\mu\nu\lambda)} \ dx^\mu \otimes dx^\nu \otimes dx^\lambda \qquad A(t) = t_{[\mu\nu\lambda]} \ dx^\mu \otimes dx^\nu \otimes dx^\lambda \tag{15.5.14}$$

with $t_{[\mu\nu\lambda]}$ and $t_{(\mu\nu\lambda)}$ containing 6 terms corresponding to the six permutations in Π_3.

Lemma (15.5.15): *whenever two symmetric indices in an object are contracted with the antisymmetric indices of another object, the product vanishes.*

Proof: *Let us consider an object $S_{\alpha\beta\dots}$ with two symmetric indices $(\alpha\beta)$, an object $A^{\alpha\beta\dots}$ with two symmetric indices $[\alpha\beta]$ and consider the product*

$$A^{\alpha\beta\dots} S_{\alpha\beta\dots} = A^{\beta\alpha\dots} S_{\beta\alpha\dots} = -A^{\alpha\beta\dots} S_{\alpha\beta\dots} \tag{15.5.16}$$

where, in the first step we just renamed dumb indices, while in the second step we used (anti)symmetries. The quantity $A^{\alpha\beta\dots} S_{\alpha\beta\dots}$ is equal to its opposite, then it is zero. ∎

Then using this Lemma we have

$$S(t) = t_{(\mu_1 \dots \mu_p)} \ dx^{(\mu_1} \otimes \dots \otimes dx^{\mu_p)} \qquad A(t) = t_{[\mu_1 \dots \mu_p]} \ dx^{[\mu_1} \otimes \dots \otimes dx^{\mu_p]} \tag{15.5.17}$$

Now we know how to modify the tensor product so that it preserves symmetry properties. The tensor product of two (anti)symmetric tensors is not (anti)symmetric. Then let us define two new products

$$\vee : S^p(V) \times S^{p'}(V) \to S^{p+p'}(V) \qquad \wedge : A^p(V) \times A^{p'}(V) \to A^{p+p'}(V) \tag{15.5.18}$$

defined by

$$t \vee r = (p + p')! \, S(t \otimes s) \qquad t \wedge r = (p + p')! \, A(t \otimes s) \tag{15.5.19}$$

The product $\wedge$ is also called *wedge product*. The product $\vee$ is called *symmetrised product* and it is also denoted by $\odot$.

Let us define $S^*(V) = \oplus_p S^p(V)$ and $A^*(V) = \oplus_p A^p(V)$ and extend on them the products by linearity. These two products endow $S^*(V)$ and $A^*(V)$ with the structure of algebra. They are called the algebra of *symmetric forms* and *antisymmetric forms*.

Then (anti)symmetric tensors can be expressed as

$$S(t) = \tfrac{1}{p!} t_{(\mu_1 \dots \mu_p)} \ dx^{\mu_1} \vee \dots \vee dx^{\mu_p} \qquad A(t) = \tfrac{1}{p!} t_{[\mu_1 \dots \mu_p]} \ dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{15.5.20}$$

The new products can be expanded as ordinary tensors as

$$dx^{\mu_1} \vee \ldots \vee dx^{\mu_p} = \sum_{\sigma \in \Pi_p} dx^{\sigma(\mu_1} \otimes \ldots \otimes dx^{\mu_p)} \qquad dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \sum_{\sigma \in \Pi_p} (-1)^{\sharp(\sigma)} dx^{\sigma(\mu_1} \otimes \ldots \otimes dx^{\mu_p)} \qquad (15.5.21)$$

which makes manifest the action on vectors.

Since both the maps $S : T_0^p(V) \to S^p(V)$ and $A : T_0^p(V) \to A^p(V)$ are surjective, the image of generators (in particular of the natural bases of tensor spaces) are generators. Thus we have that $dx^{\mu_1} \vee \ldots \vee dx^{\mu_p}$ are generators of $S^p(V)$ and $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ are generators of $A^p(V)$. However, they are not a basis since there are linear relations among generators.

Let us consider $S^3(V)$. We have for example

$$dx^\mu \vee dx^\nu \vee dx^\rho = dx^\mu \vee dx^\rho \vee dx^\nu = dx^\nu \vee dx^\rho \vee dx^\mu \qquad (15.5.22)$$

The generators are symmetric objects. Only generators with not decreasing ordered indices are independent. If $\dim(V) = 2$ then

$$dx^1 \vee dx^1 \vee dx^1, \quad dx^1 \vee dx^1 \vee dx^2, \quad dx^1 \vee dx^2 \vee dx^2, \quad dx^2 \vee dx^2 \vee dx^2 \qquad (15.5.23)$$

are independent and $\dim(S^3(V)) = 4$. Let us consider $A^2(V)$. We have, for example

$$dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu \qquad\qquad dx^1 \wedge dx^1 = 0 \qquad (15.5.24)$$

The generators are antisymmetric objects. Only generators with strictly growing ordered indices are independent. If $\dim(V) = 3$ then

$$dx^1 \wedge dx^2, \quad dx^1 \wedge dx^3, \quad dx^2 \wedge dx^3 \qquad (15.5.25)$$

are independent and $\dim(A^2(V)) = 3$.

In general, if $\dim(V) = n$ then $\dim(A^k(V)) = \binom{n}{k}$. In particular, $\dim(A^k(V)) = 0$ for $k > n$ and the dimension of the space of all antisymmetric forms $\dim(A^*(V)) = 2^n$ is finite. We leave to the reader to check that defining (anti)symmetric forms on manifolds is just a matter of notation.

6. Exterior algebra

Of particular interest are antisymmetric forms, especially in view of their integration properties.

We already defined 0-forms and 1-forms.

Since 0-forms have no indices the ordinary product of two 0-forms (namely, $F \cdot G$) is the same as the wedge product $F \cdot G$ (any function is a zero form and the projector A acts trivially).

The multiplication of a 0-form with a 1-form is a 1-form and the projector A acts again trivially. Then the multiplication of a 0-form with a 1-form coincides again with the wedge product

$$F \wedge \boldsymbol{\omega} = F \cdot \boldsymbol{\omega} \qquad (15.6.1)$$

2-forms

A 2-form is a field of antisymmetric forms of rank 2, i.e. locally

$$\boldsymbol{\omega} = \tfrac{1}{2}\omega_{\mu\nu}(x)\, dx^\mu \wedge\, dx^\nu \tag{15.6.2}$$

The set of all 2-forms will be denoted by $\Omega^2(M)$.

> Whatever the functions $\omega_{\mu\nu}$ are, they are contracted with the antisymmetric basis $dx^\mu \wedge dx^\nu$ so that only the antisymmetric part $\omega_{[\mu\nu]}(x)$ contributes to the product. It is not restrictive to assume that $\omega_{\mu\nu}(x)$ are antisymmetric. When $\omega_{\mu\nu}(x)$ are antisymmetric they are called the *components* of the 2-form $\boldsymbol{\omega}$.
>
> The factor $\tfrac{1}{2}$ is added as a convention. Some authors define $\tfrac{1}{2}\omega_{\mu\nu}(x)$ to be the components, some include the factor in the definition of the wedge product, some restrict summations to ordered pairs. We should not worry too much about notation, just choose the one you please and get stuck to it.

A 2-form is an antisymmetric bilinear form which takes two vector fields and give a function.

> The basis element $dx^\mu \wedge dx^\nu$ are simply
>
> $$dx^\mu \wedge\, dx^\nu \,=\, dx^\mu \otimes\, dx^\nu \,-\, dx^\nu \otimes\, dx^\mu \tag{15.6.3}$$
>
> so that we have the action of a 2-form on to vector fields
>
> $$\omega(X,Y) = \tfrac{1}{2}\omega_{\mu\nu}(X^\mu Y^\nu - X^\nu Y^\mu) = \omega_{[\mu\nu]}X^\mu Y^\nu = X^\mu Y^\nu \omega_{\mu\nu} \tag{15.6.4}$$
>
> where we used antisymmetry of $\omega_{\mu\nu}$ in the last step.

If we consider two 1-forms $\boldsymbol{\theta}^1 = \theta^1_\mu\, dx^\mu$ and $\boldsymbol{\theta}^2 = \theta^2_\mu\, dx^\mu$ we can get the wedge product of them

$$\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 = \theta^1_\mu \theta^2_\nu\, dx^\mu \wedge\, dx^\nu = \tfrac{1}{2}2\theta^1_{[\mu}\theta^2_{\nu]}\, dx^\mu \wedge\, dx^\nu \tag{15.6.5}$$

which is the 2-form with components $2\theta^1_{[\mu}\theta^2_{\nu]}$.

> Let us stress once and for all that components are not that important when working with forms as long as one keeps the bases. In fact, $\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 = \theta^1_\mu \theta^2_\nu\, dx^\mu \wedge\, dx^\nu$ is a perfect expression for the 2-form, even if the function coefficients are not correctly normalised and not antisymmetric. The symmetric part of it in fact does not contribute to the product since it is multiplied by an antisymmetric object.

From (15.6.4) we also get the formula for the *contraction* of a 2-form $\boldsymbol{\omega}$ along a vector field X

$$X\lrcorner\boldsymbol{\omega} := i_X\omega = X^\mu \omega_{\mu\nu}\, dx^\nu \tag{15.6.6}$$

> The contraction $X\lrcorner\boldsymbol{\omega}$ of a k-form $\boldsymbol{\omega}$ along X is a $(k-1)$-form. It is a linear operation and it obeys (graded) Leibniz rule, i.e.
>
> $$X\lrcorner(\boldsymbol{\omega} \wedge \boldsymbol{\theta}) = (X\lrcorner\boldsymbol{\omega}) \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}\boldsymbol{\omega} \wedge (X\lrcorner\boldsymbol{\theta}) \tag{15.6.7}$$

Since any 1-form is written as linear combinations of wedge products of 0-forms and 1-forms (in the form dx^μ), we can extend the differential to 1-forms by requiring that *graded Leibniz formula* holds true, i.e.

$$d(\boldsymbol{\omega} \wedge \boldsymbol{\theta}) = d\boldsymbol{\omega} \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}\boldsymbol{\omega} \wedge d\boldsymbol{\theta} \tag{15.6.8}$$

and that $d\, dx^\mu = 0$.

One can easily obtain the general formula for $\boldsymbol{\omega} = \omega_\mu(x)\, dx^\mu \in \Omega^1(M)$ as

$$d\boldsymbol{\omega} = d(\omega_\mu) \wedge dx^\mu + \boxed{\omega_\mu d\, dx^\mu} = \partial_\lambda \omega_\mu\, dx^\lambda \wedge dx^\mu \tag{15.6.9}$$

Let us stress that, for a 0-form F, one has $ddF = 0$, in fact

$$d(\partial_\mu F\, dx^\mu) = \partial_{\lambda\mu} F\, dx^\lambda \wedge dx^\mu = 0 \tag{15.6.10}$$

since $\partial_{\lambda\mu} F$ is symmetric in $(\lambda\mu)$, while $dx^\lambda \wedge dx^\mu$ is antisymmetric in $[\lambda\mu]$.

We can prolong the augmented de Rham complex to

$$0 \to \mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \tag{15.6.11}$$

by adding $d : \Omega^1(M) \to \Omega^2(M)$.

k-**forms**

A k-form is a field of antisymmetric forms of rank k (with $0 \le k \le m = \dim(M)$), i.e. locally

$$\boldsymbol{\omega} = \tfrac{1}{k!}\omega_{\mu_1\dots\mu_k}(x)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \tag{15.6.12}$$

The set of all k-forms will be denoted by $\Omega^k(M)$.

The wedge product of a k-form $\boldsymbol{\omega}$ with a h-form $\boldsymbol{\theta} = \tfrac{1}{h!}\theta_{\nu_1\dots\nu_h}(x)\, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_h}$ is a $(k+h)$-form obtained simply by

$$\boldsymbol{\omega} \wedge \boldsymbol{\theta} = \tfrac{1}{k!h!}\omega_{\mu_1\dots\mu_k}(x)\theta_{\nu_1\dots\nu_h}(x)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_h} \tag{15.6.13}$$

The differential of a k-form $\boldsymbol{\omega}$ is the $(k+1)$-form

$$d\boldsymbol{\omega} = \tfrac{1}{k!}\partial_\lambda \omega_{\mu_1\dots\mu_k}(x)\, dx^\lambda \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \tag{15.6.14}$$

For any $(k-1)$-form $\boldsymbol{\omega}$ one has $dd\boldsymbol{\omega} = 0$, in fact

$$dd\boldsymbol{\omega} = \tfrac{1}{(k-1)!}\partial_{\alpha\beta}\omega_{\mu_1\dots\mu_{k-1}}(x)\, dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-1}} \tag{15.6.15}$$

which vanishes since the component functions are symmetric in $(\alpha\beta)$, while the basis is antisymmetric in $[\alpha\beta]$.

The contraction of a k-form $\boldsymbol{\omega}$ along a vector field X gives a $(k-1)$-form

$$i_X\boldsymbol{\omega} = \tfrac{1}{(k-1)!}X^\mu \omega_{\mu\mu_2\dots\mu_k}(x)\, dx^{\mu_2} \wedge \dots \wedge dx^{\mu_k} \tag{15.6.16}$$

and the complete evaluation along k vector fields $(X_1, \ldots, X_k)$ gives a function which reads as

$$\boldsymbol{\omega}(X_1, \ldots, X_k) = \tfrac{1}{k!}\omega_{\mu_1 \ldots \mu_k}(x)X_1^{\mu_1} \ldots X_k^{\mu_k} \tag{15.6.17}$$

where we used the antisymmetry of component functions $\omega_{\mu_1 \ldots \mu_k}$. One still has graded Leibniz rule for contractions

$$X \lrcorner (\boldsymbol{\omega} \wedge \boldsymbol{\theta}) = (X \lrcorner \boldsymbol{\omega}) \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}\boldsymbol{\omega} \wedge (X \lrcorner \boldsymbol{\theta}) \tag{15.6.18}$$

> In fact, one can regard a k-form as the wedge product of a local 1-form dx^{μ_1} with $\tfrac{1}{k!}\omega_{\mu_1\mu_2\ldots\mu_k}(x)\,dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_k}$ and extend the definition of contraction to k-forms once it is defined on lower order forms in such a way the Leibniz rule holds true.

m-**forms**

When $k = m = \dim(M)$ the space of m-forms is generated by a single element $d\sigma = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$. Any m-form is in the form

$$\boldsymbol{\omega} = \omega(x)\, d\sigma \tag{15.6.19}$$

> If one changes coordinates $x'^\mu = x'^\mu(x)$ the bases of vectors and covectors changes accordingly, in particular $dx'^\mu = J^\mu_\nu(x)\,dx^\nu$. Thus the basis of m-forms changes as
>
> $$d\sigma' = J^1_{\nu_1} \ldots J^m_{\nu_m}\,dx^{\nu_1} \wedge \ldots dx^{\nu_m} = J^1_{\nu_1} \ldots J^m_{\nu_m}\,\epsilon^{\nu_1 \ldots \nu_m}\,d\sigma = \det(J)\,d\sigma \tag{15.6.20}$$
>
> Accordingly, the coefficient transforms as a scalar density of weight 1, i.e.
>
> $$\omega' = \det(\bar{J})\omega \tag{15.6.21}$$
>
> so that the product $\boldsymbol{\omega}$ is invariant.

The space of m-forms is denoted by $\Omega^m(M)$.

The contraction of an m-form $\boldsymbol{\omega}$ along a vector field X is an $(m-1)$-form given by

$$X \lrcorner \boldsymbol{\omega} = \tfrac{1}{(m-1)!}X^\mu \omega_{\mu\mu_2\ldots\mu_m}(x)\,dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m} \tag{15.6.22}$$

For $k > m$, on the space of k-forms $\Omega^k(M)$ is made of one element, namely the 0. Accordingly, the differential of an m-form is obviously $d\boldsymbol{\omega} = 0$.

We can prolong the augmented de Rham complex to

$$0 \to \mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \ldots \to \Omega^{m-1}(M) \to \Omega^m(M) \to 0 \tag{15.6.23}$$

> At each step $d^2 = 0$, hence $\mathrm{Im}(d) \subset \ker(d)$. As we shall discuss in a while, exactness of this sequence contains important information about the topology of M.

Exterior algebra

We can now consider the set of all forms

$$\Omega^*(M) = \oplus_k \Omega^k(M) \tag{15.6.24}$$

which is endowed with a structure of (graded) algebra by the wedge product. This algebra is called the *exterior algebra*.

The vector space of k-forms at the point $x \in M$ is denoted by $A^k(M)_x$. It is a vector space of dimension $\dim(A^k(M)_x) = \binom{m}{k}$.

Theorem: *let M and N be two manifolds and $\Phi : M \to N$ a diffeomorphism. For any k-form $\boldsymbol{\omega} \in \Omega^k(N)$ one has*

$$d\left(\Phi^* \boldsymbol{\omega}\right) = \Phi^* d\boldsymbol{\omega} \tag{15.6.25}$$

Proof: *let us prove the theorem for $k = 2$. The general case is more complicated just for notation.*

Let us compute both hand sides. For the left hand side one has

$$
\begin{aligned}
d\left(\Phi^* \boldsymbol{\omega}\right) =& d\left(\frac{1}{2}\omega_{ij}(y(x))J^i_\mu J^j_\nu \, dx^\mu \wedge \, dx^\nu\right) = \\
=& \frac{1}{2}\partial_k\omega_{ij}(y(x))J^k_\lambda J^i_\mu J^j_\nu \, dx^\lambda \wedge \, dx^\mu \wedge \, dx^\nu + \boxed{\frac{1}{2}\omega_{ij}(y(x))J^i_{\mu\lambda}J^j_\nu \, dx^\lambda \wedge \, dx^\mu \wedge \, dx^\nu} + \boxed{\frac{1}{2}\omega_{ij}(y(x))J^i_\mu J^j_{\nu\lambda} \, dx^\lambda \wedge \, dx^\mu \wedge \, dx^\nu} = \\
=& \frac{1}{2}\partial_k\omega_{ij}(y(x))J^k_\lambda J^i_\mu J^j_\nu \, dx^\lambda \wedge \, dx^\mu \wedge \, dx^\nu
\end{aligned} \tag{15.6.26}
$$

For the right hand side, one has

$$\Phi^* d\boldsymbol{\omega} = \Phi^*\left(\frac{1}{2}\partial_k\omega_{ij}(y) \, dy^k \wedge \, dy^i \wedge \, dy^j\right) = \frac{1}{2}\partial_k\omega_{ij}(y(x))J^k_\lambda J^i_\mu J^j_\nu \, dx^\lambda \wedge \, dx^\mu \wedge \, dx^\nu \tag{15.6.27}$$

which proves the thesis. ∎

The theorem above shows that differential and pull-back commute, i.e.

$$
\begin{array}{ccc}
\Omega^k(N) & \xrightarrow{\ d\ } & \Omega^{k+1}(N) \\
\downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi^*} \\
\Omega^k(M) & \xrightarrow{\ d\ } & \Omega^{k+1}(M)
\end{array} \tag{15.6.28}
$$

The operations which commute with the pull-back along diffeomorphisms are called *natural operations*. We just showed that differential is natural.

The *augmented de Rham complex* then stops after $m + 2$ spaces and it reads as

$$0 \to \mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \ldots \to \Omega^{m-1}(M) \to \Omega^m(M) \to 0 \tag{15.6.29}$$

At each step the composition of two maps is the zero map, i.e. $d^2 = 0$ including $i : \mathbb{R} \to \Omega^0(M)$ as a trivial differential. Hence one has that $\text{Im}(d) \subset \ker(d)$ at each step.

A k-form $\boldsymbol{\omega} \in \Omega^k(M)$ is called *exact* if it belongs to the image of the differential $d : \Omega^{k-1}(M) \to \Omega^k(M)$, i.e. if there exists a $\boldsymbol{\theta} \in \Omega^{k-1}(M)$ for which $\boldsymbol{\omega} = d\boldsymbol{\theta}$.

A k-form $\boldsymbol{\omega} \in \Omega^k(M)$ is called *closed* if it belongs to the kernel of the differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$, i.e. if $d\boldsymbol{\omega} = 0$.

Since $d^2 = 0$ (or, equivalently, $\text{Im}(d) \subset \ker(d)$) any exact form is also closed, while the opposite does not always hold true.

Let us consider the 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{15.6.30}$$

on the space $\mathbb{R}^2 - \{0\}$ with (x, y) Cartesian coordinates. The form $\boldsymbol{\omega}$ is closed since

$$d\boldsymbol{\omega} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}\,dx \wedge dy - \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2}\,dy \wedge dx = \left(\frac{2(x^2 + y^2) - 2(x^2 + y^2)}{(x^2 + y^2)^2}\right)\,dx \wedge dy = 0 \tag{15.6.31}$$

However, $\boldsymbol{\omega}$ is not exact. In fact, let us suppose there is a smooth function $f(x, y)$ such that $df = \boldsymbol{\omega}$. Then one would have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{x}{x^2 + y^2} & \Rightarrow \frac{\partial^2 f}{\partial x \partial y} &= \frac{-2xy}{(x^2 + y^2)^2} \\
\frac{\partial f}{\partial y} &= \frac{-y}{x^2 + y^2} & \Rightarrow \frac{\partial^2 f}{\partial x \partial y} &= \frac{2xy}{(x^2 + y^2)^2}
\end{aligned} \tag{15.6.32}$$

Thus there cannot exist any such function f.

Hence being closed is a necessary, thus not sufficient, condition to be exact.

This is a generalisation to arbitrary order k of the well-know discussion about when a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ (or, by using the canonical metric structure on $\mathbb{R}^3$, the 1-form $\boldsymbol{\omega} = X^\flat$) admits a potential $U \in \Omega^1(\mathbb{R}^3)$ (i.e. $\boldsymbol{\omega} = dU$). We know that necessary condition is $d\boldsymbol{\omega} = 0$, which is usually called to be *conservative* or, by an abuse of language, being *irrotational*. We also know this condition to be necessary but not in general sufficient.

Here we are *only* stating the general framework in which expressing the problem in any rank k (and any manifold, and any chart).

As for conservative vector fields in $\mathbb{R}^3$, we have a *generalised Poincaré Lemma* which we here state without proof. It says:

Lemma: *if M is contractible, then closed k-forms on M are exact for any k.*

Being *contractible* means there is no hole in any dimension from 0 to m.

The hypothesis of usual Poincaré lemma is that M is simply connected, which means there is no hole of codimension $m - 2$, i.e. in dimension 2 no hole points. This ensures that closed 1-forms are also exact, i.e. it ensures potentials 0-forms.

Technically, being contractible means that the constant map $\Phi_{x_0} : M \to M : x \mapsto x_0$ and the identity map $\text{id}_M : M \to M : x \mapsto x$ are homotopic, i.e. there exists a continuous function $F : [0, 1] \times M \to M$ such that

$$F(0, x) = x \qquad F(1, x) = x_0 \tag{15.6.33}$$

The functions $F_s : M \to M : x \mapsto F(s, x)$ provides a (somehow continuous) family of functions connecting the constant map and the identity map.

Also notice that the curves

$$\gamma_x : \mathbb{R} \to M : s \mapsto F(s, x) \tag{15.6.34}$$

provides a (somehow continuous) family of curves connecting any point $x \in M$ to the point $x_0 \in M$. When this is possible, we say that M is *star shaped*. In this sense, being star shaped is equivalent to be contractible.

Since any ball of $\mathbb{R}^m$ is contractible, any chart map $\varphi : U \to \mathbb{R}^m$ is a homeomorphism, and being contractible is invariant with respect to homeomorphisms, then around any point in a manifold there is a neighborhood U which is contractible. The open set U is a sub-manifold of M which is embedded by a canonical embedding $i : U \to M$ (which is a diffeomorphism on its image $\mathrm{Im}(U) \subset M$) and any k-form $\boldsymbol{\omega} \in \Omega^k(M)$ can be restricted by pull-back to a k-form $i^*(\boldsymbol{\omega}) \in \Omega^k(U)$. If $\boldsymbol{\omega}$ is closed then $i^*(\boldsymbol{\omega})$ is closed and since U is contractible, it is also exact, i.e. one has $i^*(\boldsymbol{\omega}) = d\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^{k-1}(U)$. Hence $\boldsymbol{\omega}$ is exact on U. In other words:

Theorem: *any closed form is locally exact.*

In augmented de Rham complex, at any point one has the space of closed forms $K^k(M) \subset \Omega^k(M)$ and the space of exact forms $I^k(M) \subset \Omega^k(M)$, with $I^k(M) \subset K^k(M)$.

We say that two closed forms are equivalent iff their difference is exact. This defines an equivalence relation in $K^k(M)$ and we define the kth *cohomology group* by

$$H_{dR}^k(M) = K^k(M)/I^k(M) \tag{15.6.35}$$

These are commutative groups with the sum operation. Also the product by a scalar is compatible with the quotient and defines a vector space structure on $H_{dR}^k(M)$.

Cohomology groups count how many closed forms there are which are not exact. Since closed non-exact forms exist when holes are present in M, cohomology groups extract informations about holes in M.

Let us stress that cohomology groups are differential objects. One needs a smooth manifold to define them. On the contrary, holes are topological objects. Holes are preserved by homeomorphisms.

This is a first instance of something which happens in many different instances. Sometimes it happens that differential objects are able to extract topological information about the space on which they are defined.

Let us now consider two manifolds M and N with a diffeomorphism $\Phi : M \to N$.

Let $\boldsymbol{\omega} \in K^k(N)$ be a closed k-form. We already know that $\Phi^*\boldsymbol{\omega} \in K^k(M)$ is closed as well and that $\Phi^* : K^k(N) \to K^k(M)$.

This map Φ^* is compatible with the equivalence relation defined on closed forms.

Let us consider two equivalent closed forms $\boldsymbol{\omega}, \boldsymbol{\omega}' \in K^k(N)$ (i.e. $\boldsymbol{\omega}' - \boldsymbol{\omega} = d\boldsymbol{\theta} \in I^k(N)$ for some $\boldsymbol{\theta} \in \Omega^{k-1}(N)$). Since the differential is a natural operator, one also has

$$\Phi^*\boldsymbol{\omega}' - \Phi^*\boldsymbol{\omega} = d(\Phi^*\boldsymbol{\theta}) \in I^k(M) \tag{15.6.36}$$

and $\Phi^*\boldsymbol{\omega}', \Phi^*\boldsymbol{\omega} \in K^k(M)$ are equivalent.

Then the map $\Phi^* : K^k(N) \rightarrow K^k(M)$ induces a map $\Phi^* : H_{dR}^k(N) \rightarrow H_{dR}^k(M)$ in the cohomology groups simply defined as

$$\Phi^* : H_{dR}^k(N) \rightarrow H_{dR}^k(M) : [\boldsymbol{\omega}] \mapsto [\Phi^*\boldsymbol{\omega}] \tag{15.6.37}$$

Since Φ is a diffeomorphism, such a map is a group isomorphism and then $H_{dR}^k(N) \simeq H_{dR}^k(M)$. Thus the cohomology groups are invariant by diffeomorphisms. Two manifolds cannot be diffeomorphic if they have different cohomology groups.

Contracted bases for $\Omega^k(M)$

We defined $d\sigma := dx^1 \wedge \ldots \wedge dx^m$ as a basis for m-forms. Since there are no $(m+1)$-forms on a manifold M which is m-dimensional, then we know that $dx^\mu \wedge d\sigma = 0$, identity to which we can apply a contraction along ∂_α obtaining

$$\delta_\alpha^\mu \, d\sigma - dx^\mu \wedge d\sigma_\alpha = 0 \qquad \Rightarrow \; dx^\mu \wedge d\sigma_\alpha = \delta_\alpha^\mu \, d\sigma \tag{15.6.38}$$

where we defined $d\sigma_\alpha := \partial_\alpha \lrcorner \, d\sigma$. The $(m-1)$-forms $d\sigma_\alpha$ are in fact independent and a basis for $(m-1)$-forms.

Accordingly, an $(m-1)$-form $\boldsymbol{\theta}$ can be expanded along two different basis, namely

$$\boldsymbol{\theta} = \tfrac{1}{(m-1)!}\theta_{\mu_2\ldots\mu_m} \, dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m} \qquad\qquad \boldsymbol{\theta} = \theta^\mu \, d\sigma_\mu \tag{15.6.39}$$

where we set

$$d\sigma_\mu = \tfrac{1}{(m-1)!}\epsilon_{\mu\mu_2\ldots\mu_m} \, dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m} \qquad\qquad \theta^\mu := \tfrac{1}{(m-1)!}\epsilon^{\mu\mu_2\ldots\mu_m}\theta_{\mu_2\ldots\mu_m} \tag{15.6.40}$$

This technique can be iterated, starting from the identity (15.6.38) by contracting it along ∂_β, obtaining

$$\delta_\beta^\mu \, d\sigma_\alpha - dx^\mu \wedge d\sigma_{\alpha\beta} = \delta_\alpha^\mu \, d\sigma_\beta \qquad \Rightarrow \; dx^\mu \wedge d\sigma_{\alpha\beta} = \delta_\beta^\mu \, d\sigma_\alpha - \delta_\alpha^\mu \, d\sigma_\beta \tag{15.6.41}$$

where we set $d\sigma_{\alpha\beta} := \partial_\beta \lrcorner \, d\sigma_\alpha$. Since the forms $d\sigma_{\alpha\beta}$ are defined antisymmetric in $[\alpha\beta]$, they define (restricting to ordered indices $\alpha < \beta$, of course) a basis of $(m-2)$-forms. Then an $(m-2)$-form $\boldsymbol{\omega}$ can be expanded along two different basis, namely

$$\boldsymbol{\omega} = \tfrac{1}{(m-2)!}\omega_{\mu_3\ldots\mu_m} \, dx^{\mu_3} \wedge \ldots \wedge dx^{\mu_m} \qquad\qquad \boldsymbol{\omega} = \tfrac{1}{2!}\omega^{\mu\nu} \, d\sigma_{\mu\nu} \tag{15.6.42}$$

where we set

$$d\sigma_{\mu\nu} = \tfrac{1}{(m-2)!}\epsilon_{\mu\nu\mu_3\ldots\mu_m} \, dx^{\mu_3} \wedge \ldots \wedge dx^{\mu_m} \qquad\qquad \omega^{\mu\nu} := \tfrac{1}{(m-2)!}\epsilon^{\mu\nu\mu_3\ldots\mu_m}\omega_{\mu_3\ldots\mu_m} \tag{15.6.43}$$

At lower order, we have

$$\delta_\gamma^\mu \, d\sigma_{\alpha\beta} - dx^\mu \wedge d\sigma_{\alpha\beta\gamma} = \delta_\beta^\mu \, d\sigma_{\alpha\gamma} - \delta_\alpha^\mu \, d\sigma_{\beta\gamma} \qquad \Rightarrow \; dx^\mu \wedge d\sigma_{\alpha\beta\gamma} = \delta_\gamma^\mu \, d\sigma_{\alpha\beta} - \delta_\beta^\mu \, d\sigma_{\alpha\gamma} + \delta_\alpha^\mu \, d\sigma_{\beta\gamma} \tag{15.6.44}$$

where we set $d\sigma_{\alpha\beta\gamma} := \partial_\gamma \lrcorner \, d\sigma_{\alpha\beta}$, and so on.

Hodge duality

The dimension of space of k-forms is $\dim(A^k(M)_x) = \begin{pmatrix} m \\ k \end{pmatrix}$. Thus $\dim(A^k(M)_x) = \dim(A^{m-k}(M)_x)$.

One can define an isomorphism $* : A_k \to A_{m-k}$ using a metric g.

Let us define

$$* : A_0 \to A_m : \alpha = \alpha\, \mathbb{I} \mapsto *\alpha = \sqrt{g}\, \alpha\ d\sigma = \tfrac{1}{m!}\sqrt{g}\, \alpha\, \epsilon_{\rho_1 \ldots \rho_m}\, dx^{\rho_1} \wedge \ldots \wedge\ dx^{\rho_m}$$

$$* : A_1 \to A_{m-1} : \boldsymbol{\alpha} = \alpha_\mu\ dx^\mu \mapsto *\boldsymbol{\alpha} = \sqrt{g}\, \alpha_\mu g^{\mu\rho}\ d\sigma_\rho = \tfrac{1}{(m-1)!}\sqrt{g}\, \alpha_\mu\, \epsilon^\mu_{\cdot\, \rho_2 \ldots \rho_m}\, dx^{\rho_2} \wedge \ldots \wedge\ dx^{\rho_m}$$

$$* : A_2 \to A_{m-2} : \boldsymbol{\alpha} = \tfrac{1}{2}\alpha_{\mu\nu}\ dx^\mu \wedge\ dx^\nu \mapsto *\boldsymbol{\alpha} = \tfrac{1}{2}\sqrt{g}\, \alpha_{\mu\nu} g^{\mu\rho} g^{\nu\sigma}\ d\sigma_{\rho\sigma} = \tfrac{1}{(m-2)!}\tfrac{1}{2}\sqrt{g}\, \alpha_{\mu\nu}\epsilon^{\mu\nu}_{\cdot\cdot\, \rho_3 \ldots \rho_m}\, dx^{\rho_3} \wedge \ldots \wedge\ dx^{\rho_m}$$

$$\ldots$$

$$* : A_{m-2} \to A_2 : \boldsymbol{\alpha} = \tfrac{1}{(m-2)!}\alpha_{\mu_3 \ldots \mu_m}\ dx^{\mu_3} \wedge \ldots \wedge\ dx^{\mu_m} \mapsto *\boldsymbol{\alpha} = \tfrac{1}{(m-2)!}\sqrt{g}\, \alpha_{\mu_3 \ldots \mu_m} g^{\mu_3\rho_3} \ldots g^{\mu_m\rho_m}\ d\sigma_{\rho_3 \ldots \rho_m} =$$

$$: \boldsymbol{\alpha} = \tfrac{1}{2}\sqrt{g}\alpha^{\rho\sigma}\ d\sigma_{\rho\sigma} \mapsto *\boldsymbol{\alpha} = \tfrac{1}{2}\tfrac{|g|}{g}\, \alpha^{\cdot\cdot}_{\mu\nu}\, dx^\mu \wedge\ dx^\nu \tag{15.6.45}$$

$$* : A_{m-1} \to A_1 : \boldsymbol{\alpha} = \tfrac{1}{(m-1)!}\alpha_{\mu_2 \ldots \mu_m}\ dx^{\mu_2} \wedge \ldots \wedge\ dx^{\mu_m} \mapsto *\boldsymbol{\alpha} = \tfrac{1}{(m-1)!}\sqrt{g}\, \alpha_{\mu_2 \ldots \mu_m} g^{\mu_2\rho_2} \ldots g^{\mu_m\rho_m}\ d\sigma_{\rho_2 \ldots \rho_m} =$$

$$: \boldsymbol{\alpha} = \sqrt{g}\alpha^\rho\ d\sigma_\rho \mapsto *\boldsymbol{\alpha} = (-1)^{m-1}\tfrac{|g|}{g}\, \alpha^{\cdot}_\rho\ dx^\rho$$

$$* : A_m \to A_0 : \boldsymbol{\alpha} = \tfrac{1}{m!}\alpha_{\mu_1 \ldots \mu_m}\ dx^{\mu_1} \wedge \ldots \wedge\ dx^{\mu_m} \mapsto *\boldsymbol{\alpha} = \tfrac{1}{m!}\sqrt{g}\, \alpha_{\mu_1 \ldots \mu_m} g^{\mu_1\rho_1} \ldots g^{\mu_m\rho_m}\ d\sigma_{\rho_1 \ldots \rho_m}$$

$$: \boldsymbol{\alpha} = \sqrt{g}\alpha\, d\sigma \mapsto *\boldsymbol{\alpha} = \tfrac{|g|}{g}\, \alpha\, \mathbb{I}$$

Notice that the dualities defined above maps global forms into global forms.

Let us compute the square of the dualities (notice that $|g|/g = (-1)^s$ where (r, s) is the signature of g)

$$* * (\alpha\, \mathbb{I}) = * (\sqrt{g}\, \alpha\ d\sigma) = \tfrac{|g|}{g}\, \alpha\, \mathbb{I} = (-1)^s\, \alpha\, \mathbb{I}$$

$$* * (\alpha_\mu\ dx^\mu) = * \left(\sqrt{g}\, \alpha^\rho_\cdot\ d\sigma_\rho \right) = (-1)^{m-1+s}\, \alpha_\rho\ dx^\rho \tag{15.6.46}$$

$$* * (\tfrac{1}{2}\alpha_{\mu\nu}\ dx^\mu \wedge\ dx^\nu) = * \left(\tfrac{1}{2}\sqrt{g}\, \alpha^{\mu\nu}_{\cdot\cdot}\ d\sigma_{\mu\nu} \right) = \tfrac{1}{2}\tfrac{|g|}{g}\, \alpha_{\mu\nu}\, dx^\mu \wedge\ dx^\nu = (-1)^s\tfrac{1}{2}\alpha_{\mu\nu}\, dx^\mu \wedge\ dx^\nu$$

$$\ldots$$

The general formula for the double Hodge of a k-form is then

$$* * \boldsymbol{\omega} = (-1)^{s+k(m-1)}\boldsymbol{\omega} \tag{15.6.47}$$

Given two k-forms, it is also interesting to compute the quantity

$$\boldsymbol{\alpha} \wedge *\boldsymbol{\beta} = \tfrac{1}{k!}\tfrac{1}{k!}\sqrt{g}\alpha_{\mu_1 \ldots \mu_k}\beta^{\nu_1 \ldots \nu_k}\, dx^{\mu_1} \wedge \ldots \wedge\ dx^{\mu_k} \wedge\ d\sigma_{\nu_1 \ldots \nu_k} =$$

$$= \tfrac{1}{k!}\tfrac{1}{k!}\tfrac{1}{(m-k)!}\epsilon_{\nu_1 \ldots \nu_k \rho_1 \ldots \rho_{m-k}}\epsilon^{\mu_1 \ldots \mu_k \rho_1 \ldots \rho_{m-k}}\sqrt{g}\alpha_{\mu_1 \ldots \mu_k}\beta^{\nu_1 \ldots \nu_k} = \tag{15.6.48}$$

$$= \tfrac{1}{k!}\sqrt{g}\alpha_{\mu_1 \ldots \mu_k}\beta^{\mu_1 \ldots \mu_k}\, d\sigma =: < \boldsymbol{\alpha}, \boldsymbol{\beta} > \sqrt{g}\, d\sigma$$

which usually is used to define the Hodge duality intrinsically.

Global differential operators on forms

We can generalise to an arbitrary manifold (and arbitrary coordinates) the operators of vector calculus originally defined on $\mathbb{R}^n$ (often in Cartesian coordinates).

Let M be a manifold of dimension m with a metric g of signature (r, s). If $F : M \to \mathbb{R}$ is a function, dF is a 1-form and

$$\text{grad}(F) = (dF)^\sharp \in \mathfrak{X}(M) \tag{15.6.49}$$

is a vector field which is called the *gradient* of F. Of course, the gradient (unlike the differential) depends on the metric and it is not simply the vector with partial derivatives components as it is in $\mathbb{R}^n$ in Cartesian coordinates. The local expression of gradient is instead

$$\text{grad}(F) = \partial_\mu F \, g^{\mu\nu} \partial_\nu \tag{15.6.50}$$

Let us consider $\mathbb{R}^2$ with the canonical metric in polar coordinates (r, θ), i.e. $g = dr \otimes dr + r^2 d\theta \otimes d\theta$. A function is locally expressed as $F(r, \theta)$ and its differential $dF = \partial_r F dr + \partial_\theta F d\theta$. Then the gradient is

$$\text{grad}(F) = \partial_r F \partial_r + \frac{1}{r^2} \partial_\theta F \partial_\theta \tag{15.6.51}$$

If one defines the unit vector base $u_r = \partial_r$ and $u_\theta = \frac{1}{r}\partial_\theta$, the expression of the gradient becomes

$$\text{grad}(F) = \partial_r F \, u_r + \frac{1}{r}\partial_\theta F \, u_\theta \tag{15.6.52}$$

which agrees with what is known from standard vector calculus (by a quite longer computation).

If $X \in \mathfrak{X}(M)$ is a vector field, we can apply the following chain of transformations

$$X \in \mathfrak{X}(M) \to X^\flat \in \Omega^1(M) \to *X^\flat \in \Omega^{m-1}(M) \to d * X^\flat \in \Omega^m(M) \to *d * X^\flat \in \Omega^0(M) \tag{15.6.53}$$

to obtain a function. The operator $*d * X^\flat =: \text{div}(X)$ is called the *divergence*.

By the way, the operator $\delta := *d* : \Omega^k(M) \to \Omega^{k-1}(M)$ can be extended to all k-forms and it is called the *codifferential*. One has

$$\delta^2 = *d * *d* = \pm * d^2 * = 0 \tag{15.6.54}$$

Thus we can define a dual complex

$$0 \to \Omega^m(M) \to \Omega^{m-1}(M) \to \ldots \to \Omega^2(M) \to \Omega^1(M) \to \Omega^0(M) \to 0 \tag{15.6.55}$$

Now we can combine the differential and codifferential to get a second order differential operator on k-forms

$$\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M) \tag{15.6.56}$$

which is called *Laplace-Beltrami operator.*

If we restrict to functions $F \in \Omega^0(M)$ then $d\delta F = 0$ and one has

$$\Delta F = \delta dF \tag{15.6.57}$$

which, on $\mathbb{R}^n$ with the strictly Riemannian metrics and Cartesian coordinates, reduces to the ordinary Laplacian. On Minkowski space (again in Cartesian coordinates), one obtains the ordinary wave operator.

If M is a manifold of dimension $m = 3$, then one can take a vector field and apply the following chain of transformations

$$X \in \mathfrak{X}(M) \to X^\flat \in \Omega^1(M) \to dX^\flat \in \Omega^2(M) \to *dX^\flat \in \Omega^1(M) \to (*dX^\flat)^\sharp \in \mathfrak{X}(M) \tag{15.6.58}$$

which is called the *curl* of X, $\mathrm{curl}(X) = (*dX^\flat)^\sharp$.

Also the theorems of vector analysis follows quite easily (though more generally). For example

$$\mathrm{curl}(\mathrm{grad}F) = (*ddF)^\sharp = 0 \tag{15.6.59}$$

Integration of forms

Let $\boldsymbol{\omega}$ be a k-form on M (of course, $k \le m = \dim(M)$) and $S \subset M$ be a sub-manifold (possibly with boundary) of dimension k. The sub-manifold S is canonically embedded into M by a (smooth) map

$$i : S \to M : u^i \mapsto x^\mu(u) \tag{15.6.60}$$

and let $J_i^\mu := \partial_i x^\mu$ denote the corresponding Jacobian. For example, that could be a (smooth) k-simplex $s_k = [u_0, u_1, \dots, u_{k+1}] \subset \mathbb{R}^{k+1}$.

Let us first restrict to the case $k = m$. In this case, one locally has $\boldsymbol{\omega} = f(x)\, d\sigma$ and define the *integral* of $\boldsymbol{\omega}$ over S as

$$\int_{i(S)} \boldsymbol{\omega} := \int_S i^*\boldsymbol{\omega} = \int f(x(u)) \det(J)\, du^1 \dots du^m \tag{15.6.61}$$

The definition (15.6.61) can be considered as a prescription to transform the integral of a k-form in the corresponding multiple integral. Of course, in general one needs to break the integration domain S into pieces in which the local expressions of the form and its parameterisation hold true.

This definition is invariant with respect to changes of parameterisation of the sub-manifold S. Accordingly, the value of the integral is a property of the form $\boldsymbol{\omega}$ and the integration domain S, not of the parameterisation.

Let us, in fact, consider a different parameterisation $u' = u'(u)$ (which preserves the orientation of S) of the same sub-manifold

$$i' : S \to M : u'^i \mapsto x'^\mu(u') = x^\mu(u(u')) \tag{15.6.62}$$

so that one has $J_i'^\mu = J_k^\mu \bar{X}_i^k$ where $\bar{X}_i^k$ denotes the anti-Jacobian of the change of parameterisation $u' = u'(u)$.

Then one has

$$\int_S \boldsymbol{\omega} := \int f(x'(u')) \det(J')\, du'^1 \dots du'^m = \int f(x(u)) \det(J)\det(\bar{X}) \det(X)\, du^1 \dots du^m = \int f(x(u)) \det(J)\, du^1 \dots du^m \tag{15.6.63}$$

The integral defined on m-simplices can be extended by linearity to m-chains defined on a chain complex Δ that may triangulate the integration domain, also to define the boundary operator ∂. Chain complexes provide us to a coherent orientation which is also adapted to the boundary operator. In dimension 3 that is coherent with the classical vector analysis which although needs to be generalized to arbitrary dimension.

As we noticed, a chain complex is closed by faces, in the sense that if a simplex is in Δ all its faces are as well. If m-simplex are the maximal dimension in the complex, its $(m-1)$-faces are either shared by two m-simplices or it belongs to one only and it is a boundary face of the complex.

When the faces is shared by two m-simplices, it appears with opposite orientations. If we integrate a $(m-1)$-form on the boundary these contributions cancel out and one is left only with contributions by boundary faces. As a result, one can prove Stoke's theorem

$$\int_S d\boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} \tag{15.6.64}$$

on a single simplex, then extending the result to s general m-simplicial complex.

The integral of forms is *exactly* the standard geometric integrals (the one to define lengths, areas, volumes in a parameterization and coordinates independent way) enhances with a sign from orientation.

The length (as well as the area) is defined as

$$l := \int_\gamma dl = \int_\gamma |v| ds \tag{15.6.65}$$

where dl is the *scalar* length, positive by definition.

The same for the area which is defined by

$$A := \int_S dA = \int_S |r_u \times r_v| du dv \tag{15.6.66}$$

where again dA is the scalar area element, positive by definition.

Historically, in view of the geometric meaning of these quantities, these was first addressed, often adjusting signs *ad hoc*. On the other hand, the integrals of forms come with a sign attached, as for example in flow integrals (or line integrals) as

$$\Phi_F = \int_S F \cdot dA \qquad\qquad \left(c_\gamma = \int_\gamma F \cdot dl \right) \tag{15.6.67}$$

where $dA = r_u \times r_v \, du dv$ and $dl = \vec{v} \, ds$ are now (co)vector quantities which can contribute positively or negatively to total integral.

In this sense, the sign is very important. In dimension 3 we are used to keep track and adjust signs to match them with orientations. That grows very annoying as the dimension grows and orientations becomes more difficult to be controlled.

Forms give a framework to do that, still defining invariant integrals. Because of that, integration of forms are more fundamental than the usual geometric integrals, that scales better to general dimension.

More generally, for $k \leq m$, if $\boldsymbol{\omega} = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k} \, dx^{\mu_1} \wedge \ldots \wedge \, dx^{\mu_k}$ is a k-form and S a sub-manifold (possibly with boundary) of dimension k, one can compute the pull-back of $\boldsymbol{\omega}$ on S to obtain:

$$i^*\boldsymbol{\omega} = \tfrac{1}{k!}\omega_{\mu_1\ldots\mu_k} J^{\mu_1}_{i_1} \ldots J^{\mu_k}_{i_k} \, du^{i_1} \wedge \ldots \wedge \, du^{i_k} = \tfrac{1}{k!}\omega_{\mu_1\ldots\mu_k} J^{\mu_1}_{i_1} \ldots J^{\mu_k}_{i_k} \epsilon^{i_1\ldots i_k} \, d\nu = f(u) \, d\nu \tag{15.6.68}$$

where we set $d\nu := du^1 \wedge \ldots \wedge du^k$ for the local volume element of S induced by the parameterisation and

$$f(u) := \tfrac{1}{k!}\omega_{\mu_1\ldots\mu_k}(x(u)) J^{\mu_1}_{i_1} \ldots J^{\mu_k}_{i_k} \epsilon^{i_1\ldots i_k} = \omega_{\mu_1\ldots\mu_k}(x(u)) J^{\mu_1}_1 \ldots J^{\mu_k}_k \tag{15.6.69}$$

Since $\omega_{\mu_1 \dots \mu_k}$ is antisymmetric, the products $J_1^{\mu_1} \dots J_k^{\mu_k} = J_1^{[\mu_1} \dots J_k^{\mu_k]}$ are the determinant of the square minors of the Jacobian J_i^μ.

Now we have a k-form $i^*\omega$ to be integrated on a k-sub-manifold S and applying the definition in the case above we set

$$\int_S \omega := \int_S i^*\omega = \int_S f(u)\, du^1 \wedge \dots \wedge\ du^k = \int \omega_{\mu_1 \dots \mu_k}(x(u)) J_1^{\mu_1} \dots J_k^{\mu_k}\, du^1 \dots\, du^k \tag{15.6.70}$$

Let us finally enhance the definition extending it to 0-forms. The integral of a 0-form f at a point x is simply its value $f(x)$.

Let us now consider some special cases. Let us first consider S to be an interval $[a,b]$ in $M = \mathbb{R}$. That is a 1-region, thus one integrates on it a 1-form $\omega = f(x)\, dx$ where x is both the coordinate along $\mathbb{R}$ and the parameter $u = x$ in the sub-manifold S. Then one has

$$\int_S \omega = \int_a^b f(x)\, dx \tag{15.6.71}$$

which is the standard integral of 'functions' $f : \mathbb{R} \to \mathbb{R}$.

Of course, $f(x)$ is not a function. It is not a scalar but a scalar density since it is the coefficient of a 1-form on $[a,b] \subset \mathbb{R}$. Another approach is to consider a volume form dV (e.g. $dV = \sqrt{g}\, dx$) and any other m-form is in the form $\omega = \lambda(x)\, dV$ where λ is now really a scalar field.

Of course, if one sets on $\mathbb{R}$ the standard metric $g = \delta$ and x is a Cartesian coordinate, then $\sqrt{g} = 1$ and

$$\int_S \omega = \int_a^b \lambda(x)\sqrt{g}\, dx = \int_a^b \lambda(x)\, dx \tag{15.6.72}$$

Let us stress that the expression (15.6.71) holds true only in a special coordinate system. In general, one cannot integrate functions on intervals. Only forms can be integrated in a coordinate independent fashion.

Then one can consider $M = \mathbb{R}^m$ (with Cartesian coordinates x^μ) and an (oriented) curve $\gamma : s \mapsto x^\mu(s)$. Then the integral of a 1-form $\omega = \omega_\mu\, dx^\mu$ along the curve γ is given by

$$\int_\gamma \omega = \int \omega_\mu(x(s))\dot{x}^\mu\ ds \tag{15.6.73}$$

which coincides with the definition of the *line integral*.

In $\mathbb{R}^3$, one has the standard Euclidean metric. If the coordinates x^μ are orthogonal Cartesian coordinates, the standard metric is in the form $\delta = \delta_{\mu\nu}\, dx^\mu \otimes dx^\nu$, where $\delta_{\mu\nu}$ is the unit matrix.

For any vector field $X = X^\mu \partial_\mu$, one can define a 1-form $X^\flat = X^\mu \delta_{\mu\nu}\, dx^\nu$. Vice versa, for one 1-form $\omega = \omega_\mu\, dx^\mu$, one can define a vector field $\omega^\sharp = \omega_\mu \delta^{\mu\nu} \partial_\nu$. The two operations are one the inverse of the other. Since, in orthogonal Cartesian coordinates, the components of the X and $X^\flat$ are the same functions, one often identifies the two objects in Calculus. However, in general, the two objects are different and with different components.

The integral (15.6.73) can be written in general coordinates as

$$\int_\gamma X^\flat = \int X^\mu(x(s)) g_{\mu\nu}\dot{x}^\nu\ ds = \int X \cdot \dot{\gamma}\ ds \tag{15.6.74}$$

which is the *line integral of a vector field*. This is the kind of integral to define the work of a force field along a path.

Let us now consider the integral of a 2-form on a 2-region S in $\mathbb{R}^2$. The 2-form is locally given by $\boldsymbol{\omega} = f(x)\,d\sigma$ and the surface S is given by the (orientation preserving) embedding $i : S \to \mathbb{R}^2 : (u, v) \mapsto (x(u,v), y(u,v))$. The Jacobian of the canonical embedding is

$$J = \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} \tag{15.6.75}$$

and the integral is defined by

$$\int_S \boldsymbol{\omega} = \int f(x(u))\,(\partial_u x \partial_v y - \partial_v x \partial_u y)\,du\,dv \tag{15.6.76}$$

which reproduces the *surface integral*.

For the integral of a 2-form $\boldsymbol{\omega} = \frac{1}{2}\left(\omega_x\,dy \wedge\ dz - \omega_y dx \wedge\ dz + \omega_z\,dx \wedge\ dy\right)$ on a surface S in $\mathbb{R}^3$, one has

$$\int_S \boldsymbol{\omega} = \frac{1}{2}\int \left(\omega_x\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) - \omega_y\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right) + \omega_z\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right)\,du\,dv \tag{15.6.77}$$

which reproduces the *flow integral* of the vector field $X = \omega_x\partial_x + \omega_y\partial_y + \omega_z\partial_z$ (remember that the coordinates in $\mathbb{R}^3$ are Cartesian so that $X = (*\boldsymbol{\omega})^\flat$) through the surface S.

If we want to have an integral which has to do with a vector field, we can set

$$\int_S \boldsymbol{\omega} = \int_S *(X^\sharp) = \int_S X \cdot \left(\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\right)\,du\,dv \tag{15.6.78}$$

which is in fact the flow integral of X through S.

Finally, if we consider a volume S in $\mathbb{R}^3$ and a 3-form $\boldsymbol{\omega} = f(x)\,d\sigma$, one has

$$\int_S \boldsymbol{\omega} = \int f(x)\,dx\,dy\,dz \tag{15.6.79}$$

which reproduces the *volume integral*. If we write the same integral in spherical coordinates, one has

$$\boldsymbol{\omega} = f(x)\,dx \wedge\ dy \wedge\ dz = f(x)r^2\sin\theta\,dr \wedge\ d\theta \wedge\ d\phi \qquad \Rightarrow \int_S \boldsymbol{\omega} = \int f(x)r^2\sin\theta\,dr\,d\theta\,d\phi \tag{15.6.80}$$

which is in fact the same volume integral in different coordinates. It is again clear that one is not in fact integrating a scalar function $f(x)$, but a scalar density (i.e. a 3-form). If the integrand were a scalar, one could not justify the extra factor (namely $r^2\sin\theta$) appearing when coordinates are changed.

Thus, by integration of forms, one can obtain all the integrals introduced in vector calculus tough now they are invariant with respect to reparameterisations.

Let us state without proof an important relation (called *(generalised) Stokes' theorem*) between integrals of forms.

Theorem: *Let $\boldsymbol{\omega}$ be a k-form and S a $(k+1)$-region in M. Let us denote by ∂S the boundary of S (which by definition of region is a compact sub-manifold of dimension k) and by $d\boldsymbol{\omega}$ the differential of $\boldsymbol{\omega}$.*

Then one has

$$\int_S d\boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} \qquad (15.6.81)$$

Also for Stokes' theorem, one can specialize k and m in order to obtain known results.

For a zero form $f(x)$, integrated over an interval $S = [a, b]$, one has

$$\int_a^b f'(x) \, dx = \int_{b-a} f(x) = f(b) - f(a) \qquad (15.6.82)$$

which is called the (second part of the) *fundamental theorem of integral calculus.*

Now let $X = X^\mu \partial_\mu$ be a vector field on a manifold M of dimension 3 and S be a 2-region in M. One can consider the 1-form $\boldsymbol{\omega} = X^\flat = \omega_\mu \, dx^\mu$ and its differential

$$d\boldsymbol{\omega} = \left(\partial_x \omega_y - \partial_y \omega_x\right) \, dx \wedge \, dy + (\partial_x \omega_z - \partial_z \omega_x) \, dx \wedge \, dz + \left(\partial_y \omega_z - \partial_z \omega_y\right) \, dy \wedge \, dz \qquad (15.6.83)$$

This can be transformed back into a vector field $\mathrm{curl}(X) = (*d(X^\flat))^\sharp$ which is the curl of X. Thus the left hand side of Stokes' theorem reads as

$$\int_S d\boldsymbol{\omega} = \int_S \mathrm{curl}(X) \cdot n \, d\sigma \qquad (15.6.84)$$

while the right hand side is the line integral along the boundary of S, which is a curve γ

$$\int_{\partial S} \boldsymbol{\omega} = \int_\gamma X \cdot \dot{\gamma} \, ds \qquad (15.6.85)$$

Thus Stokes' theorem, in this case, reads as

$$\int_S \mathrm{curl}(X) \cdot n \, d\sigma = \int_\gamma X \cdot \dot{\gamma} \, ds \qquad (15.6.86)$$

which is the *Stokes' theorem* (which, when everything lives on the xy-plane in $\mathbb{R}^2 \subset \mathbb{R}^3$, is called the *Green theorem*).

Finally, let S be a volume in $M = \mathbb{R}^3$. Given a vector field $X = X^\mu \partial_\mu$, this can be transformed into a 2-form $\boldsymbol{\omega} = *(X^\flat)$. The differential of such a form is a 3-form $d\boldsymbol{\omega} = d * (X^\flat)$ which by Hodge duality corresponds to the divergence $\mathrm{div}(X) = *d * (X^\flat)$.

In Cartesian coordinates, one has

$$\boldsymbol{\omega} = X^1 \, dy \wedge \, dz - X^2 \, dx \wedge \, dz + X^3 \, dx \wedge \, dy \qquad \Rightarrow d\boldsymbol{\omega} = \left(\partial_x X^1 + \partial_y X^2 + \partial_z X^3\right) \, d\sigma \qquad (15.6.87)$$

Then the Stokes' theorem reads, in this case, as

$$\int_S \mathrm{div}(X) \, d\sigma = \int_{\partial S} X \cdot n \, d\nu \qquad (15.6.88)$$

which is the *divergence theorem.*

Integration on cohomology classes

Let $\boldsymbol{\omega} \in \Omega^k(M)$ be a k-form and $S \in C_k(M, \mathbb{Z})$ be a k-chain. We can define a linear map

$$I_{\boldsymbol{\omega}} : C_k(M, \mathbb{Z}) \to \mathbb{R} : S \mapsto I_{\boldsymbol{\omega}}(S) := \int_S \boldsymbol{\omega} \tag{15.6.89}$$

which is an element of $C^k(M, \mathbb{Z})$, i.e. a k-cochain.

If we restrict to integrals of closed forms $\boldsymbol{\omega}$ on closed regions S, the integral becomes independent of cohomology representative, namely

$$\int_{S+\partial D} \boldsymbol{\omega} + d\boldsymbol{\theta} = \int_{S+\partial D} \boldsymbol{\omega} + \int_{S+\partial D} d\boldsymbol{\theta} = \int_S \boldsymbol{\omega} + \int_D d\boldsymbol{\omega} + \int_{\partial S} \boldsymbol{\theta} = \int_S \boldsymbol{\omega} \tag{15.6.90}$$

In other words, the the integral is recognized to be compatible with the (co)homological quotient. We can in fact define the integral of a cohomology de Rham class $[\boldsymbol{\omega}]$ on a homology class $[S]$ by using any representatives pair

$$\int_{[S]} [\boldsymbol{\omega}] := \int_{S+\partial D} \boldsymbol{\omega} + d\theta = \int_S \boldsymbol{\omega} \tag{15.6.91}$$

That is interpreted as a linear map $I : H^k_{dR}(M) \to H^k(M, \mathbb{Z}) : [\boldsymbol{\omega}] \mapsto I_{[\boldsymbol{\omega}]}$, where $I_{[\boldsymbol{\omega}]} : Z_k(M, \mathbb{Z}) \to \mathbb{R} : S \mapsto I_{[\boldsymbol{\omega}]}(S) = \int_S \boldsymbol{\omega}$ is a linear map, a.k.a. a k-cochain. The de Rham theorem claim this map I to be an isomorphism.

References

add

Add reference to the notes by C.Tiee

Baez-Munian

Chapter 16. Geometric structures on manifolds

1. Metric structure

The first structure introduced on a manifold M is a *metric*. A metric is a covariant, symmetric, non-degenerate tensor field of rank 2 of signature (r, s) (with $r + s = m = \dim(M)$).

A metric provides an inner product to any tangent space $T_x M$.

For metrics and notational compatibility with the literature we shall sometimes use some abuse of notation like

$$dx^2 = dx \otimes dx \qquad dy^2 = dy \otimes dy \qquad \dots \qquad dx\ dy = \tfrac{1}{2}\left(\,dx \otimes dy + \,dy \otimes dx\right) \tag{16.1.1}$$

Accordingly, we shall write the standard metric of $\mathbb{R}^2$ in polar coordinates as $g = \,dr^2 + r^2\,d\theta^2$ instead of the more correct form

$$g = \,dr \otimes \,dr + r^2\,d\theta \otimes \,d\theta \tag{16.1.2}$$

and a general metric $g = A\,dx \otimes \,dx + B\,dy \otimes \,dy + C(\,dx \otimes \,dy + \,dy \otimes \,dx)$ on a surface can be written as

$$g = A\,dx^2 + B\,dy^2 + 2C\,dx\ dy$$

To be honest, using $\boldsymbol{dr}$ (bold as for forms) in a metric is an abuse of language since $\,dr$ and $\,d\theta$ are 1-forms on their own, but they enter in a metric as covectors (in fact their product is the tensor product, not the wedge product). However, switching from bold to unbold symbols depending on where the object is used would be confusing.

Being the metric non-degenerate, one can define the inverse $g^{-1} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$, which provides an inner product on each $T_x^* M$. We set $\sqrt{g}$ for the scalar density of weight 1 defined by the square root of the absolute value of the determinant of metric g.

The metric tensor g and its inverse g^{-1} allow isomorphisms among tensor (densities) spaces so that we can raise indices by g^{-1} and lower indices by g. An index which has been moved up or down is replaced by a dot like in

$$\epsilon^{\alpha}_{.\,\beta\mu\nu} := g^{\alpha\lambda}\epsilon_{\lambda\beta\mu\nu} \tag{16.1.3}$$

The inner product on tangent space when it is definite positive allows one to define *length squared* of vectors

$$|v|^2 = v \cdot v \tag{16.1.4}$$

and the *angle* between vectors

$$\cos(\theta) = \frac{v \cdot u}{|v||u|} \tag{16.1.5}$$

For a definite-positive inner product, the quantity $|v|^2$ is always positive (and zero only for $v = 0$) so that the length of the vector v can be defined as $|v| = \sqrt{|v|^2}$ and the quantity $\frac{v \cdot u}{|v||u|}$ is in the range $[-1, 1]$ so that there exist two angles θ which satisfy (16.1.5).

For a general signature, $|v|^2$ can be positive, zero or negative and one cannot take the square root of it. Analogously, $\frac{v \cdot u}{|v||u|}$ is not restricted to the range $[-1, 1]$ and one cannot define angles. Thus, in general signature, one cannot rely too much on elementary geometric intuition.

Existence of Euclidean metrics

Also as an application of the partition of unity, we can show that any manifold M allows global Euclidean metrics.

Let us consider a smooth manifold M. It has an atlas supported on an open covering $\mathcal{U} = \{U_\alpha : \alpha \in I\}$. Since the manifold is, by definition, paracompact, the covering can be chosen, without loss of generality, to be locally finite. Let $(\varphi_\alpha, U_\alpha)_{a \in I}$ be a smooth partition of unity supported by that open covering, meaning that $\mathrm{supp}(\varphi_\alpha) \subset U_\alpha$.

Let us fix an open set in the cover $\alpha \in I$ and define on U_α a metric

$$^\alpha g = \delta_{\mu\nu} \, dx^\mu \odot \, dx^\nu \tag{16.1.6}$$

using coordinates x^μ defined on U_α. That corresponds to declare that coordinates are orthonormal for that metric, of course, locally in U_α, since there is no clue about how to extend $^\alpha g$ as a metric out of U_α.

Here the partition of unity kicks in, giving us a way to extend $^\alpha g$ out of U_α, as a symmetric tensor field of rank $(0, 2)$, if not as a metric, by setting

$$^\alpha G = \varphi_\alpha \cdot {}^\alpha g \tag{16.1.7}$$

By that we mean that, even if $^\alpha g$ is not defined out of U_α, we can extend smoothly $^\alpha G$ to be zero out of U_α since the weight φ_α vanishes smoothly before the boundary of its support is reached. Of course, $^\alpha G$ is not a global metric, being degenerate out of U_α.

Finally, let us set

$$g := \sum_{\alpha \in I} \varphi_\alpha \cdot {}^\alpha g \tag{16.1.8}$$

which is a smooth tensor field on M. It is well defined, since the cover is locally finite and the sum at any point $x \in M$ is a finite sum.

At any point $x \in M$, and for a non-zero vector $u \in T_x M$, one has

$$g(u, u) = \sum_{\alpha \in I} \varphi_\alpha(x) \cdot u^\mu \delta_{\mu\nu} u^\nu \tag{16.1.9}$$

In the sum, just a finite number of contributions are non-zero, the ones for which $\varphi_\alpha(x) > 0$.

For each of these contributions, $u^\mu \delta_{\mu\nu} u^\nu$ is positive (since we chose local Euclidean metrics ${}^\alpha g$ in the beginning) and $\varphi_\alpha(x)$ is positive as a part of definition of partition of unity. Then $g(u, u) > 0$ and the tensor g is definite-positive at each point $x \in M$. Accordingly, it is a global Euclidean metric on M.

2. Connections and curvature on manifolds

There is a particular combination of derivatives of the metric which is endowed with a special meaning.

Let us define *Christoffel symbols* as

$$\{g\}^\alpha_{\beta\nu} = \tfrac{1}{2} g^{\alpha\lambda} \left(-\partial_\lambda g_{\beta\nu} + \partial_\beta g_{\nu\lambda} + \partial_\nu g_{\lambda\beta} \right) \tag{16.2.1}$$

This combination appears naturally in a number of instances.

For example, if one considers a point constrained to move along a surface with local coordinates q^λ, the motion is described by a Lagrangian

$$L = \tfrac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \tag{16.2.2}$$

and its equations of motion reads as

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\lambda} \right) - \frac{\partial L}{\partial q^\lambda} &= \frac{d}{dt}\left(g_{\lambda\mu} \dot{q}^\mu \right) - \tfrac{1}{2}\partial_\lambda g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = g_{\lambda\mu} \ddot{q}^\mu + \partial_\nu g_{\lambda\mu} \dot{q}^\mu \dot{q}^\nu - \tfrac{1}{2}\partial_\lambda g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0 \\
&\Rightarrow \quad \ddot{q}^\sigma + \tfrac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \dot{q}^\mu \dot{q}^\nu = \ddot{q}^\sigma + \{g\}^\sigma_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0
\end{aligned}
\tag{16.2.3}
$$

We can compute transformation rules for Christoffel symbols starting form transformation rules of the metric.

The metric transforms as

$$g'_{\mu\nu}(x') = \bar{J}^\rho_\mu(x') g_{\rho\sigma}(x) \bar{J}^\sigma_\nu(x') \tag{16.2.4}$$

and we have to consider partial derivatives of it with respect to x'^λ

$$\partial'_\lambda g'_{\mu\nu} = \bar{J}^\rho_{\mu\lambda} g_{\rho\sigma} \bar{J}^\sigma_\nu + \bar{J}^\rho_\mu g_{\rho\sigma} \bar{J}^\sigma_{\nu\lambda} + \bar{J}^\rho_\mu \partial_\theta g_{\rho\sigma} \bar{J}^\sigma_\nu \bar{J}^\theta_\lambda \tag{16.2.5}$$

Then by summing on permutation of indices, one obtains

$$
\begin{aligned}
\{g'\}^\alpha_{\mu\nu} = \tfrac{1}{2} g'^{\alpha\lambda} \Big(&- \underline{\bar{J}^\rho_{\mu\lambda} g_{\rho\sigma} \bar{J}^\sigma_\nu} - \bar{J}^\rho_\mu g_{\rho\sigma} \bar{J}^\sigma_{\nu\lambda} - \bar{J}^\rho_\mu \partial_\theta g_{\rho\sigma} \bar{J}^\sigma_\nu \bar{J}^\theta_\lambda + \underline{\bar{J}^\rho_{\nu\mu} g_{\rho\sigma} \bar{J}^\sigma_\lambda} + \bar{J}^\rho_\nu g_{\rho\sigma} \bar{J}^\sigma_{\lambda\mu} + \bar{J}^\rho_\nu \partial_\theta g_{\rho\sigma} \bar{J}^\sigma_\lambda \bar{J}^\theta_\mu + \\
&+ \bar{J}^\rho_{\lambda\nu} g_{\rho\sigma} \bar{J}^\sigma_\mu + \underline{\bar{J}^\rho_\lambda g_{\rho\sigma} \bar{J}^\sigma_{\mu\nu}} + \bar{J}^\rho_\lambda \partial_\theta g_{\rho\sigma} \bar{J}^\sigma_\mu \bar{J}^\theta_\nu \Big) = \tfrac{1}{2} J^\alpha_\gamma g^{\gamma\epsilon} \left((-\partial_\epsilon g_{\rho\sigma} + \partial_\rho g_{\sigma\epsilon} + \partial_\sigma g_{\epsilon\rho}) \bar{J}^\rho_\mu \bar{J}^\sigma_\nu + 2\bar{J}^\rho_{\nu\mu} g_{\rho\epsilon} \right) = J^\alpha_\gamma \left(\{g\}^\gamma_{\rho\sigma} \bar{J}^\rho_\mu \bar{J}^\sigma_\nu + \bar{J}^\gamma_{\mu\nu} \right)
\end{aligned}
\tag{16.2.6}
$$

Thus Christoffel symbols, which transform as

$$\{g'\}^{\alpha}_{\mu\nu} = J^{\alpha}_{\gamma} \left(\{g\}^{\gamma}_{\rho\sigma} \bar{J}^{\rho}_{\mu} \bar{J}^{\sigma}_{\nu} + \bar{J}^{\gamma}_{\mu\nu} \right) \tag{16.2.7}$$

are not tensors or tensor densities.

We shall give below a better, more general, and more geometric definition.

For many years after Einstein and Hilbert defined GR, the role of connections has been obscure in nature. Christoffel and Riemann already knew the later called *Christoffel symbols* (in 1869 when covariant differentiation was introduced). However, it was Levi Civita and Ricci Curbastro who, in 1900, essentially invented tensor calculus. They worked until 1917 to define parallel transport (as well as the Levi Civita connection that has Christoffel symbols as coefficients). One could argue that parallel transport is the first hint about the true nature of connections on manifols. Then Cartan (1926) generalised to more general connection forms, and, while Koszul defined connections on vector bundles, Ehresmann defined them on principal bundles in 1950. At that point the story of gauge fields has begun.

For now, a *connection* on a manifold M is any object with a collection of components $\Gamma^{\alpha}_{\mu\nu}$ which transforms under coordinate changes as

$$\Gamma'^{\alpha}_{\mu\nu} = J^{\alpha}_{\epsilon} \left(\Gamma^{\epsilon}_{\rho\sigma} \bar{J}^{\rho}_{\mu} \bar{J}^{\sigma}_{\nu} + \bar{J}^{\epsilon}_{\mu\nu} \right) \tag{16.2.8}$$

Let us remark that if Γ is a connection, then neither $-\Gamma$ nor 3Γ is a connection.

Let us stress that transformation rules of a connection are affine. The difference of two connections transforms as tensor of rank (1.3). The space of connections is an affine space modelled on tensors of rank $(1, 2)$.

Covariant derivatives

Partial derivatives are pretty bad operators on tensor fields since they do not go along with transformation rules except few cases.

If $F : M \to \mathbb{R}$ is a function, its transformation rules are

$$F'(x') = F(x) \tag{16.2.9}$$

and we can take the partial derivative with respect to x^{μ} to obtain

$$\partial'_{\alpha} F' J^{\alpha}_{\mu} = \partial_{\mu} F \qquad \partial'_{\alpha} F' = \bar{J}^{\mu}_{\alpha} \partial_{\mu} F \tag{16.2.10}$$

Thence the partial derivatives of a function transform as the components of a 1-form. That is why the differential of a function is a 1-form.

For a vector field X, things do not go that smooth. The components of a vector field X^{μ} transform as

$$X'^{\mu}(x') = J^{\mu}_{\nu}(x) X^{\nu}(x) \tag{16.2.11}$$

and by taking partial derivatives with respect to x^{λ}, one obtains

$$\partial'_{\rho} X'^{\mu} = J^{\mu}_{\nu} \partial_{\lambda} X^{\nu} \bar{J}^{\lambda}_{\rho} + J^{\mu}_{\nu\lambda} X^{\nu} \bar{J}^{\lambda}_{\rho} \tag{16.2.12}$$

Then partial derivatives of the components of a vector field are not a tensor. They transform ugly with a piece of transformation rules which depends on the Hessian $J^\mu_{\nu\lambda}$ of the coordinate change.

Also connections have this sort of transformation rules. In fact we can solve the connection transformation rules for the Hessian

$$\bar{J}^\epsilon_{\mu\nu} = \bar{J}^\epsilon_\alpha \Gamma'^\alpha_{\mu\nu} - \Gamma^\epsilon_{\rho\sigma} \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \qquad\qquad J^\epsilon_{\mu\nu} = J^\epsilon_\alpha \Gamma^\alpha_{\mu\nu} - \Gamma'^\epsilon_{\rho\sigma} J^\rho_\mu J^\sigma_\nu \tag{16.2.13}$$

Then we can re-express (16.2.12) as

$$\partial'_\rho X'^\mu = J^\mu_\nu \partial_\lambda X^\nu \bar{J}^\lambda_\rho + \left(J^\mu_\sigma \Gamma^\sigma_{\nu\lambda} - \Gamma'^\mu_{\rho\sigma} J^\rho_\nu J^\sigma_\lambda \right) X^\nu \bar{J}^\lambda_\rho \tag{16.2.14}$$

which can be rearranged as

$$\left(\partial'_\rho X'^\mu + \Gamma'^\mu_{\rho\sigma} X'^\sigma \right) = J^\mu_\sigma \left(\partial_\lambda X^\sigma + \Gamma^\sigma_{\nu\lambda} X^\nu \right) \bar{J}^\lambda_\rho \tag{16.2.15}$$

This shows that the particular combination $\nabla_\lambda X^\sigma := \partial_\lambda X^\sigma + \Gamma^\sigma_{\nu\lambda} X^\nu$ transforms as a tensor.

The combination $\nabla_\lambda X^\sigma$ is called the *covariant derivative* of a vector field X with respect to a connection Γ.

In $\mathbb{R}^n$ with Cartesian coordinates, the components of the metric tensor $g_{\mu\nu}$ are constant, Christoffel symbols are identically zero and $\nabla_\lambda X^\sigma = \partial_\lambda X^\sigma$. The covariant derivative there coincides with partial derivatives. In that sense, the covariant derivative extends partial derivatives on general manifolds.

The same computation can be repeated for a 1-form $\boldsymbol{\omega}$. Components transform as

$$\omega'_\mu(x') = \bar{J}^\nu_\mu(x') \omega_\nu(x) \qquad \Rightarrow \partial'_\epsilon \omega'_\mu = \bar{J}^\lambda_\epsilon \bar{J}^\nu_\mu \partial_\lambda \omega_\nu + \bar{J}^\nu_{\mu\epsilon} \omega_\nu \tag{16.2.16}$$

and the Hessian can be expressed in terms of the connection as

$$\partial'_\epsilon \omega'_\mu = \bar{J}^\lambda_\epsilon \bar{J}^\nu_\mu \partial_\lambda \omega_\nu + \left(\bar{J}^\nu_\alpha \Gamma'^\alpha_{\mu\epsilon} - \Gamma^\nu_{\rho\sigma} \bar{J}^\rho_\mu \bar{J}^\sigma_\epsilon \right) \omega_\nu \qquad \Rightarrow \partial'_\epsilon \omega'_\mu - \Gamma'^\alpha_{\mu\epsilon} \omega'_\alpha = \left(\partial_\sigma \omega_\rho - \Gamma^\alpha_{\rho\sigma} \omega_\alpha \right) \bar{J}^\rho_\mu \bar{J}^\sigma_\epsilon \tag{16.2.17}$$

The combination $\nabla_\sigma \omega_\rho := \partial_\sigma \omega_\rho - \Gamma^\alpha_{\rho\sigma} \omega_\alpha$ transforms as a tensor and it is called the *covariant derivative* of a 1-form $\boldsymbol{\omega}$ with respect to a connection Γ.

For a general tensor field t, one has one (anti-)Jacobian for each index in the transformation rules, each under derivation gives a term with a Hessian, each contributing with a connection term in the covariant derivative. When the index is up it behaves as in a vector field, when it is down it behaves as a 1-form.

For example, a tensor field of rank $(1, 2)$ has a covariant derivative defined as

$$\nabla_\alpha t^\mu_{\rho\sigma} = \partial_\alpha t^\mu_{\rho\sigma} + \Gamma^\mu_{\epsilon\alpha} t^\epsilon_{\rho\sigma} - \Gamma^\epsilon_{\rho\alpha} t^\mu_{\epsilon\sigma} - \Gamma^\epsilon_{\sigma\alpha} t^\mu_{\rho\epsilon} \tag{16.2.18}$$

The covariant derivative of a tensor field of rank (p, q) is a tensor field of rank $(p + 1, q)$.

With respect to derivation, tensor densities behave like tensors, with an extra Hessian coming from the $\det(J)$ in the transformation rule.

Let us, for example, consider a tensor density of weight w and rank $(0, 1)$.

Components transform as

$$v'^\mu = (\det(J))^{-w} J^\mu_\nu v^\nu \qquad \Rightarrow \partial'_\alpha v'^\mu = (\det(J))^{-w} \left(J^\mu_\nu \partial_\lambda v^\nu \bar{J}^\lambda_\alpha + \bar{J}^\lambda_\alpha J^\mu_{\nu\lambda} v^\nu - w \bar{J}^\lambda_\alpha \bar{J}^\rho_\sigma J^\sigma_{\rho\lambda} J^\mu_\nu v^\nu \right)$$

$$\Rightarrow \quad \partial'_\alpha v'^\mu + \Gamma'^\mu_{\rho\alpha} v'^\rho - w\Gamma'^\epsilon_{\epsilon\alpha} v'^\mu = (\det(J))^{-w} J^\mu_\rho \left(\partial_\lambda v^\rho + \Gamma^\rho_{\nu\lambda} v^\nu - w\Gamma^\sigma_{\sigma\lambda} v^\rho \right) \bar{J}^\lambda_\alpha$$

(16.2.19)

The combination $\nabla_\lambda v^\rho := \partial_\lambda v^\rho + \Gamma^\rho_{\nu\lambda} v^\nu - w\Gamma^\sigma_{\sigma\lambda} v^\rho$ transforms as a tensor density of weight w and it is called the *covariant derivative* of a tensor density of weight w and rank $(0,1)$ with respect to a connection Γ.

In general, a tensor density t allows a covariant derivative with an extra $-w\Gamma^\sigma_{\sigma\lambda} \otimes t$ with respect to the corresponding tensor.

Since now all covariant derivatives we have seen except that of functions do depend on the connection Γ. This is not always the case. Consider the covariant divergence of a vector density of weight 1. One has

$$\nabla_\rho v^\rho = \partial_\rho v^\rho + \Gamma^\rho_{\nu\rho} v^\nu - \Gamma^\sigma_{\sigma\rho} v^\rho = \partial_\rho v^\rho + T^\rho_{\nu\rho} v^\nu$$

(16.2.20)

where we set $T^\rho_{\beta\nu} := \Gamma^\rho_{\beta\nu} - \Gamma^\rho_{\nu\beta}$ that is a tensor called the *torsion* of Γ.

Derive transformation laws of T.

If the connection Γ is torsionless, i.e. $T = 0$, then one simply has

$$\nabla_\rho v^\rho = \partial_\rho v^\rho$$

(16.2.21)

Consequently, the covariant divergence is equal to the ordinary divergence and it does not depend on the connection.

The same situation happens for a tensor density $A^{\alpha\beta}$ of weight 1 and rank $(0,2)$ which is antisymmetric in $[\alpha\beta]$.

The covariant derivative is

$$\nabla_\lambda A^{\alpha\beta} = \partial_\lambda A^{\alpha\beta} + \Gamma^\alpha_{\epsilon\lambda} A^{\epsilon\beta} + \Gamma^\beta_{\epsilon\lambda} A^{\alpha\epsilon} - \Gamma^\epsilon_{\epsilon\lambda} A^{\alpha\beta}$$

(16.2.22)

The covariant divergence is

$$\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + \Gamma^\alpha_{\epsilon\alpha} A^{\epsilon\beta} + \Gamma^\beta_{\epsilon\alpha} A^{\alpha\epsilon} - \Gamma^\epsilon_{\epsilon\alpha} A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + T^\epsilon_{\alpha\epsilon} A^{\alpha\beta} + \tfrac{1}{2} T^\beta_{\epsilon\alpha} A^{\alpha\epsilon}$$

(16.2.23)

which, for a torsionless connection, reads as

$$\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta}$$

(16.2.24)

Whenever one has more than one connection around, or the covariant derivative is stressed to be done with respect to a connection Γ, we shall write $\overset{\Gamma}{\nabla}_\mu t^\alpha_\beta$. Whenever covariant derivative is done with respect to Christoffel symbols of a metric g, we shall write $\overset{g}{\nabla}_\mu t^\alpha_\beta$. Whenever a covariant derivative turns out to be independent of the connection, we shall write (for example for a vector density of weight 1) $\overset{*}{\nabla}_\mu v^\mu$.

Torsion

Although Christoffel symbols are symmetric in the lower indices, we are not asking to coefficients $\Gamma^\alpha_{\mu\nu}$ of a general connection to be symmetric in $(\mu\nu)$.

Let us define the *torsion* of the connection Γ as (the object with components)

$$T^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \qquad (16.2.25)$$

which hence transform as

$$T'^\alpha_{\mu\nu} = J^\alpha_\epsilon T^\epsilon_{\rho\sigma} \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \qquad (16.2.26)$$

Then the torsion is a tensor and it is antisymmetric in the lower indices $[\mu\nu]$.

Hence Christoffel symbols are a torsionless connection induced by a metric structure g. This connection induced by g is called the *Levi Civita connection* of the metric g. The Levi Civita connection also has the following property

$$\overset{g}{\nabla}_\alpha g_{\mu\nu} = 0 \qquad (16.2.27)$$

Let us simply expand the covariant derivative

$$\overset{g}{\nabla}_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \{g\}^\epsilon_{\mu\alpha} g_{\epsilon\nu} - \{g\}^\epsilon_{\nu\alpha} g_{\mu\epsilon} = \partial_\alpha g_{\mu\nu} - \tfrac{1}{2}\left(-\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\alpha\nu} + \underline{\partial_\alpha g_{\nu\mu}} - \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} + \underline{\partial_\alpha g_{\mu\nu}} \right) = \partial_\alpha g_{\mu\nu} - \partial_\alpha g_{\mu\nu} = 0 \qquad (16.2.28)$$

When we have a connection Γ for which $\overset{\Gamma}{\nabla}_\alpha g_{\mu\nu} = 0$, we say the Γ is *compatible* with the metric g. What we just proved is that the Levi Civita connection of g is a torsionless connection compatible with g.

Theorem *(16.2.29): the Levi Civita connection of g is the only torsionless connection compatible with g.*

Proof: *let Γ be a torsionless connection which is compatible with g, i.e. such that $\overset{\Gamma}{\nabla}_\alpha g_{\mu\nu} = 0$. We have*

$$\partial_\alpha g_{\mu\nu} = \Gamma^\epsilon_{\mu\alpha} g_{\epsilon\nu} + \Gamma^\epsilon_{\nu\alpha} g_{\mu\epsilon} \qquad (16.2.30)$$

Let us permute indices

$$\partial_\mu g_{\nu\alpha} = \Gamma^\epsilon_{\nu\mu} g_{\epsilon\alpha} + \Gamma^\epsilon_{\alpha\mu} g_{\nu\epsilon} \qquad\qquad \partial_\nu g_{\alpha\mu} = \Gamma^\epsilon_{\alpha\nu} g_{\epsilon\mu} + \Gamma^\epsilon_{\mu\nu} g_{\alpha\epsilon} \qquad (16.2.31)$$

and subtract (16.2.30) from the sum of (16.2.31). One obtains

$$-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} = -\Gamma^\epsilon_{\mu\alpha} g_{\epsilon\nu} - \Gamma^\epsilon_{\nu\alpha} g_{\mu\epsilon} + \underline{\Gamma^\epsilon_{\nu\mu} g_{\epsilon\alpha}} + \Gamma^\epsilon_{\alpha\mu} g_{\nu\epsilon} + \Gamma^\epsilon_{\alpha\nu} g_{\epsilon\mu} + \underline{\Gamma^\epsilon_{\mu\nu} g_{\alpha\epsilon}} = 2\Gamma^\epsilon_{\mu\nu} g_{\alpha\epsilon} \qquad (16.2.32)$$

from which it simply follows that

$$\Gamma^\epsilon_{\mu\nu} = \tfrac{1}{2} g^{\alpha\epsilon} \left(-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} \right) \qquad (16.2.33)$$

Then $\Gamma = \{g\}$ is the Levi Civita connection of g. ∎

Levi Civita connections are special among connections not only because they are torsionless. As we shall see hereafter discussing curvature, they are also special among torsionless connections.

Parallel transport

A connection Γ on a manifold M defines a parallel transport of vectors along a curve γ.

The parallel transport relies on the notion of *horizontal vector* on TM. Let us consider TM with natural fibered coordinates (x^μ, v^μ). A vector $\hat{w} \in TTM$ at the point $(x, v) \in TM$ is *horizontal* with respect to the connection Γ iff it is in the form

$$\hat{w} = w^\mu \left(\frac{\partial}{\partial x^\mu} - \Gamma^\alpha_{\beta\mu}(x)v^\beta \frac{\partial}{\partial v^\alpha} \right) \tag{16.2.34}$$

Let us consider a family of vectors $(\gamma(s), X(s))$ defined on the curve. This family can be considered as a curve $\hat{\gamma} : \mathbb{R} \to TM : s \mapsto (\gamma(s), X(s))$. Its tangent vector is a point in TTM, namely

$$\dot{\hat{\gamma}} = \dot{\gamma}^\mu(s) \frac{\partial}{\partial x^\mu} + \dot{X}^\mu(s) \frac{\partial}{\partial v^\mu} \tag{16.2.35}$$

The vectors $X(s)$ along γ are *parallel* with respect to Γ iff the tangent vector $\dot{\hat{\gamma}}$ is horizontal, i.e.

$$\dot{X}^\mu + \Gamma^\mu_{\alpha\beta}(\gamma(s))X^\alpha(s)\dot{\gamma}^\beta(s) = 0 \tag{16.2.36}$$

Equation (16.2.36) is a system of ODE for the unknowns $X^\mu(s)$ and is called the *equation for parallel transport* for a given curve $\gamma(s)$. It is in normal form and, if one gives initial conditions $X^\mu(0)$, the solution $X^\mu(s)$ to the Cauchy problem exists and it is unique.

In $M = \mathbb{R}^m$ with Cartesian coordinates x^μ the Levi Civita connection vanishes (or to be precise Christoffel symbols do) and the equation of parallel transport reads as

$$\dot{X}^\mu = 0 \tag{16.2.37}$$

meaning that a vector is parallelly transported along a curve γ iff its components stay constant.
Of course, this is not the case for a different connection. Then changing the connection gives a different notion of parallel transport along the same curve.

If we consider a vector field X on M that defines a family of vectors along the curve γ which reads as

$$X^\mu(s) = X^\mu(\gamma(s)) \qquad \Rightarrow \dot{X}^\mu(s) = \partial_\alpha X^\mu(\gamma(s))\dot{\gamma}^\alpha(s) \tag{16.2.38}$$

then the vector $X(s)$ is parallelly transported along the curve γ with respect to the connection Γ if it satisfies the equation (16.2.36), i.e.

$$\dot{X}^\mu + \Gamma^\mu_{\alpha\beta}(\gamma(s))X^\alpha(s)\dot{\gamma}^\beta(s) = \dot{\gamma}^\alpha(s)\left(\partial_\alpha X^\mu(\gamma(s)) + \Gamma^\mu_{\sigma\alpha}(\gamma(s))X^\sigma(s)\right) = \dot{\gamma}^\alpha(s)\overset{\Gamma}{\nabla}_\alpha X^\mu = 0 \tag{16.2.39}$$

In this case, we say that the vector field X is *parallelly transported* along the curve γ with respect to the connection Γ.

Parallel transport comes with a theory attached which is called *holonomy* which is also important in some applications to gauge theories and quantum gravity. However, as a matter of fact, it is worth discussing holonomy, also holonomy of a connection on a manifold, as a particular case of holonomy for a general principal connection. For this reason, we postpone the discussion to have the most general definition of connections first.

Geodesics

Given a curve γ in M, one can consider a special family of vectors along the curve, namely the tangent vector $\dot\gamma$, which is locally described by $X^\mu(s) = \dot\gamma^\mu(s)$. Such a vector is paralelly transported along γ iff

$$\ddot\gamma^\mu(s) + \Gamma^\mu_{\alpha\beta}(\gamma(s))\dot\gamma^\alpha(s)\dot\gamma^\beta(s) = 0 \qquad (16.2.40)$$

This can be seen as a system of ODE for the unknowns $\gamma^\mu(s)$. It is in normal form, hence we expect a solution for initial conditions $(\gamma^\mu(0), \dot\gamma^\mu(0))$. A solution of equation (16.2.40) is called a *geodesic motion* and one has a geodesic motion for each initial velocity. Let us stress that equation (16.2.40) is not invariant with respect to reparameterisations, thus geodesic motions are trajectories parameterised in a specific way.

Given a curve γ, the vector

$$a_\Gamma = \left(\ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta\right)\partial_\mu \in TM \qquad (16.2.41)$$

is called *intrinsic acceleration* of the curve (with respect to Γ). Geodesic motions are thence curves with vanishing intrinsic acceleration.

This condition depends on the parameterisation. A more geometric alternative is requiring that the intrinsic acceleration is parallel to the velocity, namely that there exists a function $\lambda(s)$ such that:

$$\ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = \lambda\dot\gamma^\mu \qquad (16.2.42)$$

These equations are invariant with respect to reparameterisations and a solution is called a *geodesic trajectory.*

One can show (see here) that for any geodesic trajectory there exists a parameterisation which is a geodesic motion as well as that any reparameterisation of a geodesic motion is a geodesic trajectory. Accordingly, geodesic trajectories can be found as arbitrary reparameterisations of geodesic motions.

Consider a vector field X and one of its integral curve γ. Then γ is a geodesic motion iff

$$X^\beta\left(\partial_\beta X^\mu + \Gamma^\mu_{\alpha\beta}X^\alpha\right) = X^\beta\overset{\Gamma}{\nabla}_\beta X^\mu = 0 \qquad (16.2.43)$$

In this case, the vector field X is called a *geodesic vector field.* The integral curves of a geodesic field are geodesic motions.

> That means that a geodesic field collects *some* geodesics. If we consider a hypersurface S transverse to X, we can regard integral curves to be selected by initial conditions given on $x_0 \in S$ requiring $(\gamma(0) = x_0, \dot\gamma(0) = X(x_0))$. The surface S can be taken with a certain freedom, a geodesic field selects a specific initial velocity at each point of S.
>
> Of course, we already discussed there is a geodesic motion for each initial velocity at each point. Geodesic fields just select a congruence of geodesic motions.

One can also require that the integral curve γ is a geodesic trajectory

$$X^\beta\overset{\Gamma}{\nabla}_\beta X^\mu = \lambda X^\mu \qquad (16.2.44)$$

These vector fields are called *geodesic directions*, since if X is a geodesic direction then any $X' = F \cdot X$ is a geodesic direction as well.

For any of such field X, one can find a (everywhere positive) function F so that the vector field $F \cdot X' = X$ obeys to the condition

$$X'^\beta\overset{\Gamma}{\nabla}_\beta(FX'^\mu) = X'^\beta\left(\overset{\Gamma}{\nabla}_\beta(F)X'^\mu + F\overset{\Gamma}{\nabla}_\beta X'^\mu\right) = \lambda X'^\mu \qquad \Rightarrow X'^\beta\overset{\Gamma}{\nabla}_\beta X'^\mu = F^{-1}\left(\lambda - X'(F)\right)X'^\mu = \lambda' X'^\mu \qquad (16.2.45)$$

Thus one can rescale the field X so that $\lambda = X'(F)$ and X' is a geodesic field.

Curvature

Once a connection $\Gamma^\alpha_{\beta\mu}$ is given on M, we can show that

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]v^\alpha = R^\alpha{}_{\beta\mu\nu}v^\beta + T^\gamma_{\mu\nu}\overset{\Gamma}{\nabla}_\gamma v^\alpha \tag{16.2.46}$$

where we set

$$R^\alpha{}_{\beta\mu\nu} := \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} \qquad\qquad T^\gamma_{\mu\nu} = \Gamma^\gamma_{\mu\nu} - \Gamma^\gamma_{\nu\mu} \tag{16.2.47}$$

Let us expand the commutator

$$\begin{aligned}
[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]v^\alpha &= \left[\partial_\mu\left(\boxed{\partial_\nu v^\alpha} + \Gamma^\alpha_{\beta\nu}v^\beta\right) + \Gamma^\alpha_{\gamma\mu}\left(\partial_\nu v^\gamma + \Gamma^\gamma_{\beta\nu}v^\beta\right) - \Gamma^\gamma_{\nu\mu}\left(\partial_\gamma v^\alpha + \Gamma^\alpha_{\beta\gamma}v^\beta\right) - [\mu\nu]\right] = \\
&= \left[\partial_\mu\Gamma^\alpha_{\beta\nu}v^\beta + \textcolor{red}{\Gamma^\alpha_{\beta\nu}\partial_\mu v^\beta} + \Gamma^\alpha_{\gamma\mu}\left(\textcolor{red}{\partial_\nu v^\gamma} + \Gamma^\gamma_{\beta\nu}v^\beta\right) - [\mu\nu]\right] - T^\gamma_{\nu\mu}\overset{\Gamma}{\nabla}_\gamma v^\alpha = \\
&= \left(\partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu}\right)v^\beta + T^\gamma_{\mu\nu}\overset{\Gamma}{\nabla}_\gamma v^\alpha = R^\alpha{}_{\beta\mu\nu}v^\beta + T^\gamma_{\mu\nu}\overset{\Gamma}{\nabla}_\gamma v^\alpha
\end{aligned} \tag{16.2.48}$$

This is by definition a tensor called the *curvature tensor*. It is manifestly antisymmetric in the last two indices $[\mu\nu]$.

We can also prove directly that $R^\alpha{}_{\beta\mu\nu}$ transforms as a tensor. For,

$$\begin{aligned}
R'^\alpha{}_{\beta\mu\nu} &= \left[\partial'_\mu\Gamma'^\alpha_{\beta\nu} + \Gamma'^\alpha_{\gamma\mu}\Gamma'^\gamma_{\beta\nu} - [\mu\nu]\right] = \\
&= \left[\partial'_\mu\left(J^\alpha_\rho\left(\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu + \bar{J}^\rho_{\beta\nu}\right)\right) + J^\alpha_\rho\left(\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\gamma\bar{J}^\lambda_\mu + \bar{J}^\rho_{\gamma\mu}\right)J^\gamma_\theta\left(\Gamma^\theta_{\eta\epsilon}\bar{J}^\eta_\beta\bar{J}^\epsilon_\nu + \bar{J}^\theta_{\beta\nu}\right) - [\mu\nu]\right] = \\
&= \left[J^\alpha_{\rho\epsilon}\bar{J}^\epsilon_\mu\left(\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu + \bar{J}^\rho_{\beta\nu}\right) + J^\alpha_\rho\left(\bar{J}^\epsilon_\mu\partial_\epsilon\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu + \Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_{\beta\mu}\bar{J}^\lambda_\nu + \boxed{\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_{\nu\mu}} + \boxed{\bar{J}^\rho_{\beta\nu\mu}}\right) + \right. \\
&\quad \left. + J^\alpha_\rho\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\gamma\bar{J}^\lambda_\mu J^\gamma_\theta\Gamma^\theta_{\eta\epsilon}\bar{J}^\eta_\beta\bar{J}^\epsilon_\nu + J^\alpha_\rho\bar{J}^\rho_{\gamma\mu}J^\gamma_\theta\Gamma^\theta_{\eta\epsilon}\bar{J}^\eta_\beta\bar{J}^\epsilon_\nu + J^\alpha_\rho\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\gamma\bar{J}^\lambda_\mu J^\gamma_\theta\bar{J}^\theta_{\beta\nu} + J^\alpha_\rho\bar{J}^\rho_{\gamma\mu}J^\gamma_\theta\bar{J}^\theta_{\beta\nu} - [\mu\nu]\right] = \\
&= \left[\textcolor{blue}{J^\alpha_{\rho\epsilon}\bar{J}^\epsilon_\mu\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu} + \textcolor{red}{J^\alpha_{\rho\epsilon}\bar{J}^\epsilon_\mu\bar{J}^\rho_{\beta\nu}} + \textcolor{red}{J^\alpha_\rho\partial_\epsilon\Gamma^\rho_{\sigma\lambda}\bar{J}^\epsilon_\mu\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu} + \underline{J^\alpha_\rho\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_{\beta\mu}\bar{J}^\lambda_\nu} + \underline{J^\alpha_\rho\Gamma^\rho_{\gamma\epsilon}\Gamma^\gamma_{\sigma\lambda}\bar{J}^\epsilon_\mu\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu} - \textcolor{blue}{J^\alpha_{\epsilon\rho}\bar{J}^\epsilon_\mu\Gamma^\rho_{\sigma\lambda}\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu} + \textcolor{green}{J^\alpha_\rho\Gamma^\rho_{\sigma\lambda}\bar{J}^\lambda_\mu\bar{J}^\sigma_{\beta\nu}} - \underline{J^\alpha_{\rho\theta}\bar{J}^\rho_\mu\bar{J}^\theta_{\beta\nu}} - [\mu\nu]\right] = \\
&= \left[J^\alpha_\rho\left(\partial_\epsilon\Gamma^\rho_{\sigma\lambda} + \Gamma^\rho_{\gamma\epsilon}\Gamma^\gamma_{\sigma\lambda}\right)\bar{J}^\epsilon_\mu\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu - [\mu\nu]\right] = J^\alpha_\rho\left[\partial_\epsilon\Gamma^\rho_{\sigma\lambda} + \Gamma^\rho_{\gamma\epsilon}\Gamma^\gamma_{\sigma\lambda} - [\epsilon\lambda]\right]\bar{J}^\epsilon_\mu\bar{J}^\sigma_\beta\bar{J}^\lambda_\nu = J^\alpha_\rho R^\rho{}_{\sigma\epsilon\lambda}\bar{J}^\sigma_\beta\bar{J}^\epsilon_\mu\bar{J}^\lambda_\nu
\end{aligned} \tag{16.2.49}$$

In view of antisymmetry in the last two indices, we can define out of $R^\alpha{}_{\beta\mu\nu}$ two traces

$$R_{\beta\nu} := R^\alpha{}_{\beta\alpha\nu} \qquad r_{\mu\nu} := R^\alpha{}_{\alpha\mu\nu} \tag{16.2.50}$$

The tensor $R_{\beta\nu}$ is called the *Ricci curvature*, while $r_{\mu\nu}$ is called the *first trace* and it is skew by construction. We shall see below that $r_{\mu\nu}$ can be written in terms of $R_{\beta\nu}$ and the torsion.

Notice that in general Ricci curvature is not necessarily symmetric. Let us stress that a the third contraction $R^\alpha{}_{\beta\mu\alpha} = -R_{\beta\mu}$ is function of the Ricci curvature.

If we have a metric $g_{\mu\nu}$, we can also define the *Ricci scalar*

$$R := g^{\beta\nu} R_{\beta\nu} \tag{16.2.51}$$

while, of course, $g^{\mu\nu} r_{\mu\nu} \equiv 0$. The Ricci scalar is a function of the connection and the metric. If the connection is the Levi Civita connection of g itself, then the Ricci scalar is a function of the metric alone.

First Bianchi identities

The curvature tensor obeys the so-called *first Bianchi identity*

$$R^\alpha{}_{[\beta\mu\nu]} = -\overset{\Gamma}{\nabla}_{[\beta} T^\alpha{}_{\mu\nu]} + T^\alpha{}_{\gamma[\beta} T^\gamma{}_{\mu\nu]} \tag{16.2.52}$$

Of course, for torsionless connections (in particular Levi Civita connections) one has the usual $R^\alpha{}_{[\beta\mu\nu]} = 0$.

In fact, we have

$$
\begin{aligned}
3R^\alpha{}_{[\beta\mu\nu]} =& \partial_\mu \Gamma^\alpha_{\beta\nu} - \underline{\partial_\nu \Gamma^\alpha_{\beta\mu}} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} + \partial_\beta \Gamma^\alpha_{\nu\mu} - \underline{\partial_\mu \Gamma^\alpha_{\nu\beta}} + \Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\nu\mu} - \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\nu\beta} + \underline{\partial_\nu \Gamma^\alpha_{\mu\beta}} - \partial_\beta \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\mu\beta} - \Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\nu} = \\
=& \underline{\partial_\mu T^\alpha_{\beta\nu}} + \underline{\underline{\Gamma^\alpha_{\gamma\mu} T^\gamma_{\beta\nu}}} - \Gamma^\gamma_{\beta\mu} T^\alpha_{\gamma\nu} - \Gamma^\gamma_{\nu\mu} T^\alpha_{\beta\gamma} + \Gamma^\gamma_{\beta\mu} T^\alpha_{\gamma\nu} + \Gamma^\gamma_{\nu\mu} T^\alpha_{\beta\gamma} - \underline{\underline{\partial_\beta T^\alpha_{\mu\nu}}} - \underline{\Gamma^\alpha_{\gamma\beta} T^\gamma_{\mu\nu}} + \underline{\underline{\Gamma^\gamma_{\mu\beta} T^\alpha_{\gamma\nu}}} + \underline{\underline{\Gamma^\gamma_{\nu\beta} T^\alpha_{\mu\gamma}}} - \Gamma^\gamma_{\mu\beta} T^\alpha_{\gamma\nu} - \Gamma^\gamma_{\nu\beta} T^\alpha_{\mu\gamma} + \\
& + \underline{\underline{\partial_\nu T^\alpha_{\mu\beta}}} + \underline{\underline{\Gamma^\alpha_{\gamma\nu} T^\gamma_{\mu\beta}}} - \underline{\underline{\Gamma^\gamma_{\mu\nu} T^\alpha_{\gamma\beta}}} - \underline{\underline{\Gamma^\gamma_{\beta\nu} T^\alpha_{\mu\gamma}}} + \Gamma^\gamma_{\mu\nu} T^\alpha_{\gamma\beta} + \Gamma^\gamma_{\beta\nu} T^\alpha_{\mu\gamma} = \\
=& \overset{\Gamma}{\nabla}_\mu T^\alpha_{\beta\nu} + \overset{\Gamma}{\nabla}_\beta T^\alpha_{\nu\mu} + \overset{\Gamma}{\nabla}_\nu T^\alpha_{\mu\beta} + T^\alpha_{\gamma\nu} T^\gamma_{\beta\mu} + T^\alpha_{\gamma\mu} T^\gamma_{\nu\beta} + T^\alpha_{\gamma\beta} T^\gamma_{\mu\nu} = 3\overset{\Gamma}{\nabla}_{[\mu} T^\alpha_{\beta\nu]} + 3T^\alpha_{\gamma[\nu} T^\gamma_{\beta\mu]} = -3\overset{\Gamma}{\nabla}_{[\beta} T^\alpha_{\mu\nu]} + 3T^\alpha_{\gamma[\beta} T^\gamma_{\mu\nu]}
\end{aligned}
\tag{16.2.53}
$$

Then, by tracing the first Bianchi identity, we can write the first trace $r_{\mu\nu}$ as

$$r_{\mu\nu} = R^\alpha{}_{\alpha\mu\nu} = \tfrac{1}{2} R_{[\mu\nu]} - \overset{\Gamma}{\nabla}_{[\alpha} T^\alpha{}_{\mu\nu]} + T^\alpha{}_{\gamma[\alpha} T^\gamma{}_{\mu\nu]} \tag{16.2.54}$$

For a torsionless connection, one has

$$r_{\mu\nu} = \tfrac{1}{2} R_{[\mu\nu]} \tag{16.2.55}$$

For a Levi Civita connection, $r_{\mu\nu} = 0$.

Second Bianchi identity

The curvature tensor also obeys the so-called *second Bianchi identity*

$$\overset{\Gamma}{\nabla}_{[\rho} R^\alpha{}_{\underline{\beta}\mu\nu]} = R^\alpha{}_{\beta\gamma[\rho} T^\gamma{}_{\mu\nu]} \tag{16.2.56}$$

Of course, for torsionless connections (in particular for Levi Civita connections), one has the usual $\overset{\Gamma}{\nabla}_{[\rho} R^{\alpha}{}_{\underline{\beta}\mu\nu]} = 0$.

In fact, we have

$$
\begin{aligned}
3\overset{\Gamma}{\nabla}_{[\rho} R^{\alpha}{}_{\underline{\beta}\mu\nu]} =& \overset{\Gamma}{\nabla}_{\rho} R^{\alpha}{}_{\beta\mu\nu} + \overset{\Gamma}{\nabla}_{\nu} R^{\alpha}{}_{\beta\rho\mu} + \overset{\Gamma}{\nabla}_{\mu} R^{\alpha}{}_{\beta\nu\rho} = \\
=& \partial_{\rho} R^{\alpha}{}_{\beta\mu\nu} + \Gamma^{\alpha}_{\gamma\rho} R^{\gamma}{}_{\beta\mu\nu} - \Gamma^{\gamma}_{\beta\rho} R^{\alpha}{}_{\gamma\mu\nu} - \Gamma^{\gamma}_{\mu\rho} R^{\alpha}{}_{\beta\gamma\nu} - \Gamma^{\gamma}_{\nu\rho} R^{\alpha}{}_{\beta\mu\gamma} + \\
&+ \partial_{\nu} R^{\alpha}{}_{\beta\rho\mu} + \Gamma^{\alpha}_{\gamma\nu} R^{\gamma}{}_{\beta\rho\mu} - \Gamma^{\gamma}_{\beta\nu} R^{\alpha}{}_{\gamma\rho\mu} - \Gamma^{\gamma}_{\rho\nu} R^{\alpha}{}_{\beta\gamma\mu} - \Gamma^{\gamma}_{\mu\nu} R^{\alpha}{}_{\beta\rho\gamma} + \\
&+ \partial_{\mu} R^{\alpha}{}_{\beta\nu\rho} + \Gamma^{\alpha}_{\gamma\mu} R^{\gamma}{}_{\beta\nu\rho} - \Gamma^{\gamma}_{\beta\mu} R^{\alpha}{}_{\gamma\nu\rho} - \Gamma^{\gamma}_{\nu\mu} R^{\alpha}{}_{\beta\gamma\rho} - \Gamma^{\gamma}_{\rho\mu} R^{\alpha}{}_{\beta\nu\gamma} = \\
=& \partial_{\rho\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\rho\nu}\Gamma^{\alpha}_{\beta\mu} + \partial_{\nu\rho}\Gamma^{\alpha}_{\beta\mu} - \partial_{\nu\mu}\Gamma^{\alpha}_{\beta\rho} + \partial_{\mu\nu}\Gamma^{\alpha}_{\beta\rho} - \partial_{\mu\rho}\Gamma^{\alpha}_{\beta\nu} + \\
&+ \partial_{\rho}(\Gamma^{\alpha}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu}) - \partial_{\rho}(\Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu}) + \underline{\partial_{\nu}(\Gamma^{\alpha}_{\gamma\rho}\Gamma^{\gamma}_{\beta\mu})} - \underline{\partial_{\nu}(\Gamma^{\alpha}_{\gamma\mu}\Gamma^{\gamma}_{\beta\rho})} + \underline{\partial_{\mu}(\Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\beta\rho})} - \underline{\partial_{\mu}(\Gamma^{\alpha}_{\gamma\rho}\Gamma^{\gamma}_{\beta\nu})} + \\
&+ \textcolor{red}{\Gamma^{\alpha}_{\gamma\rho}\partial_{\mu}\Gamma^{\gamma}_{\beta\nu}} - \textcolor{red}{\underline{\Gamma^{\alpha}_{\gamma\rho}\partial_{\nu}\Gamma^{\gamma}_{\beta\mu}}} - \Gamma^{\gamma}_{\beta\rho}\partial_{\mu}\Gamma^{\alpha}_{\gamma\nu} + \textcolor{blue}{\underline{\Gamma^{\gamma}_{\beta\rho}\partial_{\nu}\Gamma^{\alpha}_{\gamma\mu}}} - \Gamma^{\gamma}_{\mu\rho}\partial_{\gamma}\Gamma^{\alpha}_{\beta\nu} + \underline{\Gamma^{\gamma}_{\mu\rho}\partial_{\nu}\Gamma^{\alpha}_{\beta\gamma}} - \underline{\Gamma^{\gamma}_{\nu\rho}\partial_{\mu}\Gamma^{\alpha}_{\beta\gamma}} + \Gamma^{\gamma}_{\nu\rho}\partial_{\gamma}\Gamma^{\alpha}_{\beta\mu} + \\
&+ \Gamma^{\alpha}_{\gamma\nu}\partial_{\rho}\Gamma^{\gamma}_{\beta\mu} - \textcolor{red}{\underline{\Gamma^{\alpha}_{\gamma\nu}\partial_{\mu}\Gamma^{\gamma}_{\beta\rho}}} - \Gamma^{\gamma}_{\beta\nu}\partial_{\rho}\Gamma^{\alpha}_{\gamma\mu} + \textcolor{blue}{\underline{\Gamma^{\gamma}_{\beta\nu}\partial_{\mu}\Gamma^{\alpha}_{\gamma\rho}}} - \Gamma^{\gamma}_{\rho\nu}\partial_{\gamma}\Gamma^{\alpha}_{\beta\mu} + \underline{\Gamma^{\gamma}_{\rho\nu}\partial_{\mu}\Gamma^{\alpha}_{\beta\gamma}} - \underline{\underline{\Gamma^{\gamma}_{\mu\nu}\partial_{\rho}\Gamma^{\alpha}_{\beta\gamma}}} + \Gamma^{\gamma}_{\mu\nu}\partial_{\gamma}\Gamma^{\alpha}_{\beta\rho} + \\
&+ \textcolor{blue}{\underline{\Gamma^{\alpha}_{\gamma\mu}\partial_{\nu}\Gamma^{\gamma}_{\beta\rho}}} - \Gamma^{\gamma}_{\gamma\mu}\partial_{\rho}\Gamma^{\gamma}_{\beta\nu} - \textcolor{blue}{\underline{\Gamma^{\gamma}_{\beta\mu}\partial_{\nu}\Gamma^{\alpha}_{\gamma\rho}}} + \Gamma^{\gamma}_{\beta\mu}\partial_{\rho}\Gamma^{\alpha}_{\gamma\nu} - \Gamma^{\gamma}_{\nu\mu}\partial_{\gamma}\Gamma^{\alpha}_{\beta\rho} + \underline{\Gamma^{\gamma}_{\nu\mu}\partial_{\rho}\Gamma^{\alpha}_{\beta\gamma}} - \underline{\Gamma^{\gamma}_{\rho\mu}\partial_{\nu}\Gamma^{\alpha}_{\beta\gamma}} + \Gamma^{\gamma}_{\rho\mu}\partial_{\gamma}\Gamma^{\alpha}_{\beta\nu} + \\
&+ \boxed{\textcolor{red}{\Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu}}} - \boxed{\textcolor{green}{\Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu}}} - \boxed{\textcolor{blue}{\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\gamma\nu}\Gamma^{\gamma}_{\beta\rho}}} + \boxed{\textcolor{red}{\Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\gamma\mu}\Gamma^{\gamma}_{\beta\rho}}} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\nu}\Gamma^{\gamma}_{\mu\rho} + \Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\mu\rho} - \Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\nu\rho} + \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\mu}\Gamma^{\gamma}_{\nu\rho} + \\
&+ \boxed{\textcolor{green}{\Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\gamma\rho}\Gamma^{\gamma}_{\beta\mu}}} - \boxed{\textcolor{red}{\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\gamma\nu}\Gamma^{\gamma}_{\beta\rho}}} - \boxed{\textcolor{blue}{\Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu}}} + \boxed{\textcolor{green}{\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\gamma\rho}\Gamma^{\gamma}_{\beta\nu}}} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\mu}\Gamma^{\gamma}_{\rho\nu} + \Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\rho\nu} - \Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\mu\nu} + \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\rho}\Gamma^{\gamma}_{\mu\nu} + \\
&+ \boxed{\textcolor{blue}{\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\gamma\nu}\Gamma^{\gamma}_{\beta\rho}}} - \boxed{\textcolor{blue}{\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\gamma\rho}\Gamma^{\gamma}_{\beta\nu}}} - \boxed{\textcolor{green}{\Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\gamma\rho}\Gamma^{\gamma}_{\beta\mu}}} + \boxed{\textcolor{green}{\Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu}}} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\rho}\Gamma^{\gamma}_{\nu\mu} + \Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\nu\mu} - \Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\gamma}_{\rho\mu} + \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\nu}\Gamma^{\gamma}_{\rho\mu} =
\end{aligned}
\tag{16.2.57}
$$

$$
\begin{aligned}
3\overset{\Gamma}{\nabla}_{[\rho} R^{\alpha}{}_{\underline{\beta}\mu\nu]} =& T^{\gamma}_{\rho\nu}\left(\partial_{\mu}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\mu}\right) + T^{\gamma}_{\nu\mu}\left(\partial_{\rho}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\rho}\right) + T^{\gamma}_{\mu\rho}\left(\partial_{\nu}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\nu}\right) + \\
&+ \left(\Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\mu}\right)T^{\gamma}_{\rho\nu} + \left(\Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\rho}\right)T^{\gamma}_{\nu\mu} + \left(\Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\nu}\right)T^{\gamma}_{\mu\rho} = \\
=& T^{\gamma}_{\rho\nu}\left(\partial_{\mu}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\epsilon\mu}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\mu}\right) + T^{\gamma}_{\nu\mu}\left(\partial_{\rho}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\rho} + \Gamma^{\alpha}_{\epsilon\rho}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\rho}\right) + \\
&+ T^{\gamma}_{\mu\rho}\left(\partial_{\nu}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\nu} + \Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\beta\gamma} - \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\nu}\right) = T^{\gamma}_{\rho\nu} R^{\alpha}{}_{\beta\mu\gamma} + T^{\gamma}_{\nu\mu} R^{\alpha}{}_{\beta\rho\gamma} + T^{\gamma}_{\mu\rho} R^{\alpha}{}_{\beta\nu\gamma} = \\
=& - R^{\alpha}{}_{\beta\gamma\mu} T^{\gamma}_{\rho\nu} - R^{\alpha}{}_{\beta\gamma\rho} T^{\gamma}_{\nu\mu} - R^{\alpha}{}_{\beta\gamma\nu} T^{\gamma}_{\mu\rho} = -3 R^{\alpha}{}_{\beta\gamma[\mu} T^{\gamma}_{\rho\nu]}
\end{aligned}
\tag{16.2.58}
$$

We can contract by δ^{μ}_{α} and again by $g^{\beta\rho}$ the second Bianchi identities

$$
2\overset{\Gamma}{\nabla}_{\beta} R^{\beta}{}_{\nu} - \overset{\Gamma}{\nabla}_{\nu} R - \overset{\Gamma}{\nabla}_{\rho} g^{\beta\rho} R_{\beta\nu} - \overset{\Gamma}{\nabla}_{\alpha} g^{\beta\rho} R^{\alpha}{}_{\beta\nu\rho} + \overset{\Gamma}{\nabla}_{\nu} g^{\beta\rho} R_{\beta\rho} = R^{\alpha}{}_{\gamma} T^{\gamma}_{\alpha\nu} - g^{\beta\rho} R_{\beta\gamma} T^{\gamma}_{\nu\rho} - R^{\alpha\beta}{}_{\gamma\nu} T^{\gamma}_{\alpha\beta}
\tag{16.2.59}
$$

That does not say much in general, due to the fact that there is no compatibility condition between metric and connection. Let us assume $\overset{\Gamma}{\nabla}_{\nu} g_{\beta\rho} = \alpha_{\nu} g_{\beta\rho}$ as it happens in view of EPS compatibility for some covector α_{ν}. That generalized the compatibility condition which is recovered for $\alpha = 0$. In this case we have

$$
\overset{\Gamma}{\nabla}_{\nu} g^{\beta\rho} = -\alpha_{\nu} g^{\beta\rho}
\tag{16.2.60}
$$

hence we have

$$
\overset{\Gamma}{\nabla}_{\beta}\left(R^{\beta}{}_{\nu} - \tfrac{1}{2}\delta^{\beta}_{\nu} R\right) = -\alpha_{\beta}\left(R^{\beta}{}_{\nu} - \tfrac{1}{2}\delta^{\beta}_{\nu} R\right) + R^{\alpha}{}_{\gamma} T^{\gamma}_{\alpha\nu} - \tfrac{1}{2} R^{\alpha\beta}{}_{\gamma\nu} T^{\gamma}_{\alpha\beta}
\tag{16.2.61}
$$

For torsionless connections (still with $\overset{\Gamma}{\nabla}_\nu g_{\beta\rho} = \alpha_\nu g_{\beta\rho}$) we have

$$\overset{\Gamma}{\nabla}_\beta \left(R^\beta{}_\nu - \tfrac{1}{2}\delta^\beta_\nu R \right) = -\alpha_\beta \left(R^\beta{}_\nu - \tfrac{1}{2}\delta^\beta_\nu R \right) \tag{16.2.62}$$

Riemann tensor

The curvature tensor of the Levi Civita connection of a metric g is called the *Riemann tensor*. Riemann tensors have extra algebraic properties. Riemann tensor of a metric g is antisymmetric in the first pair of indices. Let us define $R_{\alpha\beta\mu\nu} := g_{\alpha\lambda}R^\lambda{}_{\beta\mu\nu}$. One has

$$
\begin{aligned}
R_{(\alpha\beta)\mu\nu} =& g_{\lambda(\alpha}R^\lambda{}_{\beta)\mu\nu} = \left(\tfrac{1}{2}g_{\lambda\alpha}\left(\partial_\mu\Gamma^\lambda_{\beta\nu} - \partial_\nu\Gamma^\lambda_{\beta\mu} + \Gamma^\lambda_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\lambda_{\gamma\nu}\Gamma^\gamma_{\beta\mu} \right) + (\alpha\beta) \right) = \\
=& \Bigg(-\tfrac{1}{4}\left(\partial_\mu g_{\lambda\alpha}g^{\lambda\epsilon}\left(-\partial_\epsilon g_{\beta\nu} + \partial_\beta g_{\nu\epsilon} + \underline{\partial_\nu g_{\epsilon\beta}} \right) - \partial_\nu g_{\lambda\alpha}g^{\lambda\epsilon}\left(-\partial_\epsilon g_{\beta\mu} + \partial_\beta g_{\mu\epsilon} + \underline{\partial_\mu g_{\epsilon\beta}} \right) \right) + \\
& + \tfrac{1}{4}\left(-\partial_{\alpha\mu}g_{\beta\nu} + \partial_{\beta\mu}g_{\nu\alpha} + \partial_{\nu\mu}g_{\alpha\beta} \right) - \tfrac{1}{4}\left(-\partial_{\alpha\nu}g_{\beta\mu} + \partial_{\beta\nu}g_{\mu\alpha} + \partial_{\mu\nu}g_{\alpha\beta} \right) + \\
& + \tfrac{1}{8}\left(-\partial_\alpha g_{\lambda\mu} + \partial_\lambda g_{\mu\alpha} + \partial_\mu g_{\alpha\lambda} \right) g^{\lambda\epsilon}\left(-\partial_\epsilon g_{\beta\nu} + \partial_\beta g_{\nu\epsilon} + \partial_\nu g_{\epsilon\beta} \right) + \\
& - \tfrac{1}{8}\left(-\partial_\alpha g_{\lambda\nu} + \partial_\nu g_{\alpha\lambda} + \partial_\lambda g_{\nu\alpha} \right) g^{\lambda\epsilon}\left(-\partial_\epsilon g_{\beta\mu} + \partial_\beta g_{\mu\epsilon} + \partial_\mu g_{\epsilon\beta} \right) + (\alpha\beta) \Bigg) = \\
=& \Bigg(\Big[\tfrac{1}{4}\partial_\mu g_{\lambda\alpha}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu} - \tfrac{1}{4}\partial_\mu g_{\lambda\alpha}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon} + \boxed{\tfrac{1}{8}\partial_\alpha g_{\lambda\mu}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu}} - \boxed{\tfrac{1}{8}\partial_\alpha g_{\lambda\mu}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon}} - \underline{\tfrac{1}{8}\partial_\alpha g_{\lambda\mu}g^{\lambda\epsilon}\partial_\nu g_{\epsilon\beta}} + \\
& - \boxed{\tfrac{1}{8}\partial_\lambda g_{\mu\alpha}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu}} + \boxed{\tfrac{1}{8}\partial_\lambda g_{\mu\alpha}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon}} + \tfrac{1}{8}\partial_\lambda g_{\mu\alpha}g^{\lambda\epsilon}\partial_\nu g_{\epsilon\beta} - \tfrac{1}{8}\partial_\mu g_{\alpha\lambda}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu} + \underline{\underline{\tfrac{1}{8}\partial_\mu g_{\alpha\lambda}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon}}} + \boxed{\tfrac{1}{8}\partial_\mu g_{\alpha\lambda}g^{\lambda\epsilon}\partial_\nu g_{\epsilon\beta}} - [\mu\nu] \Big] + (\alpha\beta) \Bigg)
\end{aligned}
\tag{16.2.63}
$$

Hence we get

$$R_{(\alpha\beta)\mu\nu} = \left(\left[\tfrac{1}{4}\partial_\mu g_{\lambda\alpha}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu} - \tfrac{1}{4}\partial_\mu g_{\lambda\alpha}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon} - \tfrac{1}{4}\partial_\mu g_{\alpha\lambda}g^{\lambda\epsilon}\partial_\epsilon g_{\beta\nu} + \tfrac{1}{4}\partial_\mu g_{\alpha\lambda}g^{\lambda\epsilon}\partial_\beta g_{\nu\epsilon} - [\mu\nu] \right] + (\alpha\beta) \right) = 0 \tag{16.2.64}$$

Notice that no symmetry can be assumed for general connections nor for general torsionless connections; one really needs to resort to Levi Civita connections for it.

Riemann tensor of a metric g is also symmetric in the exchange of the first and second pair of indices, namely $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$. Notice again that no symmetry can be assumed for general connections nor for general torsionless connections; one really needs to resort to Levi Civita connections for it.

Let us set $\Gamma_{\alpha\beta\nu} := g_{\alpha\lambda}\Gamma^\lambda_{\beta\nu}$; one has

$$
\begin{aligned}
R_{\mu\nu\alpha\beta} - R_{\alpha\beta\mu\nu} =& \left[g_{\lambda\mu}\partial_\alpha\Gamma^\lambda_{\nu\beta} + g_{\lambda\mu}\Gamma^\lambda_{\epsilon\alpha}\Gamma^\epsilon_{\nu\beta} - [\alpha\beta] \right] - \left[g_{\lambda\alpha}\partial_\mu\Gamma^\lambda_{\beta\nu} + g_{\lambda\alpha}\Gamma^\lambda_{\epsilon\mu}\Gamma^\epsilon_{\beta\nu} - [\mu\nu] \right] = \\
=& \left[\partial_\alpha\Gamma_{\mu\nu\beta} + \tfrac{1}{2}\left(-\partial_\mu g_{\epsilon\alpha} + \partial_\epsilon g_{\alpha\mu} + \partial_\alpha g_{\mu\epsilon} - 2\partial_\alpha g_{\epsilon\mu} \right)\Gamma^\epsilon_{\nu\beta} - [\alpha\beta] \right] - \left[\partial_\mu\Gamma_{\alpha\beta\nu} + \tfrac{1}{2}\left(-\partial_\alpha g_{\epsilon\mu} + \partial_\epsilon g_{\mu\alpha} + \partial_\mu g_{\alpha\epsilon} - 2\partial_\mu g_{\epsilon\alpha} \right)\Gamma^\epsilon_{\beta\nu} - [\mu\nu] \right] = \\
=& \left[\partial_\alpha\Gamma_{\mu\nu\beta} - \Gamma_{\epsilon\mu\alpha}\Gamma^\epsilon_{\nu\beta} - [\alpha\beta] \right] - \left[\partial_\mu\Gamma_{\alpha\beta\nu} - \Gamma_{\epsilon\alpha\mu}\Gamma^\epsilon_{\beta\nu} - [\mu\nu] \right] = \\
=& \tfrac{1}{2}\left[-\partial_{\mu\alpha}g_{\nu\beta} + \partial_{\nu\alpha}g_{\beta\mu} + \boxed{\partial_{\beta\alpha}g_{\mu\nu}} - [\alpha\beta] \right] - \tfrac{1}{2}\left[-\partial_{\alpha\mu}g_{\beta\nu} + \partial_{\beta\mu}g_{\nu\alpha} + \boxed{\partial_{\nu\mu}g_{\alpha\beta}} - [\mu\nu] \right] = \\
=& \tfrac{1}{2}\left(-\partial_{\mu\alpha}g_{\nu\beta} + \partial_{\nu\alpha}g_{\beta\mu} + \partial_{\mu\beta}g_{\nu\alpha} - \partial_{\nu\beta}g_{\alpha\mu} \right) - \tfrac{1}{2}\left(-\partial_{\alpha\mu}g_{\beta\nu} + \partial_{\beta\mu}g_{\nu\alpha} + \partial_{\alpha\nu}g_{\beta\mu} - \partial_{\beta\nu}g_{\mu\alpha} \right) \equiv 0
\end{aligned}
\tag{16.2.65}
$$

The Ricci curvature of a Riemann tensor is also called a *Ricci tensor*; as a consequence of symmetry, by contraction, we obtain that it is symmetric.

$$R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu} = g^{\alpha\mu}R_{\mu\nu\alpha\beta} = R_{\nu\beta} \tag{16.2.66}$$

As usual nothing can be said for the Ricci curvature of a generic connection.

The second Bianchi identities

$$\overset{g}{\nabla}_{[\rho}R^{\alpha}{}_{\underline{\beta}\mu\nu]} = \tfrac{1}{3}\left(\overset{g}{\nabla}_{\rho}R^{\alpha}{}_{\beta\mu\nu} + \overset{g}{\nabla}_{\mu}R^{\alpha}{}_{\beta\nu\rho} + \overset{g}{\nabla}_{\nu}R^{\alpha}{}_{\beta\rho\mu}\right) = 0 \tag{16.2.67}$$

can be contracted by δ^{μ}_{α} and again by $g^{\beta\rho}$

$$\overset{g}{\nabla}_{\rho}R^{\alpha}{}_{\beta\alpha\nu} + \overset{g}{\nabla}_{\alpha}R^{\alpha}{}_{\beta\nu\rho} + \overset{g}{\nabla}_{\nu}R^{\alpha}{}_{\beta\rho\alpha} = 0$$

$$\overset{g}{\nabla}_{\rho}R_{\beta\nu} + \overset{g}{\nabla}_{\alpha}R^{\alpha}{}_{\beta\nu\rho} - \overset{g}{\nabla}_{\nu}R_{\beta\rho} = 0 \tag{16.2.68}$$

$$\overset{g}{\nabla}_{\rho}R^{\rho}{}_{\cdot\nu} + \overset{g}{\nabla}_{\alpha}R^{\alpha}{}_{\cdot\nu} - \overset{g}{\nabla}_{\nu}R = 0 \quad \Rightarrow \overset{g}{\nabla}_{\alpha}\left(R^{\alpha}{}_{\cdot\nu} - \tfrac{1}{2}R\delta^{\alpha}_{\nu}\right) = 0$$

This are called *contracted (second) Bianchi identities*; they claim that the Einstein tensor $G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}$ is conserved (i.e. it has zero divergence).

Counting independent components of the curvature

On a manifold M of dimension m, a metric has $\tfrac{m}{2}(m+1)$ indipendent components. A torsionless connection has $\tfrac{m^2}{2}(m+1)$, while a general connections has m^3 components.

Since the first derivatives of the metrics are $\tfrac{m^2}{2}(m+1)$, one can invert the definition of Christoffel symbols.

Once Christoffel symbols are defined, one can permute indices

$$2g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu} = -\partial_{\alpha}g_{\beta\mu} + \partial_{\mu}g_{\alpha\beta} + \partial_{\beta}g_{\mu\alpha} \qquad 2g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha} = -\partial_{\beta}g_{\mu\alpha} + \partial_{\alpha}g_{\beta\mu} + \partial_{\mu}g_{\alpha\beta} \tag{16.2.69}$$

and sum

$$2g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu} + 2g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha} = -\partial_{\alpha}g_{\beta\mu} + \partial_{\mu}g_{\alpha\beta} + \partial_{\beta}g_{\mu\alpha} - \partial_{\beta}g_{\mu\alpha} + \partial_{\alpha}g_{\beta\mu} + \partial_{\mu}g_{\alpha\beta} = 2\partial_{\mu}g_{\alpha\beta} \Rightarrow$$

$$\Rightarrow \partial_{\mu}g_{\alpha\beta} = g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu} + g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha} \tag{16.2.70}$$

The torsion $T^{\alpha}_{\beta\mu}$ has $\tfrac{m^2}{2}(m-1)$ independent components, showing that a general connection can be split into a torsionless connection and its torsion

$$\tfrac{m^2}{2}(m+1) + \tfrac{m^2}{2}(m-1) = m^3 \tag{16.2.71}$$

The (covariant) curvature tensor $R_{\alpha\beta\mu\nu}$ of a general torsionless connection obeys $\tfrac{m^2}{2}(m+1) + m\begin{pmatrix}m\\3\end{pmatrix}$ linear identities

$$R_{\alpha\beta(\mu\nu)} = 0 \qquad\qquad R_{\alpha[\beta\mu\nu]} = 0 \tag{16.2.72}$$

However, one needs to discuss whether these identities are independent in order to count for independent components. The main tool is to consider that the space of $(4,0)$-tensors $R_{\alpha\beta\mu\nu}$ supports a representation of permutation group Π_4 acting permuting indices. We can write the constraints (16.2.72) using this representation, the hope being that contraints have a meaning in terms of the decomposition into its irreducible sub-representations.

Let us set aside the index α. An object with three indices can be split in general as

$$\beta\mu\nu = \beta(\mu\nu) \oplus \beta[\mu\nu] \tag{16.2.73}$$

This sum is direct since the intersection of the two subspaces is the zero only. This splits the space of rank 3 tensors (which is of dimension m^3) into two subspaces of dimension $\frac{m^2}{2}(m+1)$ and $\frac{m^2}{2}(m-1)$, respectively.

However, in the first subspace there is still an invariant subspace with respect to the permutation group Π_3 acting on indices, namely the space of completely symmetric objects with 3 indices, which is of dimension $\binom{m+2}{3} = \frac{m}{6}(m+2)(m+1)$. Then

$$\beta(\mu\nu) = (\beta\mu\nu) \oplus (\beta(\mu\nu) - (\beta\mu\nu)) \tag{16.2.74}$$

which corresponds to the splitting into two subspaces of dimension

$$\frac{m^2}{2}(m+1) = \frac{m}{6}(m+2)(m+1) \oplus \left(\frac{m}{3}(m^2-1)\right) \tag{16.2.75}$$

Analogously, the subspace of antisymmetric objects can be split into two subspaces

$$\beta[\mu\nu] = [\beta\mu\nu] \oplus (\beta[\mu\nu] - [\beta\mu\nu]) \tag{16.2.76}$$

which are of dimension $\binom{m}{3} = \frac{m}{6}(m-1)(m-2)$ and $\frac{m^2}{2}(m-1) - \binom{m}{3} = \frac{m}{3}(m^2-1)$, respectively.

In other words, one can define a group action of Π_3 over the space of tensors of rank 3 on a manifold of dimension m, which is of dimension m^3. This representation is not irreducible but it can be written as the sum of 4 irreducible representations of Π_3 acting on subspaces

$$\beta\mu\nu = (\beta\mu\nu) \oplus (\beta(\mu\nu) - (\beta\mu\nu)) \oplus [\beta\mu\nu] \oplus (\beta[\mu\nu] - [\beta\mu\nu]) \tag{16.2.77}$$

which are, respectively, of dimension
$$m^3 = \frac{m}{6}(m+2)(m+1) \oplus \left(\frac{m}{3}(m^2-1)\right) \oplus \frac{m}{6}(m-1)(m-2) \oplus \frac{m}{3}(m^2-1) \tag{16.2.78}$$

The identities (16.2.72) tell us that the curvature tensor (for a generic torsionless connection) belongs to the 4th subspace which, accounting for the extra index α, is of dimension

$$\frac{m^2}{3}(m^2-1) \tag{16.2.79}$$

If we restrict to Levi Civita connections of some metrics, one has more linear constraints on the components of Riemann tensor, namely:

$$R_{\alpha\beta(\mu\nu)} = 0 \qquad R_{(\alpha\beta)\mu\nu} = 0 \qquad R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0 \qquad R_{\alpha[\beta\mu\nu]} = 0 \tag{16.2.80}$$

Of course, $R_{(\alpha\beta)\mu\nu} = 0$ follows from $R_{\alpha\beta(\mu\nu)} = 0$ and $R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0$, thus it can be suppressed.

Since one has

$$R_{[\alpha\beta\mu\nu]} = \tfrac{1}{4}\left(R_{\alpha[\beta\mu\nu]} - R_{\nu[\alpha\beta\mu]} + R_{\mu[\nu\alpha\beta]} - R_{\beta[\mu\nu\alpha]}\right) \tag{16.2.81}$$

then also $R_{[\alpha\beta\mu\nu]} = 0$ follows.

On the other hand, if one assumes $R_{\alpha\beta(\mu\nu)} = 0$, $R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0$, and $R_{[\alpha\beta\mu\nu]} = 0$, then one has

$$
\begin{aligned}
R_{\alpha[\beta\mu\nu]} =& \tfrac{1}{24}\left(4\alpha\beta\mu\nu + 4\alpha\mu\nu\beta + 4\alpha\nu\beta\mu\right) = \tfrac{1}{24}\left(\alpha\beta\mu\nu + \alpha\mu\nu\beta + \alpha\nu\beta\mu - 3\beta\alpha\mu\nu - 3\beta\nu\alpha\mu - 3\beta\mu\nu\alpha\right) = \\
=& \tfrac{1}{24}\left(\alpha[\beta\mu\nu] - \beta[\alpha\mu\nu] + \mu[\nu\alpha\beta] - \nu[\mu\alpha\beta]\right) = R_{[\alpha\beta\mu\nu]} = 0
\end{aligned}
\tag{16.2.82}
$$

Thus the constraints are equivalent to the conditions

$$R_{[\alpha\beta\mu\nu]} = 0 \tag{16.2.83}$$

on the space of components $R_{\alpha\beta\mu\nu}$ skew symmetric in each pair and symmetric for the exchange of the pairs, which is of dimension $\frac{m}{8}(m-1)(m^2-m+2)$. The conditions are $\frac{m}{24}(m-1)(m-2)(m-3)$ and they account for

$$\frac{m}{8}(m-1)(m^2-m+2) - \frac{m}{24}(m-1)(m-2)(m-3) = \frac{m}{24}\left[3m^2-3m+6 - m^2+5m-6\right](m-1) = \frac{m^2}{12}(m-1)^2 \tag{16.2.84}$$

independent components.

Thus the independent components of a metric Riemann tensor are

$$\frac{m^2}{12}(m^2-1) \tag{16.2.85}$$

Since for any (integer) dimension $m > 1$, one has

$$\frac{m^2}{3}(m^2-1) > \frac{m^2}{12}(m^2-1) \tag{16.2.86}$$

then metric Riemann tensors live in a smaller space than curvature tensors of torsionless connections.

If we consider tensors $S^\alpha_{(\beta\mu\nu)}$ symmetric in lower indices they live in a space of dimension $\frac{m^2}{6}(m+2)(m+1)$. Then

$$\frac{m^2}{3}(m^2-1) \oplus \frac{m^2}{6}(m+2)(m+1) = \frac{m^3}{2}(m+1) \tag{16.2.87}$$

which exactly accounts for first derivatives of torsionless connections $\partial_\mu \Gamma^\alpha_{\beta\nu}$.

That suggests that one can split $\partial_\mu \Gamma^\alpha_{\beta\nu}$ into its curvature $R^\alpha{}_{\beta\mu\nu}$ and the quantity $S^\alpha{}_{\beta\mu\nu} = \partial_{(\mu}\Gamma^\alpha_{\beta\nu)}$ in a one-to-one way.

Let us start from $(\Gamma^\alpha_{\beta\nu}, \partial_\mu\Gamma^\alpha_{\beta\nu})$ and map it to $(\Gamma^\alpha_{\beta\nu}, R^\alpha_{\beta\mu\nu}, S^\alpha_{\beta\mu\nu})$ as above by setting

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} \qquad S^\alpha_{\beta\mu\nu} = 3\partial_{(\mu}\Gamma^\alpha_{\beta\nu)} = \partial_\mu\Gamma^\alpha_{\beta\nu} + \partial_\nu\Gamma^\alpha_{\mu\beta} + \partial_\beta\Gamma^\alpha_{\nu\mu} \tag{16.2.88}$$

This transformation is one-to-one since it can be inverted as

$$
\begin{aligned}
& -2R^\alpha{}_{(\beta\mu)\nu} + S^\alpha_{\beta\mu\nu} = \underline{-\partial_\mu\Gamma^\alpha_{\beta\nu}} + \underline{\partial_\nu\Gamma^\alpha_{\beta\mu}} - \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} + \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} - \underline{\partial_\beta\Gamma^\alpha_{\mu\nu}} + \underline{\partial_\nu\Gamma^\alpha_{\beta\mu}} - \Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\nu} + \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} + \underline{\partial_\mu\Gamma^\alpha_{\beta\nu}} + \underline{\partial_\nu\Gamma^\alpha_{\mu\beta}} + \underline{\partial_\beta\Gamma^\alpha_{\nu\mu}} \qquad \Rightarrow \\
\Rightarrow\quad & 3\partial_\nu\Gamma^\alpha_{\mu\beta} = -2R^\alpha{}_{(\beta\mu)\nu} + S^\alpha_{\beta\mu\nu} + 2\Gamma^\alpha_{\gamma(\mu}\Gamma^\gamma_{\beta)\nu} - 2\Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu}
\end{aligned}
\tag{16.2.89}
$$

Let us now consider metric Riemann tensors. Let us notice that

$$\frac{m^2}{12}(m^2-1) \oplus \frac{m^2}{6}(m+2)(m+1) = \frac{m^2}{4}(m+1)^2 \qquad (16.2.90)$$

which exactly accounts for second derivatives of the metric $\partial_{\beta\nu}g_{\alpha\mu}$.

That suggests that one can split $\partial_{\beta\nu}g_{\alpha\mu}$ into its curvature $R^{\alpha}{}_{\beta\mu\nu}$ and the quantity $S^{\alpha}{}_{\beta\mu\nu} = \partial_{(\mu}\{g\}^{\alpha}_{\beta\nu)}$ in a one-to-one way.

Let us start from $(g_{\mu\nu}, \partial_{\nu}g_{\alpha\mu}, \partial_{\beta\nu}g_{\alpha\mu})$ and map it to $(g_{\mu\nu}, \{g\}^{\alpha}_{\beta\nu}, R^{\alpha}{}_{\beta\mu\nu}, S^{\alpha}{}_{\beta\mu\nu})$ as above by setting

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\{g\}^{\alpha}_{\beta\nu} - \partial_{\nu}\{g\}^{\alpha}_{\beta\mu} + \{g\}^{\alpha}_{\gamma\mu}\{g\}^{\gamma}_{\beta\nu} - \{g\}^{\alpha}_{\gamma\nu}\{g\}^{\gamma}_{\beta\mu} \qquad S^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\{g\}^{\alpha}_{\beta\nu} + \partial_{\nu}\{g\}^{\alpha}_{\mu\beta} + \partial_{\beta}\{g\}^{\alpha}_{\nu\mu} \qquad (16.2.91)$$

This transformation is one-to-one since it can be inverted as (16.2.70) and

$$3\partial_{\nu}\{g\}^{\alpha}_{\mu\beta} = S^{\alpha}_{\beta\mu\nu} - 2R^{\alpha}{}_{(\beta\mu)\nu} + 2\{g\}^{\alpha}_{\gamma(\mu}\{g\}^{\gamma}_{\beta)\nu} - 2\{g\}^{\alpha}_{\gamma\nu}\{g\}^{\gamma}_{\beta\mu} \quad \Rightarrow \partial_{\mu\nu}g_{\alpha\beta} = \partial_{\nu}g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu} + g_{\alpha\epsilon}\partial_{\nu}\{g\}^{\epsilon}_{\beta\mu} + \partial_{\nu}g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha} + g_{\beta\epsilon}\partial_{\nu}\{g\}^{\epsilon}_{\mu\alpha} =$$
$$= g_{\epsilon\lambda}\{g\}^{\lambda}_{\nu\alpha}\{g\}^{\epsilon}_{\beta\mu} + g_{\epsilon\lambda}\{g\}^{\lambda}_{\nu\beta}\{g\}^{\epsilon}_{\mu\alpha} + \frac{1}{3}S^{\cdot}_{\alpha\beta\mu\nu} - \frac{2}{3}R_{\alpha(\beta\mu)\nu} + \frac{1}{3}S^{\cdot}_{\beta\alpha\mu\nu} - \frac{2}{3}R_{\beta(\alpha\mu)\nu} + g_{\alpha\epsilon}\{g\}^{\epsilon}_{\gamma(\mu}\{g\}^{\gamma}_{\beta\nu)} + g_{\beta\epsilon}\{g\}^{\epsilon}_{\gamma(\mu}\{g\}^{\gamma}_{\alpha\nu)} \qquad (16.2.92)$$

We can summarise the number of independent components in low dimensions by the following table:

dim(M)	metric	torsionless connection	torsion	generic connection	metric Riemann	torsionless curvature	$S^{\alpha}{}_{\beta\mu\nu}$
1	1	1	0	1	0	0	1
2	3	6	2	8	1	4	8
3	6	18	9	27	6	24	30
4	10	40	24	64	20	80	80
5	15	75	50	125	50	200	175
...	...	...	...	...	...	...	...
m	$\frac{m}{2}(m+1)$	$\frac{m^2}{2}(m+1)$	$\frac{m^2}{2}(m-1)$	m^3	$\frac{m^2}{12}(m^2-1)$	$\frac{m^2}{3}(m^2-1)$	$\frac{m^2}{6}(m+2)(m+1)$

Isometries and Killing vectors

On a Riemannian manifold (M, g), a diffeomorphism $\Phi : M \to M$ is called an *isometry* if it preserves the metric, i.e. if $\Phi^*g = g$.

Rotations and translations on the Euclidean planes $(\mathbb{R}^m, \delta)$ are isometries.

If one considers a sphere $S_n \subset \mathbb{R}^{n+1}$ with the induced metrics (S_n, g) then rotations in $\mathbb{R}^{n+1}$ restrict to diffeomorphisms on the sphere and they are in fact isometries.

A 1-parameter subgroup of isometries Φ_s is called an *infinitesimal isometry*. An infinitesimal isometry, as any other infinitesimal diffeomorphism, is characterised by its infinitesimal generator ξ, which is a vector field on M, of which Φ_s is the flow. The infinitesimal generator ξ of an infinitesimal isometry is also called a *Killing vector*.

An isometry $\Phi : M \to M$ is locally described by a map $\Phi : x^\mu \mapsto x'^\mu = \Phi^\mu(x)$. It is an isometry iff

$$g'_{\mu\nu}(x') = \bar{J}^\rho_\mu g_{\rho\sigma}(x) \bar{J}^\sigma_\nu = g_{\mu\nu}(x') \tag{16.2.93}$$

which we can differentiate with respect to the subgroup parameter s to obtain

$$\partial_\mu \xi^\rho g_{\rho\nu} + g_{\mu\sigma} \partial_\nu \xi^\sigma = \xi^\epsilon d_\epsilon g_{\mu\nu} \qquad \Rightarrow \boxed{\xi^\epsilon \overset{g}{\nabla}_\epsilon g_{\mu\nu}} = \overset{g}{\nabla}_\mu \xi^\rho g_{\rho\nu} + g_{\mu\sigma} \overset{g}{\nabla}_\nu \xi^\sigma = 0 \tag{16.2.94}$$

The equation

$$\overset{g}{\nabla}_{(\mu} \xi_{\nu)} = 0 \tag{16.2.95}$$

is called the *Killing equation* and, of course, a vector field ξ is a Killing vector iff it is a solution fo the Killing equation.

The Killing equation is a linear equation with respect to ξ, thus Killing vectors forms a vector subspace space. Moreover, in view of the naturality of commutators, the commutators of Killing vectors is a Killing vector. Accordingly, Killing vectors form a Lie-subalgebra of vector fields on M.

The Killing equation sets a constraint on first derivative of the vector field ξ; the symmetric part is vanishing so first derivatives are antisymmetric. Accordingly, at a point x the m values of the components $\xi^\epsilon(x)$, as well as, possibly, the $\frac{m}{2}(m-1)$ values of the antisymmetric first derivatives $\nabla_{[\mu} \xi_{\nu]}(x)$ can be chosen at will, while the symmetric first derivatives must vanish.

All the second derivatives, as well as, consequently, the higher derivatives, are algebraically expressed in terms of $\xi^\epsilon(x)$ and $\nabla_{[\mu} \xi_{\nu]}(x)$.

By using the second derivatives $\nabla_\mu \nabla_\nu \xi_\alpha$, one has a tensor $k_{\mu\nu\alpha} := \nabla_\mu \nabla_\nu \xi_\alpha$ which is antisymmetric in the last two indices and

$$k_{\mu\nu\alpha} = k_{\nu\mu\alpha} + R^\epsilon{}_{\alpha\mu\nu} \xi_\epsilon \tag{16.2.96}$$

Then one can play the (almost) usual trick

$$\begin{aligned} k_{\mu\nu\alpha} &= k_{\nu\mu\alpha} + R^\epsilon{}_{\alpha\mu\nu} \xi_\epsilon = -k_{\nu\alpha\mu} + R^\epsilon{}_{\alpha\mu\nu} \xi_\epsilon = -k_{\alpha\nu\mu} + (-R^\epsilon{}_{\mu\nu\alpha} + R^\epsilon{}_{\alpha\mu\nu}) \xi_\epsilon = k_{\alpha\mu\nu} + (-R^\epsilon{}_{\mu\nu\alpha} + R^\epsilon{}_{\alpha\mu\nu}) \xi_\epsilon = \\ &= k_{\mu\alpha\nu} + (R^\epsilon{}_{\nu\alpha\mu} - R^\epsilon{}_{\mu\nu\alpha} + R^\epsilon{}_{\alpha\mu\nu}) \xi_\epsilon = -k_{\mu\nu\alpha} + (R^\epsilon{}_{\nu\alpha\mu} - R^\epsilon{}_{\mu\nu\alpha} + R^\epsilon{}_{\alpha\mu\nu}) \xi_\epsilon \end{aligned} \tag{16.2.97}$$

and obtain

$$k_{\mu\nu\alpha} = \nabla_\mu \nabla_\nu \xi_\alpha = \tfrac{1}{2} (R^\epsilon{}_{\nu\alpha\mu} - R^\epsilon{}_{\mu\nu\alpha} + R^\epsilon{}_{\alpha\mu\nu}) \xi_\epsilon = R_{\mu\epsilon\nu\alpha} \xi^\epsilon \tag{16.2.98}$$

Thus all second derivatives of ξ are algebraically determined by the values of the zero order components ξ^ϵ. Then, recursively, the third derivatives depend on $\xi^\epsilon, \nabla_\alpha \xi^\epsilon$, and so on.

If all derivatives of $\xi^\mu(x)$ at a point x_0 are determined by (at most) by $\xi^\epsilon(x_0), \nabla_{[\alpha} \xi_{\epsilon]}(x_0)$, it means that the space of Killing vectors is at most of dimension $m + \frac{m}{2}(m-1) = \frac{m}{2}(m+1)$.

A Riemannian manifold (M, g) on which one has $\frac{m}{2}(m+1)$ independent Killing vectors is called a *maximally symmetric* space.

On the plane $(\mathbb{R}^m, \eta)$ in signature (r, s), one has m Killing vectors related to translations, and $\frac{m}{2}(m-1)$ Killing vectors related to rotations. It is hence maximally symmetric.

On the sphere (S_n, g), one has $\frac{n}{2}(n+1)$ independent infinitesimal rotations of $\mathbb{R}^{n+1}$ which generate Killing vectors, so that the sphere is maximally symmetric as well. An ellipsoid of axes (a, a, b) (embedded in the Euclidean space $(\mathbb{R}^3, \delta)$ and endowed with the induced metric) is not maximally symmetric (if $b \neq a$) and only the rotations about the z-axis generate a Killing vector. An ellipsoid of different axes (a, b, c) has no Killing vectors.

3. Symplectic manifolds

A similar, though somehow complementary, structure on a manifold is the *symplectic structure*. With a symplectic structure, one can do things similar to the ones one does with a metric structure. A symplectic structure also captures the geometry of Hamiltonian systems.

Before dealing with symplectic manifolds, it is better to discuss some of the structures algebraically, i.e. on a vector space. As it happens on manifolds, algebraic structures are eventually realised on tangent spaces of the manifold.

Dual vector spaces

Let us consider a vector space V of dimension $\dim(V) = n$ with a basis e_i and let us denote by V^* the dual space with the dual basis e^i. Let $U \subset V$ be a subspace of dimension k and let us define the *polar* as the subspace

$$U^\circ = \{\alpha \in V^* : \forall u \in U, \ \alpha(u) = 0\} \subset V^* \tag{16.3.1}$$

Theorem: $dim(U^\circ) + dim(U) = dim(V)$

Proof: *let e_i be a basis of V adapted to U (the first elements $e_1, \ldots, e_k$ are a basis of U) and e^i the dual basis of V^*.*

Then any $\alpha \in V^*$ can be expanded as $\alpha = \alpha_i e^i$. Then α is in U° iff $\alpha(e_1) = \ldots = \alpha(e_k) = 0$, i.e. iff it is $\alpha \in Span(e^{k+1}, \ldots e^n)$.

Thus $(e^{k+1}, \ldots e^n)$ is a basis of U°, then $\dim(U^\circ) = n - k$. ∎

Theorem: *if $\alpha(v) = 0$ for all $\alpha \in U^\circ$ then $v \in U$.*

Proof: *let us suppose, for the sake of argument, that $v \notin U$.*

Since $\alpha(v) = 0$, one has that $U^\circ \subset (U \oplus Span(v))^\circ$. Since we are using vector spaces, that means $\dim(U^\circ) < \dim((U \oplus Span(v))^\circ)$, while we know that $\dim((U \oplus Span(v))^\circ) < \dim(U^\circ)$. Because of the contradiction, one then has $v \in U$. ∎

Let us denote $U^{\circ\circ} \subset V^{**} \simeq V$ and the canonical isomorphism $i : V \to V^{**} : v \mapsto \hat{v}$ iff for any $\alpha \in V^*$ one has $\hat{v}(\alpha) = \alpha(v)$. Let us show that $U^{\circ\circ} = i(U)$.

Let us consider $u \in U$ and $i(u) \in V^{**}$. The element $i(u) \in U^{\circ\circ}$ iff for all $\alpha \in U^\circ$ one has $i(u)(\alpha) = \alpha(u) = 0$ which is true since $\alpha \in U^\circ$.

Notice we do not define any map between U and U°, though we do define a canonical isomorphism between U and $U^{\circ\circ}$.

If $W \subset U$ then $U^\circ \subset W^\circ$.

One has $\alpha \in U^\circ$ iff for all u in $U : \alpha(u) = 0$. In particular, that holds true for all $w \in W \subset U$, thus $\alpha \in W^\circ$.

The polar has also the following properties

$$(U \cap W)^\circ = U^\circ \oplus W^\circ \qquad\qquad (U \oplus W)^\circ = U^\circ \cap W^\circ \tag{16.3.2}$$

Prove them!

Symplectic vector spaces

A *symplectic vector space* is a pair $(V, \boldsymbol{\omega})$ of a vector space V and a non-degenerate 2-form $\boldsymbol{\omega}$ on V. The form $\boldsymbol{\omega}$ is expressed in a basis e_i as

$$\boldsymbol{\omega} = \tfrac{1}{2}\omega_{ij}e^i \wedge e^j \tag{16.3.3}$$

One can always find a basis (e_i') in which

$$\boldsymbol{\omega} = \tfrac{1}{2}J_{ij}e'^i \wedge e'^j \qquad J_{ij} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{16.3.4}$$

Such bases are called *canonical bases*.

First of all, let e_1 be a non-zero vector in V. One can always find ϵ_1 for which $\boldsymbol{\omega}(e_1, \epsilon_1) = -1$, otherwise $\boldsymbol{\omega}$ would be degenerate.
Let $\tilde{U}$ be a complement of $\mathrm{Span}(e_1, \epsilon_1)$. Of course, there is no reason for having $\boldsymbol{\omega}(\tilde{u}, e_1) = \boldsymbol{\omega}(\tilde{u}, \epsilon_1) = 0$ for all $\tilde{u} \in \tilde{U}$. However, other complements of $\mathrm{Span}(e_1, \epsilon_1)$ are in the form $u = \alpha e_1 + \beta \epsilon_1 + \tilde{u}$ and one has

$$\boldsymbol{\omega}(u, e_1) = \beta + \boldsymbol{\omega}(\tilde{u}, e_1) \qquad \boldsymbol{\omega}(u, \epsilon_1) = -\alpha + \boldsymbol{\omega}(\tilde{u}, \epsilon_1) \tag{16.3.5}$$

Thus by choosing $\alpha = \boldsymbol{\omega}(\tilde{u}, \epsilon_1)$ and $\beta = -\boldsymbol{\omega}(\tilde{u}, e_1)$ one can find a new complement

$$U = \{u = \boldsymbol{\omega}(\tilde{u}, \epsilon_1)e_1 - \boldsymbol{\omega}(\tilde{u}, e_1)\epsilon_1 + \tilde{u}\} \tag{16.3.6}$$

foir which $\boldsymbol{\omega}(u, e_1) = \boldsymbol{\omega}(u, \epsilon_1) = 0$.
If V was $\dim(V) = 2$, then $\dim(U) = 0$ and we found a base $e_i = (e_1, \epsilon_1)$. If not, let us denote by $\boldsymbol{\omega}_U$ the restriction of $\boldsymbol{\omega}$ to U. The form $\boldsymbol{\omega}_U$ is a bilinear, antisymmetric form. It is non-degenerate. In fact, if it were degenerate there would be a non-zero $u \in U$ such that $\forall u' \in U : \boldsymbol{\omega}(u, u') = 0$. But we also know that $\boldsymbol{\omega}(u, e_1) = \boldsymbol{\omega}(u, \epsilon_1) = 0$; thus $\boldsymbol{\omega}(u, v)$ would vanish for all vectors in V and $\boldsymbol{\omega}$ would be degenerate, which is against the hypothesis. Then $(U, \boldsymbol{\omega}_U)$ is again a symplectic vector space and we can find (e_2, ϵ_2) and a symplectic complement U'. The procedure can be iterated until one find a zero-dimensional symplectic complement. At that point one has $e_i = (e_1, e_2, \ldots, e_k, \epsilon^1, \epsilon_2, \ldots \epsilon_k)$ such that $\boldsymbol{\omega}(e_i, e_j) = J_{ij}$.

Notice that a symplectic vector space is necessary even-dimensional. For future convention we shall denote by Greek indices the ones running from 1 to k and by Latin indices the ones running from 1 to $n = 2k$. Thus, for example, one has $e_i = (e_\alpha, \epsilon_\beta)$.

Given two symplectic vector spaces $(V, \boldsymbol{\omega})$ and $(V', \boldsymbol{\omega}')$, a *symplectic map* is a map $\Phi : V \to V'$ such that

$$\boldsymbol{\omega}'(\Phi(v), \Phi(w)) = \boldsymbol{\omega}(v, w) \tag{16.3.7}$$

The components of the symplectic form $\boldsymbol{\omega}$ will be denoted by $\omega_{ij} = \boldsymbol{\omega}(e_i, e_j)$. It is a non-degenerate matrix. The inverse matrix will be denoted by ω^{ij} and one has (watch out for indices order)

$$\omega_{ik}\omega^{jk} = \delta^i_j \tag{16.3.8}$$

One can define the maps

$$\flat : V \to V^* : v^i e_i \mapsto v^i \omega_{ij} e^j \qquad\qquad \sharp : V^* \to V : \alpha_i e^i \mapsto \alpha_i \omega^{ji} e_j \tag{16.3.9}$$

One can easily show that

$$\left((v)^\flat \right)^\sharp = (v^k \omega_{ki} e^i)^\sharp = v^k \omega_{ik} \omega^{ij} e_j = v^i e_i = v \tag{16.3.10}$$

The two maps are the inverse one of the other and hence are isomorphisms, canonical isomorphisms induced by the symplectic structures. They differ from the corresponding isomorphisms induced by a metric structure even though, by an abuse of notation, we denote them by the same names.

Example: Let Q be a vector space of dimension k and consider $V = Q \oplus Q^*$ with the form $\boldsymbol{\omega} = e_\alpha \wedge e^\alpha$. The pair $(V, \boldsymbol{\omega})$ is a symplectic vector space and (e_α, e^α) is a canonical basis.

Example: If $(V, \boldsymbol{\omega})$ is a symplectic space, one can define

$$\boldsymbol{\omega}^* : V^* \times V^* \to \mathbb{R} \qquad \boldsymbol{\omega}^*(\alpha, \beta) = \boldsymbol{\omega}(\alpha^\sharp, \beta^\sharp) \tag{16.3.11}$$

In a basis, one has

$$\boldsymbol{\omega}^*(\alpha, \beta) = \boldsymbol{\omega}(\alpha^\sharp, \beta^\sharp) = (\alpha_i \omega^{ji})(\beta_k \omega^{lk})\omega_{jl} = -\alpha_i \omega^{ki}\beta_k = \alpha_i \omega^{ik}\beta_k \qquad \boldsymbol{\omega}^* = \tfrac{1}{2}\omega^{ij} e_i \wedge e_j \tag{16.3.12}$$

The pair $(V^*, \boldsymbol{\omega}^*)$ is then a symplectic space. Then the isomorphisms $\flat : V \to V^*$ and $\sharp : V^* \to V$ are symplectic isomorphisms between $(V, \boldsymbol{\omega})$ and $(V^*, \boldsymbol{\omega}^*)$.

We also defined the maps

$$\flat : V \to V^* : v^i e_i \mapsto v^i g_{ij} e^j \qquad\qquad \sharp : V^* \to V : \alpha_i e^i \mapsto \alpha_i g^{ji} e_j \tag{16.3.13}$$

induced by an inner product $g : V \times V \to \mathbb{R}$.

Given a subspace $U \subset V$, one can define $U^\circ \subset V^*$ and map it back $(U^\circ)^\sharp \subset V$. A vector $w \in (U^\circ)^\sharp$ is $w = \alpha_i g^{ij} e_j$ with $\alpha \in U^\circ$. Thus for all $u \in U$, one has

$$g(w, u) = \alpha_i g^{ij} g_{jk} u^k = \alpha_i u^i = \alpha(u) = 0 \tag{16.3.14}$$

Thus $(U^\circ)^\sharp$ is made (of all the) vectors orthogonal to u and it is denoted by $(U^\circ)^\sharp = U^\perp$.

Hereafter, we shall always work in a symplectic space with no metric structure specified on it. Thus $\sharp$, $\flat$, $\perp$ will here denote the ones induced by the symplectic structure, unless otherwise specified.

Let us define the *symplectic polar* to be

$$U^\perp = (U^\circ)^\sharp = \{w \in V : \forall u \in U, \, \boldsymbol{\omega}(w, u) = 0\} \tag{16.3.15}$$

Being the symplectic polar the image through an isomorphism of the polar, one has $U^{\perp\perp} = U$, $\dim(U) + \dim(U^\perp) = \dim(V)$ and if $W \subset U$ then $U^\perp \subset W^\perp$. Moreover,

$$(U \cap W)^\perp = U^\perp \oplus W^\perp \qquad (U \oplus W)^\perp = U^\perp \cap W^\perp \tag{16.3.16}$$

Definition: *a subspace $U \in (V, \boldsymbol{\omega})$ is called:*

coisotropic iff $U^\perp \subset U$;

isotropic iff $U \subset U^\perp$;

Lagrangian if it is both isotropic and coistropic (i.e. iff $U = U^\perp$);

symplectic iff $U \cap U^\perp = \{0\}$.

Of course, if U is isotropic then $\dim(U) \le k$ ($\dim(V) = n = 2k$). If U is coisotropic then $\dim(U) \ge k$. If U is Lagrangian then $\dim(U) = k$.

A subspace U is isotropic iff $U^\perp$ is coisotropic.

A useful criterium for U to be isotropic is the following:

Theorem: *$U \subset (V, \boldsymbol{\omega})$ is isotropic iff $\forall u, u' \in U, \, \boldsymbol{\omega}(u, u') = 0$.*

Proof: *if U is isotropic, $U \subset U^\perp$. Then for any $u, u' \in U$, $u' \in U^\perp$, then $\boldsymbol{\omega}(u, u') = 0$.*

If $\forall u, u' \in U, \, \boldsymbol{\omega}(u, u') = 0$ then $u' \in U^\perp$, then $U \subset U^\perp$, then U is isotropic. ∎

Theorem: *let $L \subset (V, \boldsymbol{\omega})$ be Lagrangian and let $L \subset U \subset (V, \boldsymbol{\omega})$; then U is coisotropic. In any coisotropic subspace U there exists a Lagrangian subspace $L \subset U$.*

Proof: *if L is Lagrangian, then $U^\perp \subset L^\perp = L \subset U$, then U is coisotropic.*

If U is coisotropic, then $U^\perp \subset U$. If $U^\perp = U$ then U is Lagrangian and the end the proof. If $U^\perp \ne U$ it means one can find $v \in V$ which is U but not in $U^\perp$. Then consider $W = U^\perp \oplus \text{Span}(v)$ and $W^\perp \subset U$. One has $U^\perp \subset W \subset W^\perp \subset U$. Thus $W^\perp$ is again coisotropic (though with one dimension less than U). The procedure can be iterated until one gets $L = W = W^\perp$. ∎

Theorem: *if $U \subset (V, \boldsymbol{\omega})$ is symplectic than $(U, \boldsymbol{\omega}_U)$ is a symplectic vector space.*

Proof: *let $\boldsymbol{\omega}_U$ denote the restriction of the symplectic form to U. It is of course a bilinear, antisymmetric form on U. We have to show that it is non-degenerate.*

For, let $u \in U$ be a non-zero vector and let us assume $\boldsymbol{\omega}(u, w) = 0$ for all $w \in U$. However, since $U \cap U^\perp = \{0\}$ and $\dim(U) + \dim(U^\perp) = \dim(V)$ then one has $V = U \oplus U^\perp$. Accordingly, any vector in V can be written (in a unique way) as $v = u + \hat{u}$. We already know that $\boldsymbol{\omega}(u, w) = 0$ for all $w \in U$. And we know that $\boldsymbol{\omega}(u, \hat{u}) = 0$ for all $\hat{u} \in U^\perp$. Thus $\boldsymbol{\omega}(u, v) = 0$ for all $v \in V$. Thus the symplectic form $\boldsymbol{\omega}$ would be degenerate, which is against the original hypothesis.

Thus $\boldsymbol{\omega}_U$ is non-degenerate and $(U, \boldsymbol{\omega}_U)$ is a symplectic space. ∎

Any one dimensional subspace $U \subset (V, \omega)$ is isotropic. Thus any subspace of codimension 1 is coisotropic.

Symplectic reduction

If $U \subset (V, \boldsymbol{\omega})$ is a coisotropic subspace we can define an equivalence relation

$$u \sim u' \iff u - u' \in U^\perp \tag{16.3.17}$$

and define the quotient space $W = U/\sim$ (which is of course a vector space) and the projection map $\pi : U \to W : u \mapsto [u]$.

Then we have the sequence

$$\{0\} \longrightarrow U^\perp \xrightarrow{i} U \xrightarrow{\pi} W \xrightarrow{0} \{0\} \tag{16.3.18}$$

which is a *short exact sequence*. This sequence split by a map $\varphi : W \to U : w \mapsto u$ such that $\pi \circ \varphi = \mathrm{id}_W$. In other words, the map φ is such that $[u] = w$ which ammounts to choosing a representative $u = \varphi(w)$ of the class w), i.e. $\pi(u) = w$. Then the image $\varphi(W) \subset U$ is a subspace and $U = i(U^\perp) \oplus \varphi(W)$.

Let us define a bilinear form on W by

$$\hat{\boldsymbol{\omega}} : W \times W \to \mathbb{R} \qquad \hat{\boldsymbol{\omega}}([u], [u']) = \boldsymbol{\omega}(\varphi([u]), \varphi([u'])) \tag{16.3.19}$$

This is obviously antisymmetric. Let us show it is non-degenerate.

First of all, let us show that it is constant on equivalence classes. Let $u \in U$, and $u_1, u_2 \in [w] \subset U$.

$$\boldsymbol{\omega}(u, u_1) - \boldsymbol{\omega}(u, u_2) = \boldsymbol{\omega}(u, u_1 - u_2) = 0 \quad \Rightarrow \boldsymbol{\omega}(u, u_1) = \boldsymbol{\omega}(u, u_2) \tag{16.3.20}$$

since $u_1 - u_2 \in U^\perp$.

Let us now suppose that $\hat{\boldsymbol{\omega}}$ is degenerate; then there exists $w \in W$ with $w \neq 0$ such that $\hat{\boldsymbol{\omega}}(w, w') = 0$ for all $w' \in W$. Then $\boldsymbol{\omega}(\varphi(w), u') = 0$ for all $u' \in U$. Then $\varphi(w) \in U^\perp$. Hence $w = 0$ which is against the hypothesis. Then $\hat{\boldsymbol{\omega}}$ is non-degenerate.

Then, if $\dim(V) = 2n$ and $\dim(U) = 2n - k$, then $\dim(U^\perp) = k$ and $\dim(W) = 2(n - k)$ (with k the codimension of U in V). The map φ is injective (two different classes cannot have the same representative) thus $\dim(\varphi(W)) = 2(n - k)$.

Accordingly, $(W, \hat{\boldsymbol{\omega}})$ is a symplectic vector space. In other words, when one has a coisotropic subspace U that defines an isotropic subspace $U^\perp \subset U$ (which is called the *characteristic subspace*) and, by quotienting out the characteristic subspace, one obtains a symplectic vector space $(W = U/U^\perp, \hat{\boldsymbol{\omega}})$.

Symplectic manifolds

A *symplectic manifold* is a pair $(M, \boldsymbol{\omega})$ of a manifold M and a closed, non-degenerate 2-form $\boldsymbol{\omega}$.

We shall see hereafter that a closed $\boldsymbol{\omega}$ is necessary condition of existence of canonical coordinates.

A coordinate system x^a on M is said *canonical* if the symplectic form $\boldsymbol{\omega}$ is written in the form

$$\boldsymbol{\omega} = \tfrac{1}{2} J_{ab} \, dx^a \wedge \, dx^b \tag{16.3.21}$$

Given a generic coordinate system x^a in which $\boldsymbol{\omega} = \frac{1}{2}\omega_{ab}(x)\,dx^a \wedge dx^b$ one can find a canonical system x'^a iff

$$\bar{J}_a^c\,\omega_{cd}\,\bar{J}_b^d = J_{ab} \tag{16.3.22}$$

In order for this to be possible, one needs $\boldsymbol{\omega}$ to be closed. In fact if canonical coordinates exist then one has

$$\begin{aligned}
\bar{J}_{ae}^c\,\omega_{cd}\,\bar{J}_b^d + \bar{J}_a^c\,\omega_{cd}\,\bar{J}_{be}^d + \bar{J}_a^c\,\partial_f\omega_{cd}\,\bar{J}_e^f\bar{J}_b^d = 0 \\
\bar{J}_{ba}^c\,\omega_{cd}\,\bar{J}_e^d + \bar{J}_b^c\,\omega_{cd}\,\bar{J}_{ea}^d + \bar{J}_b^c\,\partial_f\omega_{cd}\,\bar{J}_a^f\bar{J}_e^d = 0 \\
\bar{J}_{eb}^c\,\omega_{cd}\,\bar{J}_a^d + \bar{J}_e^c\,\omega_{cd}\,\bar{J}_{ab}^d + \bar{J}_e^c\,\partial_f\omega_{cd}\,\bar{J}_b^f\bar{J}_a^d = 0
\end{aligned} \tag{16.3.23}$$

where we cyclically permuted the indices $[aeb]$. Then summing them all one obtains

$$\begin{aligned}
\bar{J}_a^c\,\partial_f\omega_{cd}\,\bar{J}_e^f\bar{J}_b^d + \bar{J}_b^c\,\partial_f\omega_{cd}\,\bar{J}_a^f\bar{J}_e^d + \bar{J}_e^c\,\partial_f\omega_{cd}\,\bar{J}_b^f\bar{J}_a^d = 0 \\
\bar{J}_a^c\,(\partial_f\omega_{cd} + \partial_c\omega_{df} + \partial_d\omega_{fc})\bar{J}_b^d\bar{J}_a^c = 0 \qquad \Rightarrow \bar{J}_a^c\partial_{[f}\omega_{cd]}\bar{J}_b^d\bar{J}_a^c = 0 \qquad \Rightarrow d\boldsymbol{\omega} = 0
\end{aligned} \tag{16.3.24}$$

The condition $d\boldsymbol{\omega} = 0$ has been proven in a coordinate system but, if it holds true in a coordinate system, it is true in any coordinate system.

Theorem: *(Darboux) on any symplectic manifold $(M, \boldsymbol{\omega})$ there exist canonical coordinates.*

Example: for any manifold Q, the manifold $M = T^*Q$ can be equipped with natural coordinates $x^a = (q^\lambda, p_\lambda)$ and the closed 2-form $\boldsymbol{\omega} = dp_\lambda \wedge dq^\lambda$. The pair $(T^*Q, \boldsymbol{\omega})$ is a symplectic manifold and natural coordinates are always canonical.

This can be shown otherwise, by noticing that, in natural coordinates, $\boldsymbol{\theta} = p_\mu\,dq^\mu$ is a global 1-form, which is called the *Liouville form*. Liouville form is a characteristic structure on cotangent spaces. Then we set $\boldsymbol{\omega} = d\boldsymbol{\theta}$, which is closed (and exact) by construction.

In other words, cotangent spaces are symplectic manifolds in which the symplectic form is, not only closed, but also exact. ∎

Let us remark that, in a symplectic manifold $(M, \boldsymbol{\omega})$, any tangent space is a symplectic vector space $(T_xM, \boldsymbol{\omega}_x)$. Then all the definitions given on symplectic vector spaces somehow extend to symplectic manifolds.

A sub-manifold $S \subset M$ is (co)isotropic (Lagrangian, symplectic, respectively) iff its tangent space $T_xS \subset (T_xM, \boldsymbol{\omega}_x)$ is a (co)isotropic (Lagrangian, symplectic, respectively) vector space.

Hence, if S is a coisotropic sub-manifold, one has $(T_xS)^\perp \subset T_xS \subset (T_xM, \boldsymbol{\omega}_x)$ which defines a distribution on S which is called the *characteristic distribution*. A vector field on S which belongs to the characteristic distribution is called a *characteristic field*.

Theorem: *characteristic distributions are always integrable.*

Let X, Y be two characteristic fields. Being characteristic fields means that

$$\forall v \in TS: \ \boldsymbol{\omega}(X, v) = 0 \qquad \Longleftrightarrow \quad X^a\omega_{ab}v^b = 0 \qquad \Longleftrightarrow \quad i_X\boldsymbol{\omega}|_S = 0 \tag{16.3.25}$$

Let us consider the commutator $[X, Y]$; this is characteristic iff $\left(i_{[X,Y]}\boldsymbol{\omega}\right)|_S = 0$. In fact, by using the formula proven in Appendix A, one has

$$i_{[X,Y]}\boldsymbol{\omega}|_S = \pounds_X\left(i_Y\boldsymbol{\omega}\right)|_S - i_Y\pounds_X\boldsymbol{\omega}|_S = -\boxed{i_Y i_X d\boldsymbol{\omega}}|_S - i_Y d\left(i_X\boldsymbol{\omega}\right)|_S = 0 \tag{16.3.26}$$

and the commutator is characteristic. Then the characteristic distribution is involutive, then it is integrable.

This is great! That means that one can foliate a coisotropic sub-manifold S, the leaves of the foliation being called *characteristics* of S.

Then one can define an equivalence relation on S

$$x \sim y \iff \quad \text{both } x \text{ and } y \text{ belongs to the same characteristics of } S \tag{16.3.27}$$

and one can take the quotient $N = S/\sim$. The tangent space $T_xN = T_xS/(T_xS)^\perp$ is a symplectic space with the induced symplectic form $\hat{\boldsymbol{\omega}}_x$. Then $(N, \hat{\boldsymbol{\omega}})$ is a candidate to be a symplectic manifold; one just needs to show that the form $\hat{\boldsymbol{\omega}}$ is closed.

By the way, the form $\hat{\boldsymbol{\omega}}$ is obtained by pulling back $\boldsymbol{\omega}$ along the map $i : N \to M$ which chooses a representative in M of each point on N. Since differential is a natural operator, it commutes with pull-back, then one has

$$d\hat{\boldsymbol{\omega}} = di^*\boldsymbol{\omega} = i^*d\boldsymbol{\omega} = 0 \tag{16.3.28}$$

and $\hat{\boldsymbol{\omega}}$ *is* closed!

Appendix A. A useful formula

We shall here prove the formula

$$i_{[X,Y]}\boldsymbol{\omega} = \pounds_X i_Y\boldsymbol{\omega} - i_Y\pounds_X\boldsymbol{\omega} \tag{16.A.1}$$

where $\pounds_X := d \circ i_X + i_X d$ denotes the Lie derivative of forms along the vector field X.

Let us start by checking that $\pounds_X$ is a derivative. It is, of course, linear and one needs to show Leibniz rule.

$$\begin{aligned}\pounds_X(\boldsymbol{\omega} \wedge \boldsymbol{\theta}) =&d(i_X\boldsymbol{\omega} \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}\boldsymbol{\omega} \wedge i_X\boldsymbol{\theta}) + i_X(d\boldsymbol{\omega} \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}\boldsymbol{\omega} \wedge d\boldsymbol{\theta}) = \\ =&di_X\boldsymbol{\omega} \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}d\boldsymbol{\omega} \wedge i_X\boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})-1}i_X\boldsymbol{\omega} \wedge d\boldsymbol{\theta} + \boldsymbol{\omega} \wedge di_X\boldsymbol{\theta} + i_X d\boldsymbol{\omega} \wedge \boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})}i_X\boldsymbol{\omega} \wedge d\boldsymbol{\theta} + (-1)^{\deg(\boldsymbol{\omega})+1}d\boldsymbol{\omega} \wedge i_X\boldsymbol{\theta} + \boldsymbol{\omega} \wedge i_X d\boldsymbol{\theta} = \\ =&\pounds_X\boldsymbol{\omega} \wedge \boldsymbol{\theta} + \boldsymbol{\omega} \wedge \pounds_X\boldsymbol{\theta}\end{aligned} \tag{16.A.2}$$

Second step is to show that the formula holds true when $\deg(\boldsymbol{\omega}) = 0$ (which is trivial) and $\deg(\boldsymbol{\omega}) = 1$. For $\deg(\boldsymbol{\omega}) = 1$, let us assume $\boldsymbol{\omega} = \omega_\mu\, dx^\mu$. Then

$$\begin{aligned}i_Y\boldsymbol{\omega} &= Y^\alpha\omega_\alpha \\ \pounds_X\boldsymbol{\omega} &= d(X^\alpha\omega_\alpha) + i_X(\partial_\beta\omega_\alpha\, dx^\beta \wedge dx^\alpha) = [\partial_\mu(X^\alpha\omega_\alpha) + X^\alpha(\partial_\alpha\omega_\mu - \partial_\mu\omega_\alpha)]\, dx^\mu = [\partial_\mu X^\alpha\omega_\alpha + X^\alpha\partial_\alpha\omega_\mu]\, dx^\mu \\ \pounds_X i_Y\boldsymbol{\omega} &= i_X d(i_Y\boldsymbol{\omega}) = X^\beta\partial_\beta(Y^\alpha\omega_\alpha) = X^\beta\partial_\beta Y^\alpha\omega_\alpha + X^\beta Y^\alpha\partial_\beta\omega_\alpha\end{aligned} \tag{16.A.3}$$

Then one has

$$\pounds_X i_Y\boldsymbol{\omega} - i_Y\pounds_X\boldsymbol{\omega} =X^\beta\partial_\beta Y^\alpha\omega_\alpha + X^\beta Y^\alpha\partial_\beta\omega_\alpha - Y^\mu[\partial_\mu X^\alpha\omega_\alpha + X^\alpha\partial_\alpha\omega_\mu] = (X^\beta\partial_\beta Y^\alpha - Y^\mu\partial_\mu X^\alpha)\omega_\alpha = i_{[X,Y]}\boldsymbol{\omega} \tag{16.A.4}$$

Finally, any form of degree $k \geq 2$ can be written (locally) as wedge product of lower degree forms. Let us suppose that we showed that the result holds true for all $k-1$ forms $\boldsymbol{\theta}$ and show that it holds true for k forms $\boldsymbol{\omega} \wedge \boldsymbol{\theta}$ with $\deg(\boldsymbol{\omega}) = 1$.

$$
\begin{aligned}
(\pounds_X i_Y - i_Y \pounds_X)(\boldsymbol{\omega} \wedge \boldsymbol{\theta}) &= \pounds_X i_Y \boldsymbol{\omega} \wedge \boldsymbol{\theta} - {\color{red}\pounds_X \boldsymbol{\omega} \wedge i_Y \boldsymbol{\theta}} + {\color{blue}i_Y \boldsymbol{\omega} \wedge \pounds_X \boldsymbol{\theta}} - \boldsymbol{\omega} \wedge \pounds_X i_Y \boldsymbol{\theta} - i_Y \pounds_X \boldsymbol{\omega} \wedge \boldsymbol{\theta} - {\color{blue}i_Y \boldsymbol{\omega} \wedge \pounds_X \boldsymbol{\theta}} + {\color{red}\pounds_X \boldsymbol{\omega} \wedge i_Y \boldsymbol{\theta}} + \boldsymbol{\omega} \wedge i_Y \pounds_X \boldsymbol{\theta} = \\
&= (\pounds_X i_Y - i_Y \pounds_X)(\boldsymbol{\omega}) \wedge \boldsymbol{\theta} - \boldsymbol{\omega} \wedge (\pounds_X i_Y - i_Y \pounds_X)(\boldsymbol{\theta}) = i_{[X,Y]} \boldsymbol{\omega} \wedge \boldsymbol{\theta} - \boldsymbol{\omega} \wedge i_{[X,Y]} \boldsymbol{\theta} = i_{[X,Y]}(\boldsymbol{\omega} \wedge \boldsymbol{\theta})
\end{aligned}
\tag{16.A.5}
$$

Lagrange multipliers

As an application of polar spaces, we wish to briefly review the Lagrange multipliers. Let us consider a manifold M with coordinates x^μ and with a sub-manifold S with coordinates k^A and embedded into M by the canonical embedding

$$
\iota : S \to M : k^A \mapsto x^\mu(k) \qquad\qquad J_A^\mu(k) = \frac{\partial x^\mu}{\partial k^A}
\tag{16.A.6}
$$

The embedding extends to its tangent map

$$
T_k \iota : T_k S \to T_{\iota(k)} M : v^A \partial_A \mapsto v^A J_A^\mu \partial_\mu
\tag{16.A.7}
$$

and we can identify the abstract tangent bundle TS with its image $T\iota(TS) \subset TM$.

If the sub-manifold is also identified as a level surface $S := \{x \in M : \chi^a(x) = 0\}$ a number of constraints $\chi^a : M \to \mathbb{R}$, we can define a sub-space N_k of covectors spanned by the differentials $d\chi^a$, namely $\lambda_a d\chi^a$ at a point $k \in S$. They are sub-spaces $N_k \subset T^*_{\iota(k)} M$.

If we consider a curve $\gamma : \mathbb{R} \to S$ on the constraint $S \subset M$, we have $\chi^a(\iota(k)) = 0$, hence

$$
d\chi^a(\dot\gamma) = \dot k^A \frac{\partial \chi^a}{\partial x^\mu} J_A^\mu = 0
\tag{16.A.8}
$$

That means that each $d\chi^a$ (hence any linear combination $\lambda_a d\chi^a \in N$) is in the polar tangent space $N_k = (T_k S)^\circ$.

Now consider a function $F : M \to \mathbb{R}$. Critical points of F are found as $dF_x = 0$. These can be on or off $S \subset M$. If $x = \iota(k) \in S$ is a critical point of F we call it a *free critical point*.

On the other hand, we have the *restriction* $f = F \circ \iota : S \to \mathbb{R}$ of F on S and its critical points, which are found as $df_k = 0$, and are called *constrained critical points*. A constrained critical point can be found as

$$
\partial_A f(k) = J_A^\mu \partial_\mu F(\iota(k)) = 0
\tag{16.A.9}
$$

Any free critical point on S is also a constrained critical point, although F can also have critical points which are not on S. However, there may be constrained critical points which are not free critical points at all. That is because for the constrained critical points the function F is stationary in the directions tangent to the constraint, regardless how they behave in the transverse directions. In other words, F is stationary along the tangent directions, i.e.

$$
\forall v \in TS : dF(v) = 0 \qquad \Rightarrow \qquad dF(v) = v^A J_A^\mu \partial_\mu F = df(v) = 0
\tag{16.A.10}
$$

One can check that this is in fact equivalent to the condition $dF_x \in (TS)^\circ$, namely

$$dF = \lambda_a d\chi^a \qquad \Rightarrow \qquad d(F - \lambda_a d\chi^a) = 0 \tag{16.A.11}$$

namely, the constraint critical points of F on S are free critical points of the function $F - \lambda_a \chi^a$. The 2 functions F and $F - \lambda_a \chi^a$ agree on S, there they are equal to f, and the are both different extensions of f out of the constraint S.

 The Lagrange multipliers methods is that. One can search for constrained critical points of F on S by searching free critical points of the extension $F - \lambda_a \chi^a$.

 One can solo look for constrained critical points of F as free critical points of f in S. To map one method into the other one can write the condition in adapted coordinates (k^A, χ^a) on M in which the constraint is obtained by $\chi^a = 0$.

 Then we have

$$\partial_A F - \lambda_a \partial_A \chi^a = \partial_A f = 0 \qquad \partial_b F - \lambda_b = 0 \tag{16.A.12}$$

The first condition reads exactly as that for the free critical points of f on S. The second condition, gives the Lagrange multipliers $\lambda_b = \partial_b F(k, \chi)$.

 Which method is the best depends on which description of the constraint S is easier to get, if a parameterization $x(k)$ or a description as a level surface $\chi^a(x) = 0$. One can go from one method to the other easily.

> Unfortunately, often the Lagrange multipliers are explained in terms of vectors.
> On a Riemannian manifold (M, g) one can write $dF \in (TS)^\circ$ by saying that $dF^\sharp = \partial_\mu F g^{\mu\nu} \partial_\nu$ is orthogonal to TS. In fact, we have
>
> $$\forall v \in TS : dF(v) = 0 \qquad \Rightarrow \qquad \partial_\mu F J_A^\mu v^A = 0 \qquad \Rightarrow \qquad \partial_\mu F g^{\mu\nu} g_{\nu\epsilon} J_A^\epsilon v^A = 0 \qquad \Rightarrow \qquad g(dF^\sharp, v) = 0 \tag{16.A.13}$$
>
> i.e. the vector $dF^\sharp$ is orthogonal to S. Also $(d\chi^a)^\sharp$ is a basis of normal vectors, then we can express $dF^\sharp = \lambda_a (d\chi^a)^\sharp$.
> That is unfortunate, since the structure is completely independent of a metric, which of corse is relevant in relativistic theories in which spacetimes come with no structure fixed on it.

4. Hamiltonian systems

 Let $(M, \boldsymbol{\omega})$ be a symplectic manifold. A vector field X is called a *(global) Hamiltonian field* iff there exists a function $f : M \to \mathbb{R}$ such that $X = (df)^\sharp$; ▷*Next Section* in that case, f is called a *potential* (or a *Hamiltonian*) for the field X and the field itself is denoted by X_f.

 A vector field X is Hamiltonian if the 1-form $X^\flat$ is exact and the Hamiltonian f for it is the potential, i.e. iff $(X_f)^\flat = df$.

> For $X_f = X^a \partial_a$, one has
>
> $$X^\flat = X^a \omega_{ab} dx^b \qquad df = \partial_a f \, dx^a \tag{16.4.1}$$
>
> Thus one has in general coordinates
>
> $$X_f = \partial_a f \omega^{ba} \partial_b \tag{16.4.2}$$

However, in canonical coordinates, one has $\omega_{ab} = J_{ab}$ so that one can split the coordinates into two blocks $x^a = (q^\lambda, p_\lambda)$ and one has

$$X_f = \partial_\lambda f \partial^\lambda - \partial^\lambda f \partial_\lambda = -\frac{\partial f}{\partial q^\lambda}\frac{\partial}{\partial p_\lambda} + \frac{\partial f}{\partial p_\lambda}\frac{\partial}{\partial q^\lambda} \tag{16.4.3}$$

(with obvious notation!).

Any Hamiltonian field X_H defines differential equations for integral curves (which are called *Hamilton equations* for H) and a family of integral curves which are their solutions. The intrinsic form for Hamilton equations is

$$\dot\gamma = X_H = (dH)^\sharp \tag{16.4.4}$$

These equations, in canonical coordinates, become the usual

$$\begin{cases} \dot q^\lambda = \dfrac{\partial f}{\partial p_\lambda} \\[2mm] \dot p_\lambda = -\dfrac{\partial f}{\partial q^\lambda} \end{cases} \tag{16.4.5}$$

Thus we can, in general, define a *Hamiltonian system* as a symplectic manifold $(M, \boldsymbol\omega)$ with a function H on it, meaning that from the set $(M, \boldsymbol\omega, H)$ one can always define the Hamiltonian field X_H on M which defines Hamilton equations as in (16.4.4).

Exercise: Consider a Hamiltonian system $(M, \boldsymbol\omega, H)$ for which one knows an observable $f(q, p)$ which is a first integral. Then the evolution of the system lies on the level surfaces of f, namely on $S_c = \{x \in M : f(x) = c\}$. Show that the hypersurface $S_c \subset M$ is coisotropic. Find the characteristic distribution and define a new Hamiltonian system with one less degrees of freedom which is equivalent to the original.

Poisson brackets

One can define Poisson brackets of two functions $f, g : M \to \mathbb{R}$ as

$$\{f, g\} := X_f(g) = -X_g(f) = \boldsymbol\omega(df, dg) \quad \Rightarrow \quad \{f, g\} = \partial_a f\, \omega^{ab}\, \partial_b g \tag{16.4.6}$$

That is trivially bilinear and antisymmetric. Moreover, one has

$$\{f, \{g, h\}\} = X_f(\{g, h\}) = X_f(X_g(h)) \quad \Rightarrow$$
$$\{f, \{g, h\}\} = \partial_a f \omega^{ab}\partial_b\left(\partial_c g\omega^{cd}\partial_d h\right) = \partial_a f\omega^{ab}\partial_{bc}g\omega^{cd}\partial_d h + \partial_a f\omega^{ab}\partial_c g\partial_b\omega^{cd}\partial_d h + \partial_a f\omega^{ab}\partial_c g\omega^{cd}\partial_{bd}h \tag{16.4.7}$$

Thus expanding the Jacobi identities $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$, one obtains

$$\begin{aligned} &\color{red}{\partial_a f\omega^{ab}\partial_{bc}g\omega^{cd}\partial_d h} + \partial_a f\omega^{ab}\partial_c g\partial_b\omega^{cd}\partial_d h + \color{blue}{\partial_a f\omega^{ab}\partial_c g\omega^{cd}\partial_{bd}h} +\\ &+ \partial_a g\omega^{ab}\partial_{bc}h\omega^{cd}\partial_d f + \partial_a g\omega^{ab}\partial_c h\partial_b\omega^{cd}\partial_d f + \partial_a g\omega^{ab}\partial_c h\omega^{cd}\partial_{bd}f +\\ &+ \partial_a h\omega^{ab}\partial_{bc}f\omega^{cd}\partial_d g + \partial_a h\omega^{ab}\partial_c f\partial_b\omega^{cd}\partial_d g + \color{red}{\partial_a h\omega^{ab}\partial_c f\omega^{cd}\partial_{bd}g} =\\ =&\partial_a f\partial_c g\partial_d h\left(\omega^{ae}\omega^{pc}\omega^{dq}\partial_e\omega_{pq} + \omega^{ce}\omega^{pd}\omega^{aE}\partial_e\omega_{pq} + \omega^{de}\omega^{pa}\omega^{cq}\partial_e\omega_{pq}\right) =\\ =& -\partial_a f\partial_c g\partial_d h\omega^{ae}\omega^{cp}\omega^{dq}\left(\partial_e\omega_{pq} + \partial_p\omega_{qe} + \partial_q\omega_{ep}\right) = -\partial_a f\partial_c g\partial_d h\omega^{ae}\omega^{cp}\omega^{dq}\partial_{[e}\omega_{pq]} = 0 \end{aligned} \tag{16.4.8}$$

where we used the identity $\partial_e \omega^{af} = -\omega^{ac} \partial_e \omega_{bc} \omega^{bf} = \omega^{ba} \omega^{fc} \partial_e \omega_{bc}$.

In canonical coordinates, all terms with derivatives of ω_{ab} vanish. Moreover, if the result is zero in canonical coordinates it is zero in all coordinates since Poisson brackets are intrinsic operations.

Thus the functions on M with Poisson brackets form a Lie algebra, called the *algebra of observables*. Notice that a constant observable has zero Poisson brackets with anything. For that reason, we can define an equivalence relation between observables $\mathcal{F}(M)$

$$f \sim g \iff f - g \in \mathbb{R} \tag{16.4.9}$$

Let us denote the quotient space $\mathcal{F}_R(M) = \mathcal{F}(M)/\sim$ and Poisson brackets are induced on the quotient.

Theorem: $[X_f, X_g] = X_{\{f,g\}}$

Proof: *we have*

$$[X_f, X_g](h) = X_f(X_g(h)) - X_g(X_f(h)) = \{f, \{g, h\}\} - \{g, \{f, h\}\} = \{f, \{g, h\}\} + \{g, \{h, f\}\} = \{\{f, g\}, h\} = X_{\{f,g\}}(h) \tag{16.4.10}$$

∎

This theorem proves two things at a time. First, Hamiltonian fields are a Lie subalgebra of the Lie algebra of vector fields. Second, the association between observables and their Hamiltonian fields is a Lie algebra morphism.

First and second class constraints

Let us consider a Hamiltonian system H on a symplectic manifold $(M, \boldsymbol{\omega})$ and a sub-manifold $S_1 \subset M$, called the *primary constraint*. Generically, the dynamics defined by H is not compatible with the primary constraint. It may happen that a solution starting on S_1 stays on S_1, which means X_H is tangent to S_1. When the primary constraint is described as a surface level $S_1 := \{x \in M : \chi_1^a(x) = 0\}$, then the compatibility condition in terms of the Poisson brackets reads as

$$\{H, \chi_1^a\}|_{S_1} = 0 \tag{16.4.11}$$

If $\{H, \chi_1^a\}$ does not vanish on S_1 one can restrict to the sub-set $S_2 \subset S$ on which $\{H, \chi_1^a\}|_{S_2} = 0$. We assume $S_2 \subset M$ is a sub-manifold of dimension greater than 0, in particular non-empty, and we call it the *secondary constraint*.
Then we know that a solution starting on S_2 stays in S_1, however we need it to stay in S_2. For it, if S_2 is set as $S_2 = \{x \in S_1 : \chi_2^a(x) = 0\}$, we need to check that

$$\{H, \chi_2^a\}|_{S_2} = 0 \tag{16.4.12}$$

This may be identically satisfied, never satisfied or satisfied on a sub-manifold $S_3 \subset S_2$ which is called the *tertiary constraint*.
We assume eventually we found a constraint S compatible with dynamics. If S is described as $\chi^a = 0$ ($\chi^a = (\chi_1^a, \chi_2^b, \chi_3^c, \ldots)$), one has $\{H, \chi^a\}|_S = 0$. The dynamics enforces solutions starting on S to stay on S. As usual we assume S is a sub-manifold of dimension $n \geq 1$.

A sub-manifold $S \subset M$ is called a *first class constraint* if it is a coisotropic sub-manifold, it is called a *second class constraint* otherwise.

Let us define the set $O_S = \{F : M \to \mathbb{R} : F|_S = \text{ constant}\}$ of observables which are constant on S. If $F \in O_S$ and $x \in S$, then the differential $dF_x(v) = 0$ for all vectors $v \in T_x S$ tangent to S, i.e. $T_x S \subset \text{Span}(dF_x)^\circ =: V_F$.

Let us assume we have $\dim(\text{Span}(dF_x)) = 1$, namely the differential dF_x is a non-zero covector. Hence, for each such observable F, we have $\dim(V_F) = \dim(M) - 1$. Accordingly, the tangent space $T_x S$ is found as the intersection $T_x S = \cap_{F \in O_S} V_F$.

For each $F \in O_F$, its differential dF_x at a point $x \in S$ defines a 1-dimensional vector space $\text{Span}(dF_x) \subset T_x^* M$ (unless x is a critical point for F). We say that 2 observables $F, G \in O_S$ are *functionally independent* iff the corresponding differentials $dF_x, dG_x \in T_x^* M$ are linearly independent at any $x \in S$.

If $\dim(S) = s \le m = \dim(M)$, then locally one can find $m - s = \text{codim}(S)$ observables $\chi^a \in O_S$ which are functionally independent and locally $S = \{x \in M : \chi^a(x) = 0\}$.

Let us consider a first class constraint, i.e. a coisotropic sub-manifold $S \subset (M, \boldsymbol{\omega})$; the sub-manifold S is coisotropic iff $T_k S \subset T_x M$ is coisotroipic iff $(T_k S)^\circ \subset T_x^* M$ is isotropic, iff

$$\forall \alpha, \beta \in (T_k S)^\circ : \boldsymbol{\omega}(\alpha, \beta) = 0 \tag{16.4.13}$$

The space $(T_k S)^\circ$ is generated by the differentials of observables $F \in O_S$, i.e. functions $F : M \to \mathbb{R}$ constant on S. Since F is constant on S, we have $dF_x \in (T_k S)^\circ : \forall v \in T_k S : v(F) = dF(v) = 0$. Hence, $\forall F, G \in O_S$ observables constant on S, one has

$$\boldsymbol{\omega}(dF_x, dG_x) = 0 \quad \Longleftrightarrow \quad (X_F(dG))|_S = 0 \quad \Longleftrightarrow \quad \{F, G\}|_S = 0 \tag{16.4.14}$$

Moreover, we can find a maximal set of functionally independent observables χ^a so that the corresponding differentials $(d\chi^a)_x$ are a basis in $(T_x S)^\circ$.

Let us consider $F \in O_S$. We have $dF_x = \alpha^a(x)(d\chi^a)_x$ and let us set $\hat{F} := \alpha^a(x)\chi^a$. Then we have

$$(d\hat{F})_x = \alpha^a(x)(d\chi^a)_x + (d\alpha^a)_x \chi^a = dF_x + (d\alpha^a)_x \chi^a \quad \Rightarrow \quad (d\hat{F})_x|_S = dF_x|_S \tag{16.4.15}$$

Accordingly, $(\hat{F} - F)|_S = \text{ constant}$. If we add the condition $\hat{F}(x) = F(x)$ at a single point $x \in S$, then we have that for all $F \in O_S$ it can be written as $F|_S = \alpha^a(x)\chi^a$. In other words, we are showing that O_S is a module freely generated by χ^a.

Let us remark that $\text{codim}(S) = \dim(T_x S)^\circ$ and we have

$$\{\chi^a, \chi^b\} = f^{ab}{}_c(x)\chi^c \tag{16.4.16}$$

for some local functions $f^c{}_{ab}(x)$ which are called the *structure functions* of the distribution $(T_x S)^\circ$. Then, since S is obtained by constraints $\chi^c = 0$, we have

$$\{\chi^a, \chi^b\}|_S = 0 \tag{16.4.17}$$

This is why one cannot have more than k first integrals in involution on a $2k$ dimensional symplectic manifold. Otherwise, the level surface $\chi^a = 0$ would be coisotropic although too low dimensional.

The constraint observables χ^a also produce a Hamiltonian field $X_a = -(d\chi^a)^\sharp$, which is local or global depending on whether χ^a are given as local or global functions. The vector field X_a is, by construction, in $T_x S$, i.e. it is tangent to S. The flows of χ^a are called *gauge flows*. The fields X^a are not only tangent to S, they are in fact characteristic vectors since $\forall v \in TS : \boldsymbol{\omega}(X^a, v) = d\chi^a(v) = 0$. Hence gauge flows leave the characteristic surfaces invariant.

When we consider a first class constraint $S \subset M$, it comes with a characteristic distribution $TS^\perp \subset TS$ and we can perform a symplectic reduction to project Hamiltonian system onto $(Q, \tilde{\boldsymbol{\omega}})$.

Again we have a fibration $[M \to Q]$ and vertical vectors on it are generated by the characteristic fields X_a. The characteristic flow is then vertical and it leaves the Hamiltonian system on Q invariant. Accordingly, X_a generate gauge transformations on S and Q is identified with the phase space.

When we have a second class constraint $S \subset M$ ...

Locally Hamiltonian fields

Of course, necessary condition for being exact is being closed. A vector field X is called *locally Hamiltonian* iff the 1-form $(X)^\flat$ is closed, i.e. $d(X)^\flat = 0$.

Of course, a locally Hamiltonian vector field is Hamiltonian when restricted to a small enough open set $U \subset M$ (any closed form is locally exact).

Locally Hamiltonian vector fields form a Lie subalgebra of vector fields (which contains globally Hamiltonian vector fields). In fact, any locally Hamiltonian vector field is locally a globally Hamiltonian vector field. If X, Y are locally Hamiltonian vector fields, then there exists an open set U on which $X = X_f$ and $Y = Y_g$. Then, in U, one has $[X, Y] = X_{\{f,g\}}$; then the commutator is a locally Hamiltonian vector field.

Theorem: *X is locally Hamiltonian iff its flow is canonical.*

Proof: *X is locally Hamiltonian iff $dX^\flat = 0$ iff $di_X\boldsymbol{\omega} = 0$ iff $\pounds_X\boldsymbol{\omega} = (i_X d + di_X)\boldsymbol{\omega} = 0$ iff $\Phi_s^*\boldsymbol{\omega} = \boldsymbol{\omega}$.* ∎

Generating functions

Let us consider a symplectic map $\Phi : M \to M$ on a symplectic manifold $(M, \boldsymbol{\omega})$. In canonical coordinates, the map reads as $\Phi : (q^\lambda, p_\lambda) \mapsto (q'^\lambda, p'_\lambda)$ is locally given by

$$\begin{cases} q'^\lambda = q'^\lambda(q, p) \\ p'_\lambda = p'_\lambda(q, p) \end{cases} \tag{16.4.18}$$

Since the map Φ is invertible, the partial Jacobians

$$\left(\frac{\partial q'}{\partial q}\right) \qquad \left(\frac{\partial q'}{\partial p}\right) \qquad \left(\frac{\partial p'}{\partial q}\right) \qquad \left(\frac{\partial p'}{\partial p}\right) \tag{16.4.19}$$

cannot be all degenerate at the same time.

If the first partial Jacobian is non-degenerate, one can invert $q = q(p, q')$, and use (p, q') as independent (non-canonical) coordinates on M. The symplectic map is then called a *type III transformation*.

If the second partial Jacobian is non-degenerate, one can invert $p = p(q, q')$, and use (q, q') as independent (non-canonical) coordinates on M. The symplectic map is then called a *type I transformation*.

If the third partial Jacobian is non-degenerate, one can invert $q = q(p, p')$, and use (p, p') as independent (non-canonical) coordinates on M. The symplectic map is then called a *type IV transformation*.

Finally, if the fourth partial Jacobian is non-degenerate, one can invert $p = p(q, p')$, and use (q, p') as independent (non-canonical) coordinates on M. The symplectic map is then called a *type II transformation*.

> Each symplectic map is at least of one of these types, possibly (even generically) of all four of them. For example, the identity map is type III and II, not a type I or IV transformation.

Since the map Φ is symplectic, it leaves the symplectic form $\boldsymbol{\omega}$ invariant. Since the symplectic form is closed, it is locally exact. Let us denote by $\boldsymbol{\theta}$ a local potential of $\boldsymbol{\omega} = d\boldsymbol{\theta}$. On a cotangent space, such a potential is global and it is the Liouville form.

Accordingly, the map Φ leaves the Liouville form, being it local or global, invariant up to an exact differential dS, i.e. one has

$$p'_\lambda dq'^\lambda = p_\lambda dq^\lambda + dS \tag{16.4.20}$$

For a type I transformation, one can express everything in terms of coordinates (q, q') as

$$p'_\lambda dq'^\lambda = p_\lambda dq^\lambda + \frac{\partial S}{\partial q^\lambda} dq^\lambda + \frac{\partial S}{\partial q'^\lambda} dq'^\lambda \qquad \begin{cases} p'_\lambda = \dfrac{\partial S}{\partial q'^\lambda}(q, q') \\[2mm] p_\lambda = -\dfrac{\partial S}{\partial q^\lambda}(q, q') \end{cases} \qquad \left| \frac{\partial S}{\partial q \partial q'} \right| \neq 0 \tag{16.4.21}$$

Then, the function $F_I(q, q') = -S(q, q')$ is called a *generating function of type I*. Under the regularity condition $\left| \frac{\partial S}{\partial q \partial q'} \right| \neq 0$, one can invert the second to obtain $q' = q'(q, p)$, which can then be replaced into the first equation to obtain $p' = p'(q, p)$, i.e. the symplectic transformation.

For a type II transformation, one can express everything in terms of coordinates (q, p') as

$$p'_\lambda \left(\frac{\partial q'^\lambda}{\partial q^\mu} dq^\mu + \frac{\partial q'^\lambda}{\partial p'_\mu} dp'_\mu \right) = p_\lambda dq^\lambda + \frac{\partial S}{\partial q^\lambda} dq^\lambda + \frac{\partial S}{\partial p'_\lambda} dp'_\lambda \qquad \begin{cases} p'_\mu \dfrac{\partial q'^\mu}{\partial q^\lambda} = p_\lambda + \dfrac{\partial S}{\partial q^\lambda}(q, p') \\[2mm] p'_\mu \dfrac{\partial q'^\mu}{\partial p'_\lambda} = \dfrac{\partial S}{\partial p'_\lambda}(q, p') \end{cases} \tag{16.4.22}$$

Then, the function $F_{II}(q, p') = p'_\lambda q'^\lambda(q, p') - S(q, p')$ is called a *generating function of type II*. The conditions can be written in terms of $F_{II}(q, p')$ as

$$\begin{cases} p_\lambda = \dfrac{\partial F_{II}}{\partial q^\lambda}(q, p') \\[2mm] q'^\lambda = \dfrac{\partial F_{II}}{\partial p'_\lambda}(q, p') \end{cases} \qquad \left| \frac{\partial F_{II}}{\partial q \partial p'} \right| \neq 0 \tag{16.4.23}$$

Under the regularity condition $\left| \frac{\partial F_{II}}{\partial q \partial p'} \right| \neq 0$, one can invert the first to obtain $p' = p''(q, p)$, which can then be replaced into the second equation to obtain $q' = q'(q, p)$, i.e. the symplectic transformation.

For a type III transformation, one can express everything in terms of coordinates (p, q') as

$$p'_\lambda dq'^\lambda = p_\lambda \left(\frac{\partial q^\lambda}{\partial p_\mu} dp_\mu + \frac{\partial q^\lambda}{\partial q'^\mu} dq'^\mu \right) + \frac{\partial S}{\partial p_\lambda} dp_\lambda + \frac{\partial S}{\partial q'^\lambda} dq'^\lambda \qquad \begin{cases} p'_\lambda = p_\mu \dfrac{\partial q^\mu}{\partial q'^\lambda} + \dfrac{\partial S}{\partial q'^\lambda}(p, q') \\[2ex] 0 = p_\mu \dfrac{\partial q^\mu}{\partial p_\lambda} + \dfrac{\partial S}{\partial p_\lambda}(p, q') \end{cases} \tag{16.4.24}$$

Then, the function $F_{III}(p, q') = p_\lambda q^\lambda(p, q') + S(p, q')$ is called a *generating function of type III*. The conditions can be written in terms of $F_{III}(p, q')$ as

$$\begin{cases} p'_\lambda = \dfrac{\partial F_{III}}{\partial q'^\lambda}(p, q') \\[2ex] q^\lambda = \dfrac{\partial F_{III}}{\partial p_\lambda}(p, q') \end{cases} \qquad \left| \frac{\partial F_{III}}{\partial p \partial q'} \right| \neq 0 \tag{16.4.25}$$

Under the regularity condition $\left| \frac{\partial F_{III}}{\partial p \partial q'} \right| \neq 0$, one can invert the second to obtain $q' = q'(q, p)$, which can then be replaced into the first equation to obtain $p' = p'(q, p)$, i.e. the symplectic transformation.

For a type IV transformation, one can express everything in terms of coordinates (p, p') as

$$p'_\lambda \left(\frac{\partial q'^\lambda}{\partial p_\mu} dp_\mu + \frac{\partial q'^\lambda}{\partial p'_\mu} dp'_\mu \right) = p_\lambda \left(\frac{\partial q^\lambda}{\partial p_\mu} dp_\mu + \frac{\partial q^\lambda}{\partial p'_\mu} dp'_\mu \right) + \frac{\partial S}{\partial p_\lambda} dp_\lambda + \frac{\partial S}{\partial p'_\lambda} dp'^\lambda \qquad \begin{cases} p'_\mu \dfrac{\partial q'^\mu}{\partial p'_\lambda} = p_\mu \dfrac{\partial q^\mu}{\partial p'_\lambda} + \dfrac{\partial S}{\partial p'_\lambda}(p, p') \\[2ex] p'_\mu \dfrac{\partial q'^\mu}{\partial p_\lambda} = p_\mu \dfrac{\partial q^\mu}{\partial p_\lambda} + \dfrac{\partial S}{\partial p_\lambda}(p, p') \end{cases} \tag{16.4.26}$$

Then, the function $F_{IV}(p, p') = p_\lambda q^\lambda(p, p') - p'_\lambda q'^\lambda(p, p') + S(p, p')$ is called a *generating function of type IV*. The conditions can be written in terms of $F_{IV}(p, p')$ as

$$\begin{cases} q'^\lambda = -\dfrac{\partial F_{IV}}{\partial p'_\lambda}(p, p') \\[2ex] q^\lambda = \dfrac{\partial F_{IV}}{\partial p_\lambda}(p, p') \end{cases} \qquad \left| \frac{\partial F_{IV}}{\partial p \partial p'} \right| \neq 0 \tag{16.4.27}$$

Under the regularity condition $\left| \frac{\partial F_{IV}}{\partial p \partial p'} \right| \neq 0$, one can invert the second to obtain $p' = p'(q, p)$, which can then be replaced into the first equation to obtain $q' = q'(q, p)$, i.e. the symplectic transformation.

The good thing of generating functions is that by an almost generic function, they generate *algebraically*, i.e. with no integration, all symplectic maps.

References

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add

Chapter 17. Lie groups and their Lie algebras

1. Lie groups and their Lie algebra

A Lie group is a group G which is also a smooth manifold and the product and inverse operations are smooth maps with respect to the manifold structure.

> The set of real numbers $\mathbb{R}$ with the sum is a Lie group.
> The group $\mathrm{GL}(m, \mathbb{K})$ is an open set in $\mathbb{K}^{(m^2)}$ (identified with the vector space of all $m \times m$ matrices) from which we have to delete the zeros of the equation $\det(S) = 0$. Here, depending on the case, $\mathbb{K}$ can be specified to $\mathbb{R}$ or $\mathbb{C}$. The function $\det(S) = 0$ is polynomial so that its zeros are a closed set. Cartesian coordinates on $\mathbb{R}^{(m^2)}$ simply restrict to $\mathrm{GL}(m, \mathbb{K})$ to give global coordinates. For short, we set $\mathrm{GL}(m) := \mathrm{GL}(m, \mathbb{R})$.
> The orthogonal group $O(m) \subset \mathrm{GL}(m)$ (and $\mathrm{SO}(m) \subset \mathrm{GL}(m)$) is given by solutions of the constraint ${}^{t}SS = \mathbb{I}$ (and $\det(S) = 1$) which are polynomial equations in $\mathrm{GL}(m)$. They define a smooth sub-manifold.
> The group of unimodular complex numbers $U(1)$ is a Lie group with the product.
> The groups $\mathrm{SU}(n)$ of special unitary complex matrices are Lie groups.

The issue about Lie groups is that, in general, it is quite hard to choose good coordinates on a group. One can use their Lie algebra though usually the expression obtained for the product is not simple (plus one needs a preliminary coordinate notation for developing the theory), or can choose coordinates on an *ad hoc* basis which makes it difficult to discuss in general even groups in the same family (e.g. on orthogonal groups the formulae depend on the dimension).

The way out is often to resort to matrices. Most groups can be defined as subgroups of some $\mathrm{GL}(n, \mathbb{K})$. We shall do it below in the next Section. For now, one can consider a coordinate system g^a defined on a neighbourhood U of the identity of the group G, defined by the map $\varphi : U \to \mathbb{R}^n$.

On a Lie group G, one has two canonical smooth maps (which are not group homomorphisms, though they are manifold morphisms)

$$L_g : G \to G : k \mapsto g \cdot k \qquad R_g : G \to G : k \mapsto k \cdot g \tag{17.1.1}$$

These maps are called *left (right) translations* on the group G.

By using translations, one can define coordinates around any point $g \in G$ using a chart (U, φ) around the identity. In fact, one can define a neighbourhood $V = L_g(U)$ of $g \in G$ and a chart $\varphi \circ L_{g^{-1}} : V \to \mathbb{R}^n$. Analogously, one can define new coordinates using the right translation R_g.

One can also define the product $\pi : G \times G \to G : (g, k) \mapsto \pi(g, k)$ which is also smooth. We shall denote by $\partial_a^1 \pi(g, k)$ and $\partial_a^2 \pi(g, k)$ the partial derivatives with respect to g^a and k^a, respectively.

Let us remark that the tangent maps of the left (right) translations are expressed as

$$T_k L_g : T_k G \to T_{g \cdot k} G : v^a \partial_a \mapsto v^b \partial_b^2 \pi^a(g, k) \partial_a \qquad T_k R_g : T_k G \to T_{g \cdot k} G : v^a \partial_a \mapsto v^b \partial_b^1 \pi^a(k, g) \partial_a \tag{17.1.2}$$

Accordingly, the partial derivatives $\partial^1 \pi(g, k)$ and $\partial^2 \pi(g, k)$ can be considered as the matrix expressions for the tangent maps $T_k R_g$ and $T_k L_g$, respectively. Regarding these matrices as local expressions of tangent maps is sometimes useful since one can use functorial properties. For example, associativity in the group, namely $g \cdot (k \cdot h) = (g \cdot k) \cdot h$, can be written in one of these three ways

$$L_g \circ L_k(h) = L_{g \cdot k}(h) \qquad L_g \circ R_h(k) = R_h \circ L_g(k) \qquad R_{k \cdot h}(g) = R_h \circ R_k(g) \tag{17.1.3}$$

The corresponding relations induced on tangent maps read as

$$T_{k \cdot h} L_g \circ T_h L_k = T_h L_{g \cdot k} \qquad T_{k \cdot h} L_g \circ T_k R_h = T_{g \cdot k} R_h \circ T_k L_g \qquad T_g R_{k \cdot h} = T_{g \cdot k} R_h \circ T_g R_k \tag{17.1.4}$$

In view of the correspondence described above, one has the identities

$$\partial_c^2 \pi^a(g, k \cdot h) \partial_b^2 \pi^c(k, h) = \partial_b^2 \pi^a(g \cdot k, h) \qquad \partial_c^2 \pi^a(g, k \cdot h) \partial_b^1 \pi^c(k, h) = \partial_c^1 \pi^a(g \cdot k, h) \partial_b^2 \pi^c(g, k) \qquad \partial_b^1 \pi^a(g, k \cdot h) = \partial_c^1 \pi^a(g \cdot k, h) \partial_b^1 \pi^c(g, k) \tag{17.1.5}$$

in which one can specialise g, k, h according to situations. For example, L_e and R_e are the identity map in the group, thus $T_g L_e$ and $T_g R_e$ are the identity map in the tangent spaces, hence

$$\partial_b^2 \pi^a(e, g) = \delta_b^a \qquad \partial_b^1 \pi^a(g, e) = \delta_b^a \tag{17.1.6}$$

are the identity matrices. Accordingly, one has

$$\partial_c^2 \pi^a(g^{-1}, g \cdot h) \partial_b^2 \pi^c(g, h) = \delta_b^a \qquad \partial_c^1 \pi^a(g \cdot k, k^{-1}) \partial_b^1 \pi^c(g, k) = \delta_b^a \tag{17.1.7}$$

and

$$\partial_c^2 \pi^a(g^{-1}, g \cdot h) = \left(\partial_a^2 \pi^c(g, h) \right)^{-1} \qquad \partial_c^1 \pi^a(g \cdot k, k^{-1}) = \left(\partial_a^1 \pi^c(g, k) \right)^{-1} \tag{17.1.8}$$

which are simply the local expression of the fact that $T_{g \cdot h} L_{g^{-1}} = (T_h L_g)^{-1}$ and $T_{g \cdot k} R_{k^{-1}} = (T_g R_k)^{-1}$. By further specialising, one has

$$\partial_c^2 \pi^a(g^{-1}, g) = \left(\partial_a^2 \pi^c(g, e) \right)^{-1} \qquad \partial_c^1 \pi^a(k, k^{-1}) = \left(\partial_a^1 \pi^c(e, k) \right)^{-1} \tag{17.1.9}$$

which, in fact, correspond to the fact that $T_g L_{g^{-1}} = (T_e L_g)^{-1}$ and $T_k R_{k^{-1}} = (T_e R_k)^{-1}$.

For future convenience, we shall also set

$$T_e L_g(v) = v^b \partial_b^2 \pi^a(g, e) \partial_a =: v^b L_b^a(g) \partial_a \qquad T_e R_g(v) = v^b \partial_b^1 \pi^a(e, g) \partial_a =: v^b R_b^a(g) \partial_a \tag{17.1.10}$$

as well as

$$\bar{L}_b^a(g) := \partial_b^2 \pi^a(g^{-1}, g) \qquad \bar{R}_b^a(g) := \partial_b^1 \pi^a(g, g^{-1}) \tag{17.1.11}$$

for the inverse matrices.

An interesting general result about Lie groups comes form the fact that translations are diffeomorphisms. In fact, given a basis e_a of the tangent space to the identity T_eG, one can define a basis

$$e'_a := T_eL_g(e_a) \qquad \left(e''_a := T_eR_g(e_a)\right) \tag{17.1.12}$$

at any point g. The basis depends smoothly on the point, i.e. we just defined $n = \dim(G)$ vector fields $X_a(g) := e'_a$ which are pointwise independent.

A manifold M on which one can find $m = \dim(M)$ vector fields pointwise independent is called *parallelisable*. For example, $\mathbb{R}^n$ is parallelisable, S_2 is not (since any vector field on a 2-sphere vanishes somewhere). Accordingly, any Lie group is parallelisable.

In view of this result, S_2 cannot be a Lie group since it is not parallelisable, while $S_3 \simeq \mathrm{SU}(2)$ is parallelizabe since it is a Lie group.

A vector field X on G is called *left-invariant* (*right-invariant*) iff, for any $g \in G$, $(L_g)_*X = X$ (or $(R_g)_*X = X$, respectively). The set of all left-invariant (right-invariant, respectively) vector fields on G is denoted by $\mathfrak{X}_L(G)$ ($\mathfrak{X}_R(G)$), respectively).

Since the commutator is a natural operation and it commutes with push-forwards, one has for $X, Y \in \mathfrak{X}_L(G)$

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y] \tag{17.1.13}$$

and the commutator $[X, Y]$ is left-invariant as well. Similarly, for right-invariant fields. This shows that left-invariant fields $\mathfrak{X}_L(G)$ (and right-invariant fields $\mathfrak{X}_R(G)$) form a Lie subalgebra of $\mathfrak{X}(G)$. The subalgebra $\mathfrak{X}_L(G)$ of left-invariant vector fields on G is called *the Lie algebra of G* and it is denoted by $\mathfrak{g}$.

One can associate a left-invariant (and right-invariant) vector field on G to a vector at the identity of the group $v \in T_eG$ by

$$l_v(g) = T_eL_g(v) \qquad r_v(g) = T_eR_g(v) \tag{17.1.14}$$

The field l_v is left-invariant. In fact,

$$(L_g)_*l_v(k) = T_kL_g \circ l_v(k) = T_kL_g \circ T_eL_k(v) = T_e(L_g \circ L_k)(v) = T_e(L_{gk})(v) = l_v(gk) \qquad \Rightarrow (L_g)_*l_v = l_v \tag{17.1.15}$$

Similarly, for right-invariant fields $r_v(g) = T_eR_g(v)$.

Then we defined two maps $l : T_eG \to \mathfrak{X}_L(G)$ and $r : T_eG \to \mathfrak{X}_R(G)$ so that $l(v) = l_v$ and $r(v) = r_v$. These maps are one-to-one since $l^{-1} : \mathfrak{X}_L(G) \to T_eG : X \mapsto X(e)$ and $r^{-1} : \mathfrak{X}_R(G) \to T_eG : X \mapsto X(e)$ are the inverse maps.

In fact,

$$l^{-1} \circ l : T_eG \to T_eG : v \mapsto l_v(e) = T_eL_e(v) = T_e\mathrm{id}_G(v) = v \qquad l \circ l^{-1} : \mathfrak{X}_L(G) \to \mathfrak{X}_L(G) : X(g) \mapsto T_eL_g(X(e)) = X(g) \tag{17.1.16}$$

and similarly for the right maps.

Then we have that $T_eG \simeq \mathfrak{X}_L(G) \simeq \mathfrak{X}_R(G)$ as vector spaces. In particular, if $\dim(G) = n$, we have that $\dim(\mathfrak{X}_L(G)) = \dim(\mathfrak{X}_R(G)) = n$ are finite dimensional subalgebras of the infinite dimensional Lie algebra $\mathfrak{X}(G)$ of vector fields.

Let us consider a basis $T_A \in T_e G$ and $l(T_A) =: l_A$ the corresponding left-invariant fields. Since l is an isomorphism, it sends a basis into a basis. Then any left-invariant field $X_L \in \mathfrak{X}_L(G)$ can be expanded with *real* (constant not functions!) coefficients X^A as

$$X_L = X^A l_A = l(X^A T_A) \tag{17.1.17}$$

By evaluation at the identity, one has $X_L(e) = X^A l_A(e) = X^A T_A$, so that $X_L = l(X^A T_A) = X^A l_A$.

Analogously, $r_A = r(T_A)$ is a basis for $\mathfrak{X}_R(G)$ and any right-invariant field $X_R \in \mathfrak{X}_R(G)$ can be expanded with real coefficients X^A as

$$X_R = X^A r_A \qquad \Rightarrow X_R = r(X^A T_A) \tag{17.1.18}$$

The commutators $[l_A, l_B]$ are again a real linear combination of the basis l_C of left-invariant fields. One has

$$[l_A, l_B] = c^C{}_{AB} l_C \tag{17.1.19}$$

for some set of *real* constants $c^C{}_{AB}$ which are called the *structure constants* of the group (or of the algebra). Of course, structure constants depend on the basis T_A chosen in the algebra.

By expanding it in coordinates, one obtains

$$[l_A, l_B] = \left(T_A^a T_B^b - T_B^a T_A^b \right) L_a^c(g) \partial_c^1 L_b^d(g) \partial_d = T_A^a T_B^b \left(L_a^c(g) \partial_c L_b^d(g) - L_b^c(g) \partial_c L_a^d(g) \right) \bar{L}_d^e(g) T_e^C l_C \tag{17.1.20}$$

so that the structure constants are given by

$$c^C{}_{AB} = T_A^a T_B^b \left(L_a^c(g) \partial_c L_b^d(g) - L_b^c(g) \partial_c L_a^d(g) \right) \bar{L}_d^e(g) T_e^C \tag{17.1.21}$$

Despite appearances, we know that the structure constants cannot depend on the point g in the group. Accordingly, one has

$$c^C{}_{AB} = T_A^a T_B^b \left(\partial_a L_b^c(e) - \partial_b L_a^c(e) \right) T_c^C \tag{17.1.22}$$

In fact, one can also prove this directly.

$$
\begin{aligned}
c^C{}_{AB} =\; & T_A^a T_B^b \left(L_a^c(g) \partial_c L_b^d(g) - L_b^c(g) \partial_c L_a^d(g) \right) \bar{L}_d^e(g) T_e^C = T_A^a T_B^b \left(\partial_a^2 \pi^c(g,e) \partial_c^1 \partial_b^2 \pi^d(g,e) - \partial_b^2 \pi^c(g,e) \partial_c^1 \partial_a^2 \pi^d(g,e) \right) \partial_d^2 \pi^e(\bar{g},g) T_e^C = \\
=\; & T_A^a T_B^b \left(\partial_b^2 \partial_a^1 \pi^c(e,e) \partial_c^2 \pi^d(g,e) + {\color{red}\partial_b^2 \partial_a^2 \pi^d(g,e)} - \partial_a^2 \partial_c^1 \pi^c(e,e) \partial_c^2 \pi^d(g,e) - {\color{red}\partial_a^2 \partial_b^2 \pi^d(g,e)} \right) \partial_d^2 \pi^e(\bar{g},g) T_e^C = \\
=\; & T_A^a T_B^b \left(\partial_a^1 \partial_b^2 \pi^c(e,e) - \partial_b^1 \partial_a^2 \pi^c(e,e) \right) T_c^C = T_A^a T_B^b \left(\partial_a L_b^c(e) - \partial_b L_a^c(e) \right) T_c^C
\end{aligned}
\tag{17.1.23}
$$

where we used the identity

$$\partial_a^2 \pi^c(g,k) \partial_c^1 \partial_b^2 \pi^d(gk,h) = \partial_a^1 \partial_b^2 \pi^c(k,h) \partial_c^2 \pi^d(g,kh) + \partial_d^2 \partial_c^2 \pi^d(g,kh) \partial_a^1 \pi^d(k,h) \partial_b^2 \pi^c(k,h) \tag{17.1.24}$$

then specialised to $k = h = e$ which is a consequence of the associative property $(gk)h = g(kh)$ obtained by taking the derivative with respect to k^a and h^b.

The same argument holds for right-invariant vector fields. One has

$$[r_A, r_B] = \hat{c}^C{}_{AB} r_C \tag{17.1.25}$$

for some set of *real* constants $\hat{c}^C{}_{AB}$. By expanding it in coordinates one obtains

$$[r_A, r_B] = \left(T_A^a T_B^b - T_B^a T_A^b\right) R_a^c(g) \partial_c^1 R_b^d(g) \partial_d = T_A^a T_B^b \left(R_a^c(g) \partial_c R_b^d(g) - R_b^c(g) \partial_c R_a^d(g)\right) \bar{R}_d^e(g) T_e^C r_C \tag{17.1.26}$$

so that the constants are given by

$$\hat{c}^C{}_{AB} = T_A^a T_B^b \left(R_a^c(g) \partial_c R_b^d(g) - R_b^c(g) \partial_c R_a^d(g)\right) \bar{R}_d^e(g) T_e^C \tag{17.1.27}$$

Again, we know that the structure constants cannot depend on the point g in the group. Accordingly, one has

$$\hat{c}^C{}_{AB} = T_A^a T_B^b \left(\partial_a R_b^c(e) - \partial_b R_a^c(e)\right) T_c^C \tag{17.1.28}$$

By direct computation, one has:

$$\begin{aligned}
\hat{c}^C{}_{AB} =& T_A^a T_B^b \left(R_a^c(g) \partial_c R_b^d(g) - R_b^c(g) \partial_c R_a^d(g)\right) \bar{R}_d^e(g) T_e^C = T_A^a T_B^b \left(\partial_a^1 \pi^c(e,g) \partial_c^2 \partial_b^1 \pi^d(e,g) - \partial_b^1 \pi^c(e,g) \partial_c^2 \partial_a^1 \pi^d(e,g)\right) \partial_d^1 \pi^e(g,\bar{g}) T_e^C = \\
=& T_A^a T_B^b \left(\partial_a^2 \partial_b^1 \pi^c(e,e) \partial_c^1 \pi^d(e,g) + {\color{red}\partial_a^1 \partial_b^1 \pi^d(e,g)} - \partial_b^2 \partial_a^1 \pi^c(e,e) \partial_c^1 \pi^d(e,g) - {\color{red}\partial_b^1 \partial_a^1 \pi^d(e,g)}\right) \partial_d^1 \pi^e(g,\bar{g}) T_e^C = \\
=& T_A^a T_B^b \left(\partial_a^2 \partial_b^1 \pi^c(e,e) - \partial_b^2 \partial_a^1 \pi^c(e,e)\right) T_c^C = T_A^a T_B^b \left(\partial_a R_b^c(e) - \partial_b R_a^c(e)\right) T_c^C
\end{aligned} \tag{17.1.29}$$

where we used the identity

$$\partial_a^1 \pi^c(k,g) \partial_c^2 \partial_b^1 \pi^d(h,kg) = \partial_a^2 \partial_b^1 \pi^c(h,k) \partial_c^1 \pi^d(hk,g) + \partial_e^1 \partial_b^1 \pi^d(hk,g) \partial_a^2 \pi^e(h,k) \partial_b^1 \pi^c(h,k) \tag{17.1.30}$$

then specialised to $k = h = e$ which is a consequence of the associative property $(hk)g = h(kg)$ obtained by taking the derivative with respect to k^a and h^b.

Let us also remark that one has $\hat{c}^C{}_{AB} = -c^C{}_{AB}$ as one can check by comparing equations (17.1.23) and (17.1.29) (in their last but one step). That means that the map

$$\varphi : \mathfrak{X}_L(G) \to \mathfrak{X}_R(G) : l_A \mapsto -r_A \tag{17.1.31}$$

is an isomorphism of Lie algebras. However, it also means that the right and left Lie algebras induce two different structures of Lie algebras on $T_e G$. That is why it is convenient to make a choice and select the *left* Lie algebra as *the* Lie algebra of the group.

Finally, let us prove that

$$[l_A, r_B] = 0 \tag{17.1.32}$$

By expanding, we have

$$\begin{aligned}
[l_A, r_B] =& T_A^a T_B^b \left(L_a^c(g) \partial_c R_b^d(g) - R_b^c(g) \partial_c L_a^d(g)\right) \partial_d = T_A^a T_B^b \left(\partial_a^2 \pi^c(g,e) \partial_c^2 \partial_b^1 \pi^d(e,g) - \partial_b^1 \pi^c(e,g) \partial_c^2 \partial_a^2 \pi^d(g,e)\right) \partial_d = \\
=& T_A^a T_B^b \left({\color{red}\partial_c^1 \partial_a^2 \pi^d(g,e) \partial_b^1 \pi^c(e,g)} - {\color{red}\partial_b^1 \pi^c(e,g) \partial_c^1 \partial_a^2 \pi^d(g,e)}\right) \partial_d = 0
\end{aligned} \tag{17.1.33}$$

where we used the identity $\partial_a^2 \pi^c(g,e) \partial_c^2 \partial_b^1 \pi^d(e,ge) = \partial_c^1 \partial_a^2 \pi^d(eg,e) \partial_b^1 \pi^c(e,g)$.

All these identities among second derivatives of the product map is why one would like to avoid coordinates in the general theory of Lie groups. Essentially, the only group in which this can be done easily is $GL(m)$. Luckily enough, this then extends to all matrix groups.

For future convenience and accordingly to (17.1.10) and (17.1.11), let us set

$$L_A^a(g) = T_A^c L_c^a(g) = T_A^c \partial_c^2 \pi^a(g, e) \qquad R_A^a(g) = T_A^c R_c^a(g) = T_A^c \partial_c^1 \pi^a(e, g)$$
$$\bar{L}_a^A(g) = T_c^A \bar{L}_a^c(g) = T_c^A \partial_a^2 \pi^c(g^{-1}, g) \qquad \bar{R}_a^A(g) = T_c^A \bar{R}_a^c(g) = T_c^A \partial_a^1 \pi^c(g, g^{-1}) \tag{17.1.34}$$

Let us also remark that one has a canonical left action of the group G on itself (as a manifold)

$$\text{Ad} : G \times G \to G : (g, k) \mapsto g \cdot k \cdot g^{-1} \tag{17.1.35}$$

which is called the *adjoint action*. One has

$$\text{Ad}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g \tag{17.1.36}$$

The tangent map of the adjoint action at the identity $\text{ad}_g = T_e \text{Ad}_g$ reads as

$$\text{ad}_g = T_{g^{-1}} L_g \circ T_e R_{g^{-1}} = T_g R_{g^{-1}} \circ T_e L_g \tag{17.1.37}$$

i.e.

$$\text{ad}_g : T_e G \to T_e G : T_A \mapsto \text{ad}_A^B(g) T_B \qquad \text{ad}_A^B(g) = \bar{R}_c^B(g) L_A^c(g) = \bar{L}_d^B(\bar{g}) R_A^d(\bar{g}) \tag{17.1.38}$$

Among other things, the adjoint action is useful to change left-invariant to right-invariant fields and vice versa. In fact

$$l_A(g) = L_A^a(g) \partial_a = L_A^a(g) \bar{R}_a^B(g) r_B(g) = \text{ad}_A^B(g) r_B(g) \qquad r_A(g) = R_A^a(g) \partial_a = R_A^a(g) \bar{L}_a^B(g) l_B(g) = \text{ad}_A^B(\bar{g}) l_B(g) \tag{17.1.39}$$

which are one the inverse of the other, i.e. $\text{ad}_A^B(\bar{g}) =: \bar{\text{ad}}_A^B(g)$ is the inverse of $\text{ad}_A^B(g)$.

Then, in view of (17.1.32), one has

$$0 = [r_A, \text{ad}_B^C(g) r_C] = \left(L_A^a(g) \partial_a \text{ad}_B^D(g) - \text{ad}_B^C(g) c^D{}_{AC} \right) r_D \qquad \Rightarrow T_A^a \partial_a \text{ad}_B^D(e) = c^D{}_{AB} \tag{17.1.40}$$

One also can define the adjoint action of the Lie algebra $\mathfrak{g}$ on itself which reads as $\text{ad}(\xi) : \mathfrak{g} \to \mathfrak{g} : \zeta \mapsto [\xi, \zeta] = \xi^A \zeta^B c^C{}_{AB} T_C$.

The Cartan–Killing form

We can define a symmetric bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$K(\xi, \zeta) = \text{Tr}(\text{ad}(\xi) \circ \text{ad}(\zeta)) = \xi^E c^B{}_{EA} \xi^F c^A{}_{FB} = K_{EF} \xi^E \zeta^F \qquad K_{EF} := c^B{}_{EA} c^A{}_{FB} \tag{17.1.41}$$

which is called the *Cartan–Killing form*. It is non-degenerate on semisimple Lie algebras (i.e. Lie algebras with no non-zero commutative ideal). Then it is an inner product on $\mathfrak{g}$.

That is a bit complicated, however, one should not worry too much about the details of the computation of the Cartan-Killing form. What is important is to know it is a metric on the group. Once you know it, remember its expression still depends on a basis chosen in the Lie algebra. What one does, in practice, is the restrict to orthogonal bases, so that $K_{EF} = \delta_{EF}$.

Notice that, in the Yang-Mills theories, one needs the Cartan-Killing form to contract the curvatures in the Lagrangian density. That is why one restricts to semisimple groups and orthonormal bases in the Lie algebra.

Nightmare identities on the Lie algebra

With that, we are ready to find a number of identities. For example,

$$0 = [l_A, r_B] = [l_A, \mathrm{ad}_B^C(\bar{g})l_C] = L_A^a(g)\partial_a\mathrm{ad}_B^C(\bar{g})l_C + \mathrm{ad}_B^C(\bar{g})c^D{}_{AC}l_D \qquad \Rightarrow L_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g) = -c^C{}_{AD}\mathrm{ad}_B^D(\bar{g}) \tag{17.1.42}$$

$$0 = [l_A, r_B] = [\mathrm{ad}_A^C(g)r_C, r_B] = -R_B^b(g)\partial_b\mathrm{ad}_A^C(g)r_C - \mathrm{ad}_A^C(g)c^D{}_{CB}r_D \qquad \Rightarrow R_B^b(g)\partial_b\mathrm{ad}_A^C(g) = -c^C{}_{DB}\mathrm{ad}_A^D(g) \tag{17.1.43}$$

$$0 = [l_A, r_B] = [\mathrm{ad}_A^C(g)r_C, \mathrm{ad}_B^D(\bar{g})l_D] = L_A^c(g)\partial_c\bar{\mathrm{ad}}_B^D(g)l_D - R_B^d(g)\partial_d\mathrm{ad}_A^C(g)\bar{\mathrm{ad}}_C^D(g)l_D \qquad \Rightarrow$$
$$\Rightarrow \quad L_A^a(g)\partial_a\bar{\mathrm{ad}}_B^D(g) = R_B^b(g)\partial_b\mathrm{ad}_A^C(g)\bar{\mathrm{ad}}_C^D(g) \tag{17.1.44}$$

as well as

$$[l_A, l_B] = [l_A, \mathrm{ad}_B^C(g)r_C] = L_A^a(g)\partial_a\mathrm{ad}_B^C(g)r_C = c^C{}_{AB}\mathrm{ad}_C^D(g)r_D \qquad \Rightarrow L_A^a(g)\partial_a\mathrm{ad}_B^C(g) = c^D{}_{AB}\mathrm{ad}_D^C(g) \tag{17.1.45}$$

$$[l_A, l_B] = [\mathrm{ad}_A^C(g)r_C, \mathrm{ad}_B^D(g)r_D] = L_A^c(g)\partial_c\mathrm{ad}_B^D(g)r_D - L_B^d(g)\partial_d\mathrm{ad}_A^C(g)r_C - \mathrm{ad}_A^C(g)\mathrm{ad}_B^D(g)c^E{}_{CD}r_E = c^C{}_{AB}\mathrm{ad}_C^D(g)r_D \qquad \Rightarrow$$
$$\Rightarrow \quad \underline{c^C{}_{AB}\mathrm{ad}_C^E(g)} - \underline{c^C{}_{BA}\mathrm{ad}_C^E(g)} - \mathrm{ad}_A^C(g)\mathrm{ad}_B^D(g)c^E{}_{CD} = \underline{c^C{}_{AB}\mathrm{ad}_C^E(g)} \qquad \Rightarrow$$
$$\Rightarrow \quad c^E{}_{CD}\mathrm{ad}_A^C(g)\mathrm{ad}_B^D(g) = \mathrm{ad}_C^E(g)c^C{}_{AB} \tag{17.1.46}$$

and

$$[r_A, r_B] = [r_A, \bar{\mathrm{ad}}_B^C(g)l_C] = R_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g)l_C = -c^C{}_{AB}\bar{\mathrm{ad}}_C^D(g)l_D \qquad \Rightarrow R_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g) = -c^D{}_{AB}\bar{\mathrm{ad}}_D^C(g) \tag{17.1.47}$$

$$[r_A, r_B] = [\bar{\mathrm{ad}}_A^C(g)l_C, \bar{\mathrm{ad}}_B^D(g)l_D] = R_A^c(g)\partial_c\bar{\mathrm{ad}}_B^D(g)l_D - R_B^d(g)\partial_d\bar{\mathrm{ad}}_A^C(g)l_C + \bar{\mathrm{ad}}_A^C(g)\bar{\mathrm{ad}}_B^D(g)c^E{}_{CD}l_E = -c^C{}_{AB}\bar{\mathrm{ad}}_C^D(g)l_D \qquad \Rightarrow$$
$$\Rightarrow \quad -\underline{c^C{}_{AB}\bar{\mathrm{ad}}_C^E(g)} + \underline{c^C{}_{BA}\bar{\mathrm{ad}}_C^E(g)} + \bar{\mathrm{ad}}_A^C(g)\bar{\mathrm{ad}}_B^D(g)c^E{}_{CD} = -\underline{c^C{}_{AB}\mathrm{ad}_C^E(g)} \qquad \Rightarrow$$
$$\Rightarrow \quad c^E{}_{CD}\bar{\mathrm{ad}}_A^C(g)\bar{\mathrm{ad}}_B^D(g) = \bar{\mathrm{ad}}_C^E(g)c^C{}_{AB} \tag{17.1.48}$$

Of course, these are not all independent. For example, (17.1.46) and (17.1.48) are the same and they both express the fact that structure constants are ad_g-invariant. Moreover, (17.1.43) is equivalent to (17.1.47) and (17.1.45) is equivalent to (17.1.42).

In fact

$$R_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g) = -c^D{}_{AB}\bar{\mathrm{ad}}_D^C(g) \quad \Longleftrightarrow \quad R_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g)\mathrm{ad}_C^D(g) = -c^D{}_{AB} \quad \Longleftrightarrow \quad R_A^a(g)\partial_a\mathrm{ad}_C^D(g) = -c^D{}_{BA}\mathrm{ad}_C^B(g) \tag{17.1.49}$$

and

$$L_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g) = -c^C{}_{AD}\mathrm{ad}_B^D(\bar{g}) \quad \Longleftrightarrow \quad L_A^a(g)\partial_a\bar{\mathrm{ad}}_B^C(g)\mathrm{ad}_D^B(g) = -c^C{}_{AD} \quad \Longleftrightarrow \quad L_A^a(g)\partial_a\mathrm{ad}_D^B(g) = c^C{}_{AD}\mathrm{ad}_C^B(g) \tag{17.1.50}$$

In view of ad-invariance of structure constants, also (17.1.43) and (17.1.45) are equivalent.

In fact

$$L^a_A(g)\partial_a \mathrm{ad}^C_B(g) = c^D{}_{AB}\mathrm{ad}^C_D(g) \quad \Longleftrightarrow \mathrm{ad}^D_A(g)R^d_D(g)\partial_d \mathrm{ad}^C_B(g) = c^D{}_{AB}\mathrm{ad}^C_D(g) \quad \Longleftrightarrow$$
$$\Longleftrightarrow R^d_E(g)\partial_d \mathrm{ad}^C_B(g) = \bar{\mathrm{ad}}^A_E(g)c^D{}_{AB}\mathrm{ad}^C_D(g) = -c^D{}_{BA}\mathrm{ad}^B_E(g) \tag{17.1.51}$$

Then we can keep as independent identities

$$c^C{}_{AB} = \bar{\mathrm{ad}}^C_F(g)c^F{}_{ED}\mathrm{ad}^E_A(g)\mathrm{ad}^D_B(g) \qquad\qquad L^a_A(g)\partial_a \mathrm{ad}^C_B(g) = \mathrm{ad}^C_D(g)c^D{}_{AB} \tag{17.1.52}$$

Let me stress that these identities are difficult to find, difficult to remember, as well as difficult to use when needed. Moreover, when working on a specific group, one should first find the specific form of $ad^A_B(g)$ and the structure constants $c^A{}_{BC}$ which, in fact, are group dependent. These objects, in turn, depend on the matrices $L^a_A(g)$ and $R^a_A(g)$ (together with their inverses) which eventually depend on the derivative of the product $\pi(g, k)$, which finally depend on the coordinates chosen on the group.

Let me state it once again: although this is more general and it applies to any Lie group, it is here meant more as motivation to restrict to matrix groups, which cover most applications and are way simpler.

Invariant forms

Before specialising to the general linear group and matrix groups, let us define left and right-invariant 1-forms.

A 1-form $\boldsymbol{\theta}_L$ ($\boldsymbol{\theta}_R$, respectively) on G is called *left-invariant* (or *right-invariant*) if one has

$$(L_g)^* \boldsymbol{\theta}_L = \boldsymbol{\theta}_L \qquad\qquad (R_g)^* \boldsymbol{\theta}_R = \boldsymbol{\theta}_R \tag{17.1.53}$$

Locally $\boldsymbol{\theta} = \theta_a(g)\, dg^a$ and it is left-invariant iff $T^*_{gh} L_g \boldsymbol{\theta}(gh) = \boldsymbol{\theta}(h)$, i.e. iff

$$\theta_a(gh)\partial^2_b \pi^a(g, h) = \theta_b(h) \tag{17.1.54}$$

This essentially determines the coefficients as a function of $\theta_a := \theta_a(e)$, i.e.

$$\theta_b(g) = \theta_a \partial^2_b \pi^a(\bar{g}, g) = \theta_a \bar{L}^a_b(g) \tag{17.1.55}$$

Thus a left-invariant 1-form has the form

$$\boldsymbol{\theta} = \theta_a \bar{L}^a_b(g)\, dg^b = \theta_A \mathbf{l}^A \tag{17.1.56}$$

where we set $\mathbf{l}^A := T^A_a \bar{L}^a_b(g)\, dg^b$ and $\theta_A := T^a_A \theta_a$.

The forms $\mathbf{l}^A$ are a basis for left-invariant 1-forms. This is the dual basis of the basis of left-invariant vector fields l_A in fact

$$\mathbf{l}^A(l_B) = L^b_B(g)\bar{L}^A_b(g) = \delta^A_B \tag{17.1.57}$$

Analogously, for right-invariant forms we can define

$$\mathbf{r}^A := T^A_a \bar{R}^a_b(g)\, dg^b \tag{17.1.58}$$

which is the dual basis to the fields r_A. Any right-invariant 1-form can be written as

$$\boldsymbol{\theta} = \theta_A \mathbf{r}^A \tag{17.1.59}$$

Of course, one has

$$\mathbf{r}^A = \bar{R}_b^A(g) L_B^b(g) \mathbf{l}^B = \mathrm{ad}_B^A(g) \mathbf{l}^B \qquad\qquad \mathbf{l}^A = \bar{\mathrm{ad}}_B^A(g) \mathbf{r}^B \tag{17.1.60}$$

2. General linear group

The general linear group is at the same time an example of how to actually compute what we did above in general and the starting point of how to
deal with a general matrix group.

For the general linear group $\mathrm{GL}(m) := \mathrm{GL}(m, \mathbb{R})$ the group product is $\pi : \mathrm{GL}(m) \times \mathrm{GL}(m) \to \mathrm{GL}(m)$ is given by $\pi_j^i(a, b) = a_k^i b_j^k$ which is smooth and the inverse is given by algebraic functions of the entries (which are smooth as well).

A good property of this product is that

$$\partial^1{}_m^l \pi_j^i(h, g) = \delta_m^i g_j^l \qquad\qquad \partial^2{}_m^l \pi_j^i(g, h) = g_m^i \delta_j^l \tag{17.2.1}$$

so that neither $\partial^1 \pi(h, g)$ or $\partial^2 \pi(g, h)$ depend on h.

Taking one more derivative, we get

$$\partial^1{}_m^l \partial^2{}_h^k \pi_j^i(h, g) = \delta_m^i \delta_h^l \delta_j^k \qquad \text{(or, equivalently, } \partial^1{}_h^k \partial^2{}_m^l \pi_j^i(g, h) = \delta_h^i \delta_m^k \delta_j^l) \tag{17.2.2}$$

These simplify a lot the computations of the previous Section.

The left and right translations are

$$L_g : \mathrm{GL}(m) \to \mathrm{GL}(m) : k_j^i \mapsto g_k^i k_j^k \qquad\qquad R_g : \mathrm{GL}(m) \to \mathrm{GL}(m) : k_j^i \mapsto k_k^i g_j^k \tag{17.2.3}$$

Since the natural basis of vectors is ∂_i^j, the basis for left-invariant (and right-invariant) vector fields is

$$l_j^i(g) = g_j^k \partial_k^i \qquad\qquad r_j^i(g) = g_k^i \partial_j^k \tag{17.2.4}$$

The vector field l_j^i is left-invariant. In fact

$$(L_k)^* F(g) = F(k \cdot g) \quad \Rightarrow (L_k)_*(\partial_i^j) = k_i^k \partial_k^j \quad \Rightarrow (L_k)_*(l_j^i) = g_j^k (L_k)_*(\partial_k^i) = k_k^l g_j^k \partial_l^i = (k \cdot g)_j^k \partial_k^i = l_j^i(k \cdot g) \tag{17.2.5}$$

Similarly, for right-invariant fields.

The structure constants are

$$[l_j^i, l_k^h] = g_a^a \partial_a^i g_k^b \partial_b^h - g_k^a \partial_a^h g_j^b \partial_b^i = \delta_k^i l_j^h - \delta_j^h l_k^i = \left(\delta_k^i \delta_m^h \delta_j^n - \delta_j^h \delta_m^i \delta_k^n \right) l_n^m \qquad c_{mjk}^{nih} = \delta_k^i \delta_m^h \delta_j^n - \delta_j^h \delta_m^i \delta_k^n \qquad (17.2.6)$$

The other commutators are easy to be computed

$$[l_j^i, r_k^h] = g_j^a \partial_a^i g_b^h \partial_b^b - g_b^h \partial_k^b g_j^a \partial_a^i = g_j^h \partial_k^i - g_j^h \partial_k^i = 0 \qquad (17.2.7)$$

$$[r_j^i, r_k^h] = g_a^i \partial_j^a g_b^h \partial_k^b - g_b^h \partial_k^b g_a^i \partial_j^a = \delta_j^h r_k^i - \delta_k^i r_j^h = - \left(\delta_k^i \delta_m^h \delta_j^n - \delta_j^h \delta_m^i \delta_k^n \right) r_n^m = -c_{mjk}^{nih} r_n^m \qquad (17.2.8)$$

One can also define the adjoint action

$$l_j^i(g) = g_j^k \bar{g}_n^i g_h^n \partial_k^h = \bar{g}_n^i r_k^n(g) g_j^k \qquad \Rightarrow \mathrm{ad}_{jn}^{ik}(g) = g_j^k \bar{g}_n^i \qquad r_k^n(g) = g_i^n l_j^i(g) \bar{g}_k^j \qquad \Rightarrow \bar{\mathrm{ad}}_{ki}^{nj}(g) = \bar{g}_k^j g_i^n = \mathrm{ad}_{ki}^{nj}(\bar{g}) \qquad (17.2.9)$$

The identities about the adjoint action can be now proven directly:

$$\bar{\mathrm{ad}}_{bj}^{ai}(g) c_{adf}^{bce} \mathrm{ad}_{kc}^{hd}(g) \mathrm{ad}_{ne}^{mf}(g) = \bar{g}_b^i g_j^a \left(\delta_f^c \delta_a^e \delta_d^b - \delta_d^e \delta_a^c \delta_f^b \right) g_k^d \bar{g}_c^h g_n^f \bar{g}_e^m = \bar{g}_d^i g_j^e g_n^c g_k^d \bar{g}_c^h \bar{g}_e^m - \bar{g}_f^i g_j^c g_k^d \bar{g}_c^h g_n^f \bar{g}_d^m = \delta_n^h \delta_j^m \delta_k^i - \delta_k^m \delta_j^h \delta_n^i = c_{jkn}^{ihm} \qquad (17.2.10)$$

$$L_{ia}^{jb}(g) \partial_b^a (\mathrm{ad}_{mh}^{nk}(g)) = g_i^b (\delta_m^j \delta_b^k \bar{g}_h^n - g_m^k \bar{g}_c^n \delta_d^j \delta_b^c \bar{g}_h^d) = g_c^k \bar{g}_h^d \left(\delta_m^j \delta_d^n \delta_i^c - \delta_i^n \delta_d^j \delta_m^c \right) = \mathrm{ad}_{ch}^{dk}(g) c_{djm}^{cin} \qquad (17.2.11)$$

For invariant 1-forms, let us set

$$\mathbf{l}_j^i = \bar{g}_k^i \, dg_j^k \qquad \mathbf{r}_j^i = \, dg_k^i \bar{g}_j^k \qquad (17.2.12)$$

for left-invariant and right-invariant 1-forms on $\mathrm{GL}(m)$.

The Cartan–Killing form specialises to

$$K_{mk}^{ij} = \left(\delta_m^a \delta_c^i \delta_b^d - \delta_b^i \delta_c^a \delta_m^d \right) \left(\delta_k^c \delta_a^j \delta_d^b - \delta_d^j \delta_a^c \delta_k^b \right) = \delta_m^a \delta_c^i \delta_b^d \delta_k^c \delta_a^j \delta_d^b - \delta_m^a \delta_c^i \delta_b^d \delta_d^j \delta_a^c \delta_k^b - \delta_b^i \delta_c^a \delta_m^d \delta_k^c \delta_a^j \delta_d^b + \delta_b^i \delta_c^a \delta_m^d \delta_d^j \delta_a^c \delta_k^b =$$
$$= m\delta_m^j \delta_k^i - \delta_m^i \delta_k^j - \delta_k^j \delta_m^i + m\delta_k^i \delta_m^j = 2 \left(m\delta_m^j \delta_k^i - \delta_m^i \delta_k^j \right) \qquad (17.2.13)$$

Using matrices

All we did here for the group $\mathrm{GL}(m)$ is in terms of what we have done in general in the previous Section. However, since group elements now are matrices, we can also repeat it in terms of matrices.

We have $\mathrm{GL}(m) \subset \mathrm{Mat}(m, m) = \mathbb{R}^{(m^2)}$, the matrix entries g_j^i are good coordinates on the group $\mathrm{GL}(m)$ and the group identity corresponds to the identity matrix δ_j^i.

If you want to be more precise, you can think you are using an embedding $i : \mathrm{GL}(m) \hookrightarrow \mathrm{Mat}(m, m)$ (which, in fact, expresses group elements as matrices) and decide not to understand it as we shall systematically do.

The coordinates induce a basis ∂_i^j in $T_e\mathrm{GL}(m)$ which, in fact, can be regarded as a matrix T_i^j.

This can be seen in two different ways.

We can consider a coordinate curve in the group $c_i^j : \mathbb{R} \to \mathrm{Mat}(m,m) : s \mapsto (c_i^j)_h^k$ passing through the identity (i.e. $c_i^j(0) = e \in \mathrm{GL}(m)$) in which all components are constant (and equal to the components of the identity matrix) except the component (j,i) which is $\delta_i^j + s$. The tangent vector to this curve is a matrix T_i^j which is all zero except for a 1 in position (j,i), i.e.

$$(T_i^j)_h^k = \delta_h^j \delta_i^k \tag{17.2.14}$$

Alternatively, we can define $T_j^j = Ti(\partial_i^j)$ by explicitly using the embedding $i : \mathrm{GL}(m) \hookrightarrow \mathrm{Mat}(m,m)$. In fact, being $\mathrm{Mat}(m,m) = \mathbb{R}^{(m^2)}$ one has $T_e\mathrm{Mat}(m,m) \simeq \mathrm{Mat}(m,m)$. In both cases, we have a natural way of considering a basis T_i^j in the Lie algebra $\mathfrak{gl}(m)$.

Consequently, any element $l = l_j^i T_i^j \in \mathfrak{gl}(m)$ can be regarded as a matrix $l = |l_j^i|$. This is specifically the reason why the group $\mathrm{GL}(m)$ is simple to deal with: one has an (associative) algebra $\mathrm{Mat}(m,m)$ in which both the group $\mathrm{GL}(m) \subset \mathrm{Mat}(m,m)$ and the algebra $\mathfrak{gl}(m) \subset \mathrm{Mat}(m,m)$ can be embedded. This gives us a way to multiply elements of the algebra with elements of the group, which usually we do not have.

For example, we can write the (tangent map of the) left translations as

$$T_e L_g(\xi) = \xi_j^i T_e L_g(T_i^l) = \xi_j^i g_i^k T_k^j = (g \cdot \xi)_j^k T_k^j \qquad \Rightarrow T_e L_g : |\xi_j^i| \mapsto |(g \cdot \xi)_j^k| \tag{17.2.15}$$

Thus the left translation L_g amounts on the algebra to multiplication on the left by the multiplication by the matrix g_j^i.

Accordingly, the right translation R_g amounts on the algebra to multiplication on the right by the multiplication by the matrix g_j^i, i.e.

$$T_e R_g(\xi) = \xi \cdot g \tag{17.2.16}$$

and the adjoint action reads as

$$\mathrm{ad}_g(\xi) = g \cdot \xi \cdot g^{-1} \tag{17.2.17}$$

Notice as this formula has no meaning at all for a general group.

The adjoint action of the algebra on the algebra reads as

$$\mathrm{ad}(\xi) = \xi_k^i \delta_j^h - \delta_k^i \xi_j^h \qquad \mathrm{ad}(\xi)(\zeta) = \left(\xi_k^i \delta_j^h - \delta_k^i \xi_j^h \right) \zeta_h^k = \xi_k^i \zeta_j^k - \zeta_k^i \xi_j^k = [\xi, \zeta]_j^i \tag{17.2.18}$$

The trace of a commutator is

$$K(\xi, \zeta) = \mathrm{Tr}\left(\mathrm{ad}(\xi) \circ \mathrm{ad}(\zeta) \right) = \mathrm{Tr}\left((\xi_k^i \delta_j^h - \delta_k^i \xi_j^h)(\zeta_n^k \delta_h^p - \delta_n^k \zeta_h^p) \right) = \left(\xi_k^i \zeta_n^k \delta_j^p - \xi_j^p \zeta_n^i - \xi_n^i \zeta_j^p + \xi_j^h \zeta_h^p \delta_n^i \right) \delta_i^n \delta_p^j =$$

$$= m \xi_k^n \zeta_n^k - \xi_j^j \zeta_i^i - \xi_i^i \zeta_j^j + m \xi_j^h \zeta_h^j = 2m \xi_m^i \zeta_i^m - 2\xi_i^i \zeta_j^j = 2 \left(m \delta_j^m \delta_i^k - \delta_i^m \delta_j^k \right) \xi_m^i \zeta_k^j \tag{17.2.19}$$

Then we have in matrix form

$$K(\xi, \zeta) = 2 \left(m\mathrm{Tr}(\xi \cdot \zeta) - \mathrm{Tr}(\xi)\mathrm{Tr}(\zeta) \right) \tag{17.2.20}$$

The basis of left-(right-)invariant 1-forms are

$$\mathbf{l} = \bar{g} \cdot dg \qquad\qquad \mathbf{r} = dg \cdot \bar{g} \tag{17.2.21}$$

which can be considered as a matrix of 1-forms as well as a 1-form valued in matrices.

3. Matrix groups

We shall hereafter restrict to Lie groups which have finite dimensional representations. Such groups can be identified with subgroups of $m \times m$ real or complex matrices, i.e. $G \subset \mathrm{GL}(m, \mathbb{K})$ where $\mathbb{K}$ can be specified to $\mathbb{R}$ or $\mathbb{C}$.

The group embedding $i : G \hookrightarrow \mathrm{GL}(m, \mathbb{K})$ induces a Lie algebra map $T_e i : \mathfrak{g} \hookrightarrow \mathfrak{gl}(m, \mathbb{K}) \simeq \mathrm{Mat}(m, m)$. Thus also elements in the Lie algebra $\mathfrak{g}$ are represented by matrices.

If we set a basis T_A in the Lie algebra $\mathfrak{g}$, then the matrices $T_e i(T_A) = (T_A)^i_j$ is a basis of the image of the Lie algebra $T_e i(\mathfrak{g}) \subset \mathrm{Mat}(m, m)$.

For an element in the Lie algebra $\mathfrak{g}$, we can consider a curve $S : \mathbb{R} \to G \hookrightarrow \mathrm{GL}(m)$ in G (thought as a curve in $\mathrm{GL}(m)$) passing through the identity (namely, $S(0) = \mathbb{I}$) and consider its tangent vector $\dot{S}(0)$ which is by construction tangent to G. On the other hand, a general element in te Lie algebra $\mathfrak{g}$ can be obtained this way.

Constraints determining G in $\mathrm{GL}(m)$ can be differentiated to determine $\mathfrak{g}$ in $\mathfrak{gl} \simeq \mathrm{Mat}(m, m)$.

Orthogonal groups (Euclidean signature)

Orthogonal matrices are the ones satisfying the conditions

$$O^k_i \delta_{kh} O^h_j = \delta_{ij} \qquad ({}^t O O = \mathbb{I}) \tag{17.3.1}$$

and they form a matrix group, denoted by $O(m) \subset \mathrm{GL}(m)$.

If $O_1, O_2 \in O(m)$ then one has

$$(O_2)^m_i (O_1)^k_m \delta_{kh} (O_1)^h_n (O_2)^n_j = (O_2)^m_i \delta_{mn} (O_2)^n_j = \delta_{ij} \qquad \Rightarrow O_1 \cdot O_2 \in O(m) \tag{17.3.2}$$

Of course, one also has $\mathbb{I} \in O(m)$.

By taking the determinant of the defining property (17.3.1), one has the condition $\det^2(O) = 1$, i.e. $\det(O) = \pm 1$.

The orthogonal groups $O(m)$ are made of two disconnected components.

One can, in fact, consider the function $\det : O(m) \to \mathbb{K}$ which is a continuous function (since it is a polynomial) on the group, which is never vanishing (since $O(m) \subset \mathrm{GL}(m)$). Actually, it restricts to a function $\det : O(m) \to \mathbb{Z}_2 = \{\pm 1\}$. Then the preimages $O_+(m) = \{O \in O(m) : \det(O) = 1\}$ and $O_-(m) = \{O \in O(m) : \det(O) = -1\}$ are mutually disconnected.

Suppose, in fact, that one can continuously join two elements S_+ and S_- (such that $\det(S_\pm) \in O_\pm(m)$). Let us call S_t the curve joining them. Then $\det(S_t)$ would continuously join $\det(S_+) = 1 \in \mathbb{Z}_2$ to $\det(S_-) = -1 \in \mathbb{Z}_2$. Of course, there is no such a continuous curve in $\mathbb{Z}_2$, contradicting the fact that $O(m)$ is (arcwise) connected.

Another (better) way of saying it is that $\mathbb{Z}_2$ in endowed with the discrete topology so that $\{1\} \subset \mathbb{Z}_2$ and $\{-1\} \subset \mathbb{Z}_2$ are both open sets (as well closed, i.e. $\mathbb{Z}_2$ is not connected). Being the function det : $O(m) \to \mathbb{Z}_2$ continuous the preimages of open sets are open, thus both $O_\pm(m) \subset O(m)$ are open (and since $O(m) = O_+(m) \cup O_-(m)$ they are also both closed). Thus $O(m)$ is not connected.

Another (shorter, equivalent) way of saying the same thing is that a continuous function det in a discrete set $\mathbb{Z}_2$ is locally constant, so that preimages of different values are disconnected.

The preimage of $\{1\}$ is called $\mathrm{SO}(m) \subset O(m)$, it is a subgroup of $O(m)$.

The product of two elements in $SO(m)$ is still in $\mathrm{SO}(m)$ as well as $\mathbb{I} \in \mathrm{SO}(m)$.

Since the function det : $O(m) \to \mathbb{Z}_2$ is locally constant there is a whole neighbourhood $U \subset O(m)$ of the identity $\mathbb{I} \in O(m)$ which is contained in $\mathrm{SO}(m)$; thus $\mathrm{SO}(m)$ has the same Lie algebra of the orthogonal group $O(m)$.

The other connected component $O_-(m)$ is not a subgroup, since the product of an even number of elements is in $\mathrm{SO}(m)$, not in $O_-(m)$. As a manifold, however, we have that $O_-(m) \simeq \mathrm{SO}(m)$.

Let us consider the matrix $M = \mathrm{diag}(-1, 1, \ldots, 1) \in O_-(m)$ and notice that $M^2 = \mathbb{I}$. Then consider the map $m : \mathrm{SO}(m) \to O_-(m) : S \mapsto S \cdot M$. Check that $m^{-1} : O_-(m) \to \mathrm{SO}(m) : R \mapsto R \cdot M$. Thus the map m is a diffeomorphism.

Accordingly, the orthogonal group is made as a manifold by two disconnected copies of $\mathrm{SO}(m)$, so that studying $\mathrm{SO}(m)$ tells us all essential things about the whole orthogonal group.

More generally, if one has a Lie group G which is the union of disjoint components $G = \cup_i G_i$ there is one of such components, namely G_0, which contains the group identity. Such a component is also called the *connected component to the identity* and also denoted by G_e.

For any other connected component G_i, one can select an element $S_i \in G_i$ and consider the map $m_i : G_e \to G_i : S \mapsto S \cdot S_i$. The map $m_i^{-1} : G_i \to G_e : S \mapsto S \cdot S_i^{-1}$ is its inverse so that $G_e \simeq G_i$. Thus the group G as a manifold is the union of copies of its connected component to the identity G_e, which contains all essential information about the whole group G.

Let us consider a curve S_t on $\mathrm{SO}(m)$ which is passing through the identity $S_0 = \mathbb{I}$. Since $S_t \in \mathrm{SO}(m)$ for any t it obeys the conditions

$$(S_t)_i^k \delta_{kh} (S_t)_j^h = \delta_{ij} \qquad \det(S_t) = 1 \tag{17.3.3}$$

Its tangent vector is obtaining by differentiation at $t = 0$ as

$$(\dot{S}_0)_i^k \delta_{kj} + \delta_{ih}(\dot{S}_0)_j^h = 0 \qquad (\dot{S}_0)_i^i = 0 \tag{17.3.4}$$

which implies

$$(\dot{S}_0)_i^k = -\delta_{ih}(\dot{S}_0)_j^h \delta^{jk} =: -{}^t(\dot{S}_0)_i^k \tag{17.3.5}$$

Thus an element of the Lie algebra $\dot{S}_0 \in \mathfrak{so}(m)$ is an antisymmetric (hence traceless) matrix.

Notice that the space of $m \times m$ antisymmetric matrices is a vector space of dimension $\dim(\mathfrak{so}(m)) = \frac{m}{2}(m-1)$, so that also the group is of dimension $\dim(\mathrm{SO}(m)) = \frac{m}{2}(m-1)$.

If we now consider a matrix $\xi = |\xi_j^i| \in \mathfrak{so}(m)$, it is associated to a tangent vector $\xi = \xi_j^i T_i^j = \xi_j^i \partial_i^j \in T_{\mathbb{I}}\mathrm{SO}(m)$.

To it we can associate a right-(left-)invariant vector field on the group

$$r(\xi)(S) = \xi_j^i S_k^j \partial_i^k = \xi_j^i r_i^j(S) \qquad (l(\xi) = \xi_j^i l_i^j = \xi_j^i S_i^k \partial_k^j) \tag{17.3.6}$$

These are regarded as special right-(left-)invariant vector fields on $\mathrm{GL}(m)$; they are tangent to the subgroup $\mathrm{SO}(m) \subset \mathrm{GL}(m)$ iff ξ_j^i satisfy the properties (17.3.5). If they do then

$$r(\xi) = \xi_j^i r_i^j = \xi_k^i \delta^{kl} \delta_{lj} r_i^j = \xi^{ik} r_{ki} \tag{17.3.7}$$

where we set $r_{ki} := \delta_{kj} r_i^j$ and $\xi^{ik} := \xi_l^i \delta^{kl}$. In view of property (17.3.5), ξ^{ik} is antisymmetric. Thus we have

$$r(\xi) = \xi^{ik} \left(r_{(ki)} + r_{[ki]} \right) = \xi^{ik} \sigma_{ki} \tag{17.3.8}$$

where we set $\sigma_{ki} := r_{[ki]}$. These are antisymmetric by construction, thus we discovered that any right-invariant vector field on $\mathrm{SO}(m)$ can be written as linear combination of the $\frac{m}{2}(m-1) = \dim(\mathrm{SO}(m))$ independent right-invariant fields σ_{ki}. Accordingly, the right-invariant vector fields σ_{ki} (where $[ki]$ are considered an antisymmetric pair) are a basis of right-invariant vector fields on the group $\mathrm{SO}(m)$, i.e. a basis of $\mathfrak{X}_R(\mathrm{SO}(m))$. A very similar derivation can be performed for left-invariant vector fields.

Let us remark as the embedding $i : \mathrm{SO}(m) \to \mathrm{GL}(m)$ is used (though the properties (17.3.5)) to find right-(left-)invariant vector fields, without selecting a coordinate system on the group $SO(m)$, relying instead on the coordinates on $GL(m)$.

Orthogonal groups (general signature)

Let us now fix a signature (r, s) (with $r + s = m$) and the corresponding canonical matrix $\eta = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ with r plus and s minus. We can generalise the orthogonal group to any signature. Let us denote by $O(r, s)$ the group of the orthogonal matrices with respect to the signature η_{ij}, i.e.

$$S_i^k \eta_{kh} S_j^h = \eta_{ij} \qquad \Longleftrightarrow \qquad S_i^k \hat{S}_k^n = \delta_i^n \tag{17.3.9}$$

where we set $\hat{S}_k^n := \eta_{kh} S_j^h \eta^{jn}$ to generalise the transpose of the Euclidean case. It is striking that by introducing the new transpose then almost everything become independent of the signature. Of course, if either $r = 0$ or $s = 0$ the group $O(r, s)$ reduces to the Euclidean orthogonal group $O(m)$.

The difference is that we have more (4 when both r and s are non-zero) connected components.

At a point $x \in M$, we consider the tangent space $V = T_x M$ with its inner product η, an orthonormal basis e_a, and the standard Euclidean inner product δ defined by that basis. In other words, we have two inner products on V, which, in the basis e_a, read as $\eta_{ab} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ and $\delta_{ab} = \mathrm{diag}(1, \ldots, 1, 1, \ldots, 1)$.

Then we can put in canonical form the bilinear form η with respect to the metric δ. In particular, we obtain two orthogonal eigenspaces $S = \mathrm{Span}(e_1, \ldots, e_s)$ and $R = \mathrm{Span}(e_{s+1}, \ldots, e_{s+r})$ such that $V = R \oplus S$. Each subspace receives, by restriction of η, a definite-positive metric on it.

For any orthogonal transformation $\Phi \in O(r, s)$, we can restrict to $v \in R$ (or $v \in S$) and project the result vector $\pi_R \circ \Phi(v) \in R$ (or $\pi_S \circ \Phi(v) \in S$). We then have two linear maps $\Phi_R = \pi_R \circ \Phi : R \to R$ (or $\Phi_S = \pi_S \circ \Phi : S \to S$).

That is a convoluted way to define the A and D blocks of the matrix

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{17.3.10}$$

representing the map Φ in the orthonormal basis e_a.

In general, the transformations Φ_R and Φ_S are not orthogonal, but they are invertible. Thus, their determinant is either positive (in which case we set $P(A) = 1$) or negative (in which case we set $P(A) = -1$).

We can define a map $s : O(r,s) \to \mathbb{Z}_2 \times \mathbb{Z}_2 : S \mapsto (P(A), P(D))$. It is a continuous map onto a discrete set, hence the preimages are disconnected.

An orthogonal transformation which preserves the orientation of the whole space is called *special* or *proper*. Special transformations form a subgroup denoted by $\mathrm{SO}(r,s)$. The orthogonal transformations preserving the orientation of S are called *orthochronous*. They form a subgroup $O^+(r,s)$.

Finally, proper, orthochronous, transformations form a subgroup $\mathrm{SO}^+(r,s)$ which is the connected components to the identity. Sometimes, by an abuse of notation, we write $\mathrm{SO}(r,s)$ for its connected components to the identity.

The subgroup of $O(r,s)$ with a positive determinant is denoted by $\mathrm{SO}(r,s)$ which is still the union of two connected components. The connected component to the identity is denoted by $\mathrm{SO}^+(r,s)$, which is a subgroup with the same Lie algebra.

An element $\xi \in \mathfrak{so}(r,s)$ is obtained by considering a curve $S_t : \mathbb{R} \to \mathrm{SO}^+(r,s)$ passing through the identity, i.e. with $S_0 = \mathbb{I}$. By differentiating the embedding one obtains

$$(\dot{S}_0)_i^k \eta_{kj} + \eta_{ih}(\dot{S}_0)_j^h = 0 \qquad\qquad (\dot{S}_0)_i^i = 0 \tag{17.3.11}$$

which implies that an element of the Lie algebra $\dot{S}_0 \in \mathfrak{so}(r,s)$ satisfies the following property

$$(\dot{S}_0)_i^k = -\eta_{ih}(\dot{S}_0)_j^h \eta^{jk} =: -\hat{\ }(\dot{S}_0)_i^k \tag{17.3.12}$$

This does not mean that $\dot{S}_0$ is antisymmetric (because of the signs in η) but something like. If we raise an index by η^{ij} to it, however, we obtain

$$(\dot{S}_0)^{ki} := (\dot{S}_0)_j^k \eta^{ij} = -(\dot{S}_0)_j^i \eta^{jk} = -(\dot{S}_0)^{ik} \tag{17.3.13}$$

Hence $\xi^{ij} := (\dot{S}_0)_k^i \eta^{jk}$ is antisymmetric, regardless the signature. Thus an element in the Lie algebra reads as $\xi = \xi^{ij}\sigma_{ij} \in \mathfrak{so}(r,s)$, where we set

$$\sigma_{ij} = \eta_{k[i} r_{j]}^k \tag{17.3.14}$$

for a basis of $\mathfrak{X}_R(\mathrm{SO}(r,s))$. The basis σ_{ij} is meant to be antisymmetric in the $[ij]$ indices and the Lie algebras (as well the groups) are of dimension $\dim(\mathrm{SO}(r,s)) = \frac{m}{2}(m-1)$ in any dimension.

4. Clifford algebras and spin groups

Consider a (real) vector space V of dimension m, fix a signature $\eta = (r,s)$ with $m = r + s$. Let us also denote by η_{ab} the matrix of the canonical diagonal matrix $\mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ with r plus and s minus, which defines an "inner product" η on V which is associated to the matrix η_{ab} in an orthonormal basis e_a.

Let $T^*(V) = \oplus_{p=0}^\infty T_0^p(V)$, where $T_0^p(V) = \otimes^p V$, so that an element $t \in T_0^p(V)$ is a p-linear map $t : V^* \times V^* \times \ldots \times V^* \to \mathbb{R}$. That is called the *covariant tensor algebra*. It is a $\mathbb{Z}$-graded algebra with the tensor product.

Let us remark that $V = T_0^1(V) \subset T^*(V)$. For any vector $v \in V$, one can define the generator $v \otimes v + \eta(v,v)\mathbb{I}$ and let $\mathcal{I}(\eta)$ be the bilateral ideal defined by these generators. Thus the quotient

$$\mathcal{C}(\eta) = \frac{T^*(V)}{\mathcal{I}(\eta)} \tag{17.4.1}$$

is an algebra, called the *Clifford algebra* of signature η. The product is defined as

$$[r][s] = [r \otimes s] \tag{17.4.2}$$

and sometimes, when there is no confusion, we understand the equivalence classes and simply write $\mathbf{rs} = [r][s]$. Since $\mathcal{C}(\eta)$ is defined as a quotient of a $\mathbb{Z}$-graded algebra by a ideal generated by even elements, the Clifford algebra retains a $\mathbb{Z}_2$-graduation. In fact, it splits as

$$\mathcal{C}(\eta) = \mathcal{C}^+(\eta) \oplus \mathcal{C}^-(\eta) \tag{17.4.3}$$

where $\mathcal{C}^+(\eta)$ is called the *even Clifford sub-algebra*.

For example, in the Clifford algebra, we have

$$\mathbf{v}^2 := \mathbf{vv} = -\eta(v,v)\mathbb{I} \tag{17.4.4}$$

for any vector $v \in V$. Also, for any two vectors $v, w \in V$, one has

$$\begin{aligned}
(\mathbf{v} + \mathbf{w})^2 = {}& -\eta(v+w, v+w)\mathbb{I} \quad \Rightarrow \\
\Rightarrow{}& \mathbf{v}^2 + \mathbf{vw} + \mathbf{wv} + \mathbf{w}^2 = -\eta(v,v+)\mathbb{I} - 2\eta(v,w)\mathbb{I} - \eta(w,w)\mathbb{I} \quad \Rightarrow \quad \{\mathbf{v}, \mathbf{w}\} := \mathbf{vw} + \mathbf{wv} = -2\eta(v,w)\mathbb{I}
\end{aligned} \tag{17.4.5}$$

From now on, e_a will be an orthonormal basis of V with respect to η. The first s elements e_A are $\eta(e_A, e_A) = -1$, for the last r elements e_i one has $\eta(e_i, e_j) = 1$. They are also Clifford elements and they obey $\{\mathbf{e}_a, \mathbf{e}_b\} = -2\eta_{ab}\mathbb{I}$. In particular, whenever $a \neq b$, then $\mathbf{e}_a\mathbf{e}_b = -\mathbf{e}_b\mathbf{e}_a$. Moreover, when $a = b$ one has $\mathbf{e}_a\mathbf{e}_a = \pm\mathbb{I}$.

Elements in the tensor algebra are finite linear combinations of finite products of elements in the basis. In each term of a tensor, one can use $\{\mathbf{e}_a, \mathbf{e}_b\} = -2\eta_{ab}\mathbb{I}$ to order the elements in the product and simplify the equal factors once they are side by side. Accordingly, any Clifford element is a finite linear combinations of products of different elements in the basis with ordered indices. In other words, a basis of Clifford algebra is in the form

$$\mathbb{I} \quad \mathbf{e}_a \quad \mathbf{e}_{ab} := \underset{(a<b)}{\mathbf{e}_a\mathbf{e}_b} \quad \mathbf{e}_{abc} := \underset{(a<b<c)}{\mathbf{e}_a\mathbf{e}_b\mathbf{e}_c} \quad \ldots \quad \mathbf{e} := \mathbf{e}_0\mathbf{e}_1 \ldots \mathbf{e}_{m-1} \tag{17.4.6}$$

and the Clifford algebra is of dimension 2^m.

The Clifford product is completely described by computing the products of the elements of the Clifford basis. For example, if we consider the product $\mathbf{e}_2\mathbf{e}_{12}$ we have

$$\mathbf{e}_2\mathbf{e}_{12} = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\eta_{22}\mathbf{e}_1 = \mp\mathbf{e}_1 \tag{17.4.7}$$

where the sign depends on the signature, in particular on η_{22} which is $\eta_{22} = \pm 1$ depending on the signature.

From this, we learn two things:

    1) we expressed the product $\mathbf{e}_2\mathbf{e}_{12}$ as a linear combination of elements in the Clifford basis.

    2) the number of factors is not preserved; when we write the product $\mathbf{e}_2\mathbf{e}_{12}$ in the Clifford basis we get it as an element which is the (product of 1) element of the basis $\mp\mathbf{e}_1$.

Item 2 shows that the number of vectors involved in expressing a Clifford element is not preserved by products: if we consider the product of an element which is a linear combination of products of 1 vector with an element which is linear combinations of products of 2 vectors, then the product is not a linear combinations of products of $3 = 1 + 2$ vectors.

However, the parity of the vectors in preserved: the product of an element which is a linear combination of products of odd vectors with an element which is linear combinations of products of even vectors, is a linear combinations of products of odd = odd + even vectors. That is just because when the number of factors changes, it changes due to the rule $\{\mathbf{e}_a, \mathbf{e}_b\} = -2\eta_{ab}\mathbb{I}$, which preserves the parity.

In other words, the Clifford algebra is $\mathbb{Z}_2$-graded, it contains an *even part* $\mathcal{C}^+(\eta)$, which is a sub-algebra and it contains all finite linear combinations of product of an even number of elements of the basis of V. The even part has a basis

$$\mathbb{I} \quad \mathbf{e}_{ab} \quad \mathbf{e}_{abcd} \quad \ldots \tag{17.4.8}$$

i.e. the even part is of dimension 2^{m-1}.

The Clifford algebra contains also an *odd part* $\mathcal{C}^-(\eta)$, which is *not* a sub-algebra and it contains all finite linear combinations of products of an odd number of elements of the basis of V. The odd part has a basis

$$\mathbf{e}_a \quad \mathbf{e}_{abc} \quad \mathbf{e}_{abcde} \quad \ldots \tag{17.4.9}$$

which is again of dimension 2^{m-1}. Whether $\mathbf{e}$ is in the even or odd part depends on the parity of the dimension m.

Thus let us now consider some examples in low dimension. We are eventually interested in particular in signatures $(3,0)$, $(4,0)$, and $(3,1)$ for 3d Euclidean signature and 4d Euclidean and Lorentzian signatures.

Clifford algebras for $m = 1$

We have a vector space V of dimension 1. There are essentially two signatures, namely $\eta_+ = (1,0)$ and $\eta_- = (0,1)$. The orthonormal basis of V is e and the products are $(\mathbf{e})^2 = -\mathbb{I}$ and $(\mathbf{e})^2 = \mathbb{I}$, respectively depending on the signature.

Then the basis of the Clifford algebra is in both cases $(\mathbb{I}, \mathbf{e})$, since $\mathbf{e}^2 = \pm\mathbb{I}$ is not independent. The most general Clifford element is

$$a\mathbb{I} + b\mathbf{e} \in \mathcal{C}(\eta) \tag{17.4.10}$$

For $\eta = (1,0)$ the product is defined so that $(\mathbf{e})^2 = -1$, it is instead defined so that $(\mathbf{e})^2 = 1$ for the signature $\eta = (0,1)$.

In the first case $(\eta = (1,0))$, the map $\Phi_+ : \mathcal{C}(1,0) \rightarrow \mathbb{C} : a\mathbb{I} + b\mathbf{e} \mapsto a + ib$ is an algebra (iso)morphism, it preserves linear combinations and products.

In the second case ($\eta = (0,1)$), we can consider a matrix representation of the Clifford algebra obtained by the correspondence $i : \mathbf{e} \mapsto \sigma_3$. That extends by linearity to the map

$$i(a\mathbb{I} + b\mathbf{e}) = \begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix} \tag{17.4.11}$$

which suggests to set $q_\pm := a \pm b$. The correspondence $\Phi_- : \mathcal{C}(0,1) \to \mathbb{R} \oplus \mathbb{R} : a\mathbb{I} + b\mathbf{e} \mapsto (q_+, q_-)$ is finally an algebra isomorphism.

Notice that the even Clifford sub-algebra $\mathcal{C}^+(0,1) = \mathbb{R}$ is embedded in the decomposition $\mathcal{C}(0,1) \simeq \mathbb{R} \oplus \mathbb{R}$ as the diagonal $d : \mathcal{C}^+(0,1) \to \mathbb{R} \oplus \mathbb{R} : q \mapsto (q,q)$.

> Let us stress that Clifford algebras are not even invariant with respect to inversion of the signature; e.g., $\mathcal{C}(1,0) \simeq \mathbb{C}$, while $\mathcal{C}(0,1) \simeq \mathbb{R} \oplus \mathbb{R}$. To increase confusion, in the literature sometimes the generators of the ideal $\mathcal{I}(\eta)$ with the opposite sign. Thus, depending on the book, sometimes the algebras $\mathcal{C}(r,s)$ and $\mathcal{C}(s,r)$ are exchanged!

Clifford algebras for $m = 2$

For a vector space V of dimension $m = 2$, we shall consider a basis $(e_A) = (e_1, e_2)$ for the signatures $(2,0)$ and $(0,2)$, while we shall use a basis $(e_A) = (e_0, e_1)$ for the signature $(1,1)$.

Thus, the Clifford basis is

$$\mathbb{I} \qquad \mathbf{e}_A \qquad \mathbf{e} \tag{17.4.12}$$

where we set $\mathbf{e} := \mathbf{e}_1\mathbf{e}_2$ or $\mathbf{e} := \mathbf{e}_0\mathbf{e}_1$, respectively. The Clifford algebra is of dimension $2^2 = 4$, while the even Clifford algebra is of dimension 2.

For the signature $(2,0)$, we have $(\mathbf{e}_1)^2 = -1$, $(\mathbf{e}_2)^2 = -1$, $(\mathbf{e})^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -1$. One also has $\mathbf{e}\mathbf{e}_1 = \mathbf{e}_2$, $\mathbf{e}_2\mathbf{e} = \mathbf{e}_1$. Accordingly, we can denote $\mathbf{e}_1 = i$, $\mathbf{e}_2 = j$, $\mathbf{e} = k$ and recover the product in the algebra of quaternions, i.e. we get $\mathcal{C}(2,0) = \mathbb{H}$. The even Clifford algebra is spanned by $(\mathbb{I}, \mathbf{e})$, thus it is $\mathcal{C}^+(2,0) \simeq \mathbb{C}$.

For the signature $(0,2)$, we have essentially the same basis, though the product is defined so that $(\mathbf{e}_1)^2 = 1$, $(\mathbf{e}_2)^2 = 1$, $(\mathbf{e})^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -1$ as well as $\mathbf{e}_1\mathbf{e} = \mathbf{e}_2$, $\mathbf{e}_2\mathbf{e} = -\mathbf{e}_1$.

We can fix the representation given by the correspondence $\mathbf{e}_1 \mapsto \sigma_3$, $\mathbf{e}_2 \mapsto \sigma_1$, $\mathbf{e} \mapsto i\sigma_2$ which extends linearly to the map

$$\Phi : \mathcal{C}(0,2) \to M(2,\mathbb{R}) : a\mathbb{I} + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e} \mapsto \begin{pmatrix} a + b & c + d \\ c - d & a - b \end{pmatrix} \tag{17.4.13}$$

That restricts to the even sub-algebra as $\Phi_+ : \mathcal{C}^+(0,2) \to \mathbb{C}$, where $\mathbb{C}$ is identified within $M(2,\mathbb{R})$ as

$$i : \mathbb{C} \to M(2,\mathbb{R}) : a + id \mapsto \begin{pmatrix} a & d \\ -d & a \end{pmatrix} \tag{17.4.14}$$

Finally, for the signature $(1,1)$, we consider the basis (e_0, e_1) with the Clifford product defined by

$$(\mathbf{e}_0)^2 = 1 \qquad (\mathbf{e}_1)^2 = -1 \qquad (\mathbf{e})^2 = 1 \tag{17.4.15}$$

Accordingly, also $\mathcal{C}(1,1) \simeq \mathcal{C}(0,2) \simeq M(2,\mathbb{R})$, this time by the correspondence $\mathbf{e}_0 \mapsto \sigma_1$, $\mathbf{e}_1 \mapsto i\sigma_3$, $\mathbf{e} \mapsto \sigma_2$. The even part, this time, is $\mathcal{C}^+(1,1) \simeq \mathbb{R} \oplus \mathbb{R}$.

The Clifford algebra $\mathcal{C}(3,0)$

Although Clifford algebras are classified and they exhibit a beautiful periodicity, we are interested in three more cases, in particular. For the vector space V of dimension 3 and Euclidean signature, the corresponding Clifford algebra is denoted by $\mathcal{C}(3) := \mathcal{C}(3,0)$ and it is spanned by a Clifford basis

$$\mathbb{I} \quad \mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_{12} \quad \mathbf{e}_{13} \quad \mathbf{e}_{23} \quad \mathbf{e} \tag{17.4.16}$$

The products of the elements of the basis are $(\mathbf{e}_i)^2 = -\mathbb{I}$, $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12}$ and similars. Moreover, $\mathbf{e}_1\mathbf{e}_{12} = -\mathbf{e}_2$, $\mathbf{e}_1\mathbf{e}_{23} = \mathbf{e}$, and similars. Also, $\mathbf{e}_1\mathbf{e} = -\mathbf{e}_{23} = \mathbf{e}\mathbf{e}_1$, $-\mathbf{e}_1 = \mathbf{e}_{12}\mathbf{e}$ and $(\mathbf{e})^2 = \mathbb{I}$.

Accordingly, a general element $C \in \mathcal{C}(3)$ can be recast as

$$C = a\mathbb{I} - b\mathbf{e}\mathbf{e}_{23} + c\mathbf{e}\mathbf{e}_{13} - d\mathbf{e}\mathbf{e}_{12} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} + h\mathbf{e} = u + \mathbf{e}v \tag{17.4.17}$$

where we set

$$u = a\mathbb{I} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} \qquad v = h\mathbb{I} - b\mathbf{e}_{23} + c\mathbf{e}_{13} - d\mathbf{e}_{12} \tag{17.4.18}$$

Since we have

$$(\mathbf{e}_{12})^2 = -\mathbb{I} \qquad \mathbf{e}_{12}\mathbf{e}_{13} = \mathbf{e}_{23} \quad \text{(and so on)} \tag{17.4.19}$$

we can map $\mathbf{e}_{12} \mapsto i$, $\mathbf{e}_{13} \mapsto j$, $\mathbf{e}_{23} \mapsto k$, and extend it to become an algebra isomorphism, $\Phi : \mathcal{C}(3) \to \mathbb{H} \oplus \mathbb{H} : C \mapsto (u,v)$. This time, we have the isomorphism on the even Clifford algebra $\Phi_+ : \mathcal{C}^+(3) \to \mathbb{H} \subset \mathbb{H} \oplus \mathbb{H} : C \mapsto (u,0)$.

The Clifford algebra $\mathcal{C}(4,0)$

Let us now consider a vector space V in dimension 4, fix the Euclidean signature $\eta = (4,0)$ and denote an orthonormal basis in V by e_1, e_2, e_3, e_4.

The Clifford algebra is of dimension 16. We are not too interested in the whole algebra, which, however, one can prove to be isomorphic to $\mathcal{C}(4,0) = M(2,\mathbb{H})$ (4 components times 4 quaternions, i.e. $\dim(\mathcal{C}(4,0)) = 16$).

Here we are more interested in the even Clifford sub-algebra which is spanned by the basis

$$\mathbb{I} \quad \mathbf{e}_{12} \quad \mathbf{e}_{13} \quad \mathbf{e}_{14} \quad \mathbf{e}_{23} \quad \mathbf{e}_{24} \quad \mathbf{e}_{34} \quad \mathbf{e} \tag{17.4.20}$$

We have $(\mathbf{e}_{ij})^2 = -\mathbb{I}$, as well as $\mathbf{e}_{12}\mathbf{e}_{13} = \mathbf{e}_{23}$ (and similar ones). Accordingly, we can map $\mathbf{e}_{23} \mapsto i$, $\mathbf{e}_{24} \mapsto j$, $\mathbf{e}_{34} \mapsto k$ to recover a number of these products.

Moreover, we have $\mathbf{e}\mathbf{e}_{23} = -\mathbf{e}_{14} = \mathbf{e}_{23}\mathbf{e}$, $\mathbf{e}\mathbf{e}_{24} = \mathbf{e}_{13} = \mathbf{e}_{24}\mathbf{e}$, $\mathbf{e}\mathbf{e}_{34} = -\mathbf{e}_{12} = \mathbf{e}_{34}\mathbf{e}$, so that the general even element $C \in \mathcal{C}^+(4,0)$ can be recast as

$$C = a\mathbb{I} + b\mathbf{e}_{12} + c\mathbf{e}_{13} + d\mathbf{e}_{14} + e\mathbf{e}_{23} + f\mathbf{e}_{24} + g\mathbf{e}_{34} + h\mathbf{e} = a\mathbb{I} - b\mathbf{e}\mathbf{e}_{34} + c\mathbf{e}\mathbf{e}_{24} - d\mathbf{e}\mathbf{e}_{23} + e\mathbf{e}_{23} + f\mathbf{e}_{24} + g\mathbf{e}_{34} + h\mathbf{e} = u + \mathbf{e}v \tag{17.4.21}$$

where we set

$$u = a + ei + fj + gk \qquad v = h - di + cj - bk \tag{17.4.22}$$

Finally, since we have $(\mathbf{e})^2 = \mathbb{I}$, then we can define $q_\pm = u \pm v$ and we have $\mathcal{C}^+(4,0) \simeq \mathbb{H} \oplus \mathbb{H}$.

The Clifford algebra $\mathcal{C}(3,1)$

Let us now consider a vector space V in dimension 4, fix the Lorentzian signature $\eta = (3,1)$ and denote an orthonormal basis in V by e_0, e_1, e_2, e_3.

The Clifford algebra is of dimension 16. We are not too interested in the whole algebra, which, however, one can prove to be isomorphic to $\mathcal{C}(3,1) = M(2, \mathbb{H})$ (4 components times 4 quaternions, i.e. $\dim(\mathcal{C}(3,1)) = 16$).

Here we are more interested in the even Clifford sub-algebra which is spanned by the basis

$$\mathbb{I} \quad \mathbf{e}_{01} \quad \mathbf{e}_{02} \quad \mathbf{e}_{03} \quad \mathbf{e}_{12} \quad \mathbf{e}_{13} \quad \mathbf{e}_{23} \quad \mathbf{e} \tag{17.4.23}$$

We have $(\mathbf{e}_{12})^2 = -\mathbb{I}$, $(\mathbf{e}_{13})^2 = -\mathbb{I}$, $(\mathbf{e}_{23})^2 = -\mathbb{I}$, as well as $\mathbf{e}_{12}\mathbf{e}_{13} = \mathbf{e}_{23}$ (and similar ones). Accordingly, we can map $\mathbf{e}_{12} \mapsto i$, $\mathbf{e}_{13} \mapsto j$, $\mathbf{e}_{23} \mapsto k$ to recover a number of these products.

Moreover, we have $\mathbf{e}\mathbf{e}_{12} = -\mathbf{e}_{03} = \mathbf{e}_{12}\mathbf{e}$, $\mathbf{e}\mathbf{e}_{13} = \mathbf{e}_{02} = \mathbf{e}_{13}\mathbf{e}$, $\mathbf{e}\mathbf{e}_{23} = -\mathbf{e}_{01} = \mathbf{e}_{23}\mathbf{e}$, so that the general even element $C \in \mathcal{C}^+(3,1)$ can be recast as

$$C = a\mathbb{I} + b\mathbf{e}_{01} + c\mathbf{e}_{02} + d\mathbf{e}_{03} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} + h\mathbf{e} = a\mathbb{I} - b\mathbf{e}\mathbf{e}_{23} + c\mathbf{e}\mathbf{e}_{13} - d\mathbf{e}\mathbf{e}_{12} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} + h\mathbf{e} = u + \mathbf{e}v \tag{17.4.24}$$

where we set

$$u = a + ei + fj + gk \qquad v = h - di + cj - bk \tag{17.4.25}$$

Finally, since we have $(\mathbf{e})^2 = -\mathbb{I}$, then we have $\mathcal{C}^+(3,1) \simeq \mathbb{C} \otimes \mathbb{H}$.

Universal property of Clifford algebras

Given a vector space V with a non-degenerate, symmetric, bilinear form η, one can define the Clifford algebra $\mathcal{C}(\eta)$ as well the embedding $i : V \to \mathcal{C}(\eta) : v \mapsto \mathbf{v}$.

Then, for any associative algebra A and any $f : V \to A$ such that $f(v) \cdot f(v) = -\eta(v,v)\mathbb{I}$, there exists a unique extension $\hat{f} : \mathcal{C}(\eta) \to A$ such that

$$
\begin{array}{ccc}
V & \xrightarrow{\ i\ } & \mathcal{C}(\eta) \\
& f \searrow & \big\downarrow \hat{f} \qquad\qquad \hat{f} \circ i = f \\
& & A
\end{array}
\tag{17.4.26}
$$

That is called the *universal property of the Clifford algebras*. As usual, with universal properties, one can prove that Clifford algebras are unique up to isomorphisms. In fact, let us suppose that there exists an associative algebra $\mathcal{C}$ which also has the universal property, then

$$V \xrightarrow{\quad i \quad} \mathcal{C}(\eta)$$

$$\downarrow^{j}$$

$$\mathcal{C} \qquad\qquad (17.4.27)$$

Then, since $\mathcal{C}(\eta)$ has the universal property, there exists a unique map $f : \mathcal{C}(\eta) \to \mathcal{C}$ such that $f \circ i = j$. On the other hand, since $\mathcal{C}$ has the universal property, then there exists a unique map $\bar{f} : \mathcal{C} \to \mathcal{C}(\eta)$ such that $\bar{f} \circ j = i$.

That implies $\bar{f} \circ f \circ i = i$ and $f \circ \bar{f} \circ j = j$. Again for universal properties, this time for the uniqueness part, one has $\bar{f} \circ f = \mathrm{id}$ and $f \circ \bar{f} = \mathrm{id}$, since for example $\bar{f} \circ f : \mathcal{C}(\eta) \to \mathcal{C}(\eta)$ is associated to the map $i : V \to \mathcal{C}(\eta)$. Thus, one has $\mathcal{C}(\eta) \simeq \mathcal{C}$.

Another application of universal property is the following: consider the map $f = i \circ L : V \to \mathcal{C}(\eta)$ where $L \in O(V, \eta)$ is an orthogonal transformation on V. That preserves the norms of vectors, i.e. $f(v) \cdot f(v) = -\eta(v,v)\mathbb{I}$, then it extends to the Clifford algebra $\hat{f} : \mathcal{C}(\eta) \to \mathcal{C}(\eta)$.

General structures on Clifford algebras

We have some structures which are useful to be defined on Clifford algebras in order then to discuss further in a way which is more or less independent of the signature and dimension. In other words, these structures and maps encapsulate the differences which come with dimension and signature.

In view of the universal property and the map $a = -i : V \to \mathcal{C}(\eta) : v \mapsto -\mathbf{v}$, which, in fact, has the property $a(v)a(v) = -\eta(v,v)\mathbb{I}$, one can extend it uniquely to a map $\alpha : \mathcal{C}(\eta) \to \mathcal{C}(\eta)$ which restricts to the identity on even elements, to minus identity on odd elements. This map is called the *grading* since it essentially defines the $\mathbb{Z}_2$-grading on $\mathcal{C}(\eta)$. Of course, α is an involution since it squares to the identity, i.e. one has $\alpha \circ \alpha = \mathbb{I}$.

We can also define a *transpose* map $t : \mathcal{C}(\eta) \to \mathcal{C}(\eta)$, obtained by extending by linearity the map defined on products of vectors as

$$^{t}(\mathbf{v}_1\mathbf{v}_2 \dots \mathbf{v}_k) = \mathbf{v}_k \dots \mathbf{v}_2\mathbf{v}_1 \qquad\qquad (17.4.28)$$

That is well defined since the generator of the Clifford ideal are invariant with respect to transpose. Moreover, it is an anti-automorphism since it reverses the product order and it is an involution (thus it is invertible). The transpose map is also denoted as $^{t}(S) = \bar{S}$.

Since the transpose preserves the grading, one also has $\alpha(^{t}S) = {}^{t}(\alpha(S))$.

From the examples of Clifford algebras above, one can see that, depending on dimension and signature, not all elements in the Clifford algebras are always invertible. For example, $\mathcal{C}(2,0) \simeq \mathbb{H}$ is a corp, while $\mathcal{C}(0,2) \simeq M(2,\mathbb{R})$ is not. Let us denote by $C(\eta) \subset \mathcal{C}(\eta)$ the multiplicative sub-group of invertible elements. Let us also define the sub-group $S(\eta) \subset C(\eta) \subset \mathcal{C}(\eta)$ of elements for which the inverse is proportional to the transpose, i.e. such that $S\bar{S} \propto \mathbb{I}$.

Now, let us consider unit vectors $v \in V$ (i.e. such that $\eta(v,v) = \pm 1$). These define elements $\mathbf{v} \in S(\eta)$, since $\mathbf{v}\bar{\mathbf{v}} = -\mathbf{v}\mathbf{v} = -\eta(v,v)\mathbb{I} = \mp\mathbb{I}$. Let us denote by $\mathrm{Pin}(\eta) \subset S(\eta) \subset \mathcal{C}(\eta)$ the sub-group generated by unit vectors, by $\mathrm{Spin}(\eta) \subset \mathcal{C}^{+}(\eta)$ the intersection of it with the even Clifford sub-algebra. Elements $S \in \mathrm{Spin}(\eta)$ are products of an even number of unit vectors, i.e.

$$S = \mathbf{u}_1\mathbf{u}_2 \dots \mathbf{u}_{2k} \qquad\qquad (17.4.29)$$

Of course, in this case, one has

$$S^{-1} = \frac{(-1)^n}{|u_n|^2 \dots |u_2|^2 |u_1|^2} \mathbf{u}_n \dots \mathbf{u}_2 \mathbf{u}_1 = \pm^t S \tag{17.4.30}$$

Let us also define $N : \mathcal{C}(\eta) \to \mathcal{C}(\eta)$ as $N(A) = A\, \alpha(^tA)$, as well as $N_0 : \mathcal{C}(\eta) \to \mathbb{R}$ to be the (uniquely determined) component $N_0(A)$ along $\mathbb{I}$ of $N(A) \in \mathcal{C}(\eta)$.

Notice that if $N(B) = N_0(B)\mathbb{I}$ then

$$N(AB) = AB\, \alpha(^t(AB)) = AB\, \alpha(^tB\, ^tA)) = AB\, \alpha(^tB)\alpha(^tA) = AN(B)\alpha(^tA) = A\alpha(^tA)N(B) = N(A)N(B) \tag{17.4.31}$$

If $\mathbf{v} \in V \subset \mathcal{C}(\eta)$, then $N(\mathbf{v}) = \mathbf{v}\, \alpha(^t\mathbf{v}) = \mathbf{v}\, \alpha(\mathbf{v}) = -\mathbf{v}\mathbf{v} = |v|^2\mathbb{I}$ (i.e. $V \subset S(\eta)$) and $N_0(\mathbf{v}) = |v|^2$.

The covering maps of Spin(η) groups

Let us consider an element $S \in \mathrm{Pin}(\eta)$, i.e. $S = \mathbf{u}_1\mathbf{u}_2 \dots \mathbf{u}_k$ and a vector $w \in V$. We want to compute $\alpha(S)\mathbf{w}S^{-1}$, where of course $S^{-1} = \pm\mathbf{u}_k \dots \mathbf{u}_2\mathbf{u}_1$. For any unit vector u, let us denote the norm of u by $|u|^2 = \eta(u,u) = \pm 1$, the parallel component of w to u by $w^\parallel = \frac{\eta(w,u)}{\eta(u,u)}u$, the perpendicular component of w to u by $w^\perp = w - w^\parallel$.

The perpendicular component is orthogonal to u, in fact

$$\eta(w^\perp, u) = \eta(w,u) - \eta(w^\parallel, u) = \eta(w,u) - \frac{\eta(w,u)}{\eta(u,u)}\eta(u,u) = 0 \tag{17.4.32}$$

Moreover, we have $w = w^\parallel + w^\perp$. As usual, we shall consider the corresponding elements in the Clifford algebra which are denoted by $\mathbf{w}^\parallel$ and $\mathbf{w}^\perp$.

Thus, we have $\mathbf{u}\mathbf{u} = -|u|^2\mathbb{I}$ and

$$\alpha(\mathbf{u})\mathbf{w}\mathbf{u}^{-1} = -\mathbf{u}\mathbf{w}\mathbf{u}^{-1} = |u|^{-2}\mathbf{u}\mathbf{w}\mathbf{u} = |u|^{-2}\left(\mathbf{u}\mathbf{w}^\perp\mathbf{u} + \mathbf{u}\mathbf{w}^\parallel\mathbf{u}\right) = |u|^{-2}\left(-\mathbf{u}\mathbf{u}\mathbf{w}^\perp - \eta(u,w^\perp)\mathbf{u} - |u|^2\mathbf{w}^\parallel\right) = \mathbf{w}^\perp - \mathbf{w}^\parallel = \ell(u)(w) \in V \tag{17.4.33}$$

We can then define the linear map $\ell(u) : V \to V : w \mapsto w^\perp - w^\parallel$, which is in fact a *reflection* about the plane orthogonal to the unit vector u. Being a reflection, in particular, it is an orthogonal transformation $\ell(u) \in O(V, \eta)$.

Then, for any $S \in \mathrm{Pin}(\eta)$, one has $\ell(S) : V \to V \in O(V, \eta)$

$$\alpha(S)\mathbf{w}S^{-1} = \left(\ell(u_1) \circ \ell(u_2) \circ \dots \circ \ell(u_k)\right)(w) = \ell(S)(w) \tag{17.4.34}$$

We can check directly that the map $\ell : \mathrm{Pin}(\eta) \to O(V, \eta)$ is a group homomorphism, called the *covering* of the pin group.

Since reflections are transformations with determinant -1, when we restrict to $\mathrm{Spin}(\eta)$ and we compose an even number of reflections, the map restricts to $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(V, \eta)$. The covering homomorphism $\ell : \mathrm{Spin}(\eta) \to \mathrm{SO}(V, \eta)$ is called the *covering map* of the spin group.

Here we have to highlight an abuse of notation. Depending on dimension and signature the groups may or may not be connected. When we write the covering map, we always restrict to components connected to the identity.

It is easy to show that $\ell(\pm \mathbb{I}) = \mathbb{I}$, thus $\pm \mathbb{I} \in \ker(\ell)$. Accordingly, the covering map is not injective.

If $S \in \mathrm{Pin}(\eta)$ and $\ell(S) = \mathbb{I}$, that can be regarded as a condition in the whole Clifford algebra, namely as $[S, e_a] = 0$ in the even Clifford algebra, $\{S, e_a\} = 0$ in the odd part.

For $S \in \mathcal{C}^+(\eta)$, we can expand it in the basis $\mathbb{I}, e_{ab}, \ldots$. The first one, $[\mathbb{I}, e_a] = 0$ is identically satisfied and it gives no condition.

The second one, $[e_{bc}, e_a] = 0$ can be written as $2\eta_{a[b}e_{c]} = 0$; that is the only contribution of degree 2 and it says S has no component along e_{bc}. The same is happening at higher degrees, so eventually $S = s\mathbb{I}$ are the only elements in the even Clifford algebra which are allowed are proportional to $\mathbb{I}$. If we restrict to $\mathrm{Spin}(\eta)$, we are left only with $S = \pm \mathbb{I}$.

For the odd part, we have the first condition $\{e_b, e_a\} = -2\eta_{ab}\mathbb{I} = 0$, which is the only contribution to zero degree, so no component of S along e_b.

At third degree we have $\{e_{bcd}, e_a\} = 0$ which has components only on degree two. And so on.

So, eventually, the only elements in the kernel are $\ker(\ell) = \{\mathbb{I}, -\mathbb{I}\} = \mathbb{Z}_2 \subset \mathrm{Pin}(\eta)$.

The map $\ell : \mathrm{Pin}(\eta) \to O(V, \eta)$ is surjective.

Any orthogonal transformation $O \in O(V, \eta)$ is the composition of reflections. As an argument for plausibility, let us consider a rotation in a 2d Euclidean space $V = \mathbb{R}^2$. We can consider an orthogonal transformation or a rotation

$$R_1 := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \qquad\qquad R_2 := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{17.4.35}$$

The reflection about a plane orthogonal to a unit vector $\vec{u} = \cos(\alpha)e_1 + \sin(\alpha)e_2$ can be written as

$$\ell(\vec{u}_\alpha) = \begin{pmatrix} \cos(2\alpha + \pi) & \sin(2\alpha + \pi) \\ \sin(2\alpha + \pi) & -\cos(2\alpha + \pi) \end{pmatrix} \tag{17.4.36}$$

If we require $\ell(\vec{u}_\alpha) = R_1$, we get the relations $\alpha_1 = \frac{1}{2}(\theta - \pi)$ or $\alpha_2 = \frac{1}{2}(\theta + \pi)$ which correspond to the unit vectors $\vec{u}_1 = \cos(\alpha_1)e_1 + \sin(\alpha_1)e_2$ and $\vec{u}_2 = \cos(\alpha_2)e_1 + \sin(\alpha_2)e_2 = -\vec{u}_1$, which in turn define the same reflection.

For R_2, we try to write it as a composition of two reflections, i.e.

$$\ell(\vec{u}_\alpha)\ell(\vec{u}_\beta) = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} \begin{pmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{pmatrix} = \begin{pmatrix} \cos(2\alpha - 2\beta) & -\sin(2\alpha - 2\beta) \\ \sin(2\alpha - 2\beta) & \cos(2\alpha - 2\beta) \end{pmatrix} \tag{17.4.37}$$

By setting $\ell(\vec{u}_\alpha)\ell(\vec{u}_\beta) = R_2$, we get the relations $\alpha_1 = \beta_1 + \frac{1}{2}\theta$ or $\alpha_2 = \beta_2 + \frac{1}{2}\theta + \pi$ which correspond to the unit vectors $\vec{u}_\alpha = \cos(\beta_1 + \frac{1}{2}\theta)e_1 + \sin(\beta_1 + \frac{1}{2}\theta)e_2$, $\vec{u}_\beta = \cos(\beta_1)e_1 + \sin(\beta_1)e_2$ (or $(-\vec{u}_\alpha, \vec{u}_\beta)$ which in fact define the same reflections). These give us infinitely many ways of writing R_2 as the composition of 2 reflections, although we are looking for *one* way of doing it. Thus let us choose $\beta_1 = 0$ to obtain

$$\vec{u}_\alpha = \cos(\tfrac{1}{2}\theta)e_1 + \sin(\tfrac{1}{2}\theta)e_2 \qquad\qquad \vec{u}_\beta = e_1 \tag{17.4.38}$$

More generally, in any dimensions we can write any orthogonal transformations as a composition of orthogonal transformations acting on a 2d sub-space, essentially by using Euler angles, and each of these transformations as a product of reflections. That completely solves the Euclidean signature.

Anyway, when the covering map is surjective, then we have a short exact sequence of groups

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(\eta) \to \mathrm{SO}(\eta) \to 0$$

These are examples of group exact sequences which do not always split, actually almost ever.

If it does split, one gets that $\mathrm{Spin}(\eta) \simeq \mathrm{SO}(\eta) \times \mathbb{Z}_2$ which is true only for $\eta = (1,1)$.

Let us consider $S \in \mathrm{Pin}(\eta)$, then $S = \mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_k$ is a product of unit vectors. Hence we have $N(S) = N_0(S)\mathbb{I} = \pm\mathbb{I}$. Moreover, when restricted to $\mathrm{Pin}(\eta)$, one has $N(RS) = N(R)N(S)$.

Then let us consider the sub-group $G \in \mathcal{C}(\eta)$ of elements S such that

$$N(S) = N_0(S)\mathbb{I}, \qquad N_0(S) = \pm 1, \qquad \exists \ell_S \in O(V, \eta) : \alpha(S)wS^{-1} = \ell_S(w) \tag{17.4.39}$$

In fact, G is a sub-group. Of course, $\mathbb{I} \in G$ and for all $R, S \in G$ one has $N(RS) = N(R)N(S) = N_0(R)N_0(S)\mathbb{I} = N_0(RS)\mathbb{I}$, $N_0(RS) = N_0(R)N_0(S) = \pm 1$.
Moreover, if $\alpha(S)wS^{-1} = \ell_S(w)$ and $\alpha(R)wR^{-1} = \ell_R(w)$, then $\alpha(RS)w(RS)^{-1} = \ell_R \circ \ell_S(w)$. Thus $RS \in G$ and G is a sub-group.

Of course, $\mathrm{Pin}(\eta) \subset G$ and we now prove that $G = \mathrm{Pin}(\eta)$.

Let us take $S \in G$. Then there exists $\ell_S \in O(V, \eta)$ such that $\alpha(S)wS^{-1} = \ell_S(w)$. We also know that ℓ_S can be written as a compositions of reflections about planes normal to unit vectors $(u_1, u_2, \ldots, u_k)$, i.e. that $\ell_S = \ell(u_1)\ell(u_2)\ldots\ell(u_k)$. Thus set $T = \mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_k \in \mathcal{C}(\eta)$ and consider $S \cdot T^{-1} \in G$. For it we have

$$\alpha(ST^{-1})w(ST^{-1})^{-1} = w \qquad \Rightarrow \ell(ST^{-1}) = \mathbb{I} \qquad \Rightarrow ST^{-1} = s\mathbb{I} = \pm\mathbb{I} \tag{17.4.40}$$

Hence we have $S = \pm T \in \mathrm{Pin}(\eta)$.

Accordingly, to determine pin and spin groups, we can proceed in two ways: either we consider products of unit vectors, or we look for elements S in the (even) Clifford algebra which obey the definition of G.

The group Spin(2)

Consider V of dimension 2, an Euclidean metric δ, an orthonormal basis e_1, e_2. A basis of a Clifford algebra is $\mathbb{I}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}$ and they have the products

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e} \quad \mathbf{e}_1\mathbf{e} = -\mathbf{e}_2 = -\mathbf{e}\mathbf{e}_1 \quad \mathbf{e}_2\mathbf{e} = e_1 = -\mathbf{e}\mathbf{e}_2 \qquad \mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}\mathbf{e} = -\mathbb{I} \tag{17.4.41}$$

In this case, unit vectors are $u_\theta = \cos(\theta)e_1 + \sin(\theta)e_2$. The products of two unit vectors are in the form $u_\theta u_\phi = -\cos(\theta - \phi)\mathbb{I} - \sin(\theta - \phi)\mathbf{e}$, so let us define the set $\Sigma = \{\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}\} \subset \mathrm{Spin}(2)$. The product of two elements in Σ is in the form

$$(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e})\,(\cos(\phi)\mathbb{I} + \sin(\phi)\mathbf{e}) = \cos(\theta + \phi)\mathbb{I} + \sin(\theta + \phi)\mathbf{e} \in \Sigma \tag{17.4.42}$$

Accordingly, $\mathrm{Spin}(2) = \Sigma \simeq U(1)$ by the isomorphism

$$i : \mathrm{Spin}(2) \to U(1) : \cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e} \mapsto e^{i\theta} \tag{17.4.43}$$

In order to close the group $\text{Pin}(\eta)$, we also need to compute

$$(\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2)\,(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}) = \cos(\theta - \phi)\mathbf{e}_1 + \sin(\theta - \phi)\mathbf{e}_2 \tag{17.4.44}$$

These are unit vectors, so $\text{Pin}(\eta) \simeq U(1) \times \mathbb{Z}_2$, which is then disconnected in two pieces.

The covering map is then obtained by setting $S = \cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}$ and computing

$$\begin{aligned}
\alpha(S)\mathbf{w}S^{-1} &= (\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e})\,(w^1\mathbf{e}_1 + w^2\mathbf{e}_2)\,(\cos(\theta)\mathbb{I} - \sin(\theta)\mathbf{e}) = \\
&= \left(w^1\cos(2\theta) - w^2\sin(2\theta)\right)\mathbf{e}_1 + \left(w^1\sin(2\theta) + w^2\cos(2\theta)\right)\mathbf{e}_2
\end{aligned} \tag{17.4.45}$$

Hence, we have the covering map defined as

$$\ell : \text{Spin}(2) \to \text{SO}(2) : S = (\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}) \mapsto \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \tag{17.4.46}$$

On the other hand, we already know the even Clifford algebra is $\mathcal{C}^+(2) \simeq \mathbb{C}$ generated by $\mathbb{I}$ and $\mathbf{e} = \mathbf{e}_1\mathbf{e}_2$. Let us consider an element $S = a\,(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}) \in \mathcal{C}^+(2)$. We have

$$N(S) = S\alpha(^tS) = S\,^tS = a^2\,(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e})\,(\cos(\theta)\mathbb{I} - \sin(\theta)\mathbf{e}) = a^2\mathbb{I} \tag{17.4.47}$$

Thus elements of the spin group should be looked among $S = \cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}$. And actually they *are* the elements in $\text{Spin}(\eta)$ since we already checked that $\alpha(S)\mathbf{w}S^{-1} = \ell(S)(w)$ for a matrix $\ell(S) \in \text{SO}(\eta)$.

Already in this simple case, the second method is much faster and simpler than the first direct approach.

The group Spin$(0, 2)$

Let is consider the same case with reversed signature. A basis of the Clifford algebra is $\mathbb{I}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}$ with the products

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e} \quad \mathbf{e}_1\mathbf{e} = \mathbf{e}_2 = -\mathbf{e}\mathbf{e}_1 \quad \mathbf{e}_2\mathbf{e} = -\mathbf{e}_1 = -\mathbf{e}\mathbf{e}_2 \qquad \mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = \mathbb{I} \qquad \mathbf{e}\mathbf{e} = -\mathbb{I} \tag{17.4.48}$$

An element in the even Clifford algebra is $S = a\,(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}) \in \mathcal{C}^+(0, 2)$. We have

$$N(S) = S\alpha(^tS) = S\,^tS = a^2\,(\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e})\,(\cos(\theta)\mathbb{I} - \sin(\theta)\mathbf{e}) = a^2\mathbb{I} \tag{17.4.49}$$

Thus elements of the spin group should be looked among $S = \cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}$. The covering map is

$$\begin{aligned}
\alpha(S)\mathbf{w}S^{-1} &= (\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e})\,(w^1\mathbf{e}_1 + w^2\mathbf{e}_2)\,(\cos(\theta)\mathbb{I} - \sin(\theta)\mathbf{e}) = \\
&= \left(w^1\cos^2(\theta) + w^2\cos(\theta)\sin(\theta)\right)\mathbf{e}_1 + \left(w^2\cos^2(\theta) - w^1\cos(\theta)\sin(\theta)\right)\mathbf{e}_2 + \\
&\quad - \left(w^1\cos(\theta)\sin(\theta) + w^2\sin^2(\theta)\right)\mathbf{e}_2 + \left(w^2\cos(\theta)\sin(\theta) - w^1\sin^2(\theta)\right)\mathbf{e}_1 = \\
&= \left(w^1\cos(2\theta) + w^2\sin(2\theta)\right)\mathbf{e}_1 + \left(-w^1\sin(2\theta) + w^2\cos(2\theta)\right)\mathbf{e}_2
\end{aligned} \tag{17.4.50}$$

Hence, we have the covering map defined as

$$\ell : \mathrm{Spin}(0,2) \to \mathrm{SO}(0,2) : S = (\cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e}) \mapsto \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix} \tag{17.4.51}$$

Accordingly, when considering $\mathrm{Spin}(2)$ and $\mathrm{Spin}(0,2)$ they are both $U(1)$, though with an different covering map. One can also notice that the map

$$i : \mathrm{Spin}(2) \to \mathrm{Spin}(0,2) : \cos(\theta)\mathbb{I} + \sin(\theta)\mathbf{e} \mapsto \cos(\theta)\mathbb{I} - \sin(\theta)\mathbf{e} \tag{17.4.52}$$

is a group isomorphism. Let us remark that in these cases, the spin group is not simply connected, i.e. it is *not* the universal covering of the orthogonal group.

The group Spin$(1,1)$

For the signature $\eta = (1,1)$, we choose a basis (e_0, e_1) in V, the corresponding basis in the Clifford algebra is

$$\mathbb{I}, \qquad \mathbf{e}_0, \qquad \mathbf{e}_1, \qquad \mathbf{e} = \mathbf{e}_0\mathbf{e}_1 \tag{17.4.53}$$

with the products

$$\mathbf{e}_0\mathbf{e}_1 = \mathbf{e} \quad \mathbf{e}_0\mathbf{e} = \mathbf{e}_1 = -\mathbf{e}\mathbf{e}_0 \quad \mathbf{e}_1\mathbf{e} = \mathbf{e}_0 = -\mathbf{e}\mathbf{e}_1 \qquad \mathbf{e}_1\mathbf{e}_1 = -\mathbb{I} \qquad \mathbf{e}\mathbf{e} = \mathbf{e}_0\mathbf{e}_0 = \mathbb{I} \tag{17.4.54}$$

An element in the even Clifford algebra is $S = a\,(\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e}) \in \mathcal{C}^+(1,1) \simeq \mathbb{R} \oplus \mathbb{R}$. We have

$$N(S) = S\alpha(^tS) = S\,^tS = a^2\,(\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e})\,(\mathrm{ch}(\theta)\mathbb{I} - \mathrm{sh}(\theta)\mathbf{e}) = a^2\mathbb{I} \tag{17.4.55}$$

Thus elements of the spin group should be looked for among $S = \pm(\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e})$. The covering map is

$$
\begin{aligned}
\alpha(S)\mathbf{w}S^{-1} &= (\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e})\,(w^0\mathbf{e}_0 + w^1\mathbf{e}_1)\,(\mathrm{ch}(\theta)\mathbb{I} - \mathrm{sh}(\theta)\mathbf{e}) = \\
&= (\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e})\left(\left(w^0\mathrm{ch}(\theta) - w^1\mathrm{sh}(\theta)\right)\mathbf{e}_0 + \left(w^1\mathrm{ch}(\theta) - w^0\mathrm{sh}(\theta)\right)\mathbf{e}_1 \right) = \\
&= \left(\left(w^0\mathrm{ch}^2(\theta) - w^1\mathrm{ch}(\theta)\mathrm{sh}(\theta)\right)\mathbf{e}_0 + \left(w^1\mathrm{ch}^2(\theta) - w^0\mathrm{ch}(\theta)\mathrm{sh}(\theta)\right)\mathbf{e}_1 \right) + \\
&\quad - \left(\left(w^0\mathrm{ch}(\theta)\mathrm{sh}(\theta) - w^1\mathrm{sh}^2(\theta)\right)\mathbf{e}_1 - \left(w^1\mathrm{ch}(\theta)\mathrm{sh}(\theta) - w^0\mathrm{sh}^2(\theta)\right)\mathbf{e}_0 \right) = \\
&= \left(w^0\mathrm{ch}(2\theta) - w^1\mathrm{sh}(2\theta)\right)\mathbf{e}_0 + \left(-w^0\mathrm{sh}(2\theta) + w^1\mathrm{ch}(2\theta)\right)\mathbf{e}_1
\end{aligned}
\tag{17.4.56}
$$

Hence we have the covering map defined as

$$\ell : \mathrm{Spin}(1,1) \to \mathrm{SO}(1,1) : S = \pm\,(\mathrm{ch}(\theta)\mathbb{I} + \mathrm{sh}(\theta)\mathbf{e}) \mapsto \begin{pmatrix} \mathrm{ch}(2\theta) & -\mathrm{sh}(2\theta) \\ -\mathrm{sh}(2\theta) & \mathrm{ch}(2\theta) \end{pmatrix} \tag{17.4.57}$$

Accordingly, $\mathrm{Spin}_e(1,1)$ is $\mathrm{GL}(1,\mathbb{R}) \simeq \mathbb{R}$.

The group Spin(*3*)

Let us consider a Euclidean, 3d, vector space V, with an orthonormal basis e_1, e_2, e_3. The basis of even Clifford $\mathcal{C}^+(3) \simeq \mathbb{H}$ is

$$\mathbb{I}, \qquad \mathbf{e}_{12} = i, \qquad \mathbf{e}_{13} = j, \qquad \mathbf{e}_{23} = k \tag{17.4.58}$$

An element in the even Clifford algebra is $S = a\mathbb{I} + xi + yj + zk \in \mathcal{C}^+(3)$. We have

$$N(S) = S\alpha(^tS) = S\,{}^tS = SS^\dagger = |S|^2\mathbb{I} \tag{17.4.59}$$

where $S^\dagger = a\mathbb{I} - xi - yj - zk$ denotes the conjugation in $\mathbb{H}$ and $N_0(S) = |S|^2$ coincides with the norm in $\mathbb{H}$. Thus elements of the spin group should be looked among the unit quaternions $S_3 \simeq \mathrm{SU}(2)$.

The covering map is

$$
\begin{aligned}
&\alpha(S)\mathbf{w}S^{-1} = S\left(w^1\mathbf{e}_1 + w^2\mathbf{e}_2 + w^3\mathbf{e}_3\right)S^\dagger = \\
&= S\Big(\left(w^1a - w^2x - w^3y\right)\mathbf{e}_1 + \left(w^1x + w^2a - w^3z\right)\mathbf{e}_2 + \left(w^1y + w^2z + w^3a\right)\mathbf{e}_3 + \left(w^2y - w^1z - w^3x\right)\mathbf{e}\Big) = \\
&= a\left(w^1a - w^2x - w^3y\right)\mathbf{e}_1 + a\left(w^1x + w^2a - w^3z\right)\mathbf{e}_2 + a\left(w^1y + w^2z + w^3a\right)\mathbf{e}_3 + a\left(w^2y - w^1z - w^3x\right)\mathbf{e} + \\
&+ x\left(w^1a - w^2x - w^3y\right)\mathbf{e}_2 - x\left(w^1x + w^2a - w^3z\right)\mathbf{e}_1 + x\left(w^1y + w^2z + w^3a\right)\mathbf{e} - x\left(w^2y - w^1z - w^3x\right)\mathbf{e}_3 + \\
&+ y\left(w^1a - w^2x - w^3y\right)\mathbf{e}_3 - y\left(w^1x + w^2a - w^3z\right)\mathbf{e} - y\left(w^1y + w^2z + w^3a\right)\mathbf{e}_1 + y\left(w^2y - w^1z - w^3x\right)\mathbf{e}_2 + \\
&+ z\left(w^1a - w^2x - w^3y\right)\mathbf{e} + z\left(w^1x + w^2a - w^3z\right)\mathbf{e}_3 - z\left(w^1y + w^2z + w^3a\right)\mathbf{e}_2 - z\left(w^2y - w^1z - w^3x\right)\mathbf{e}_1 = \\
&= \left(w^1(a^2 - x^2 - y^2 + z^2) - 2(ax + yz)w^2 + 2(xz - ay)w^3\right)\mathbf{e}_1 + \\
&+ \left(2(ax - yz)w^1 + w^2(a^2 - x^2 + y^2 - z^2) - 2(az + xy)w^3\right)\mathbf{e}_2 + \\
&+ \left(2(ay + xz)w^1 + 2(az - xy)w^2 + w^3(a^2 + x^2 - y^2 - z^2)\right)\mathbf{e}_3 + \\
&\boxed{+ \left(w^1(xy - az - xy + az) + w^2(ay + xz - ay - xz) - w^3(ax - ax + zy - yz)\right)\mathbf{e}}
\end{aligned}
\tag{17.4.60}
$$

Hence we have the covering map defined as

$$
\ell : \mathrm{Spin}(3) \to \mathrm{SO}(3) : S \mapsto \begin{pmatrix} a^2 - x^2 - y^2 + z^2 & -2(ax + yz) & -2(ay - xz) \\ 2(ax - yz) & a^2 - x^2 + y^2 - z^2 & -2(az + xy) \\ 2(ay + xz) & 2(az - xy) & a^2 + x^2 - y^2 - z^2 \end{pmatrix} \in \mathrm{SO}(3) \tag{17.4.61}
$$

Accordingly, Spin(3) is SU(2).

The group Spin(*4*)

Let us consider an Euclidean, 4d, vector space V, with an orthonormal basis e_0, e_1, e_2, e_3. The basis of even Clifford $\mathcal{C}^+(4) \simeq \mathbb{H} \otimes \mathbb{H}$ is

$$
\begin{aligned}
&\mathbb{I}, \qquad \mathbf{e}_{12} = i, \qquad \mathbf{e}_{13} = j, \qquad \mathbf{e}_{23} = k, \qquad \mathbf{e}_{01} = -k\mathbf{e}, \qquad \mathbf{e}_{02} = j\mathbf{e}, \qquad \mathbf{e}_{03} = -i\mathbf{e}, \qquad \mathbf{e} \\
&k\mathbf{e} = \mathbf{e}k, \qquad j\mathbf{e} = \mathbf{e}j, \qquad i\mathbf{e} = \mathbf{e}i, \qquad \mathbf{e}\mathbf{e} = \mathbb{I}, \qquad {}^t\mathbf{e} = \mathbf{e}
\end{aligned}
\tag{17.4.62}
$$

An element in the even Clifford algebra is $S = a\mathbb{I} + b\mathbf{e}_{01} + c\mathbf{e}_{02} + d\mathbf{e}_{03} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} + h\mathbf{e} = u + \mathbf{e}v \in \mathcal{C}^+(4)$, with $u = a\mathbb{I} + ei + fj + gk$ and $v = h\mathbb{I} - di + cj - bk$ two quaternions. We have

$$N(S) = S\alpha(^tS) = (u + \mathbf{e}v)(u^\dagger + \mathbf{e}v^\dagger) = (|u|^2 + |v|^2)\mathbb{I} + (uv^\dagger + vu^\dagger)\mathbf{e} \tag{17.4.63}$$

where $u^\dagger = a\mathbb{I} - ei - fj - gk$ and $v^\dagger = h\mathbb{I} + di - cj + bk$ denote the conjugation in $\mathbb{H}$.

One has $N(S) = \pm\mathbb{I}$ iff $|u|^2 + |v|^2 = 1$ and $uv^\dagger + vu^\dagger = 0$. Let us set $q_\pm := u \pm v$ and compute

$$|q_\pm|^2 = (u \pm v)(u^\dagger \pm v^\dagger) = |u|^2 + |v|^2 \pm (uv^\dagger + vu^\dagger) = 1 \tag{17.4.64}$$

For the covering map we need to compute $\alpha(S)\mathbf{w}S^{-1} = (u + \mathbf{e}v)\mathbf{w}(u^\dagger + \mathbf{e}v^\dagger)$. That product expands to 256 terms. We know it must be a vector, so we can expand the result on the basis of the odd Clifford algebra, so that we know that 4 of the 8 coefficients vanishes on their own. Moreover, we collect by w^α.

In this way, we can analyse 64 terms at a time, 4 groups of 8 terms cancel, the other 32 split in groups of 8 terms for the 4 entries of $\ell(S)$ which correspond to a fixed w^α. In other words we can expand $(u + \mathbf{e}v)\mathbf{w}(u^\dagger + \mathbf{e}v^\dagger)$ as

$$
\begin{aligned}
S\mathbf{w}\bar{S} =& w^0\Big[(+a^2 + e^2 + f^2 + g^2 - h^2 - d^2 - c^2 - b^2)e_0 - 2(-ad - eh - fb - gc)e_3 + 2(+ac + eb - fh - gd)e_2 - 2(-ab + ec + fd - gh)e_1\Big] + \\
&+ w^1\Big[(+a^2 - e^2 - f^2 + g^2 - h^2 + d^2 + c^2 - b^2)e_1 + 2(+ae - fg + hd - cb)e_2 + 2(+af + eg - hc - db)e_3 + 2(-ab - ec - fd - gh)e_0\Big] + \\
&+ w^2\Big[(+a^2 - e^2 + f^2 - g^2 - h^2 + d^2 - c^2 + b^2)e_2 - 2(+ae + fg + hd + cb)e_1 + 2(+ag - ef + hb - dc)e_3 - 2(+ac - eb - fh + gd)e_0\Big] + \\
&+ w^3\Big[(+a^2 + e^2 - f^2 - g^2 - h^2 - d^2 + c^2 + b^2)e_3 - 2(+af - eg - hc + db)e_1 - 2(+ag + ef + hb + dc)e_2 + 2(-ad - eh + fb + gc)e_0\Big] = w^\beta \ell^\alpha_\beta(S)e_\alpha
\end{aligned}
\tag{17.4.65}
$$

That corresponds to the covering map $\ell : \mathrm{Spin}(4) \to \mathrm{SO}(4) : S \mapsto \ell(S)$, which associate the orthogonal matrix $\ell(S)$, given by

$$
\ell(S) = \begin{pmatrix}
a^2 + e^2 + f^2 + g^2 - h^2 - d^2 - c^2 - b^2 & 2(-ab - ec - fd - gh) & -2(ac - eb - fh + gd) & 2(-ad - eh + fb + gc) \\
-2(-ab + ec + fd - gh) & a^2 - e^2 - f^2 + g^2 - h^2 + d^2 + c^2 - b^2 & -2(ae + fg + hd + cb) & -2(af - eg - hc + db) \\
2(ac + eb - fh - gd) & 2(ae - fg + hd - cb) & a^2 - e^2 + f^2 - g^2 - h^2 + d^2 - c^2 + b^2 & -2(ag + ef + hb + dc) \\
-2(-ad - eh - fb - gc) & 2(af + eg - hc - db) & 2(ag - ef + hb - dc) & a^2 + e^2 - f^2 - g^2 - h^2 - d^2 + c^2 + b^2
\end{pmatrix}
\tag{17.4.66}
$$

to the element $S \in \mathrm{Spin}(4)$.

One can check directly that the columns are an orthonormal basis, under the constraint which defines the group $\mathrm{Spin}(4)$.

Accordingly, $\mathrm{Spin}(4)$ is $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

The group Spin(*3, 1*)

Let us consider a Lorentzian, 4d, vector space V, with an orthonormal basis e_0, e_1, e_2, e_3. The basis of even Clifford $\mathcal{C}^+(3, 1) \simeq \mathbb{H} \otimes \mathbb{C}$ is

$$
\begin{aligned}
&\mathbb{I}, \quad \mathbf{e}_{12} = i, \quad \mathbf{e}_{13} = j, \quad \mathbf{e}_{23} = k, \quad \mathbf{e}_{01} = -k\mathbf{e}, \quad \mathbf{e}_{02} = j\mathbf{e}, \quad \mathbf{e}_{03} = -i\mathbf{e}, \quad \mathbf{e} \\
&k\mathbf{e} = \mathbf{e}k, \quad j\mathbf{e} = \mathbf{e}j, \quad i\mathbf{e} = \mathbf{e}i, \quad \mathbf{e}\mathbf{e} = -\mathbb{I}, \quad {}^t\mathbf{e} = \mathbf{e}
\end{aligned}
\tag{17.4.67}
$$

An element in the even Clifford algebra is $S = a\mathbb{I} + b\mathbf{e}_{01} + c\mathbf{e}_{02} + d\mathbf{e}_{03} + e\mathbf{e}_{12} + f\mathbf{e}_{13} + g\mathbf{e}_{23} + h\mathbf{e} = u + \mathbf{e}v \in \mathcal{C}^+(3,1)$, with $u + \mathbf{e}v = A\mathbb{I} + Bi + Cj + Dk$ where we set

$$A = a + \mathbf{e}h \qquad B = e - \mathbf{e}d \qquad C = f + \mathbf{e}c \qquad D = g - \mathbf{e}b \tag{17.4.68}$$

We have

$$N(S) = S\alpha(^tS) = (A\mathbb{I} + Bi + Cj + Dk)(A\mathbb{I} - Bi - Cj - Dk) = (A^2 + B^2 + C^2 + D^2)\mathbb{I} \tag{17.4.69}$$

One has $N(S) \propto \mathbb{I}$. For S to be in $\mathrm{Spin}(3,1)$, one has

$$N_0(S) = A^2 + B^2 + C^2 + D^2 = \pm 1 \tag{17.4.70}$$

One has a representation $i \mapsto \mathbf{e}\sigma_1$, $j \mapsto -\mathbf{e}\sigma_2$, $k \mapsto \mathbf{e}\sigma_3$, by which an element S is in the form

$$S = \begin{pmatrix} A + \mathbf{e}D & -C + \mathbf{e}B \\ C + \mathbf{e}B & A - \mathbf{e}D \end{pmatrix} = M(2, \mathbb{C}) \tag{17.4.71}$$

The constraint from $N_0(S) = \pm 1$ reads as

$$\det(S) = (A + \mathbf{e}D)(A - \mathbf{e}D) - (C + \mathbf{e}B)(-C + \mathbf{e}B) = A^2 + D^2 + C^2 + B^2 = \pm 1 \tag{17.4.72}$$

With respect to signature $(4,0)$, the only things changing are the sign of some vectors (as for example, $\mathbf{e}ie_0 = e_3$) and $E^1 = -1$.
We have

$$
\begin{aligned}
S\mathbf{w}\bar{S} = & w^0\Big[(+a^2 + e^2 + f^2 + g^2 + h^2 + d^2 + c^2 + b^2)e_0 + 2(-ad - eh - fb - gc)e_3 - 2(+ac + eb - fh - gd))e_2 + 2(-ab + ec + fd - gh)e_1\Big] + \\
& + w^1\Big[(+a^2 - e^2 - f^2 + g^2 + h^2 - d^2 - c^2 + b^2)e_1 + 2(+ae - fg - hd + cb)e_2 + 2(+af + eg + hc + db)e_3 + 2(-ab - ec - fd - gh)e_0\Big] + \\
& + w^2\Big[(+a^2 - e^2 + f^2 - g^2 + h^2 - d^2 + c^2 - b^2)e_2 - 2(+ae + fg - hd - cb)e_1 + 2(+ag - ef - hb + dc)e_3 - 2(+ac - eb - fh + gd)e_0\Big] + \\
& + w^3\Big[(+a^2 + e^2 - f^2 - g^2 + h^2 + d^2 - c^2 - b^2)e_3 - 2(+af - eg + hc - db)e_1 - 2(+ag + ef - hb - dc)e_2 + 2(-ad - eh + fb + gc)e_0\Big] = w^\beta \ell^\alpha_\beta(S)e_\alpha
\end{aligned}
\tag{17.4.73}
$$

That corresponds to the covering map $\ell : \mathrm{Spin}(3,1) \to \mathrm{SO}(3,1) : S \mapsto \ell(S)$, which associate the orthogonal matrix $\ell(S)$, given by

$$\ell(S) = \begin{pmatrix} a^2 + e^2 + f^2 + g^2 + h^2 + d^2 + c^2 + b^2 & 2(-ab - ec - fd - gh) & -2(+ac - eb - fh + gd) & 2(-ad - eh + fb + gc) \\ 2(-ab + ec + fd - gh) & a^2 - e^2 - f^2 + g^2 + h^2 - d^2 - c^2 + b^2 & -2(+ae + fg - hd - cb) & -2(+af - eg + hc - db) \\ -2(+ac + eb - fh - gd)) & 2(+ae - fg - hd + cb) & a^2 - e^2 + f^2 - g^2 + h^2 - d^2 + c^2 - b^2 & -2(+ag + ef - hb - dc) \\ 2(-ad - eh - fb - gc) & 2(+af + eg + hc + db) & 2(+ag - ef - hb + dc) & a^2 + e^2 - f^2 - g^2 + h^2 + d^2 - c^2 - b^2 \end{pmatrix} \tag{17.4.74}$$

to the element $S \in \mathrm{Spin}(3,1)$.
One can check directly that the columns are an orthonormal basis, under the constraint which defines the group $\mathrm{Spin}(3,1)$.

Accordingly, $\mathrm{Spin}(3,1)$ is $\mathrm{SL}(2, \mathbb{C})$.

5. Lie group representations

A *representation* of a Lie group G on a (complex) vector space V is a group homomorphism

$$\rho : G \to \text{Aut}(V) \tag{17.5.1}$$

An *invariant* subspace $U \subset V$ is a subspace such that, for any $g \in G$, one has $\rho(g)(U) \subset U$. It is called *isotropic* iff, for all $u \in U$, one has $\rho(g)(u) = u$.

Of course, an isotropic subspace is invariant, not necessarily the other way around. If $U \subset V$ is invariant, then one can restrict $\rho_U : U \to U$ which is again a representation, on U this time.

For any representation, the subspaces $\{0\}$ and V are always invariant subspaces. They are called *trivial* invariant subspaces. A representation is *irreducible* iff V has no non-trivial invariant subspaces.

On a Lie group G, any representation induces a representation on its Lie algebra $\mathfrak{g}$, which are easier to classify as we showed above. On the other hand, given a representation of the algebra, one can exponentiate it to obtain back the corresponding group representation (if the group is simply connected). The process is often plagued with computational difficulties so it is often convenient to actually look for shortcuts, as we shall show in some particular case of interest.

If V is endowed with a Hermitian inner product, a linear automorphism $\Phi : V \to V$ is called *unitary* iff

$$\eta(\Phi(v), \Phi(w)) = \eta(v, w) \tag{17.5.2}$$

The group of unitary transformations is denoted by $U(V, \eta) \subset \text{Aut}(V)$.

In an orthonormal basis,

$$(v^a \Phi_a^c)^\dagger \eta_{cd} w^b \Phi_b^d = (v^c)^\dagger \eta_{cd} w^d \quad \Rightarrow (\eta^{ae}(\Phi_a^c)^\dagger \eta_{cd})\Phi_b^d =: (\Phi^\dagger)_d^e \Phi_b^d = \delta_b^e \tag{17.5.3}$$

Then, in an orthonormal basis, unitary transformations are represented in terms of unitary matrices, namely $\|\Phi_b^a\| \in U(n) \subset \text{GL}(n, \mathbb{C})$.

A representation $\rho : G \to \text{Aut}(V)$ is called *unitary* iff $\rho :\to U(V, \eta) \subset \text{Aut}(V)$.

Let $\rho : G \to \text{Aut}(U)$ and $\lambda : G \to \text{Aut}(W)$ two representations of G.

A *map between the representations* is a linear map $\varphi : U \to W$ which makes the following diagram commutative

$$
\begin{array}{ccc}
U & \xrightarrow{\ \varphi\ } & W \\
{\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \lambda(g)} \\
U & \xrightarrow{\ \varphi\ } & W
\end{array}
\tag{17.5.4}
$$

i.e. such that (for all $g \in G$) one has $\varphi(g \cdot v) = g \cdot \varphi(v)$.

Two representations are called *equivalent* if the map between representation is invertible. A map between representations $\varphi : V \to W$ is also a tensor in $V^* \otimes W$. One can easily see that equivariance $\varphi(g \cdot v) = g \cdot \varphi(v)$ corresponds to the tensor to be isotropic. Isotropic tensors are called *intertwines*. Accordingly, two repsentations are equivalent if there exists an intertwiner in $V^* \otimes W$.

Give a map $\varphi : U \to W$ between representations, we can always define two invariant subspaces $\ker(\varphi) \subset U$ and $\mathrm{Im}(\varphi) \subset W$.

The subspace $\ker(\varphi) \subset U$ is defined to be

$$\ker(\varphi) = \{u \in U : \varphi(u) = 0\} \tag{17.5.5}$$

For any element $g \in G$, we can consider the vector $g \cdot u$ which is still in $\ker(\varphi)$ since

$$\varphi(g \cdot u) = g \cdot \varphi(u) = g \cdot 0 = 0 \tag{17.5.6}$$

Thus $\ker(\varphi)$ is an invariant subspace of V.

Similarly, the subspace $\mathrm{Im}(\varphi) \subset W$ is defined to be

$$\mathrm{Im}(\varphi) = \{w \in W : \exists v \in V : w = \varphi(u)\} \tag{17.5.7}$$

For any element $g \in G$, we can consider the vector $g \cdot w$ which is still in $\mathrm{Im}(\varphi)$ since

$$g \cdot w = g \cdot \varphi(u) = \varphi(g \cdot u) = \varphi(u') \tag{17.5.8}$$

so that $g \cdot w \in \mathrm{Im}(\varphi)$ since it has a preimage of $u' = g \cdot u \in U$. Thus $\mathrm{Im}(\varphi)$ is an invariant subspace of W.

We can define a new representation $\rho_1 \oplus \rho_2$ on $U \oplus W$ as $\rho_1 \oplus \rho_2 : G \to \mathrm{Aut}(U \oplus W)$ which is called the *sum representation* where

$$(\rho_1 \oplus \rho_2)(S) : U \oplus W \to U \oplus W : (u, w) \mapsto (\rho_1(S)(u), \rho_2(S)(w)) \tag{17.5.9}$$

Check that it defines a representation.

In the algebra, one has $T(\rho_1 \oplus \rho_2) : \mathfrak{g} \to \mathrm{End}(U \oplus W)$

$$T(\rho_1 \oplus \rho_2)(\dot{S}) : U \oplus W \to U \oplus W : (u, w) \mapsto (T\rho_1(\dot{S})(u), T\rho_2(\dot{S})(w)) \tag{17.5.10}$$

If u is an eigenvector $T\rho_1(\dot{S})(u) = \mu u$ and w is an eigenvector $T\rho_2(\dot{S})(w) = \nu w$, then $(u, 0)$ and $(0, w)$ are eigenvectors

$$T(\rho_1 \oplus \rho_2)(\dot{S})(u, 0) = \mu(u, 0) \qquad T(\rho_1 \oplus \rho_2)(\dot{S})(0, w) = \nu(0, w) \tag{17.5.11}$$

Accordingly, with respect to the direct sum, one gets the union of eigenvalues, if two eigenvalues are the same in U and W, they sum multiplicities.

We can also define a representation ρ^* associated to ρ this time on U^*.

$$\rho^*(S) : U^* \to U^* : \alpha \mapsto \rho^*(S)(\alpha) \tag{17.5.12}$$

where

$$\rho^*(S)(\alpha) : U \to \mathbb{C} \qquad\qquad \rho^*(S)(\alpha)(u) = \alpha(\rho(S^{-1})(u)) \tag{17.5.13}$$

The quantity $\rho^*(S)(\alpha)(u) = \alpha(\rho(S^{-1})(u))$ is linear in u (thus $\rho^*(S)(\alpha) \in V^*$), and linear with respect to α (thus $\rho^*(S) \in \mathrm{Aut}(V^*)$).

If $S = \mathbb{I}$, one obtains the identity $\rho^*(\mathbb{I}) \in \mathrm{Aut}(V^*)$. For the compositions one has

$$(\rho^*(R) \circ \rho^*(S))(\alpha)(u) = \rho^*(S)(\alpha)(\rho(R^{-1})(u)) = \alpha((\rho(S^{-1}) \circ \rho(R^{-1}))(u)) = \alpha(\rho((RS)^{-1})(u)) = \rho^*(RS)(\alpha)(u) \tag{17.5.14}$$

Accordingly, ρ^* is a group homomorphism, hence a representation.

In the algebra, one has $T(\rho^*) : \mathfrak{g} \to \mathrm{End}(U^*)$

$$T(\rho^*)(\dot{S}) : U^* \to U^* : \alpha \mapsto T\rho^*(\dot{S})(\alpha) \qquad T\rho^*(\dot{S})(\alpha)(u) = -\alpha(T\rho(\dot{S})(u)) \tag{17.5.15}$$

If ρ is a unitary representation, then $T\rho(\dot{S})$ is anti-hermitian, i.e. $T\rho(\dot{S})^\dagger = -T\rho(\dot{S})$. Then, for an eigenvector $T\rho(\dot{S})(u) = \mu u$, we have

$$\mu^\dagger \eta(u, u) = \eta(T\rho(\dot{S})(u), u) = \eta(u, T\rho(\dot{S})^\dagger(u)) = -\eta(u, T\rho(\dot{S})(u)) = -\mu\eta(u, u) \tag{17.5.16}$$

Hence μ is purely imaginary.

Let us fix the non-canonical anti-isomorphism $* : U \to U^* : u \mapsto u^*$ defined as

$$u^*(u') = \eta(u, u') \tag{17.5.17}$$

It is linear in u' (hence $u^* \in U^*$) and anti-linear in u (hence it is an anti-isomorphism).

If u is an eigenvector $T\rho(\dot{S})(u) = \mu u$, then u^* is eigenvector of $T\rho^*(\dot{S})$

$$T\rho^*(\dot{S})(u^*)(u') = -u^*(T\rho(\dot{S})(u')) = -\eta(u, T\rho(\dot{S})(u')) = \eta(T\rho(\dot{S})(u), u') = \mu^\dagger \eta(u, u') = \mu^\dagger u^*(u') \qquad \Rightarrow T\rho^*(\dot{S})(u^*) = \mu^\dagger u^* = -\mu u^* \tag{17.5.18}$$

Accordingly, with respect to the duality, one gets opposite eigenvalues, at least for unitary representations.

Given two vector spaces U and W, we can consider the set of bi-linear maps $U \otimes W = \{t : U^* \times W^* \to \mathbb{C},\ \text{bi-linear}\}$ which is again a vector space. Two representations $\rho_1 : G \to \mathrm{Aut}(U)$ and $\rho_2 : G \to \mathrm{Aut}(W)$ then induce a representation $\rho_1 \otimes \rho_2 : G \to \mathrm{Aut}(U \otimes W)$ by setting $(\rho_1 \otimes \rho_2)(S)(t) \in U \otimes W$ defined by

$$(\rho_1 \otimes \rho_2)(S)(t)(\alpha, \beta) = t(\rho_1^*(S^{-1})(\alpha), \rho_2^*(S^{-1})(\beta)) \qquad \forall \alpha \in U^*, \beta \in W^* \tag{17.5.19}$$

The number $(\rho_1 \otimes \rho_2)(S)(t)(\alpha, \beta)$ so defined is linear in t as well as bi-linear in (α, β). Accordingly, $(\rho_1 \otimes \rho_2)(S)(t) \in U \otimes W$ and $(\rho_1 \otimes \rho_2)(S) \in \mathrm{Aut}(U \otimes W)$.

Moreover, for $S = \mathbb{I}$ one obtains the identity, and

$$\begin{aligned} (\rho_1 \otimes \rho_2)(R) \circ (\rho_1 \otimes \rho_2)(S)(t)(\alpha, \beta) &= (\rho_1 \otimes \rho_2)(S)(t)(\rho_1^*(R^{-1})(\alpha), \rho_2^*(R^{-1})(\beta)) = t(\rho_1^*(S^{-1} \circ \rho_1^*(R^{-1})(\alpha), \rho_2^*(S^{-1} \circ \rho_2^*(R^{-1})(\beta)) = \\ &= t(\rho_1^*((RS)^{-1})(\alpha), \rho_2^*((RS)^{-1})(\beta)) = (\rho_1 \otimes \rho_2)(RS)(t)(\alpha, \beta) \end{aligned} \tag{17.5.20}$$

Accordingly, $\rho_1 \otimes \rho_2$ is a group homomorphism, i.e. a representation.

In the algebra, one has $T(\rho_1 \otimes \rho_2) : \mathfrak{g} \to \mathrm{End}(U \otimes W)$

$$T(\rho_1 \otimes \rho_2)(\dot{S}) : U \otimes W \to U \otimes W : t \mapsto T(\rho_1 \otimes \rho_2)(\dot{S})(t) \qquad T(\rho_1 \otimes \rho_2)(\dot{S})(t)(\alpha, \beta) = -t(T\rho_1^*(\dot{S})(\alpha), \beta) - t(\alpha, T\rho_2^*(\dot{S})(\beta)) \tag{17.5.21}$$

If now u is an eigenvector of $T\rho(\dot{S})$ (i.e. $T\rho(\dot{S})(u) = \mu u$) and w is an eigenvector of $T\lambda(\dot{S})$ (i.e. $T\lambda(\dot{S})(w) = \nu w$), then we can define $u \otimes w \in U \otimes W$ which is an eigenvector of $T(\rho_1 \otimes \rho_2)(\dot{S})$. Then we have

$$
\begin{aligned}
T(\rho_1 \otimes \rho_2)(\dot{S})(u \otimes w)(\alpha, \beta) = &- (u \otimes w)(T\rho_1^*(\dot{S})(\alpha), \beta) - (u \otimes w)(\alpha, T\rho_2^*(\dot{S})(\beta)) = -u(T\rho_1^*(\dot{S})(\alpha))w(\beta) - u(\alpha)w(T\rho_2^*(\dot{S})(\beta)) = \\
= &- (T\rho_1^*(\dot{S})(\alpha)(u))\beta(w) - \alpha(u)(T\rho_2^*(\dot{S})(\beta)(w)) = \alpha(T\rho_1(\dot{S})(u))\beta(w) + \alpha(u)\beta(T\rho_2(\dot{S})(w)) = \\
= &(\mu + \nu)\alpha(u)\beta(w) = (\mu + \nu)u(\alpha)w(\beta) = (\mu + \nu)(u \otimes w)(\alpha, \beta) \qquad \Rightarrow \\
\Rightarrow \quad T(\rho_1 \otimes \rho_2)(\dot{S})(u \otimes w) = &(\mu + \nu)(u \otimes w)
\end{aligned}
\tag{17.5.22}
$$

Accordingly, $T(\rho_1 \otimes \rho_2)(\dot{S})$ has as eigenvalues all possible sums of eigenvalues of $T\rho_1(\dot{S})$ and $T\rho_2(\dot{S})$.

If we specify to spin groups, there is a special class of representations of the spin groups which are induced by representations of orthogonal groups and are called *tensorial representations*.

Let us consider a representation $\rho : \mathrm{SO}(\eta) \to \mathrm{Aut}(V)$ and consider $\hat{\rho} := \rho \circ \ell : \mathrm{Spin}(\eta) \to \mathrm{Aut}(V)$. That is automatically a representation, since it is the composition of two group homomorphisms.

As we shall see, the spin groups also have other representations, sometimes called *spinor representations* (and sometimes even adding "of the orthogonal group"!), which are not tensorial in the above sense.

Representations of Spin(*3*, *1*) and Spin(*3*)

Let us now consider the representations of the groups $\mathrm{Spin}(3, 1) \simeq \mathrm{SL}(2, \mathbb{C})$ and $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$

Any group representation $\rho : G \to \mathrm{Aut}(V)$ induces an algebra representation $T\rho : \mathfrak{g} \to \mathrm{End}(V)$. We already discussed classification of algebra representations and we found that finite-dimensional irreducible representations of $\mathfrak{spin}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{spin}(3) \simeq \mathfrak{su}(2)$ are labelled by a semi-integer j, which is called the *spin*.

The irreducible algebra representation of spin j is denoted by $T\rho^{(j)} : \mathfrak{g} \to \mathrm{End}(V)$, the corresponding group representation is denoted by $\rho^{(j)} : G \to \mathrm{Aut}(V)$.

Let us start quite trivially from $j = 0$. We know that $T\rho^{(0)}(\dot{S}) : \mathbb{C} \to \mathbb{C}$ with the basis v_0. We have $h(v_0) = 0$, $\tau_+(v_0) = 0$, $\tau_-(v_0) = 0$. Accordingly, $T\rho^{(0)}(\dot{S}) = 0$. That is called the *trivial representation* and any element $\dot{S} \in \mathfrak{sl}(2, \mathbb{C})$ is mapped to the 0 (1×1) matrix. That corresponds to the group representation $\rho^{(0)} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Aut}(\mathbb{C})$ defined on $S = a\mathbb{I} + ib^i \sigma_i$ (with $a^2 + (b^1)^2 + (b^2)^2 + (b^3)^2 = 1$) as $\rho^{(0)}(S) = \mathbb{I}$. That, of course, restricts to the trivial group representation of $\mathrm{SU}(2)$, which also maps all group elements to $\mathbb{I}$.

The next case is $j = \frac{1}{2}$ which is called the *fundamental representation* and denoted by $T\rho^{(1/2)}$. At the level of algebra, it is described in terms of Pauli matrices. For the group representation, by exponentiation, we obtain for $S = \exp(i\vec{\theta} \cdot \vec{\sigma}) = a\mathbb{I} + i\vec{a} \cdot \vec{\sigma}$.

In fact, we have

$$\rho^{(1/2)}(S) = \mathbb{I} + i\vec{\theta} \cdot \vec{\sigma} - \frac{1}{2!}(\vec{\theta} \cdot \vec{\sigma})^2 - \frac{i}{3!}(\vec{\theta} \cdot \vec{\sigma})^3 + \ldots = \left(1 - \frac{1}{2!}|\theta|^2 + \frac{1}{4!}|\theta|^4 + \ldots\right)\mathbb{I} + \frac{i}{|\theta|}\left(\vec{\theta} \cdot \vec{\sigma}\right)(|\theta| - \frac{1}{3!}|\theta|^3 + \ldots) = \cos|\theta|\mathbb{I} + i\sin|\theta|\left(\frac{\vec{\theta} \cdot \vec{\sigma}}{|\theta|}\right) \tag{17.5.23}$$

In other words, $\rho^{(1/2)}(S) = a\mathbb{I} + i\vec{a} \cdot \vec{\sigma}$ with

$$\cos|\theta| = a \qquad \sin|\theta|\left(\frac{\vec{\theta}}{|\theta|}\right) = \vec{a} \tag{17.5.24}$$

One can easily check that $a^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 = 1$. This is called *fundamental representation*, since one has

$$\rho^{(1/2)}(S) = a\mathbb{I} + i\vec{a} \cdot \vec{\sigma} = S \tag{17.5.25}$$

We introduce $V = \mathbb{C}^2$ as a vector space, with coordinates ψ^A and $\rho^{(1/2)}(S)(\psi) = S^A_B\psi^B e_A$. The dual representation $(\rho^{(1/2)})^*$ acts on $(\mathbb{C}^2)^*$, with coordinates ψ_A, and it is defined as $(\rho^{(1/2)})^*(S)(\psi)(u) = \psi(\rho^{(1/2)}(\bar{S})(u)) = \psi_A\bar{S}^A_B u^B$, i.e. $(\rho^{(1/2)})^*(S)(\psi) = \psi_B\bar{S}^B_A e^A$.

From the invariant density, we obtain

$$S^C_A\epsilon^{AB}S^D_B = \epsilon^{CD} \quad \Rightarrow \bar{S}^A_D = \epsilon^{AB}S^C_B\epsilon_{DC} \quad \Rightarrow \psi_B\bar{S}^B_D e^D = \psi_B\epsilon^{BA}S^C_A\epsilon_{DC}e^D \tag{17.5.26}$$

which corresponds to the map $* : \mathbb{C}^2 \to (\mathbb{C}^2)^* : e_A \mapsto e_{\dot{A}} = \epsilon_{BA}e^B$ (contraction on first index when lowering) on the spinor space. That also corresponds to the inverse map $* : (\mathbb{C}^2)^* \to (\mathbb{C}^2) : e^A \mapsto e^{\dot{A}}_. = \epsilon^{AB}e_B$ (contraction on second index when raising) on the dual spinor space.

They are one the inverse of the other. In fact, we have

$$e_{\dot{A}} = \epsilon_{BA}e^B \qquad \Rightarrow (e_{\dot{A}})^* = \epsilon^{AB}e_{\dot{B}} = \epsilon^{AB}\epsilon_{CB}e^C = e^A \tag{17.5.27}$$

Given two spinors $\psi_1, \psi_2 \in \mathbb{C}^2$, we can define an invariant number out of them by

$$(\psi_1, \psi_2) = \psi_1^A\psi_2^B\epsilon_{AB} \tag{17.5.28}$$

which is $SL(2, \mathbb{C})$ invariant.

That should not be confused with the inner product $< \psi_1, \psi_2 > = (\psi_1^A)^\dagger\delta_{AB}\psi_2^B$ which is $SU(2)$ invariant, only.

Since we know how to compute the inverse $\bar{S}^B_D = (S^\dagger)^B_D = \epsilon^{BA}S^C_A\epsilon_{DC}$, we can also obtain again the covering map as $\lambda^i_j(S) = S^A_C S^B_D(\sigma_j)^{CD}(\sigma^i)_{AB}$ where we set $(\sigma_j)^{CD} := \frac{\sqrt{2}}{2}(\sigma_j)^C_E\epsilon^{ED}$ and $(\sigma^i)_{DC} := \frac{\sqrt{2}}{2}\delta^{ij}(\sigma_j)^B_D\epsilon_{BC}$.

Let us consider the Clifford algebra in dimension 3 and the Dirac matrices $\gamma_i = i\sigma_i$ which in fact obeys $\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbb{I}$. They do not induce a faithful representation of the whole Clifford algebra, though it is faithful when restricted to the even Clifford algebra. Hence that induces a faithful representation of the spin group.

Accordingly, for any $S \in \mathrm{Spin}(3) \simeq \mathrm{SU}(2)$ we have

$$S\gamma_i S^{-1} = \lambda_i^j(S)\gamma_j \qquad \Rightarrow S\gamma_i S^{-1}\gamma_k = \lambda_i^j(S)(-\delta_{jk}\mathbb{I} + \epsilon_{jk}{}^l\gamma_l) \tag{17.5.29}$$

so that we can take the trace of both hand sides to obtain

$$\lambda_i^j(S) = \tfrac{1}{2}\mathrm{Tr}\left(S\sigma_i S^{-1}\sigma^j\right) = \tfrac{1}{2}S_B^A(\sigma_i)_C^B(S^{-1})_D^C(\sigma^j)_A^D = \tfrac{1}{2}S_B^A(\sigma_i)_C^B\epsilon^{CE}S_E^F\epsilon_{DF}(\sigma^j)_A^D = S_B^A S_E^F(\sigma_i)^{BE}(\sigma^j)_{AF} \tag{17.5.30}$$

The vector space $\mathbb{C}^2$ which supports the fundamental representation is also called *q-bit space*.

Let us denote by Sym^k the symmetric product of k copies of the fundamental *group* representation. By definition, we set $V = \mathbb{C}^{k+1}$ to be the space of symmetric tensors $\psi^{A_1 A_2 \dots A_k}$ of rank k (indices A_i range on $A_i = 0..1$) and the group representation

$$\mathrm{Sym}^k(U)\psi^{A_1 A_2 \dots A_k} = \psi'^{A_1 A_2 \dots A_k} = U_{B_1}^{A_1}U_{B_2}^{A_2}\cdots U_{B_k}^{A_k}\psi^{B_1 B_2 \dots B_k} \tag{17.5.31}$$

for $U \in \mathrm{SL}(2,\mathbb{C})$. This group representation induces the following algebra representation

$$\mathrm{sym}^k := T\mathrm{Sym}^k(\dot U)\psi^{A_1 A_2 \dots A_k} = k! \dot U_{B_1}^{(A_1}\psi^{\underline{B_1}A_2 \dots A_k)} \tag{17.5.32}$$

which is the algebra representation sym^k we already discussed. In particular, by setting $\dot U = h = \tfrac{1}{2}\sigma_3$ we obtain ($k = 2j$)

$$\mathrm{sym}^{2j}(h) = T\mathrm{Sym}^{2j}(h) = \begin{pmatrix} -j & 0 & 0 & \dots & 0 \\ 0 & -j+1 & 0 & \dots & 0 \\ 0 & 0 & -j+2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & j \end{pmatrix} \tag{17.5.33}$$

Hence, the representation $T\mathrm{Sym}^{2j}$ is of rank $n = 2j + 1 = k + 1$ and has n eigenspaces of dimension 1. Consequently, the representation $T\mathrm{Sym}^k$ is irreducible and thence $\mathrm{Sym}^{2j} \simeq \rho^{(j)}$.

The advantage of doing so, is that Sym^{2j} is the *group* representation, avoiding us to exponentiate the algebra representations.

We can also consider symmetric tensors $\psi_{A_1 A_2 \dots A_k}$ in V^* on which the representation $(\rho^{(j)})^*$ is defined. There we define the group representation

$$\mathrm{Sym}_k(U)\psi_{A_1 A_2 \dots A_k} = \psi'_{A_1 A_2 \dots A_k} = \bar U_{A_1}^{B_1}\bar U_{A_2}^{B_2}\cdots \bar U_{A_k}^{B_k}\psi_{B_1 B_2 \dots B_k} \tag{17.5.34}$$

which in the algebra reads as

$$\mathrm{Sym}_k(\dot U)\psi_{A_1 A_2 \dots A_k} = -n! \dot U_{(A_1}^{B_1}\psi_{\underline{B_1}A_2 \dots A_k)} \tag{17.5.35}$$

In particular, by setting $\dot U = h$ we obtain ($k = 2j$)

$$T\mathrm{Sym}_{2j}(h) = \begin{pmatrix} j & 0 & 0 & \dots & 0 \\ 0 & j-1 & 0 & \dots & 0 \\ 0 & 0 & j-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -j \end{pmatrix} \tag{17.5.36}$$

Consequently, the representation $T\mathrm{Sym}_k$ is irreducible and thence $\mathrm{Sym}_{2j} \simeq (\rho^{(j)})^*$.

Let us remark that, for an integer spin j, Sym^{2j} representation produces the same transformation for h and $-h$, i.e. it just depends on $\lambda(h)$, i.e. it is a tensor representation. On the other hand, the odd representations Sym^{2j+1} are spinor representations.

Let us finally remark, that according to a theorem, unitary representations of a non-compact group are infinite dimensional. As a consequence, $\mathrm{SL}(2, \mathbb{C})$, which is not compact, has no finite dimensional unitary representations. In fact, the irreducible representations we listed for $\mathrm{SL}(2, \mathbb{C})$, labeled by the spin j, are not unitary, since they do not preserve the hermitian scalar product, contrary to what happens with $\mathrm{SU}(2)$.

Reducible representations of SU(2)

More generally, if we consider representations which are not irreducible, we can have more general situations as shown in Fig. 12.2.

We can repeat the argument for that reducible representation $\rho : \mathrm{SU}(2) \to \mathrm{GL}(V)$.

$$\underline{\quad} \circ \underline{\quad} \circ \underline{\quad} \underset{2}{\odot} \underline{\quad} \underset{3}{\odot} \underline{\quad} \underset{2}{\odot} \underline{\quad} \circ \underline{\quad} \circ \underline{\quad} \tag{17.5.37}$$

Consider v_0, the eigenvector for the minimal eigenvalue for h, i.e. $m = -3/2$, and define for it a Fock strip and its associated invariant sub-space $V^{(3/2)} = \mathrm{Span}(v_0, v_1 = \tau_+(v_0), v_2 = \tau_+(v_1), v_3 = \tau_+(v_2))$. These are all eigenvectors of τ^2 for the eigenvalue $\frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$ and eigenvectors of h for different eigenvalues

$$h(v_0) = -\tfrac{3}{2}v_0 \qquad h(v_1) = -\tfrac{1}{2}v_1 \qquad h(v_2) = \tfrac{1}{2}v_2 \qquad h(v_3) = \tfrac{3}{2}v_3 \tag{17.5.38}$$

An alternative notation for eigenvectors is to identify them by their eigenvalues $|j, m\rangle$ of τ^2 and h, respectively. In this way we denote

$$v_0 = |3/2, -3/2\rangle \qquad v_1 = |3/2, -1/2\rangle \qquad v_2 = |3/2, 1/2\rangle \qquad v_3 = |3/2, 3/2\rangle \tag{17.5.39}$$

Once we define $V^{(3/2)}$, we are left with an invariant subspace that supports the representation

$$\underline{\quad} \circ \underline{\quad} \circ \underline{\quad} \underset{3}{\odot} \underline{\quad} \circ \underline{\quad} \circ \underline{\quad} \tag{17.5.40}$$

Its least eigenvalue is now -1 and we define a Fock strip

$$w_0 = |1, -1\rangle \qquad w_1 = |1, 0\rangle \qquad w_2 = |1, 1\rangle \tag{17.5.41}$$

that defines another invariant subspace $W^{(1)} = \mathrm{Span}(w_0, w_1, w_2)$ and we are left with a representation

$$\underline{\quad} \circ \underline{\quad} \underset{2}{\odot} \underline{\quad} \circ \underline{\quad} \tag{17.5.42}$$

Its least eigenvalue is now $-1/2$ and we define a Fock strip

$$u_0 = |\tfrac{1}{2}, -\tfrac{1}{2}> \qquad u_1 = |\tfrac{1}{2}, \tfrac{1}{2}> \tag{17.5.43}$$

that defines another invariant subspace $U^{(1/2)} = \mathrm{Span}(u_0, u_1)$ and we are left with a representation

$$\overset{\underset{2}{\odot}}{\rule{1cm}{0.4pt}\ \odot\ \rule{1cm}{0.4pt}} \tag{17.5.44}$$

We are left with a 2-dimensional subspace spanned by 2 eigenvectors both with a eigenvalue $m = 0$. Each of them is a Fock strip on its own, each of them spans a trivial representation. Let us denote them by $T_1^{(0)} = \mathrm{Span}(t_1)$ and $T_2^{(0)} = \mathrm{Span}(t_2)$, $t_i = |0, 0>$.

All in all, we have that $\rho = \rho^{(3/2)} \oplus \rho^{(1)} \oplus \rho^{(1/2)} \oplus \rho^{(0)} \oplus \rho^{(0)}$ is defined on $V = V^{(3/2)} \oplus W^{(1)} \oplus U^{(1/2)} \oplus T_1^{(0)} \oplus T_2^{(0)}$.

If we consider the tensor product $\rho^{(1/2)} \otimes \rho^{(1/2)}$ that has eigenvalues which are sums, in all possible ways, of two elements in $\{-\tfrac{1}{2}, \tfrac{1}{2}\}$, i.e. $(-1, 0, 0, 1)$ i.e. precisely

$$\rule{1cm}{0.4pt}\ \circ\ \rule{1cm}{0.4pt} \qquad \underset{2}{\rule{1cm}{0.4pt}\ \odot\ \rule{1cm}{0.4pt}} \qquad \rule{1cm}{0.4pt}\ \circ\ \rule{1cm}{0.4pt} \tag{17.5.45}$$

Accordingly, we have $\rho^{(1/2)} \otimes \rho^{(1/2)} \simeq \rho^{(1)} \oplus \rho^{(0)}$, which is defined on $\mathbb{C}^3 \oplus \mathbb{C} = \mathbb{C}^4$ which is precisely the dimension of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

As long as the choice of a basis is concerned, we have an original basis e_A in the fundamental representation which is formed by eigenvectors of h (for the eigenvalue $m = \pm\tfrac{1}{2}$) which are also eigenvectors of τ^2 (for the eigenvalue $j(j+1) = \tfrac{3}{4}$) and an orthonormal basis.

Let us stress that eigenvectors are defined up to a factor. These factors do not appear in the action of τ^2 and τ_3, however, they can affect the action of τ_+ and τ_-. It is just a basis, tuning the coefficients is just notation, although notation makes our life much easier.

Let us remark that for the Fock basis (e_0, e_1) of the fundamental representation $\rho^{(1/2)}$ we have

$$\tau_+ e_0 = e_1 \qquad \tau_+ e_1 = 0 \qquad \tau_- e_0 = 0 \qquad \tau_- e_1 = -2\alpha_0 e_0 = e_0 \tag{17.5.46}$$

This suggest to set simply $|->= e_0$ and $|+>= e_1$, so to have

$$\begin{array}{llll} \tau^2|+>= -\tfrac{3}{4}|+> & h|+>= \tfrac{1}{2}|+> & \tau_+|+>= 0 & \tau_-|+>= |-> \\ \tau^2|->= -\tfrac{3}{4}|+> & h|->= -\tfrac{1}{2}|+> & \tau_+|->= |-> & \tau_-|->= 0 \end{array} \tag{17.5.47}$$

which are still defined up to a *common* factor. This basis is not only adapted to τ^2 and h, it is also adapted to the ladder operators.

In particular, $|->$ corresponds to the eigenvalues $(j = \tfrac{1}{2}, m = -\tfrac{1}{2})$, thus it is also denoted as $|\tfrac{1}{2}, -\tfrac{1}{2}>=: |->$, while $|+>$ corresponds to the eigenvalues $(j = \tfrac{1}{2}, m = \tfrac{1}{2})$, thus it is also denoted as $|\tfrac{1}{2}, \tfrac{1}{2}>=: |+>$.

That gives us a basis of $\mathbb{C}^4$ as the space for the representation $\rho^{(1/2)} \otimes \rho^{(1/2)}$ which is denoted as

$$| + +>= |+> \otimes |+> \qquad | + ->= |+> \otimes |-> \qquad | - +>= |-> \otimes |+> \qquad | - ->= |-> \otimes |-> \tag{17.5.48}$$

On $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}$ we also have another basis as the space which supports the representation $\rho^{(1)} \oplus \rho^{(0)}$. The basis is denoted as a basis for $\mathbb{C}^3$ denoted by

$$|1, 1>= | + +> \qquad |1, 0>= \tfrac{\sqrt{2}}{2}(| + -> + | - +>) \qquad |1, -1>= | - -> \tag{17.5.49}$$

and a basis for $\mathbb{C}$ denoted as $|0,0>= \frac{\sqrt{2}}{2}(|+->-|-+>)$. These are clearly two bases in the same space $\mathbb{C}^4$. The relation between these two bases needs to be shown in detail. Let us compute $E_i := \frac{\sqrt{2}}{2}(\sigma_i)^{AB}e_A \otimes e_B = \frac{\sqrt{2}}{2}(\sigma_i)^A_C \epsilon^{CB} e_A \otimes e_B$

$$
\begin{aligned}
E_1 &= \tfrac{\sqrt{2}}{2}\left(-e_0 \otimes e_0 + e_1 \otimes e_1\right) \\
E_2 &= i\tfrac{\sqrt{2}}{2}\left(e_0 \otimes e_0 + e_1 \otimes e_1\right) \\
E_3 &= \tfrac{\sqrt{2}}{2}\left(e_0 \otimes e_1 + e_2 \otimes e_0\right)
\end{aligned}
\quad\Rightarrow\quad
\begin{cases}
E_+ := \tfrac{1}{2}(E_1 + iE_2) = e_1 \otimes e_1 = |1,1> \\
E_- := \tfrac{1}{2}(E_1 - iE_2) = e_0 \otimes e_0 = |1,-1> \\
E_3 := |1,0>
\end{cases}
\tag{17.5.50}
$$

Accordingly, any representation of SU(2) can be written as the sum of irreducible representations. In particular, any tensor product of irreducible representations can be written as a sum of irreducible representations.

Clebsch-Gordan coefficients

When we have the tensor product of 2 (or more) irreducible representations, $\rho^{(j_1)} \otimes \rho^{(j_2)}$, the basis elements τ_i act as

$$
\hat{\tau}_i := \rho^{j_1 \otimes j_2}(\tau_i) = \rho^{(j_1)}(\tau_i) \otimes \mathbb{I} + \mathbb{I} \otimes \rho^{(j_2)}(\tau_i) = \tau_i \otimes \mathbb{I} + \mathbb{I} \otimes \tau_i
\tag{17.5.51}
$$

by definition of product of representations.

In physics literature $\rho^{j_1 \otimes j_2}(\tau_i)$ is often called the *total angular momentum*, in contrast with $\rho^{(j_1)}(\tau_i)$ and $\rho^{(j_2)}(\tau_i)$ which just act on the 1-particle subspaces.

Accordingly, the corresponding Casimir τ^2 can be expressed in terms of $(\tau_\pm, h)$ (acting on the factor representations) as

$$
\begin{aligned}
\hat{\tau}^2 &= (\tau_1 \otimes \mathbb{I} + \mathbb{I} \otimes \tau_1)^2 + (\tau_2 \otimes \mathbb{I} + \mathbb{I} \otimes \tau_2)^2 + (\tau_3 \otimes \mathbb{I} + \mathbb{I} \otimes \tau_3)^2 = \tau^2 \otimes \mathbb{I} + \mathbb{I} \otimes \tau^2 + 2(\tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 + \tau_3 \otimes \tau_3) = \\
&= (h - h^2 - \tau_+\tau_-) \otimes \mathbb{I} + \mathbb{I} \otimes (h - h^2 - \tau_+\tau_-) - \tau_+ \otimes \tau_- - \tau_- \otimes \tau_+ - 2h \otimes h
\end{aligned}
\tag{17.5.52}
$$

The eigenvectors of the operators $(\hat{\tau}^2, \hat{h})$ are adapted to the decomposition $\rho^{(j_2+j_1)} \oplus \ldots \oplus \rho^{(|j_2-j_1|)}$.

In fact, we have two bases in the same space: one basis is $|j_1 j_2, m_1, m_2>:= |j_1, m_1> \otimes |j_2, m_2>$ and the other is $|J,M>$. They are 2 bases in the same space so we have a transition matrix which is made with the inner products

$$
C\begin{pmatrix} j_1, j_2, m_1, m_2 \\ J, M \end{pmatrix} = <j_1, j_2, m_1, m_2 | J, M>
\tag{17.5.53}
$$

which are called *Clebsch-Gordan coefficients*. Of course, in order for the CG coefficient to be non-zero, (j_1, j_2, J) needs to satisfy CG conditions.

These are non-zero iff $J \in \{j_1 + j_2, \ldots, |j_1 - j_2|\}$, $M = m_1 + m_2$.

For example, let us consider $\rho^{(1/2)} \otimes \rho^{(1/2)} = \rho^{(1)} \oplus \rho^{(0)}$ on $\mathbb{C}^4$. We have two bases, one for $|j_1, m_1> \otimes |j_2, m_2>$

$$
|-->:= |\tfrac{1}{2}, -\tfrac{1}{2}> \otimes |\tfrac{1}{2}, -\tfrac{1}{2}> \qquad |-+>:= |\tfrac{1}{2}, -\tfrac{1}{2}> \otimes |\tfrac{1}{2}, \tfrac{1}{2}> \qquad |+->:= |\tfrac{1}{2}, \tfrac{1}{2}> \otimes |\tfrac{1}{2}, -\tfrac{1}{2}> \qquad |++>:= |\tfrac{1}{2}, \tfrac{1}{2}> \otimes |\tfrac{1}{2}, \tfrac{1}{2}>
\tag{17.5.54}
$$

and one for $|J, M>$

$$|1, -1> \qquad |1, 0> \qquad |1, 1> \qquad |0, 0> \tag{17.5.55}$$

As a matter of fact, the vectors in the basis $|\pm, \pm>$ are eigenvectors of $\hat{h}$. In fact,

$$\hat{h}| + +> = | + +> \qquad \hat{h}| + -> = 0 \qquad \hat{h}| - +> = 0 \qquad \hat{h}| - -> = -| - -> \tag{17.5.56}$$

If we apply to them the operator $\hat{\tau}^2$ we have

$$\hat{\tau}^2| + +> = -\left(\tfrac{3}{4} + \tfrac{3}{4} + \tfrac{1}{2}\right)| + +> = -2| + +> \qquad \hat{\tau}^2| - -> = -\left(\tfrac{3}{4} + \tfrac{3}{4} + \tfrac{1}{2}\right)| - -> = -2| - -> \qquad (J = 1 \Rightarrow -J(J+1) = -2)$$
$$\hat{\tau}^2| + -> = -\left(\tfrac{3}{4} + \tfrac{3}{4} - \tfrac{1}{2}\right)| + -> -| - +> = -(| + -> +| - +>) \qquad \hat{\tau}^2| - +> = -\left(\tfrac{3}{4} + \tfrac{3}{4} - \tfrac{1}{2}\right)| - +> +| + -> = -(| + -> +| - +>) \tag{17.5.57}$$

Thus $|1, -1> := | - -> $ and $|1, 1> := | + +>$ are eigenvectors of both $\hat{\tau}^2$ and $\hat{h}$. On the contrary, $| - +>$ and $| + ->$ are not. However, we can define

$$|1, 0> := \tfrac{\sqrt{2}}{2}\left(| + -> +| - +>\right) \qquad |0, 0> := \tfrac{\sqrt{2}}{2}\left(| + -> -| - +>\right) \tag{17.5.58}$$

which are both eigenvectors of both $\hat{\tau}^2$ and $\hat{h}$ in fact

$$\hat{\tau}^2|1, 0> = -2|1, 0> \qquad \hat{h}|1, 0> = 0 \qquad \hat{\tau}^2|0, 0> = 0 \qquad \hat{h}|0, 0> = 0 \tag{17.5.59}$$

That defines the relation between the two bases discussed in $\mathbb{C}^4$.

Notice that, in view of (17.5.50), the relation between the two basis are encoded in the transformation matrices

$$|1, 1> = \tfrac{\sqrt{2}}{4}(\sigma_1 + i\sigma_2)^{AB} e_A \otimes e_B \qquad |1, -1> = \tfrac{\sqrt{2}}{4}(\sigma_1 - i\sigma_2)^{AB} e_A \otimes e_B \qquad |1, 0> = (\sigma_3)^{AB} e_A \otimes e_B \tag{17.5.60}$$

The non-zero Clebsch-Gordan coefficients are then

$$C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2} \\ 1, -1 \end{matrix}\right) = 1 \qquad C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \\ 1, -1 \end{matrix}\right) = 1$$
$$C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2} \\ 1, 0 \end{matrix}\right) = \tfrac{\sqrt{2}}{2} \qquad C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \\ 1, 0 \end{matrix}\right) = \tfrac{\sqrt{2}}{2} \tag{17.5.61}$$
$$C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2} \\ 0, 0 \end{matrix}\right) = -\tfrac{\sqrt{2}}{2} \qquad C\left(\begin{matrix} \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \\ 0, 0 \end{matrix}\right) = \tfrac{\sqrt{2}}{2}$$

Of course, Clebsch-Gordan coefficients are defined up to a complex factor, which may be partially fixed by requiring the bases to be orthonormal. A phase is still to be fixed.

If we consider the tensor product $\rho^{(j_1)} \otimes \rho^{(j_2)}$ of two irreducible representations, we can ask whether any other irreducible representation $\rho^{(j_3)}$ sits in it, i.e. if there is some invariant sub-space $V^{(j_3)} \subset V^{(j_1)} \otimes V^{(j_2)}$ which supports a representation $\rho^{(j_3)}$.

As a matter of fact we can prove that the following properties are equivalent:

(1) $\rho^{(j_3)}$ sits in the tensor product $\rho^{(j_1)} \otimes \rho^{(j_2)}$;

(2) $\rho^{(0)}$ sits in the tensor product $\rho^{(j_1)} \otimes \rho^{(j_2)} \otimes \rho^{(j_3)}$;

(3) $|j_2 - j_1| \le j_3 \le j_1 + j_2$ and $2(j_1 + j_2 + j_3) \in \mathbb{N}$ is even.

These conditions can be generalised to any finite set of spins and they are called *Clebsch-Gordan conditions* (CG conditions).

Graphic representations of tensor products

For future convenience, let us introduce a graphic representation for tensor products of representations of SU(2). If we consider a tensor product of k irreducible representations of SU(2), labelled by the spins $(j_1, j_2, \ldots, j_k)$ we represent the resulting representation by a node with k threads, one for each irreducible factor representation, each labelled with its spin. The node is called *a node of valence k.*

Since each irreducible representation is classified by its spin j_k and it is the symmetrized product of $2j$ copies of the fundamental representation, the corresponding thread can be represented by $2j$ lines. We add a box to represent the symmetrization to remind that a permutation of the lines threaded together by a box is irrelevant.

Also, we want to keep distinct an irreducible representation from its dual, which is isomorphic, though not canonically. For, we shall pick an orientation for the corresponding line, ingoing for the direct irreducible representation (the one acting by a group element), outgoing for its dual, which acts by the inverse of the group element.
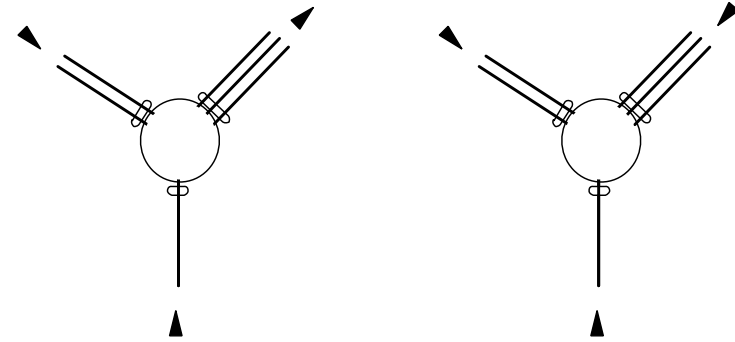

Fig.. 16.1 Tensor product of representations with $j_1 = \frac{1}{2}$, $j_2 = 1$ and $j_3 = \frac{3}{2}$ at a node of valence 3. Two possible orientations.

Intertwiners

An *intertwiner* in a representation (e.g. the tensor product representation of a family of irreducible representations) is an isotropic vector. We need to be able to determine the dimension of the isotropic subspace, to single out a basis for intertwiners, and to list linear relations among intertwiners.

If we have an element $w = w^{b_1 b_2 \ldots b_n} e_{b_1} \otimes e_{b_2} \otimes \ldots \otimes e_{b_n}$ in an isotropic subspace $W \subset V_1 \otimes V_2 \otimes \ldots \otimes V_n$, it means that

$$(\rho^{(j_1)} \otimes \ldots \otimes \rho^{(j_k)})(\varphi)(w) = w^{b_1 b_2 \ldots b_n} \varphi^{a_1}_{b_1} \varphi^{a_2}_{b_2} \ldots \varphi^{a_n}_{b_n} e_{a_1} \otimes e_{a_2} \otimes \ldots \otimes e_{a_n} = w^{a_1 a_2 \ldots a_n} e_{a_1} \otimes e_{a_2} \otimes \ldots \otimes e_{a_n} = w \qquad (17.5.62)$$

since we restrict the representation $\rho^{(j_1)} \otimes \ldots \otimes \rho^{(j_k)}$ to the isotropic subspace W, which is an isotropic subspace.

If we focus on the components we have

$$w^{b_1 b_2 \ldots b_n} \varphi^{a_1}_{b_1} \varphi^{a_2}_{b_2} \ldots \varphi^{a_n}_{b_n} = w^{a_1 a_2 \ldots a_n} \qquad (17.5.63)$$

We already know that, in the fundamental representation we have

$$\epsilon^{AB} \varphi^C_A \varphi^D_B = \epsilon^{CD} \qquad (17.5.64)$$

Then, for example, $w = \epsilon^{AB} e_A \otimes e_B$ is then an intertwiner for $\rho^{(1/2)} \otimes \rho^{(1/2)}$.

What we said above about Clebsh-Gordon decomposition allows us to find out, for a given set of irreducible representations $\rho^{(j_1)}, \ldots, \rho^{(j_k)}$, whether their tensor product $\rho^{(j_1)} \otimes \ldots \otimes \rho^{(j_k)}$ contains an isotropic subspace and to determine its multiplicity.

Case $k = 1$: the representation $\rho^{(j)}$ on $V = \mathbb{C}^{2j+1}$ contains an isotropic subspace only iff $j = 0$. If so, the isotropic subspace is of dimension 1 coinciding with V.

The trivial representation of spin $j = 0$ is graphically represented by 0 threads, so the corresponding node is an isolated node of valence zero, with no thread attached to it.

Case $k = 2$: we now have two irreducible representations $\rho^{(j_1)}$ and $\rho^{(j_2)}$ and we consider the product representation $\rho^{(j_1)} \otimes \rho^{(j_2)} : \mathrm{SU}(2) \to V = V_1 \otimes V_2 = \mathbb{C}^{(2j_1+1)(2j_2+1)}$.

This allows an isotropic subspace $W \subset V$ iff $j_1 = j_2 = \frac{n}{2}$, so that $\dim(W) = 1$. In V, we have a basis $e_{A_1} \odot \ldots \odot e_{A_n} \otimes e_{B_1} \odot \ldots \odot e_{B_n}$, hence the intertwiner will be something like

$$w = w^{(A_1 \ldots A_n)(B_1 \ldots B_n)} \, e_{A_1} \odot \ldots \odot e_{A_n} \otimes e_{B_1} \odot \ldots \odot e_{B_n} \tag{17.5.65}$$

Of course, we have that

$$w = \epsilon^{A_1 B_1} \ldots \epsilon^{A_n B_n} \, e_{A_1} \odot \ldots \odot e_{A_n} \otimes e_{B_1} \odot \ldots \odot e_{B_n} \tag{17.5.66}$$

is an isotropic element, i.e.

Fig. 16.2 *The different orientations of intertwiners at a valence 2 node and $j_1 = j_2 = 1$.*

$$\epsilon^{A_1 B_1} \ldots \epsilon^{A_n B_n} \, \varphi^{C_1}_{A_1} \ldots \varphi^{C_n}_{A_n} \varphi^{D_1}_{B_1} \ldots \varphi^{D_n}_{B_n} = \epsilon^{C_1 D_1} \ldots \epsilon^{C_n D_n} \tag{17.5.67}$$

In this case, we have two threadseach made of n lines. The graphic representation of the intertwiner is of n lines continuing through the node. To each line, it corresponds a ϵ^{AB} or a ϵ_{AB} or δ^A_B depending of the orientation of the lines at the node.

For example, let us consider the case $j_1 = j_2 = 1$ (i.e. $n = 2$). We have 3 cases, two outgoing lines, a line passing through the node, and two incoming lines.
In the first case (two outgoing lines), we choose and intertwiner

$$\bar{\varphi}^{A_1}_{C_1} \bar{\varphi}^{A_2}_{C_2} (\epsilon_{A_1 B_1} \epsilon_{A_2 B_2}) \bar{\varphi}^{B_1}_{D_1} \bar{\varphi}^{B_2}_{D_2} = \epsilon_{C_1 D_1} \epsilon_{C_2 D_2} \tag{17.5.68}$$

In the second case (one line in, one line out), we choose and intertwiner

$$\varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} (\delta^{C_1}_{B_1} \delta^{C_2}_{B_2}) \bar{\varphi}^{B_1}_{D_1} \bar{\varphi}^{B_2}_{D_2} = \delta^{A_1}_{D_1} \delta^{A_2}_{D_2} \tag{17.5.69}$$

In the last case (two incoming lines), we choose and intertwiner

$$\varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} (\epsilon^{C_1 D_1} \epsilon^{C_2 D_2}) \varphi^{B_1}_{D_1} \varphi^{B_2}_{D_2} = \epsilon^{A_1 B_1} \epsilon^{A_2 B_2} \tag{17.5.70}$$

In view of the fact that $\bar{\varphi}^A_B = \epsilon_{BD} \varphi^D_C \epsilon^{AC}$ this can be turned one into the other. For example, the second one is

$$
\begin{aligned}
\varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} \left(\delta^{C_1}_{B_1} \delta^{C_2}_{B_2} \right) \bar{\varphi}^{B_1}_{D_1} \bar{\varphi}^{B_2}_{D_2} &= \varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} \left(\delta^{C_1}_{B_1} \delta^{C_2}_{B_2} \epsilon^{B_1 E_1} \epsilon_{D_1 F_1} \epsilon^{B_2 E_2} \epsilon_{D_2 F_2} \right) \varphi^{F_1}_{E_1} \varphi^{F_2}_{E_2} = \\
&= \varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} \left(\epsilon^{C_1 E_1} \epsilon^{C_2 E_2} \right) \varphi^{F_1}_{E_1} \varphi^{F_2}_{E_2} \epsilon_{D_1 F_1} \epsilon_{D_2 F_2} = \delta^{A_1}_{D_1} \delta^{A_2}_{D_2} \\
\Rightarrow \quad & \varphi^{A_1}_{C_1} \varphi^{A_2}_{C_2} \left(\epsilon^{C_1 E_1} \epsilon^{C_2 E_2} \right) \varphi^{B_1}_{E_1} \varphi^{B_2}_{E_2} = \epsilon^{A_1 B_1} \epsilon^{A_2 B_2}
\end{aligned}
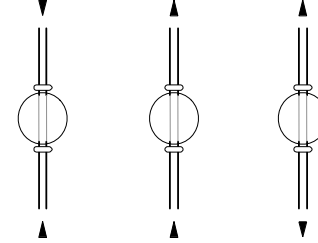\tag{17.5.71}
$$

which is the third one.

The orientation of the links affect whether the representation acts by a group element on the right or the inverse on the left (i.e., equivalently, a lower or an upper index). In order to make it systematic, we can think to consider representations on V_i or V_i^* depending on the orientation of the link. This amounts to consider the intertwiners in the 3 examples above as

$$
\begin{aligned}
w_1 &= \epsilon_{(A_1\underline{B_1}} \ldots \epsilon_{A_n)B_n} \, e^{A_1} \odot \ldots \odot e^{A_n} \otimes e^{B_1} \odot \ldots \odot e^{B_n} \\
w_2 &= \delta^{(A_1}_{B_1} \ldots \delta^{A_n)}_{B_n} \, e_{A_1} \odot \ldots \odot e_{A_n} \otimes e^{B_1} \odot \ldots \odot e^{B_n} \\
w_3 &= \epsilon^{(A_1\underline{B_1}} \ldots \epsilon^{A_n)B_n} \, e_{A_1} \odot \ldots \odot e_{A_n} \otimes e_{B_1} \odot \ldots \odot e_{B_n}
\end{aligned}
\tag{17.5.72}
$$

so that the outgoing links contribute by the dual basis e^A, the incoming links by the base e_A. Let us stress that, of course, the diagram determines in fact the 1 dimensional subspace $U = \mathrm{Span}(w)$, not w. For example, the sign we get for w is essentially up to us.

In any case, there is no much choice to do with $k = 2$. We have an isotropic space of dimension 1 iff $j_1 = j_2$, and 1 intertwiner (up to orientation). Any other isotropic element is a multiple of that intertwiner.

Case $k = 3$: we now have 3 irreducible representations $\rho^{(j_1)}$, $\rho^{(j_2)}$, and $\rho^{(j_3)}$ and we consider the product representation $\rho^{(j_1)} \otimes \rho^{(j_2)} \otimes \rho^{(j_3)} : \mathrm{SU}(2) \to V = V_1 \otimes V_2 \otimes V_3 = \mathbb{C}^{(2j_1+1)(2j_2+1)(2j_3+1)}$. Let us assume $j_2 \leq j_3$.

The product $\rho^{(j_2)} \otimes \rho^{(j_3)}$ decomposes into a sum of irreducible representations

$$
\rho^{(j_1)} \otimes \left(\rho^{(j_3+j_2)} \oplus \rho^{(j_3+j_2)-1} \oplus \ldots \oplus \rho^{(j_3-j_2)} \right)
\tag{17.5.73}
$$

and it allows an isotropic space iff $j_1 = j_3 + j_2 - k_{23}$ for some integer $0 \leq k_{23} \leq 2j_2$, i.e. iff $k_{23} = j_3 + j_2 - j_1$ is an integer $0 \leq k_{23} \leq 2j_2$. We can define $k_{13} := j_3 - j_2 + j_1 = k_{23} - 2(j_2 - j_1)$ and $k_{12} := -j_3 + j_2 + j_1 = k_{23} - 2(j_3 - j_1)$ which are also integers.

$$
\begin{cases}
k_{23} = j_3 + j_2 - j_1 \\
k_{13} = j_3 - j_2 + j_1 \\
k_{12} = -j_3 + j_2 + j_1
\end{cases}
\qquad \Longleftrightarrow \qquad
\begin{cases}
k_{23} + k_{13} = 2j_3 \\
k_{13} + k_{12} = 2j_1 \\
k_{23} + k_{12} = 2j_2
\end{cases}
\tag{17.5.74}
$$

We also have

$$
k_{12} + k_{23} + k_{13} = j_3 + j_2 + j_1 \in \mathbb{Z}
\tag{17.5.75}
$$

Moreover, they are positive integers. Precisely, since $k_{13} = 2j_3 - k_{23}$ and $0 \leq k_{23} \leq 2j_2$, then we have $0 \leq 2(j_3 - j_2) \leq k_{13} \leq 2j_3$; since $k_{12} = 2j_2 - k_{23}$ and $0 \leq k_{23} \leq 2j_2$, then we have $0 \leq k_{12} \leq 2j_2$.

The interpretation is pretty obvious. We have 3 links at the node, with $2j_1$, $2j_2$, and $2j_3$ lines. We split each thread as $2j_1 = k_{13} + k_{12}$, $2j_2 = k_{23} + k_{12}$, and $2j_3 = k_{23} + k_{13}$ and then we connect k_{ab} lines to the threads a and b.

Also in this case, the isotropic space, when it exists, is always of dimension 1, and again we have only one way of intertwining the threads. For example, if we consider $j_1 = \frac{1}{2}$, $j_2 = 1$, $j_3 = \frac{3}{2}$, we have that $j_3 + j_2 - j_1 = 2$ is a positive integer (as well as $2 \le 2j_2$). Then we have an isotropic subspace of dimension 1 and we get

$$k_{12} = 0 \qquad k_{13} = 1 \qquad k_{23} = 2 \qquad (17.5.76)$$

The two orientations shown in the Figure 16.3 correspond to choosing the intertwiners as

$$w = \delta^{B_1}_{(A_1} \delta^{B_2}_{A_2} \delta^C_{A_3)} \, e^{A_1} \odot e^{A_2} \odot e^{A_3} \otimes e_{B_1} \odot e_{B_2} \otimes e_C \qquad (17.5.77)$$

and

$$w = \epsilon^{(A_1 \underline{B_1}} \epsilon^{A_2 \underline{B_2}} \epsilon^{A_3) C} \, e_{A_1} \odot e_{A_2} \odot e_{A_3} \otimes e_{B_1} \odot e_{B_2} \otimes e_C \qquad (17.5.78)$$

We can use the same results to see the problem under a different perspective. We can fix $j_3 = \frac{3}{2}$ and $j_2 = 1$ as above, and look for all possible values of j_1 for which we have an intertwiner.

Since $j_3 + j_2 - j_1 = k_{23}$ ($\frac{5}{2} - j_1 = k_{23}$) has to be an integer between $0 \le k_{23} \le 2j_2 = 2$ we obviously can have $j_1 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ (which, in fact, correspond to $k_{23} = 2, 1, 0$, respectively). Those correspond to the intertwiners shown in Figure 16.4.

Thus, if we fix two legs out of three, we can list all the possibilities for the third leg together with the unique intertwiner allowed in each case.

Fig.. 16.3 all possible intertwiners with $j_1 = \frac{1}{2}$, $j_2 = 1$, and $j_3 = \frac{3}{2}$ at a node of valence 3.

Fig.. 16.4 all possible j_1 with an intertwiner when we set $j_2 = 1$, and $j_3 = \frac{3}{2}$, at a node of valence 3.

Case $k = 4$: we now have 4 irreducible representations $\rho^{(j_1)}$, $\rho^{(j_2)}$, $\rho^{(j_3)}$, and $\rho^{(j_4)}$ and we consider the product representation $\rho^{(j_1)} \otimes \rho^{(j_2)} \otimes \rho^{(j_3)} \otimes \rho^{(j_4)} : \mathrm{SU}(2) \to V = V_1 \otimes V_2 \otimes V_3 \otimes V_4 = \mathbb{C}^{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}$. Let us assume $j_1 \le j_2$ and $j_3 \le j_4$.

This case brings the first surprises, namely the possibility of having higher dimensional isotropic subspaces and the need to choose independent intertwiners among generators.

By multiplying the representations $\rho^{(j_4)} \otimes \rho^{(j_3)}$ and $\rho^{(j_2)} \otimes \rho^{(j_1)}$ we get the sum of the following representations:

$$((j_4+j_3) \oplus (j_4+j_3-1) \oplus \ldots \oplus (j_4-j_3+1) \oplus (j_4-j_3)) \otimes ((j_2+j_1) \oplus (j_2+j_1-1) \oplus \ldots \oplus (j_2-j_1+1) \oplus (j_2-j_1)) \quad (17.5.79)$$

Fig.. 16.5 splitting a 4-valence node into two 3-valence nodes.

Then, for having an isotropic subspace, there must be 2 integers $0 \le k_{34} \le 2j_3$ and $0 \le k_{12} \le 2j_1$ such that

$$j_4 + j_3 - k_{34} = j = j_2 + j_1 - k_{12} \qquad (17.5.80)$$

From these conditions, we can list the allowed possibilities, each gives a different isotropic subspace in a direct sum, thus they are independent.
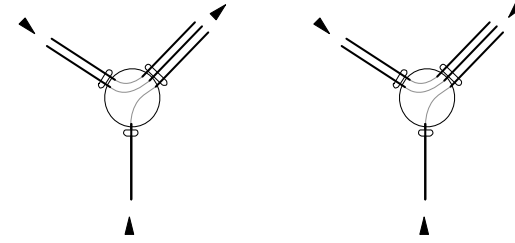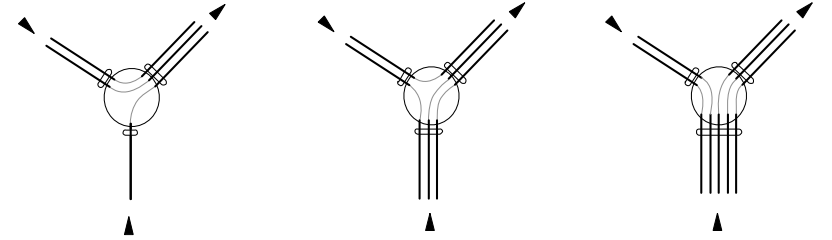
This corresponds to temporarily replace a 4-valence node with two 3-valence nodes as in the Figure 16,5. Let us fix at will the orientation of the j link. In view of the case $k = 3$ above, we are able to define a intertwiner in each node, and associate to it an isotropic vector w_{34} and w_{12}.

From this we can go back to the 4-valence node by associating the identity of the group element to the internal link j in the representation $\rho^{(j)}(e) = \delta_{B_1}^{A_1} \ldots \delta_{B_n}^{A_n}$ with $n = 2j$. This defines a way of multiplying $w_{34} \cdot w_{12} = w$ to get an isotropic element of the original 4-valence node. And we can repeat the procedure for each allowed value of j.

For example, let us consider the case $j_4 = j_3 = j_2 = j_1 = \frac{1}{2}$. Let us group (j_4, j_3) and (j_2, j_1). We have $1 - k_{34} = j = 1 - k_{12}$ with $0 \le k_{34} \le 1$ and $0 \le k_{12} \le 1$. We hence find 2 allowed solutions

$$j = 0 \quad k_{12} = k_{34} = 1 \qquad\qquad j = 1 \quad k_{12} = k_{34} = 0 \qquad\qquad (17.5.81)$$

which correspond to the intertwiners in Figure 16.6.

> Let us stress that one does not really replace a 4 valence node with 2 valence 3 nodes. We just split the node and then paste them back again. The splitting and the link j are temporary. This is just an algebraic way of listing intertwiners at a 4-valence node. If you want, we can say the link j we add is a *virtual link*.

In the first case, we have

$$w_{34} = \delta_A^B \, e^A \otimes e_B \qquad w_{12} = \delta_D^C \, e^D \otimes e_C \qquad\qquad (17.5.82)$$

which can be recombined to the intertwiner of the 4-valence node as

Fig.. 16.6 splitting a 4-valence node with $j_4 = j_3 = j_2 = j_1 = \frac{1}{2}$.

$$w_{(j=0)} = \delta_A^B \delta_D^C \, e^A \otimes e_B \otimes e^D \otimes e_C \qquad\qquad (17.5.83)$$

In the second case, we have

$$w_{34} = \delta_E^B \epsilon_{AF} \, e^A \otimes e_B \otimes e^E \odot e^F \qquad w_{12} = \delta_D^G \epsilon^{CH} \, e^D \otimes e_C \otimes e_G \odot e_H \qquad\qquad (17.5.84)$$

which can be recombined to the intertwiner of the 4-valence node as

$$
\begin{aligned}
w_{(j=1)} &= \delta_E^B \epsilon_{AF} \delta_G^E \delta_H^F \delta_D^G \epsilon^{CH} \, e^A \otimes e_B \otimes e^D \otimes e_C = \delta_D^B \epsilon_{AF} \epsilon^{CF} \, e^A \otimes e_B \otimes e^D \otimes e_C = \\
&= \delta_D^B \delta_A^C \, e^A \otimes e_B \otimes e^D \otimes e_C
\end{aligned}
\qquad (17.5.85)
$$
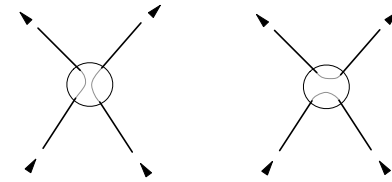
Fig.. 16.7 a basis for intertwiners at a 4-valence node with $j_4 = j_3 = j_2 = j_1 = \frac{1}{2}$.

> Of course, we could have contracted the 3-nodes by using $\delta_H^E \delta_G^F$ or $\delta_G^{(E} \delta_H^{F)}$, ending in a different intertwiner, always forming a (different) basis with the intertwiner (17.5.83) which we found for $j = 0$.
> In any event, we define a basis for the isotropic space in the tensor product with $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$. All the other intertwiners we can define are linear combinations of these two. For example, the element

$$\tilde{w} = \epsilon_{AD} \epsilon^{BC} \, e^A \otimes e_B \otimes e^D \otimes e_C \qquad\qquad (17.5.86)$$
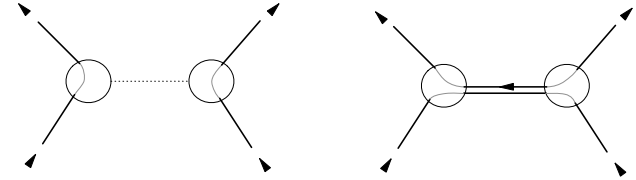
is clearly an intertwiner. However, we have $\epsilon_{AD}\epsilon^{BC} = \delta_A^B\delta_D^C - \delta_D^B\delta_A^C$ which implies that

$$\tilde{w} = w_{(j=0)} - w_{(j=1)} \tag{17.5.87}$$

That shows that we have a linear combination of intertwiners that shows they are not independent. That is often represented graphically as in the following Figure 16.8.

As a further example, we can discuss the intertwiners for the other case $j_4 = j_3 = \frac{3}{2}$ and $j_2 = j_1 = 1$, again by pairing (j_4, j_3) and (j_2, j_1). We have $3 - k_{34} = j = 2 - k_{12}$ with $0 \le k_{34} \le 3$ and $0 \le k_{12} \le 2$. This time we get 3 different cases

Fig.. 16.8 Linear relation between intertwiners at nodes

$$k_{12} = 0, j = 2, k_{34} = 1 \qquad k_{12} = 1, j = 1, k_{34} = 2 \qquad k_{12} = 2, j = 0, k_{34} = 3 \tag{17.5.88}$$

which correspond to the splitted intertwiners in Figure 16.9.

Fig. 16.9 splitted 4-valence node with $j_4 = j_3 = \frac{3}{2}$ and $j_2 = j_1 = 1$.

To the first case, we associate an intertwiner

$$\begin{aligned}
w_{(j=2)} &= (\delta_G^A\delta_H^B\delta_D^C\epsilon_{EI}\epsilon_{FL})\delta_M^G\delta_N^H\delta_O^I\delta_P^L(\delta_T^P\delta_S^O\epsilon^{MQ}\epsilon^{NR})\,e_{ABC}\otimes e^{DEF}\otimes e_{QR}\otimes e^{ST} = \\
&= \epsilon^{AQ}\epsilon^{BR}\delta_D^C\epsilon_{ES}\epsilon_{FT}\;e_{ABC}\otimes e^{DEF}\otimes e_{QR}\otimes e^{ST}
\end{aligned} \tag{17.5.89}$$

For the second case, we obtain

$$\begin{aligned}
w_{(j=1)} &= (\delta_D^C\delta_E^B\delta_G^A\epsilon_{FH})\delta_M^G\delta_N^H(\epsilon^{MQ}\delta_T^N\delta_S^R)\,e_{ABC}\otimes e^{DEF}\otimes e_{QR}\otimes e^{ST} = \\
&= \delta_D^C\delta_E^B\epsilon^{AQ}\epsilon_{FT}\delta_S^R\,e_{ABC}\otimes e^{DEF}\otimes e_{QR}\otimes e^{ST}
\end{aligned} \tag{17.5.90}$$

Finally, for $j = 0$, we get

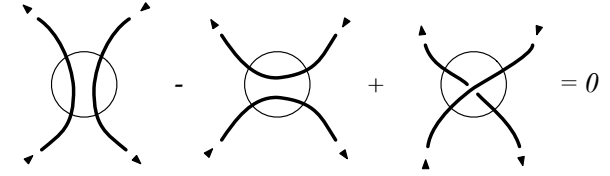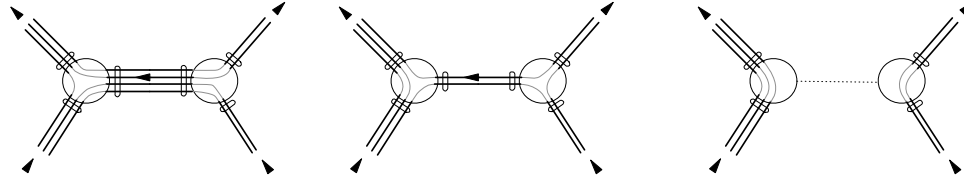$$w_{(j=0)} = \delta_D^C\delta_E^B\delta_F^A\delta_T^Q\delta_S^R\,e_{ABC}\otimes e^{DEF}\otimes e_{QR}\otimes e^{ST} \tag{17.5.91}$$

Fig.. 16.10 a basis for intertwiners at a 4-valence node with $j_4 = j_3 = \frac{3}{2}$ and $j_2 = j_1 = 1$.

They, in fact, correspond to the 4-valence intertwiners in Figure 16.10.

Let us remark that we have different choices along the process. We can pairing the legs differently, we can contract inner legs differently. Each choice produces a different basis, still a basis of the isotropic subspace.

What we discussed is enough for 4-valence nodes, which are in turn enough for discussing spin networks associating to triangulations of a 3-surface. In principle, we should discuss a method to single out a basis at any valence node, but we are satisfied with this by now.
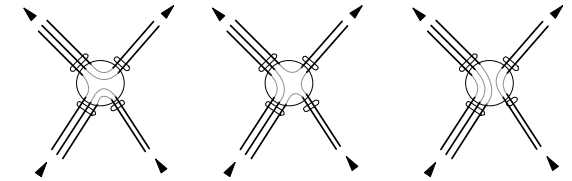
The Haar measure for SU(2)

The group $\mathrm{Spin}(3,0) \simeq \mathrm{SU}(2)$ is a compact group since it is diffeomorphic to S_3. One can think to $\mathrm{SU}(2)$ as the unit sphere in $\mathbb{H} \simeq \mathbb{R}^4$. Multiplication in the group are rotations in $\mathbb{R}^4$, since the constraint on unit vectors is preserved in $\mathbb{H}$.

Thus the standard metric induced on S_3 by the standard metric in $\mathbb{R}^4$ is invariant with respect to left and right translations and it is called the *Haar metric*. It defines a measure on $\mathrm{SU}(2)$ which can be normalized so that $\int_{SU(2)} dU = 1$.

We just have to fix some coordinates and compute the Haar measure.

Since $\mathrm{SU}(2)$ is diffeomorphic to S_3, we can fix stereographic coordinates on $\mathbb{R}^3$, i.e. the tangent space at the identity, which is the Lie algebra. An element $U \in \mathrm{SU}(2)$ can be represented by $U = a\mathbb{I} + bi + cj + dk$ (with $a^2 = 1 - b^2 - c^2 - d^2$).

If we want to cover $\mathrm{SU}(2) - \{-\mathbb{I}\}$, we can use stereographic coordinates

$$x = \frac{b}{1+a} \qquad y = \frac{c}{1+a} \qquad z = \frac{d}{1+a} \tag{17.5.92}$$

We have the embedding

$$i : \mathrm{SU}(2) \to \mathbb{H} : (x, yz) \mapsto \left(\frac{1 - |v|^2}{1 + |v|^2}, \frac{2x}{1 + |v|^2}, \frac{2y}{1 + |v|^2}, \frac{2z}{1 + |v|^2} \right) \tag{17.5.93}$$

where we set $|v|^2 = x^2 + y^2 + z^2$. By differentiating, we have that the induced invariant metric g on $\mathrm{SU}(2)$ is

$$g = \frac{2^4 \left(dx^2 + dy^2 + dz^2 \right)}{(1 + |v|^2)^2} \tag{17.5.94}$$

Thus, we can define the Haar measure

$$\int_{SU(2)} f(U)\, dU := \frac{4}{\pi^2} \int_{R^3} f(U(v)) \frac{d^3v}{(1 + |v|^2)^3} \qquad \Rightarrow \int_{SU(2)} dU = 1 \tag{17.5.95}$$

which is left and right invariant on the group and by which the group has a finite volume.

References

add

Fulton Harris

Lawson & ...

Chapter 18. Fiber bundles

1. Fiber bundles

A (smooth) *fiber bundle* is a 4-tuple $\mathcal{B} = (B, M, \pi, F)$, where B, M, F are manifolds called the *total space*, the *base*, and the *standard fiber*, respectively. The map $\pi : B \to M$ is a surjective and maximal rank map which is called the *projection*.

In order for $\mathcal{B}$ to be a fiber bundle, there must exists a *trivialisation*, i.e. a collection (U_α, t_α) of diffeomorphisms

$$t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \qquad (\pi = p_1 \circ t_\alpha) \tag{18.1.1}$$

where we set $p_1 : U_\alpha \times F \to U_\alpha : (x, f) \mapsto x$ for the projection on the first factor. A pair (U, t) is called a *local trivialisation* on U. A *trivialisation* is a collection of local trivialisations which covers the whole M.

The preimage of a point $x \in M$ is denoted by $\pi^{-1}(x) \subset B$ and it is called the *fiber* over x. One can restrict a local trivialisation to the fiber over x to obtain the map $t_x : \pi^{-1}(x) \to F$. This must be a diffeomorphism since t is.

Given two local trivialisations (U_1, t_1) and (U_2, t_2), one can define *transition maps* for $x \in U_{12} := U_1 \cap U_2$ by

$$(g_{12})_x := (t_1)_x \circ (t_2)_x^{-1} : F \to F \in \mathrm{Diff}(F) \tag{18.1.2}$$

> It may (and usually does) happen that there exist trivialisations on a bundle in which transition maps take values on a subgroup $G \subset \mathrm{Diff}(F)$. In that case we say one has a G-structure on the bundle $\mathcal{B}$. If G is a Lie group and one has a left action on F, then a G-structure restricts transition functions to act through the action (which is understood).

A fiber bundle can be denoted by $\mathcal{B} = (B, M, \pi, F)$, by $\pi : B \to M$, or by $[B \to M]$, depending on what we want to name.

A *fibered morphism* is a pair of maps (Φ, φ) such that

$$
\begin{array}{ccc}
B & \xrightarrow{\ \Phi\ } & C \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
M & \xrightarrow{\ \varphi\ } & N
\end{array}
\qquad (\pi \circ \Phi = \varphi \circ \pi)
\tag{18.1.3}
$$

Let us denote by $\mathfrak{Fib}\mathfrak{B}$ the category of fiber bundles with fibered morphisms.

A fibered morphism is called a *strong fibered morphism* if it projects onto a base diffeomorphism $\varphi : M \to N$. Let us denote by $\mathfrak{Fib}_M$ the category of fiber bundles with strong fibered morphisms.

Then a fiber bundle is a manifold B which is locally diffeomorphic to a *tube* $U \times F$. It is not necessarily a global product, exactly as a manifold is locally diffeomorphic to $\mathbb{R}^m$ though it is not necessarily globally diffeomorphic to $\mathbb{R}^m$.

As a manifold is obtained by glueing together patches of $\mathbb{R}^m$ along local diffeomorfisms, then a fiber bundle is obtained by gluing together tubes along local fibered morphisms.

From another viewpoint, the total space B is foliated by fibers $\pi^{-1}(x)$ which are all diffeomorphic to the standard fiber F. A fiber bundle is then a constant rank foliation of B (such that the quotient space is a manifold M).

A *section* of a bundle $\mathcal{B} = (B, M, \pi, F)$ is a map $\sigma : M \to B$ such that $\pi \circ \sigma = \mathrm{id}_M$. The set of all sections of $\mathcal{B}$ is denoted by $\Gamma(\pi)$. Obviously, a bundle can also allow no global sections so that $\Gamma(\pi)$ can be empty.

A *local section* is defined on $U \subset M$. It is a map $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{id}_U$. Our convention is that *section* refers to global sections, while local sections are explicitly said to be local. The set of all local sections defined in some neigborhood of $x \in M$ is denoted by $\Gamma_x(\pi)$.

As a manifold, the total space of a bundle can be covered with an atlas of *fibered coordinates* (subjected to a local trivialisation t) obtained by choosing a chart x^μ in the base around $x = \pi(b) \in M$ and a chart y^i around the point $t_x(b) \in F$. Then (x^μ, y^i) are good coordinates around $b \in B$. On bundles, we shall always use only fibered coordinates.

The most general transition maps between fibered coordinates are in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases}$$

(18.1.4)

The standard notation for Jacobians on bundles is setting

$$J_\nu^\mu = \frac{\partial x'^\mu}{\partial x^\nu}(x) \qquad J_\mu^i = \frac{\partial Y^i}{\partial x^\mu}(x, y) \qquad J_j^i = \frac{\partial Y^i}{\partial y^j}(x, y)$$

(18.1.5)

and the corresponding matrix with a bar on the top for the anti-Jacobians. Notice that the Jacobian on B is

$$\mathbb{J} = \begin{pmatrix} J_\nu^\mu & J_\mu^i \\ 0 & J_j^i \end{pmatrix}$$

(18.1.6)

and it must be non-degenerate, which implies that both J_ν^μ and J_j^i are non-degenerate.

In fibered coordinates, the projection reads as $\pi : B \to M : (x^\mu, y^i) \mapsto x^\mu$. It is clearly surjective and maximal rank since its Jacobian is $J(\pi) = (\mathbb{I}, 0)$. A section is locally expressed as a map $\sigma : M \to B : x^\mu \mapsto (x^\mu, \sigma^i(x))$. Two local expressions glue together iff

$$\sigma'^i(x'(x)) = Y^i(x, \sigma(x))$$

(18.1.7)

These are also called transformation rules of the section σ.

If σ denotes a collection of physical fields $y^i(x)$ and their value at points x of spacetime, then transformation rules constrain the readings of two observers which are observing the same situation.

We shall argue many times along this book that exactly transformation rules are what promote the *relative* knowledge of the observers to be *absolute* description of the physical reality. That is why we identify observers with the set of conventions which allow to describe the physical configuration as a set of numbers, transformation rules with changes of observer. In this case observers are identified with fibered coordinates (i.e. local trivialisations) which is what allows to describe a point in B as a set of numbers (x^μ, y^i).

An automorphism (Φ, φ) of B is also in the form

$$\begin{cases} x'^\mu = \varphi^\mu(x) \\ y'^i = \Phi^i(x, y) \end{cases} \tag{18.1.8}$$

Notice that automorphisms (18.1.8) and transition maps (18.1.4) are essentially in the same form, with different meanings of the symbols. In transition maps, different coordinates denote different names of the same point b in B in different local trivialisations.

In automorphisms, different coordinates denote different points in B, namely b and $\Phi(b)$. The first viewpoint is called *passive*, the second is called *active*. The situation is identical to what we found for manifolds in which changes of coordinates (passive) are essentially identical in form to diffeomorphisms (active).

In the end, the fact that active and passive transformations are expressed by the same formula is because global objects in the category (manifolds and fiber bundles) are obtained by glueing together trivial models (open sets in $\mathbb{R}^n$ and tubes $U \times F$) along local isomorphism of the category (diffeomorphism and automorphisms).

A vector field Ξ on B can be pushed forward along the projection map iff it is in the form

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i \tag{18.1.9}$$

and vector fields in that form are thence called *projectable*. The set of projectable vector fields is a Lie subalgebra denoted by $\mathfrak{X}(\pi) \subset \mathfrak{X}(B)$. The flow of a projectable vector field is made of fibered automorphisms.

The projection is not injective, so one cannot push-forward all vector fields. However, it can happen, for specific vector fields, that the tangent map $T\pi$ is constant on points which are projected on the same point $x \in M$. Those fields are π-projectable anyway. Here $T\pi(\Xi) = \xi^\mu(x)\partial_\mu$, which are constant along the fibers.

A tangent vector $(b, v) \in TB$ is said to be vertical iff $T_b\pi(v) = 0$, i.e. iff it is in the form (just write down in coordinates the definition $T\pi(v) = 0$)

$$v = v^i\partial_i \tag{18.1.10}$$

The set of all vertical vectors is denoted by $V(\pi) \subset TB$ and it is a sub-bundle.

A *vertical vector field* Ξ is in the form

$$\Xi = \Xi^i(x, y)\partial_i \tag{18.1.11}$$

and the set of vertical vector fields is a sub-algebra denoted by $\mathfrak{X}_V(\pi) \subset \mathfrak{X}(\pi)$.

Accordingly, a vertical vector field is a section of the bundle $(V(\pi), B, \tau, \mathbb{R}^k)$, where $\tau : V(\pi) \to B$ projects to the application point $b \in B$.

Since B is itself a bundle over M, one could also consider the bundle $(V(\pi), M, p = \pi \circ \tau, F \times \mathbb{R}^k)$. A section of this bundle is locally expressed as

$$\Xi : M \to V(\pi) : x^\mu \mapsto (x^\mu, y^i(x), \xi^i(x)) \tag{18.1.12}$$

Let us define a section $\sigma = \tau \circ \Xi : M \to B$; the section Ξ above associates a vertical vector at any point $\sigma(x)$. It is accordingly called a *vertical vector field on the section* σ. Notice it does not define any vector out of the image of the section σ; it is just defined over this subset while a vertical vector field is defined anywhere on B. Both vertical vector fields and vertical vector fields on a section are important in different contexts and one should not confuse them.

A covector $(b, \alpha) \in T^*B$ is called *horizontal* iff it vanishes on vertical vectors, i.e. it is in the form

$$\alpha = \alpha_\mu \, dx^\mu \tag{18.1.13}$$

A *horizontal* 1-*form* is in the form

$$\omega = \omega_\mu(x, y) \, dx^\mu \tag{18.1.14}$$

and the set of horizontal 1-forms is denoted by $\Omega_H^1(\pi) \subset \Omega^1(B)$.

The reader is urged to show that the local expression of projectable and vertical vector fields and horizontal 1-forms is preserved by transition maps (18.1.4). Let us stress that we did not define horizontal vector fields and vertical 1-forms, since their local form would not be preserved by transformations (18.1.4).

Vector bundles

Within fibered bundles, we can define special classes of fibered bundles with extra structures and extra properties. For example, let us consider fiber bundles with the standard fiber which is a vector space $F \equiv V$. Moreover, there is natural Lie group $\mathrm{GL}(V) \subset \mathrm{Diff}(F)$ acting naturally on V. It is natural to require that a vector bundle has a $\mathrm{GL}(V)$-structure.

Definition (18.1.15): *a vector bundle is a bundle $\mathcal{E} = (E, M, \pi, V)$ with a vector space V as the standard fiber for which there exists a trivialisation for which transition maps act linearly on V.*

That means that transition maps on vector bundles read as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = A_j^i(x) y^j \end{cases} \tag{18.1.16}$$

Compare back with (14.4.16). Think about it.
Think again about it.
Then read Section 18.2 below.

Also fibered morphism can be restricted to be linear. A linear fibered morphism is a fibered morphism which acts linearly on each fiber, i.e., in fibered coordinates, it reads as

$$\begin{cases} x'^\mu = \varphi^\mu(x) \\ y'^i = A_j^i(x) y^j \end{cases} \tag{18.1.17}$$

The category of vector bundles with linear morphisms is denoted by $\mathfrak{VecB}$.

Vector bundles always allow global sections. If we define local zero sections $\sigma : x \mapsto (x, \vec{0})$ then they glue together to define a global zero section since the zero vector $\vec{0} \in V$ is invariant under linear transformations.

> Check that (18.1.7) holds true for the zero section.

If F is a smooth function on M which is supported $\mathrm{supp}(F) \subset U$ in a trivialisation domain then, for a local section $\sigma : x \mapsto (x, v)$, one has a global section $F \cdot \sigma$ which is zero out $\mathrm{supp}(F)$ and it is different from zero in it. This as called *compactly supported sections* and any vector bundle allows infinitely many global compactly supported sections.

Let us stress that on a vector bundle (E, M, π, V) one has infinitely many global sections, though the zero section is a canonical element of $\Gamma(\pi)$. The sections of a vector bundle form a vector space and the zero section is the neutral element in such a vector space. This is pretty obvious but remember it when comparing to affine bundles in which one also has infinitely many global sections, though no one of them is canonically singled out as a preferred section.

Affine bundles

Definition (18.1.18): *an affine bundle is a bundle $\mathcal{A} = (E, M, \pi, \mathbb{A})$ with an affine space $\mathbb{A}$ as the standard fiber for which there exists a trivialisation for which transition maps act as affine maps on $\mathbb{A}$.*

That means that transition maps on affine bundles read as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = A^i_j(x) y^j + B^i(x) \end{cases} \tag{18.1.19}$$

Also fibered morphism can be restricted to be affine. An affine fibered morphism is a fibered morphism which acts as an affine map on each fiber, i.e. in fibered coordinates reads as

$$\begin{cases} x'^\mu = \varphi^\mu(x) \\ y'^i = A^i_j(x) y^j + B^i(x) \end{cases} \tag{18.1.20}$$

The category of affine bundles with affine morphisms is denoted by $\mathfrak{AffB}$.

Also affine bundles always allow global section. Let (U_i, α_i) be a partition of unity and $\sigma_i : U_i \to \pi^{-1}(U)$ be local sections. Then we can consider an affine combination

$$\sigma : x \mapsto \sum_i \alpha_i(x) \sigma_i(x) \tag{18.1.21}$$

At each point x the affine combination is a finite sum. Moreover, the weight α_i of the local section σ_i goes to zero before the local section has singularities. Accordingly, one can prove that σ is a global smooth section.

Exactly, as one can regard a vector space V as an affine space $\mathbb{A}$ where an origin $O \in \mathbb{A}$ has been selected, one can regard a vector bundle as an affine bundle with a specific section singled out to take the role of the zero section. Also in this case, one says that the affine bundle *is modelled* on the vector bundle.

Notice that connections on a manifold transform affinely as in (16.2.8). That means that they can be seen as sections of a suitable affine bundle. This immediately says that global connections always exist on any manifold. It also says that any connection can be seen as the sum of a selected preferred connection and a T_2^1 tensor field, meaning that the bundle of connections is modelled on the vector bundle $T_2^1(M)$.

Principal bundles

Definition (18.1.22): *a principal bundle is a bundle $\mathcal{P} = (P, M, \pi, G)$ with a Lie group G as the standard fiber for which there exists a trivialisation for which transition maps factorise through the left translation of the group on itself, i.e. they are in the form*

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g' = \phi(x) \cdot g \end{cases} \tag{18.1.23}$$

If a point $p \in P$ is mapped into $t_\alpha(p) = (x, g)$ by the local trivialisation t_α, then let us introduce the notation $p = [x, g]_\alpha$ where we set $[x, g]_\alpha = t_\alpha^{-1}(x, g)$. Of course, the same point in another local trivialisation t_β is mapped unto another pair $p = [x, g']_\beta$ and, in view of transition maps in the form (18.1.23), there exists a function $\phi : U_{\alpha\beta} \to G$ such that $g' = \phi(x) \cdot g$.

We can define global *canonical right action* of G on P by $R_g : P \to P : [x, k]_\alpha \mapsto [x, k \cdot g]_\alpha$.

The result does not depend on the local trivialisation, in fact

$$R_g[x, k']_\beta = R_g[x, \bar{\phi}(x) \cdot k']_\alpha = [x, \bar{\phi}(x) \cdot k' \cdot g]_\alpha = [x, k' \cdot g]_\beta \tag{18.1.24}$$

For similar reasons, one cannot define a global left action $L_g : P \to P : [x, k]_\alpha \mapsto [x, g \cdot k]_\alpha$ which would, in fact, depend on the local trivialisation.

The canonical right action is a characteristic structure on principal bundles.

It is *vertical*, i.e. $\pi(R_g p) = \pi(p)$ for all $g \in G$.

It is *transitive on the fibers*, i.e. if $p, p' \in \pi^{-1}(x)$, then $p = [x, g]_\alpha$ and $p' = [x, g']_\alpha$ and $R_{g^{-1}g'} p = p'$.

It is *free*, i.e. if there exists a $p \in P$ such that $R_g p = p$,then one necessarily has $g = e$.

If we consider two principal bundles $\mathcal{P} = (P, M, \pi, G)$ and $\mathcal{P}' = (P', M', \pi, G)$ with the same group G, then a *principal morphism* (Φ, φ) is a fibered morphism which preserves the right action, i.e.

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P' \\ \Big\downarrow{R_g} & & \Big\downarrow{R_g} \qquad (R_g \circ \Phi = \Phi \circ R_g) \\ P & \xrightarrow{\Phi} & P' \end{array} \tag{18.1.25}$$

The map Φ is also called an *equivariant map*. The category of principal bundles with principal morphisms is denoted by $\mathfrak{P}(\mathfrak{M}, \mathfrak{G})$

If we consider two principal bundles $\mathcal{P} = (P, M, \pi, G)$ and $\mathcal{P}' = (P', M', \pi, G')$ with two different groups G and G', then a *weakly equivariant morphism* subjected to a group homomorphism $i : G \to G'$ is a fibered morphism (Φ, φ) which preserves the right action modulo i, i.e.

$$
\begin{array}{ccc}
P & \xrightarrow{\ \ \Phi\ \ } & P' \\
{\scriptstyle R_g}\Big\downarrow & & \Big\downarrow{\scriptstyle R_{i(g)}} \qquad (R_{i(g)} \circ \Phi = \Phi \circ R_g) \\
P & \xrightarrow{\ \ \Phi\ \ } & P'
\end{array}
\tag{18.1.26}
$$

As a consequence of the fact that R_g is free, it means that for $p, p' \in \pi^{-1}(x)$ not only there exists a $g \in G$ such that $R_g p = p'$ but that g is unique. In fact, if there were two such g and g' then $R_{g'g^{-1}}p = p$ and, for free property, $g = g'$.

If we fix a local section $\sigma : U \to \pi^{-1}(U)$, then $\forall p \in \pi^{-1}(U)$ we can fix $x = \pi(p)$ and both $\sigma(x), p \in \pi^{-1}(x)$. Then there exists a unique $g \in G$ such that $p = R_g \sigma(x)$. We can then define a local trivialisation on U

$$
t_\sigma : \pi^{-1}(U) \to U \times G : p \mapsto (x, g)
\tag{18.1.27}
$$

In that local trivialisation, the original section reads as $\sigma : U \to \pi^{-1}(U) : x \mapsto [x, e]_\sigma$.

On principal bundles, one has a one-to-one correspondence between local sections and local trivialisations. Then non-trivial principal bundles (such as the Hopf bundle $\pi : S_3 \to S_2$, which is a principal bundle with a structure group $G = U(1)$) do not allow global sections.

Associated bundles

One can build (vector, affine, ...) fiber bundles starting from a principal bundle $\mathcal{P} = (P, M, \pi, G)$. The associated bundles come with structures which make this construction paradigmatic for us. Virtually any bundle we shall use in the physical applications will be associated to some principal bundle.

Associated bundles come with a G-structure, their sections have definite transformation rules, they come with a group of transformations (gauge transformations) singled out, and when a connection is given on the principal bundle (gauge field) then a connection is induced on associated bundles and one can define the covariant derivatives of sections in associated bundles. Since in most cases we know transformation rules of physical fields and usually they come with a group of transformations which preserves the dynamics, associated bundles provide a natural framework for gauge theories and relativistic theories.

Let us start with a principal bundle $\mathcal{P} = (P, M, \pi, G)$, a manifold F and a left action $\lambda : G \times F \to F : (g, f) \mapsto g \cdot f$.

First step is to extend the left action on the manifold $P \times F$ by

$$
\hat{\lambda} : G \times (P \times F) \to (P \times F) : (g, p, f) \mapsto (p \cdot g^{-1}, g \cdot f)
\tag{18.1.28}
$$

using the canonical right action on P.

Then one can quotient $P \times F$ with respect to the action $\hat{\lambda}$. The quotient space will be denoted by $P \times_\lambda F$ and a point will be denoted by

$$
[p, f]_\lambda = \{(p \cdot g^{-1}, g \cdot f) : g \in G\} \in P \times_\lambda F
\tag{18.1.29}
$$

We can define a projection map $p : P \times_\lambda F \to M : [p, f]_\lambda \mapsto \pi(p)$ and we can show that $(P \times_\lambda F, M, p, F)$ is a bundle.

That $P \times_\lambda F$ is a manifold and that p is maximal rank will be clear when we shall show that it can be covered with fibered coordinates.

Let us first define a local trivialisation $\hat{t}_\alpha : p^{-1}(U) \to U \times F$ of $P \times_\lambda F$ out of a local trivialisation $t_\alpha : \pi^{-1}(U) \to U \times G : p \mapsto (x, g)$ of $\mathcal{P}$. For any point $[p, f]_\lambda \in p^{-1}(U) \subset P \times_\lambda F$, one has a canonical representative $(p \cdot g^{-1}, g \cdot f)$ such that $t_\alpha(p \cdot g^{-1}) = (x, e)$. To be precise, by using transitivity on the fibers and freedom, for any p there exists a unique $g \in G$ such that $t_\alpha(p \cdot g^{-1}) = (x, e)$. Then we can define

$$\hat{t}_\alpha : p^{-1}(U) \to U \times F : [p, f]_\lambda \mapsto (x, g \cdot f) \tag{18.1.30}$$

Then the trivialisation of $\mathcal{P}$ induces a trivialisation of $P \times_\lambda F$.

By choosing coordinates x^μ around x and coordinates f^i around $g \cdot f$, one can define fibered coordinates on $P \times_\lambda F$. Since $P \times_\lambda F$ can be covered with an atlas of fibered coordinates, it is a manifold. Since the projection $p : P \times_\lambda F \to M$ in fibered coordinates reads as $p(x^\mu, f^i) = x^\mu$, it is maximal rank.

We have to compute transition maps between different fibered coordinates. Let us start by transition maps (18.1.23) on $\mathcal{P}$ between two systems of fibered coordinates (x^μ, g) and (x^μ, g'). The same point $[p, f]_\lambda \in P \times_\lambda F$ has two canonical representatives with respect to the two local trivialisations on $\mathcal{P}$, namely

$$t_\alpha(p) = (x, g) = R_g(x, e) \qquad\qquad t_\beta(p) = (x, g') = R_{g'}(x, e) \tag{18.1.31}$$

so that one defines two different local trivialisations on $P \times_\lambda F$

$$\hat{t}_\alpha([p, f]_\lambda) = (x, g \cdot f) \qquad\qquad \hat{t}_\beta([p, f]_\lambda) = (x, g' \cdot f) \tag{18.1.32}$$

Then the same point $[p, f]_\lambda \in P \times_\lambda F$ is associated to different coordinates $(x^\mu, g \cdot f)$ and $(x'^\mu, g' \cdot f) = (x'^\mu, (\phi(x) \cdot g) \cdot f)$ and transition functions on $\mathcal{P}$ reads as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ f' = \phi(x) \cdot f \end{cases} \tag{18.1.33}$$

Then, transition maps on the associated bundle are the same transition maps of the principal bundle, just represented on F by means of the action λ.

In other words, the associated bundle comes with a trivialisation in which transition maps take values in G which is embedded by the action as $\lambda : G \to \mathrm{Diff}(F)$. Then the associated bundle has a G-structure.

Then we can show that a principal automorphism on $\mathcal{P}$ induces an automorphism of $P \times_\lambda F$. Let us in fact define

$$\Phi_\lambda : P \times_\lambda F \to P \times_\lambda F : [p, f]_\lambda \mapsto [\Phi(p), f]_\lambda \tag{18.1.34}$$

We just have to check that the definition is well-posed, i.e. it does not depend on the representative. In fact

$$\Phi_\lambda([p \cdot g^{-1}, g \cdot f]_\lambda) = [\Phi(p \cdot g^{-1}), g \cdot f]_\lambda = [\Phi(p) \cdot g^{-1}, g \cdot f]_\lambda = [\Phi(p), f]_\lambda = \Phi_\lambda([p, f]_\lambda) \tag{18.1.35}$$

In coordinates, the map Φ_λ reads as

$$\begin{cases} x'^\mu = \varphi^\mu(x) \\ f' = \phi(x) \cdot f \end{cases} \tag{18.1.36}$$

which, once again, is the active form of transition maps (18.1.33).

We just defined an action $(\cdot)_\lambda : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(P \times_\lambda F) : \Phi \mapsto \Phi_\lambda$. That defines a special group of transformations $\mathrm{Aut}(\mathcal{P}) \subset \mathrm{Aut}(P \times_\lambda F)$ which will be identified with gauge transformations on fields.

Since the map $(\cdot)_\lambda$ is a group homomorphism, i.e. it preserves compositions, we, in fact, defined a covariant functor $(\cdot) \times_\lambda F : \mathfrak{P(M,G)} \to \mathfrak{FibB}$ which associates the associated bundle $\mathcal{P} \times_\lambda F$ to any principal bundle $\mathcal{P}$ and a fibered morphism $\Phi_\lambda : P \times_\lambda F \to P' \times_\lambda F$ to any equivariant morphism $\Phi : \mathcal{P} \to \mathcal{P}'$.

In the next Section, we shall show how one can define the appropriate associated bundle so to obtain the desired transformation rules.

2. Examples of fiber bundles

We already defined all one needs to define a couple of bundles, the tangent and cotangent bundles. We shall also define the frame bundle $L(M)$, which is a principal bundle with the group $\mathrm{GL}(m)$, and recover the tangent and cotangent bundle as associated bundles.

Tangent bundles

We already defined TM as a manifold and a projection map $\pi : TM \to M$. We have to define the tangent bundle $(TM, M, \pi, \mathbb{R}^m)$.

Basically, all we need is to define local trivialisations.

For any chart x^μ on M, one has a natural basis ∂_μ of the tangent spaces and any vector can be expanded as $v = v^\mu \partial_\mu$. Let us define the local trivialisations

$$t_\alpha : \pi^{-1}(U) \to U \times \mathbb{R}^m : (x, v) \mapsto (x, v^\mu) \tag{18.2.1}$$

We already computed the transition maps (see (14.4.16)). Notice that the base coordinates transform only with base coordinates and that coordinates along the fibers v^μ transform linearly among them with a linear transformation depending on x. Transformation rules (14.4.16) can be cast in the form (18.1.16) and TM is not only a fiber bundle but a vector bundle.

A section of TM is a map $X : M \to TM : x \mapsto (x, v)$ which is a vector field on M.

The tangent map $T\Phi : TM \to TM$ is defined by (15.4.20) associated to a map $\Phi : M \to M : x^\mu \mapsto \varphi^\mu(x)$. In fibered coordinates, it reads as

$$T\Phi : TM \to TM : (x^\mu, v^\mu) \mapsto (\varphi^\mu(x), J^\mu_\nu v^\nu) \tag{18.2.2}$$

Again, notice that the base coordinates transform only with base coordinates and that coordinates along the fibers v^μ transform linearly among them with a linear transformation depending on x. That means that $T\Phi$ is a fibered morphism of TM. In other words ,we defined a covariant functor $T(\cdot)$ from the category of manifolds $\mathfrak{Man}$ to the category of vector bundles $\mathfrak{VecB}$ which associates a vector bundle TM to any manifold M and a fiber morphism $T\Phi$ to any smooth map Φ. We already proved that tangent maps preserve composition; see (15.4.23).

Cotangent bundles

We already defined T^*M as a manifold and a projection map $\pi : T^*M \to M$. We have to define the cotangent bundle $(T^*M, M, \pi, \mathbb{R}^m)$.

Basically, all we need is to define local trivialisations.

For any chart x^μ on M, one has a natural basis dx^μ of the cotangent spaces and any covector can be expanded as $\alpha = \alpha_\mu \, dx^\mu$. Let us define the local trivialisations

$$t_\alpha : \pi^{-1}(U) \to U \times \mathbb{R}^m : (x, v) \mapsto (x, \alpha_\mu) \tag{18.2.3}$$

We already computed the transition maps (see (15.1.40)). Notice that the base coordinates transform only with base coordinates and that coordinates along the fibers α_μ transform linearly among them with a linear transformation depending on x. Transformation rules (15.1.40) can be cast in the form (18.1.16) and T^*M is not only a fiber bundle but a vector bundle.

A section of T^*M is a map $\omega : M \to T^*M : x \mapsto (x, \alpha)$ which is a 1-form on M.

The cotangent map $T^*\Phi : T^*N \to T^*M$ is defined by (15.1.40) to be associated to a map $\Phi : M \to N : x^\mu \mapsto \varphi^\mu(x)$. In fibered coordinates, it reads as

$$T^*\Phi : T^*N \to T^*M : (x^\mu, \alpha_\mu) \mapsto (\bar{\varphi}^\mu(x), J_\mu^\nu \alpha_\nu) \tag{18.2.4}$$

Again notice that base coordinates transform only with base coordinates and that coordinates along the fibers α_μ transform linearly among them with a linear transformation depending on x. That means that $T^*\Phi$ is a fibered morphism of T^*M. In other words, we defined a controvariant functor $T^*(\cdot)$ from the category of manifolds $\mathfrak{Man}$ to the category of vector bundles $\mathfrak{VecB}$ which associates a vector bundle T^*M to any manifold M and a fiber morphism $T^*\Phi$ to any smooth map Φ. We already proved that cotangent maps reverse composition; see (15.4.30).

Frame bundles

Let us define

$$LM = \{(x, e_a) : \text{ with } e_a \text{ a basis of } T_xM\} \tag{18.2.5}$$

as the set of basis of tangent spaces. We can define $\pi : LM \to M : (x, e_a) \mapsto x$ as the projection and prove that $(LM, M, \pi, \mathrm{GL}(m))$ is a principal bundle called the *frame bundle* of M.

A chart x^μ on M induces a natural basis ∂_μ of T_xM. For any other basis e_a, there exists a transition matrix $e_a = e_a^\mu \partial_\mu$ which by construction belongs to $\mathrm{GL}(m)$. Then we can define local trivialisations

$$t_\alpha : \pi^{-1}(U) \to U \times \mathrm{GL}(m) : (x, e_a) \mapsto (x, e_a^\mu) \tag{18.2.6}$$

One can always cover LM by these local trivialisations and $L(M)$ is a fiber bundle. Transition functions are

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e_a'^\mu = J_\nu^\mu(x) e_a^\nu \end{cases} \tag{18.2.7}$$

which act exactly by left product by $J_\nu^\mu(x)$ of the coordinates e_a^ν along the fibers. Then $L(M)$ is a principal bundle.

:Notation: :Symbols: :AIndex: :Index:

The canonical right action reads as

$$R_\alpha : P \to P : (x, e_a) \mapsto (x, e_b \alpha_a^b) \tag{18.2.8}$$

An equivariant map $\Phi : LM \to LM$ reads as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e_a'^\mu = \phi_\nu^\mu(x) e_a^\nu \end{cases} \tag{18.2.9}$$

A general fibered map is given by

$$e_a'^\mu = E_a^\mu(x, e) \tag{18.2.10}$$

and for being equivariant one must have

$$E_b^\mu(x, e)\alpha_a^b = E_a^\mu(x, e \cdot \alpha) \tag{18.2.11}$$

That equation tells that once one knows the value of the function E_a^μ at a point in the fiber, for example at $E_a^\mu(x, \mathbb{I})$, then its value is known all along the fiber. The value $E_a^\mu(x, \mathbb{I})$ just depends on x, call it $\phi(x) \in \mathrm{GL}(m)$. Then the value on a generic point is

$$e_a'^\mu = E_a^\mu(x, e_a^\mu) = \phi_\nu^\mu(x) e_a^\nu \tag{18.2.12}$$

Also on $L(M)$, one can define an automorphism $L(\varphi) : LM \to LN$ associated to a diffeomorphism $\varphi : M \to N$ by

$$\begin{cases} y^i = \varphi^i(x) \\ e_a^i = J_\nu^i(x) e_a^\nu \end{cases} \tag{18.2.13}$$

which is called the *natural lift* to the frame bundle $L(M)$ of the diffeomorphism $\varphi : M \to N$. One can easily show that $L(\mathrm{id}_M) = \mathrm{id}_{LM}$ and that $L(\varphi \circ \psi) = L(\varphi) \circ L(\psi)$. Thus we defined a covariant functor $L(\cdot)$ from the category of manifolds $\mathfrak{Man}$ to the category of principal bundles $\mathfrak{P}(\mathfrak{M}, \mathfrak{G})$ which defines frame bundles $L(M)$ for any manifold and a principal morphism $L(\varphi)$ for any smooth map.

Here is where our framework actually cracks a bit under imprecise definitions.
If we do not restrict to diffeomorphisms $\varphi : M \to N$, then $L(\varphi)(e_a)$ is not a basis of $T_x N$. However, on one hand, there is a more precise definition of natural functor, which maps foliations, not smooth maps. On the other hand, in a relativistic theories what one actually needs to lift are *transformations*, i.e. categorical isomorphisms. Forgive me the relative departure from standard mathematical notation about this.

Let us consider $F = \mathbb{R}^m$ and the following action

$$\lambda : \mathrm{GL}(m) \times \mathbb{R}^m \to \mathbb{R}^m : (J_b^a, v^a) \mapsto J_b^a v^b \tag{18.2.14}$$

We already shown that one can define an associated bundle $L(M) \times_\lambda \mathbb{R}^m$. A point in the associated bundle is $[x, e_a, v^a]_\lambda$ and its canonical representative is $(x, \mathbb{I}, e_a^\mu v^a =: v^\mu)$. Then one has coordinates on the associated bundle explicitly given by (x^μ, v^μ) which transform as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^\mu = e_a'^\mu v^a = J_\nu^\mu e_a^\nu v^a = J_\nu^\mu v^\nu \end{cases} \tag{18.2.15}$$

which are recognized to be transformation rules for tangent vectors. That means that we can define a canonical isomorphism

$$i : L(M) \times_\lambda \mathbb{R}^m \to TM : [x, e_a, v^a]_\lambda \mapsto e_a v^a \tag{18.2.16}$$

One must only check that the map is well-posed, i.e. it does not depend on the representative.

$$i([x, e_b \bar{J}_a^b, J_c^a v^c]_\lambda) = e_b \bar{J}_a^b J_c^a v^c = e_b v^b = i([x, e_a, v^a]_l) \tag{18.2.17}$$

Accordingly, the tangent bundle TM can be defined as an associated bundle to $L(M)$. As such, the tangent bundle comes with a $\mathrm{GL}(m)$-structure. Similarly, one can define $F = \mathbb{R}^m$ and the following action

$$\lambda^* : \mathrm{GL}(m) \times \mathbb{R}^m \to \mathbb{R}^m : (J_b^a, \alpha_a) \mapsto \alpha_b \bar{J}_a^b \tag{18.2.18}$$

which is in fact a left action. Then we can define the associated bundle $L(M) \times_{\lambda^*} \mathbb{R}^m$ and show that it is canonically isomorphic to the cotangent bundle by the map

$$i^* : L(M) \times_{\lambda^*} \mathbb{R}^m \to T^*M : [x, e_a, \alpha_a]_\lambda \mapsto e^a \alpha_a \tag{18.2.19}$$

where e^a is the dual basis of e_a.

In both cases, for the tangent and cotangent bundle, we just wanted to have a bundle for objects with a known transformation rule, used the transformation rule to define the action and defined the associated bundle which made the trick.

Let us consider vector densities of weight 1 which transform as

$$v'^\mu = \det(\bar{J}) J_\nu^\mu v^\nu \tag{18.2.20}$$

We can use the transformation rules to define an action

$$\lambda : \mathrm{GL}(m) \times \mathbb{R}^m \to \mathbb{R}^m : (J_b^a, v^a) \mapsto \det(\bar{J}) J_b^a v^b \tag{18.2.21}$$

and define the associated bundle $\mathcal{D} = L(M) \times_\lambda \mathbb{R}^m$. A point $[x, e_a, v^a]_\lambda$ in $\mathcal{D}$ has a canonical representative in the form $[x, \partial_\mu, e_a^\mu v^a =: v^\mu]_\lambda$ and one has fibered coordinates (x^μ, v^μ) on $\mathcal{D}$ which transforms as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^\mu = \det(\bar{J}) J_\nu^\mu v^\nu \end{cases} \tag{18.2.22}$$

A section of $\mathcal{D}$ is a map $\sigma : x^\mu \mapsto (x^\mu, v^\mu = \sigma^\mu(x))$. Two local expressions of a section glue together iff on the overlap one has

$$\sigma'^\mu(x') = \det(\bar{J}) J_\nu^\mu(x) \sigma^\nu(x) \tag{18.2.23}$$

which is the same condition for which two local expressions for a vector density to glue together.

Then we have a one-to-one correspondence between global vector densities on M and sections of the bundle $\mathcal{D}$ which is then called the *bundle of vector densities*.

This procedure is completely standard whenever one has objects which transform under the group $\mathrm{GL}(m)$. As a matter of fact, knowing the components of the objects dictates the standard fiber F and knowing the transformation rules dictates the action to define an associated bundle to $L(M)$. Then one has for free a one-to-one correspondence between global objects and sections in the associated bundle.

This common strategy needs to be improved in some cases, for example for connections on a manifold M, for which transformation rules cannot be seen as an action of the group $\mathrm{GL}(m)$. In that case, one needs to generalise the group to a bigger group $\mathrm{GL}^2(m)$ which, beside Jacobians, takes into account of the Hessians as well. The issue is that actions are *algebraic* and cannot account for higher order derivatives. In order to do the trick in general, we need jet prolongations which precisely transform differential objects into algebraic ones.

If transformation rules involve groups other than $\mathrm{GL}(m)$, one needs principal bundles other than $L(M)$ as it is necessary in gauge theories.

However, associated bundles to $L(M)$ account for all tensor fields and tensor densities we defined until now. From now on, these objects can be always seen as sections of the suitable bundles.

3. Operations with bundles

Bundles can be pasted together to obtain other bundles. There is a number of basic operations which are useful to define these bundles. We can distinguish among operations which are defined on any bundle and operations which are defined on vector bundles only.

Pull-back bundle

Let $\mathcal{B} = (B, M, \pi, F)$ be a bundle and $\varphi : N \to M$ a smooth map. We can define a bundle $\pi^* B = (\tilde{B}, N, \tilde{\pi}, F)$ with the same standard fibers F but on a different base N.

Let us define $\tilde{B} = \{(b, y) : b \in B, y \in N : \varphi(y) = \pi(b)\}$ and the new projection $\tilde{\pi} : \tilde{B} \to N : (b, y) \mapsto y$. We shall now prove that the *pull-back bundle* $\pi^* B$ is a bundle by defining a trivialisation for it.

Let (U_α, t_α) be a trivialisation of B and $V_\alpha = \varphi^{-1}(U_\alpha)$. Since φ is continuous, then V_α is an open covering of N. Let us define the local trivialisations

$$\tilde{t}_\alpha : \tilde{\pi}^{-1}(V_\alpha) \to V_\alpha \times F : (b, y) \mapsto (y, f) \qquad t_\alpha(b) = (x, f) \tag{18.3.1}$$

Then one has fibered coordinates (y^i, f^A) on $\pi^* B$ which transform as

$$\begin{cases} y'^i = y'^i(y) \\ f'^A = Y^A(y, f) \end{cases} \tag{18.3.2}$$

The transition maps $Y^A(y, f)$ are obtained from transition functions $Y^A(x, f)$ of the original bundle as

$$f'^A = Y^A(y, f) = Y^A(x(y), f) \tag{18.3.3}$$

We can define a fibered morphism $i : \tilde{B} \to B : (b, y) \mapsto b$ so that

$$
\begin{array}{ccc}
\tilde{B} & \xrightarrow{\ i\ } & B \\
\Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\
N & \xrightarrow{\ \varphi\ } & M
\end{array}
\tag{18.3.4}
$$

For example, if one has a surface $i : S \to \mathbb{R}^3$ and one needs to describe vectors of $\mathbb{R}^3$ (not necessarily tangent to S) which are applied to points on the surface S, one can define the bundle $i^* T\mathbb{R}^3$.

Fibered product

Given two bundles $\mathcal{B}_1 = (B_1, M, \pi_1, F_1)$ and $\mathcal{B}_2 = (B_2, M, \pi_2, F_2)$ over the same manifold M, we can define a bundle $\mathcal{B}_1 \times_M \mathcal{B}_2 = (B_1 \times_M B_2, M, \pi_{1\times 2}, F_1 \times F_2)$ with the same base M but the direct product of standard fibers $F_1 \times F_2$. Fields on the fibered product account for the union of fields on B_1 and fields on B_2. Accordingly, fibered product is used to paste together in the same space different objects.

Let us define the set

$$B_1 \times_M B_2 = \{(b_1, b_2) : b_1 \in B_1, b_2 \in B_2 : \pi_1(b_1) = \pi_2(b_2)\} \tag{18.3.5}$$

and the new projection $\pi_{1\times 2} : B_1 \times_M B_2 \to M : (b_1, b_2) \mapsto \pi_1(b_1) = \pi_2(b_2)$.

We shall now prove that the *fibered product* $\mathcal{B}_1 \times_M \mathcal{B}_2$ is a bundle by defining a trivialisation for it.

Let (U_α, t_α^1) be a trivialisation of B_1 and (U_α, t_α^2) a trivialisation for B_2, both defined on the same open sets.

Let us define the local trivialisations

$$t_\alpha^{12} : \tilde{\pi}^{-1}(U_\alpha) \to V_\alpha \times F : (b_1, b_2) \mapsto (x, f_1, f_2) \qquad t_\alpha^1(b_1) = (x, f_1) \quad t_\alpha^2(b_2) = (x, f_2) \tag{18.3.6}$$

Then one has fibered coordinates (x^μ, f_1^A, f_2^A) on $\mathcal{B}_1 \times_M \mathcal{B}_2$. If the old transition maps are

$$
\begin{cases} x'^\mu = x'^\mu(x) \\ f_1'^A = Y_1^A(x, f_1) \end{cases}
\qquad
\begin{cases} x'^\mu = x'^\mu(x) \\ f_2'^A = Y_2^A(x, f_2) \end{cases}
\tag{18.3.7}
$$

then the transition maps of the fibered product are

$$
\begin{cases} x'^\mu = x'^\mu(x) \\ f_1'^A = Y_1^A(x, f_1) \\ f_2'^A = Y_2^A(x, f_2) \end{cases}
\tag{18.3.8}
$$

We can define two fibered morphisms $p_i : B_1 \times_M B_2 \to B_i : (b_1, b_2) \mapsto b_i$, with i=1,2, so that

$$
\begin{array}{ccc}
B_1 \times_M B_2 & \xrightarrow{\ p_1\ } & B_1 \\
{\scriptstyle \pi_{1\times2}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
M & \longrightarrow & M
\end{array}
\qquad
\begin{array}{ccc}
B_1 \times_M B_2 & \xrightarrow{\ p_2\ } & B_2 \\
{\scriptstyle \pi_{1\times2}}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
M & \longrightarrow & M
\end{array}
\tag{18.3.9}
$$

These morphisms satisfy the so-called *universal property* of the product: for any bundle (C, M, p, F) on M and any family of morphisms $f_i : C \to B_i$ then there exists a unique bundle morphism $f_1 \times_M f_2 : C \to B_1 \times_M B_2$ such that

$$
\begin{array}{ccc}
B_1 \times_M B_2 & \xrightarrow{\ \ p_i\ \ } & B_i \\
& \nwarrow & \uparrow{\scriptstyle f_i} \\
& f_1 \times_M f_2 & \\
& & C
\end{array}
\qquad p_i \circ (f_1 \times_M f_2) = f_i
\tag{18.3.10}
$$

In fact, if we denote by $f_i : C \to B_i : v \mapsto f_i(v) = v_i$ the two bundle morphisms, it is easy to check that the morphism $f_1 \times_M f_2 : C \to B_1 \times_M B_2 : v \mapsto (f_1(v), f_2(v))$ has the required property.

Moreover, let us that suppose there is another bundle morphism $\tilde{f} : C \to B_1 \times_M B_2$ with the same property, then it is easy to check that

$$
\tilde{f}(v) = p_1 \circ \tilde{f}(v) + p_2 \circ \tilde{f}(v) = f_1(v) + f_2(v) = p_1 \circ (f_1 \times_M f_2)(v) + p_2 \circ (f_1 \times_M f_2)(v) = (p_1 + p_2) \circ (f_1 \times_M f_2)(v) = (f_1 \times_M f_2)(v)
\tag{18.3.11}
$$

so that $\tilde{f} = f_1 \times_M f_2$ and that map is unique.

As it is often the case with universal properties, we can show that given a bundle S and two maps $q_1 : S \to B_1$, $q_2 : S \to B_2$ satisfying the universal properties, then $S \simeq E_1 \times_M E_2$ and the direct sum is essentially unique.

Suppose that both (S, q_i) and $(B_1 \times_M B_2, p_i)$ obey the universal property, in particular

$$
\begin{array}{ccc}
B_1 \times_M B_2 & \xrightarrow{\ \ p_i\ \ } & B_i \\
& & \uparrow{\scriptstyle q_i} \\
& & S
\end{array}
\tag{18.3.12}
$$

Then for the universal property of $B_1 \times_M B_2$ there exists a map $\phi : S \to B_1 \times_M B_2$ such that $p_i \circ \phi = q_i$, while for the universal property of S there exists a map $\bar{\phi} : S \to B_1 \times_M B_2$ such that $q_i \circ \bar{\phi} = p_i$. Thus one has

$$
p_i \circ \phi \circ \bar{\phi} = q_i \circ \bar{\phi} = p_i
\qquad\qquad
q_i \circ \bar{\phi} \circ \phi = p_i \circ \phi = q_i
\tag{18.3.13}
$$

Now we can regard this as the universal property applied for $C = S$ and $f_i = \mathrm{id}$ and for $C = B_1 \times_M B_2$ and $f_i = \mathrm{id}$, i.e.

$$
\begin{array}{ccc}
S \xrightarrow{\ \ q_i\ \ } B_i & \qquad & B_1 \times_M B_2 \xrightarrow{\ \ p_i\ \ } B_i \\
\end{array}
$$

$$ (18.3.14) $$

However, there is one map (the identity map) which has these properties, and for the unique part of the universal property, this map is unique; thus $\bar{\phi} \circ \phi = \mathrm{id}$ and $\phi \circ \bar{\phi} = \mathrm{id}$ which shows that $S \simeq B_1 \times_M B_2$.

Whitney (or direct) sum

Let us consider two vector bundles $\mathcal{E}_1 = (E_1, M, \pi_1, \mathbb{K}^{n_1})$ and $\mathcal{E}_2 = (E_2, M, \pi_2, \mathbb{K}^{n_2})$ over the same base manifold M. Then we can define a new vector bundle on M, denoted by $\mathcal{E}_1 \oplus \mathcal{E}_2 = (E_1 \oplus E_2, M, \pi_\oplus, \mathbb{K}^{n_1+n_2})$, called the *direct sum* (or the *Whitney sum*) of the original vector bundles. The total space $E_1 \oplus E_2$ and the projection $\pi_\oplus : E_1 \oplus E_2 \to M$ are defined as

$$ E_{1+2} = \{(v_1, v_2) \in E_1 \times E_2, \pi_1(v_1) = \pi_2(v_2)\} \qquad \pi_\oplus(v_1, v_2) = \pi_1(v_1) = \pi_2(v_2) \tag{18.3.15} $$

To show that it is a bundle, we construct a local trivialisation $t_\alpha^\oplus : \pi_\oplus^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{n_1+n_2}$ as

$$ t_\alpha^\oplus(v_1, v_2) = \left(x, (v_1^i, v_2^a)\right) \qquad \Longleftrightarrow \qquad t_\alpha^1(v_1) = \left(x, v_1^i\right) \quad t_\alpha^2(v_2) = (x, v_2^a) \tag{18.3.16} $$

Then one has fibered coordinates (x^μ, v_1^i, v_2^a) on $\mathcal{E}_1 \oplus \mathcal{E}_2$. If the old transition maps are

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
v_1'^i = (Y_1)_j^i(x) v_1^j
\end{cases}
\qquad
\begin{cases}
x'^\mu = x'^\mu(x) \\
v_2'^a = (Y_2)_b^a(x) v_2^b
\end{cases}
\tag{18.3.17}
$$

then the transition maps of the direct sum are

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
v_1'^i = (Y_j)_j^i(x) v_1^j \\
v_2'^a = (Y_2)_b^a(x) v_2^b
\end{cases}
\qquad
\begin{aligned}
v^A &= (v_1^i, v_2^a) \\[4pt]
Y_B^A(x) &= \begin{pmatrix} Y_1(x) & 0 \\ 0 & Y_2(x) \end{pmatrix}
\end{aligned}
\tag{18.3.18}
$$

We can define two fibered morphisms $j_1 : E_1 \to E_1 \oplus E_2 : v_1 \mapsto (v_1, 0)$ and $j_2 : E_2 \to E_1 \oplus E_2 : v_2 \mapsto (0, v_2)$

$$ (18.3.19) $$

These morphisms satisfy the so-called *universal property* of direct sum: for any vector bundle (E, M, p, V) on M and any family of vector bundle morphisms $f_i : E_i \to E$ then there exists a unique vector bundle morphism $f_1 \oplus f_2 : E_1 \oplus E_2 \to E$ such that

$$
\begin{array}{ccc}
E_i & \xrightarrow{\ \ j_i\ \ } & E_1 \oplus E_2 \\
& f_i \searrow & \Big\downarrow f_1 \oplus f_2 \qquad (f_1 \oplus f_2) \circ j_i = f_i \\
& & E
\end{array}
\tag{18.3.20}
$$

In fact, if we denote by $f_i : E_i \to E : v_i \mapsto f_i(v_i)$ the two vector bundle morphisms it is easy to check that the vector morphism $f_1 \oplus f_2 : E_1 \oplus E_2 \to E : (v_1, v_2) \mapsto f_1(v_1) + f_2(v_2)$ has the required property (just because $f_i(0) = 0$ being linear).

Moreover, let us that suppose there is another vector bundle morphism $\tilde{f} : E_1 \oplus E_2 \to E$ with the same property, then it is easy to check that

$$
\begin{aligned}
\tilde{f}(v_1, v_2) &= \tilde{f}(v_1, 0) + \tilde{f}(0, v_2) = \tilde{f} \circ j_1(v_1) + \tilde{f} \circ j_2(v_2) = f_1(v_1) + f_2(v_2) = \\
&= (f_1 \oplus f_2) \circ j_1(v_1) + (f_1 \oplus f_2) \circ j_2(v_2) = (f_1 \oplus f_2)(v_1, 0) + (f_1 \oplus f_2)(0, v_2) = (f_1 \oplus f_2)(v_1, v_2)
\end{aligned}
\tag{18.3.21}
$$

so that $\tilde{f} = f_1 \oplus f_2$ and that map is unique.

As it is often the case with universal properties, we can show that given a vector bundle S and two maps $i_1 : E_1 \to S$, $i_2 : E_2 \to S$ satisfying the universal properties, then $S \simeq E_1 \oplus E_2$ and the direct sum is essentially unique.

Suppose that both (S, i_i) and $(E_1 \oplus E_2, j_i)$ obey the universal property, in particular

$$
\begin{array}{ccc}
E_i & \xrightarrow{\ \ j_i\ \ } & E_1 \oplus E_2 \\
& i_i \searrow & \\
& & S
\end{array}
\tag{18.3.22}
$$

Then for the universal property of $E_1 \oplus E_2$ there exists a map $\phi : E_1 \oplus E_2 \to S$ such that $\phi \circ j_i = i_i$, while for the universal property of S there exists a map $\bar{\phi} : S \to E_1 \oplus E_2$ such that $\bar{\phi} \circ i_i = j_i$. Thus one has

$$
\bar{\phi} \circ \phi \circ j_i = \bar{\phi} \circ i_i = j_i \qquad\qquad \phi \circ \bar{\phi} \circ i_i = \phi \circ j_i = i_i
\tag{18.3.23}
$$

However, there is one map (the identity map) which has these properties, and for the universal property, this map is unique; thus $\bar{\phi} \circ \phi = \mathrm{id}$ and $\phi \circ \bar{\phi} = \mathrm{id}$ which shows that $S \simeq E_1 \oplus E_2$.

Let us remark how the universal property for direct sum is somehow dual to the universal property of product.

Tensor product

Let us consider two vector bundles $\mathcal{E}_1 = (E_1, M, \pi_1, \mathbb{K}^{n_1})$ and $\mathcal{E}_2 = (E_2, M, \pi_2, \mathbb{K}^{n_2})$ over the same base manifold M. Then we can define a new vector bundle on M, denoted by $\mathcal{E}_1 \otimes \mathcal{E}_2 = (E_1 \otimes E_2, M, \pi_\otimes, \mathbb{K}^{n_1 \cdot n_2})$, called the *tensor product* of the original vector bundles. The total space $E_1 \otimes E_2$ is defined as

$$E_1 \otimes E_2 := \coprod_{x \in M} (E_1)_x \otimes (E_2)_x \tag{18.3.24}$$

Accordingly, a point in $E_1 \otimes E_2$ is a pair (x, w) where $w : (E_1)_x^* \times (E_2)_x^* \to \mathbb{K}$ is a bilinear map. The projection $\pi_\otimes : E_1 \otimes E_2 \to M$ is defined as $\pi_\otimes(x, w) = x$. From what we already know about tensor products of vector spaces, if e_i and e_a are basis of $(E_1)_x$ and $(E_2)_x$, respectively, the most general bilinear map is $w = w^{ia} e_i \otimes e_a$ and acts as $w(\alpha, \beta) = w^{ia} \alpha_i \beta_a$.

Notice that the bilinear map w depends on $n_1 \cdot n_2$ numbers, while maps in the form $v_1 \otimes v_2$ depends on $n_1 + n_2$, only. Thus, generally, there are many elements in $E_1 \otimes E_2$ which are not simply the tensor product of vectors.

To show that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a bundle, we construct a local trivialisation $t_\alpha^\otimes : \pi_\otimes^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{n_1 \cdot n_2}$ as

$$t_\alpha^\otimes(x, w) = \left(x, w^{ia}\right) \qquad \Longleftrightarrow \quad w = w^{ia} e_i \otimes e_a \tag{18.3.25}$$

where e_i and e_a are the bases associated to the local trivialisations on E_1 and E_2, respectively.

Then one has fibered coordinates (x^μ, w^{ia}) on $\mathcal{E}_1 \otimes \mathcal{E}_2$. If the old transition maps are

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v_1'^i = A_j^i(x) v_1^j \end{cases} \qquad \begin{cases} x'^\mu = x'^\mu(x) \\ v_2'^a = B_b^a(x) v_2^b \end{cases} \tag{18.3.26}$$

then the transition maps of the fibered product are

$$\begin{cases} x'^\mu = x'^\mu(x) \\ w'^{ia} = Y_{jb}^{ia}(x) w^{jb} \end{cases} \qquad Y_{jb}^{ia}(x) = A_j^i(x) B_b^a(x) \tag{18.3.27}$$

We can define a fibered morphisms $\otimes : E_1 \oplus E_2 \to E_1 \otimes E_2 : (v_1, v_2) \mapsto v_1 \otimes v_2$ (which is, of course, bilinear).

These morphisms satisfy the so-called *universal property* of tensor product (see (15.2.11)): for any vector bundle (E, M, p, V) on M and any vector bundle morphisms $f : E_1 \oplus E_2 \to E$ then there exists a unique vector bundle morphism $\bar{f} : E_1 \otimes E_2 \to E$ such that

$$
\begin{array}{ccc}
E_1 \oplus E_2 & \xrightarrow{\ \otimes\ } & E_1 \otimes E_2 \\
 & {\scriptstyle f} \searrow & \Big\downarrow {\scriptstyle \bar{f}} \\
 & & E
\end{array}
\qquad \bar{f} \circ \otimes = f
\tag{18.3.28}
$$

In fact, if we denote by $f : E_1 \oplus E_2 \to E : (v_1, v_2) \mapsto f(v_1 \oplus v_2) = F(v_1, v_2)$ the vector bundle morphism, it is easy to check that the vector morphism $\bar{f} : E_1 \otimes E_2 \to E : v_1 \otimes v_2 \mapsto F(v_1, v_2)$ has the required property.

Moreover, let us suppose that there is another vector bundle morphism $\tilde{f} : E_1 \otimes E_2 \to E$ with the same property, then it is easy to check that

$$\tilde{f}(v_1 \otimes v_2) = \tilde{f} \circ \otimes (v_1 \oplus v_2) = f(v_1 \oplus v_2) = \bar{f} \circ \otimes (v_1 \oplus v_2) = \bar{f}(v_1 \otimes v_2) \tag{18.3.29}$$

so that $\tilde{f} = \bar{f}$ and that map is unique.

As it is often the case with universal properties, we can show that given a vector bundle S and a map $\odot : E_1 \oplus E_2 \to S$ satisfying the universal properties, then $S \simeq E_1 \otimes E_2$ and the direct sum is essentially unique.

Repeat the argument for direct sum.

4. Jet prolongation

Another general construction for bundles is *jet prolongation*.

Given a bundle $\mathcal{B} = (B, M, \pi, F)$, one can define an equivalence relation among local sections $\Gamma_x(\pi)$ defined around $x \in M$,

$$\sigma \sim \sigma' \quad \Longleftrightarrow \forall f : B \to \mathbb{R} \in \mathcal{F}(B), \forall \gamma : \mathbb{R} \to M, \qquad T_0^k(f \circ \sigma \circ \gamma) = T_0^k(f \circ \sigma' \circ \gamma) \tag{18.4.1}$$

where again T_0^k denotes the kth-degree Taylor polynomial in $s = 0$ of functions $f \circ \sigma \circ \gamma$, $f \circ \sigma' \circ \gamma : \mathbb{R} \to \mathbb{R}$. This is clearly an equivalence relation and we can define $J_x^k B = \Gamma_x(\pi)/\sim$ and

$$J^k B = \coprod_{x \in M} J_x^k B \tag{18.4.2}$$

A point in $J^k B$ is denoted by $j_x^k \sigma$ and it corresponds to the equivalence class of the local section $\sigma \in \Gamma_x(\pi)$.

Let (x^μ, y^i) be fibered coordinates on B. The function f has local expression $f(x, y)$, the curve $\gamma : s \mapsto x^\mu = \gamma^\mu(s)$, the local section $\sigma : x^\mu \mapsto (x^\mu, y^i = \sigma^i(x))$. Then the functions

$$f \circ \sigma \circ \gamma : s \mapsto f(\gamma(s), \sigma(\gamma(s))) \qquad\qquad f \circ \sigma' \circ \gamma : s \mapsto f(\gamma(s), \sigma'(\gamma(s))) \tag{18.4.3}$$

are real functions of a real variable. The corresponding Taylor polynomials for $k = 1$ are

$$
\begin{aligned}
&T_0^1\left(f(\gamma(s), \sigma(\gamma(s)))\right) = f(x, \sigma(x)) + \left(\partial_\mu f(x, \sigma(x)) + \partial_i f(x, \sigma(x))\partial_\mu \sigma^i(x)\right)\dot{\gamma}^\mu(0)s \qquad &\begin{cases} \sigma(x) = \sigma'(x) \\ \partial_\mu \sigma^i(x) = \partial_\mu \sigma'^i(x) \end{cases} \\
&T_0^1\left(f(\gamma(s), \sigma'(\gamma(s)))\right) = f(x, \sigma'(x)) + \left(\partial_\mu f(x, \sigma'(x)) + \partial_i f(x, \sigma'(x))\partial_\mu \sigma'^i(x)\right)\dot{\gamma}^\mu(0)s
\end{aligned}
\tag{18.4.4}
$$

For $k = 2$, one also has to have

$$
\begin{aligned}
&\left(\partial_{\mu\nu} f(x, \sigma(x)) + 2\partial_\mu f(x, \sigma(x))\partial_\nu \sigma^i(x) + \partial_{ij} f(x, \sigma(x))\partial_\mu \sigma^i(x)\partial_\nu \sigma^j(x) + \partial_i f(x, \sigma(x))\partial_{\mu\nu}\sigma^i(x)\right)\dot{\gamma}^\mu(0)\dot{\gamma}^\nu(0) = \\
&= \left(\partial_{\mu\nu} f(x, \sigma'(x)) + 2\partial_\mu f(x, \sigma'(x))\partial_\nu \sigma'^i(x) + \partial_{ij} f(x, \sigma'(x))\partial_\mu \sigma'^i(x)\partial_\nu \sigma'^j(x) + \partial_i f(x, \sigma'(x))\partial_{\mu\nu}\sigma'^i(x)\right)\dot{\gamma}^\mu(0)\dot{\gamma}^\nu(0)
\end{aligned}
\tag{18.4.5}
$$

which, in view of the conditions for equivalence of the case $k = 1$, simply requires and extra $\partial_{\mu\nu}\sigma^i(x) = \partial_{\mu\nu}\sigma'^i(x)$.

The two local sections σ and σ' are in general equivalent iff

$$\begin{cases} \sigma(x) = \sigma'(x) \\ \partial_\mu\sigma^i(x) = \partial_\mu\sigma'^i(x) \\ \partial_{\mu\nu}\sigma^i(x) = \partial_{\mu\nu}\sigma'^i(x) \\ \dots \\ \partial_{\mu_1\dots\mu_k}\sigma^i(x) = \partial_{\mu_1\dots\mu_k}\sigma'^i(x) \end{cases} \tag{18.4.6}$$

Then to identify a point in $J^k B$, one needs x^μ for the point $x \in M$, y^i for the value of the section $\sigma^i(x)$, y_μ^i for the value of $\partial_\mu\sigma^i(x)$, $y_{\mu\nu}^i$ for the value of $\partial_{\mu\nu}\sigma^i(x)$ (though $y_{\mu\nu}^i$ are meant to be symmetric in $(\mu\nu)$), $\dots$, $y_{\mu_1\dots\mu_k}^i$ for the value of $\partial_{\mu_1\dots\mu_k}\sigma^i(x)$ (though $y_{\mu_1\dots\mu_k}^i$ are meant to be symmetric in $(\mu_1\dots\mu_k)$).

One can cover $J^k B$ with charts of natural coordinates $(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots, y_{\mu_1\dots\mu_k}^i)$.

We defined a chart on $J^k B$ for any fibered chart on B. However, to show that $J^k B$ is a smooth manifold we still have to check regularity of transition maps.

Let (x^μ, y^i) and (x'^μ, y'^i) be two sets of fibered coordinates on B related by transition maps

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \tag{18.4.7}$$

They induce first partial derivatives

$$y_\mu'^i = \partial'_\mu Y^i(x, \sigma(x)) = \bar{J}_\mu^\rho \left(J_\rho^i + J_j^i y_\rho^j \right) \tag{18.4.8}$$

and second derivatives

$$y_{\mu\nu}'^i = \bar{J}_{\mu\nu}^\rho \left(J_\rho^i + J_j^i y_\rho^j \right) + \bar{J}_\mu^\rho \bar{J}_\nu^\sigma \left(J_{\rho\sigma}^i + 2J_{j\mu}^i y_\sigma^j + J_{jk}^i y_\rho^j y_\sigma^k + J_j^i y_{\rho\sigma}^j \right) = \left(\bar{J}_{\mu\nu}^\rho J_\rho^i + \bar{J}_\mu^\rho \bar{J}_\nu^\sigma J_{\rho\sigma}^i \right) + \left(\bar{J}_{\mu\nu}^\sigma J_j^i + 2\bar{J}_\mu^\rho \bar{J}_\nu^\sigma J_{j\mu}^i \right) y_\sigma^j + \left(\bar{J}_\mu^\rho \bar{J}_\nu^\sigma J_{jk}^i \right) y_\rho^j y_\sigma^k + \left(\bar{J}_\mu^\rho \bar{J}_\nu^\sigma J_j^i \right) y_{\rho\sigma}^j \tag{18.4.9}$$

The procedure can be iterated to higher orders. Let us first notice that transition maps at any order are smooth. Accordingly, $J^k B$ is a smooth manifold. At any order, one finds that the higher k-order derivatives enter linearly (with coefficients depending on (x, y) only), and a rest which depends on $(x^\mu, y^i, y_\mu^i, \dots, y_{\mu_1\dots\mu_{k-1}}^i)$, i.e. on derivatives up to order $(k-1)$. The transformation rules of higher order derivatives are in the form

$$y_{\mu_1\dots\mu_k}'^i = A_{j\mu_1\dots\mu_k}^{i\,\nu_1\dots\nu_k}(x, y)\, y_{\nu_1\dots\nu_k}^j + B_{\mu_1\dots\mu_k}^i(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_{k-1}}^k) \tag{18.4.10}$$

Then they are affine transformations of higher order derivatives.

The other function $B_{\mu_1\dots\mu_k}^i(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_{k-1}}^k)$ is a polynomial of degree k on $(k-1)$-derivatives.

Let us define the projections π_h^k for any integer $0 \le h < k$ simply by

$$\pi_h^k : J^k B \to J^h B : (x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots, y_{\mu_1\dots\mu_k}^i) \mapsto (x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots, y_{\mu_1\dots\mu_h}^i) \tag{18.4.11}$$

which are obviously surjective and maximal rank. Let us also set $\pi^k := \pi \circ \pi_0^k : J^k B \to M$.

Since transition maps of h-order derivatives just depend on derivatives of order lower than h, we have an (infinite) chain of bundles

$$\dots \to J^k B \to J^{k-1} B \to \dots \to J^2 B \to J^1 B \to B \to M \tag{18.4.12}$$

where we identified J^0B and B since they are canonically isomorphic.

In view of the general form of transition maps discussed above, each $J^{k+1}B \to J^kB$ is an affine bundle, while for example $J^2B \to B$ is not in general.

For the bundle $\pi_0^2 : J^2B \to B$, one has (x, y) for base coordinates and $(y_\mu^i, y_{\mu\nu}^i)$ as coordinates along the fiber. In order for it to be affine, one should have

$$\begin{cases} x'^\mu = x'^\mu(x, y) \\ y'^i = Y^i(x, y) \\ y_\mu'^i = A_{j\,\mu}^{i\,\rho\sigma}(x, y)\, y_{\rho\sigma}^j + B_{j\,\mu}^{i\,\rho}(x, y) y_\rho^j + C_\mu^i(x, y) \\ y_{\mu\nu}'^i = D_{j\,\mu\nu}^{i\,\rho\sigma}(x, y)\, y_{\rho\sigma}^j + E_{j\,\mu\nu}^{i\,\rho}(x, y) y_\rho^j + F_{\mu\nu}^i(x, y) \end{cases} \tag{18.4.13}$$

while actually, in J^2B, one has somehow better behavior (namely, $x'^\mu = x'^\mu(x)$, $A_{j\,\mu}^{i\,\rho\sigma} = 0$, but also worse behaviours as an extra term $G_{jk\,\mu\nu}^{i\,\rho\sigma}(x, y) y_\rho^j y_\sigma^k$.

Dimension of jet bundles

Let us assume $\dim(M) = m$ and $\dim(F) = k$, so that $\dim(B) = m + k$ and one has fibered coordinates (x^μ, y^i) on B. First derivatives y_μ^i are km, hence $\dim(J^1B) = m + k(m + 1)$.

Second derivatives $y_{\mu\nu}^i$ are symmetric in the lower indices, thus they are $\frac{1}{2}km(m+1)$. Accordingly, we have

$$\dim(J^2B) = m + k(m+1) + \tfrac{1}{2}km(m+1) = m + \tfrac{1}{2}k(m+1)(m+2) \tag{18.4.14}$$

Third derivatives $y_{\mu\nu\sigma}^i$ are also symmetric in the lower indices. Thus they are $kd_3(m) = \frac{1}{6}km(m+1)(m+2)$.

Let us first notice that there are $d_2(m) = \frac{1}{2}m(m+1)$ triples starting with 1, $d_2(m-1)$ starting with 2, ..., $d_2(1)$ starting with m.

Thus we have $d_3(m) = d_2(m) + d_2(m-1) + \ldots + d_2(1) = \sum_{k=1}^m d_2(k)$. Accordingly, we can expand it as

$$d_3(m) = \tfrac{1}{2}\sum_{k=1}^m (k^2 + k) = \tfrac{1}{2}\sum_{k=1}^m k^2 + \tfrac{1}{4}m(m+1) \tag{18.4.15}$$

Then we just need to show that $\sum_{k=1}^m k^2 = \frac{1}{6}m(2m+1)(m+1)$. For, we just need to remember that $k^2 = \sum_{n=1}^k (2n-1)$ and reshuffle the sum as

$$\sum_{k=1}^m k^2 = \sum_{k=1}^m \sum_{n=1}^k (2n-1) = 1 \cdot m + 3 \cdot (m-1) + 5 \cdot (m-1) + \ldots + (2m-1)1 = \sum_{k=1}^m (2k-1)(m-k+1) \tag{18.4.16}$$

as in the figure alongside and then expand it as

$$3\sum_{k=1}^m k^2 = \sum_{k=1}^m ((2m+3)k - (m+1)) = \tfrac{1}{2}(2m+3)m(m+1) - m(m+1) = \tfrac{1}{2}m(2m+1)(m+1) \tag{18.4.17}$$

Then we have

$$d_3(m) = \tfrac{1}{12}m(2m+1)(m+1) + \tfrac{1}{4}m(m+1) = \tfrac{1}{6}m(m+1)(m+2) \tag{18.4.18}$$

k^2	1^2	2^2	3^2	...	k^2
k					
1	1	1	1	...	1
2		3	3	...	3
3			5	...	5
				...	
k					$2k-1$

Accordingly, we have

$$\dim(J^3 B) = m + \tfrac{1}{2}k(m+1)(m+2) + \tfrac{1}{6}km(m+1)(m+2) = m + \tfrac{1}{6}k(m+1)(m+2)(m+3) \tag{18.4.19}$$

and so on at higher orders.

Contact structure

Given a section $\sigma : M \to B$, one can define a section $j^k\sigma : M \to J^k B : x \mapsto j^k_x\sigma$ which is called the *k-jet prolongation* of the section σ.

If the section $\sigma : M \to B$ has local expression $\sigma : M \to B : x \mapsto (x, y^i = \sigma^i(x))$ then the local expression of its prolongation is

$$\sigma : M \to J^k B : x \mapsto (x, y^i = \sigma^i(x), y^i_\mu = \partial_\mu \sigma^i(x), \dots, y^i_{\mu_1\dots\mu_k} = \partial_{\mu_1\dots\mu_k}\sigma^i(x)) \tag{18.4.20}$$

The operation of prolonging a section is analogous to the tangent lift of a curve in a manifold.

Of course, sections in $\Gamma(\pi^k)$ which are the prolongation of sections $\sigma \in \Gamma(\pi)$ are a strict subset of sections on $J^k B$. The prolongation $j^k\sigma$ of sections of $\sigma \in \Gamma(\pi)$ are called *holonomic* sections of $J^k B$.

A general section $\hat\sigma$ in $\Gamma(\pi^k)$ is locally expressed as

$$\hat\sigma : M \to J^k B : x \mapsto (x, y^i(x), y^i_\mu(x), y^i_{\mu\nu}(x), \dots, y^i_{\mu_1\dots\mu_k}(x)) \tag{18.4.21}$$

for unrelated functions $(y^i(x), y^i_\mu(x), y^i_{\mu\nu}(x), \dots, y^i_{\mu_1\dots\mu_k}(x))$, while it is holonomic iff

$$y^i_{\mu\nu}(x) = \partial_\mu y^i(x), \quad \dots, \quad y^i_{\mu_1\dots\mu_k}(x) = \partial_{\mu_1\dots\mu_k} y^i(x) \tag{18.4.22}$$

A p-form $\boldsymbol{\omega}$ is a called a *contact form* iff, for any holonomic section $j^k\sigma$, one has

$$(j^k\sigma)^* \boldsymbol{\omega} = 0 \tag{18.4.23}$$

The set of all contact forms on $J^k B$ is an ideal of the exterior algebra and it is denoted by $\Omega_K(J^k B) \subset \Omega(J^k B)$.

Let us consider a 1-form $\boldsymbol{\omega} = \omega_\mu \, dx^\mu + \omega_i \, dy^i + \omega^\mu_i \, dy^i_\mu \in \Omega^1(J^1 B)$. For it to be a contact form, one should have, for any section $\sigma : x^\mu \mapsto (x^\mu, y^i(x))$

$$(j^1\sigma)^* \boldsymbol{\omega} = \left(\omega_\mu + \omega_i y^i_\mu + \omega^\nu_i y^i_{\nu\mu} \right) dx^\mu = 0 \qquad \Longleftrightarrow \qquad \omega_\mu + \omega_i y^i_\mu + \omega^\nu_i y^i_{\nu\mu} = 0 \tag{18.4.24}$$

Being the coefficients functions of (x^μ, y^i, y^i_μ), these can be satisfied by

$$\omega_\mu = -\omega_i y^i_\mu \qquad\qquad \omega^\nu_i = 0 \tag{18.4.25}$$

Thus the more general contact 1-form on $J^1 B$ is $\boldsymbol{\omega} = \omega_i \left(dy^i - y^i_\mu \, dx^\mu \right)$. Accordingly, any contact 1-form is a linear combination of the basic contact 1-forms $\omega^i = dy^i - y^i_\mu \, dx^\mu$.

That said, of course, any wedge product of ω^i with any other form, on the left and on the right is a contact form as well. However, there are more.

Since the differential is natural, any differential of a contact form is contact as well

$$(j^1\sigma)^* d\boldsymbol{\omega} = d\left((j^1\sigma)^*\boldsymbol{\omega}\right) = 0 \tag{18.4.26}$$

Thus also the 2-form $-d\omega^i = dy^i_\mu \wedge dx^\mu$ is a contact form. Notice that this is not in the form of a product of a 1-form with ω^i.

To summarise, the contact ideal $\Omega_K(J^1B)$ is generated by linear combinations, products on the left and on the right by any form and the differential of the basic contact 1-forms.

On the second jet bundle J^2B, one has basic contact 1-forms

$$\omega^i = dy^i - y^i_\mu \, dx^\mu \qquad \omega^i_\mu = dy^i_\mu - y^i_{\mu\nu} \, dx^\nu \tag{18.4.27}$$

together with their differential

$$-d\omega^i = dy^i_\mu \wedge dx^\mu \qquad -d\omega^i_\mu = dy^i_{\mu\nu} \wedge dx^\nu \tag{18.4.28}$$

Since we can write $-d\omega^i = \left(dy^i_\mu - y^i_{\mu\nu} \, dx^\nu\right) \wedge dx^\mu = \omega^i_\mu \wedge dx^\mu$, it is a product or contact 1-forms. The same cannot be done (on J^2B) with $d\omega^i_\mu$.

Thus the contact ideal $\Omega_K(J^2B)$ is generated by linear combinations, product on the left and on the right by any form of ω^i, ω^i_μ, $d\omega^i_\mu$.

This structures extends to higher orders. Thus the contact ideal $\Omega_K(J^3B)$ is generated by linear combinations, product on the left and on the right by any form of ω^i, ω^i_μ, $\omega^i_{\mu\nu}$, $d\omega^i_{\mu\nu}$.

The contact ideal is a characteristic property of jet bundles. Among other things, the contact ideal (almost) allows to split forms on J^kB into the sum of a contact and a horizontal component.

For example, if we consider a 1-form on J^1B as $\boldsymbol{\omega} = \omega_\mu \, dx^\mu + \omega_i \, dy^i$, it can be recast as

$$\boldsymbol{\omega} = \omega_\mu \, dx^\mu + \omega_i \, dy^i = (\omega_\mu + \omega_i y^i_\mu) \, dx^\mu \oplus \omega_i \left(dy^i - y^i_\mu \, dx^\nu \right) = (\omega_\mu + \omega_i y^i_\mu) \, dx^\mu \oplus \omega_i \omega^i \tag{18.4.29}$$

The same happens for higher order forms as the 2-form

$$\begin{aligned}
\boldsymbol{\theta} &= \tfrac{1}{2} \left(\theta_{\mu\nu} \, dx^\mu \wedge \, dx^\nu + \left(2\theta_{i\mu} - \theta_{ki} y^k_\mu\right) \, dy^i \wedge \, dx^\mu + \theta_{ij} \omega^i \wedge \, dy^j \right) = \\
&= \tfrac{1}{2} \left(\theta_{\mu\nu} + (\theta_{i\mu} y^i_\nu - \theta_{i\nu} y^i_\mu - \theta_{ki} y^k_\mu y^i_\nu) \right) \, dx^\mu \wedge \, dx^\nu \oplus \tfrac{1}{2} \left((2\theta_{i\mu} + \theta_{ij} y^j_\mu) \omega^i \wedge \, dx^\mu + \theta_{ij} \omega^i \wedge \omega^j \right)
\end{aligned} \tag{18.4.30}$$

However, these are not general 1-forms and 2-forms on J^1B. For example, the most general 1-form on J^1B also has a term $\omega^i_\mu \, dy^i_\mu$ which cannot be split on J^1B.

One would like to split it is as

$$\omega^\mu_i \, dy^i_\mu = \omega^\mu_i \omega^i_\mu + \omega^\mu_i y^i_{\mu\nu} \, dx^\nu \tag{18.4.31}$$

but this is not a splitting on J^1B, since it contains second order derivatives.

In general, we can consider a p-form $\boldsymbol{\omega}$ on J^kB, pull it back on $J^{k+1}B$ (so that $(\pi^{k+1}_k)^*\boldsymbol{\omega}$ does not depend on the differential of the higher derivatives, namely on $dy^i_{\mu_0...\mu_k}$) and then canonically split on $J^{k+1}B$ into a horizontal and a contact p-form. We shall denote by $\boldsymbol{\omega}_H$ the horizontal part and by $\boldsymbol{\omega}_K$ the contact part, so that we have

$$(\pi^{k+1}_k)^*\boldsymbol{\omega} = \boldsymbol{\omega}_H \oplus \boldsymbol{\omega}_K \tag{18.4.32}$$

Let us stress that $\boldsymbol{\omega}_H$, $\boldsymbol{\omega}_K$ are p-forms on $J^{k+1}B$, though.

One can also define the *horizontal differential* d_H which acts on forms on $J^k B$ by the ordinary differential, then taking the horizontal part of the result.

For example, if $\boldsymbol{\omega} = \omega_\mu\, dx^\mu + \omega_i\, dy^i$ is a 1-form on $J^0 B$, we can compute

$$d\boldsymbol{\omega} = \partial_\nu \omega_\mu\, dx^\nu \wedge\, dx^\mu + (\partial_\mu \omega_i - \partial_i \omega_\mu)\, dx^\mu \wedge\, dy^i + \partial_j \omega_i\, dy^j \wedge\, dy^i \tag{18.4.33}$$

and then take the horizontal part

$$d_H \boldsymbol{\omega} = \left(\partial_\nu \omega_\mu + \partial_i \omega_\mu y_\nu^i + (\partial_\nu \omega_i + \partial_j \omega_i y_\nu^j) y_\mu^i\right)\, dx^\nu \wedge\, dx^\mu \tag{18.4.34}$$

which is a horizontal 2-form on $J^1 B$.

That suggests to introduce the operator *total derivative* $d_\mu = \partial_\mu + y_\mu^i \partial_i + \dots$ which acts on functions on some jet, returning a function on an higher jet prolongation, which has been differentiated as if it were evaluated on a section.

Total derivative operators

As one does in mechanics (defining the total derivatives with respect to time), one can define *total derivatives* d_μ with respect to base coordinates x^μ of a function $f(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots, y_{\mu_1\dots\mu_k}^i)$ on $J^k B$ to be a function on $J^{k+1}B$, namely

$$
\begin{aligned}
&d_\mu f(x^\lambda) := \partial_\mu f(x^\lambda) \\
&d_\mu f(x^\lambda, y^k) := \partial_\mu f(x^\lambda, y^k) + \partial_i f(x^\lambda, y^k) y_\mu^i \\
&d_\mu f(x^\lambda, y^k, y_\lambda^k) := \partial_\mu f(x^\lambda, y^k, y_\lambda^k) + \partial_i f(x^\lambda, y^k, y_\lambda^k) y_\mu^i + \partial_i^\nu f(x^\lambda, y^k, y_\lambda^k) y_{\nu\mu}^i \\
&\dots \\
&d_\mu f(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_k}^k) := \partial_\mu f(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_k}^k) + \partial_i f(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_k}^k) y_\mu^i + \partial_i^\nu f(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_k}^k) y_{\nu\mu}^i + \\
&\qquad\qquad + \dots + \partial_i^{\nu_1\dots\nu_k} f(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1\dots\lambda_k}^k) y_{\nu_1\dots\nu_k\mu}^i
\end{aligned}
\tag{18.4.35}
$$

Total derivatives for any function $f \in \mathcal{F}(J^k B)$ have the property

$$(d_\mu f) \circ j^{k+1}\sigma = \partial_\mu \left(f \circ j^k \sigma\right) \tag{18.4.36}$$

As in mechanics d_t takes derivatives of functions *as if* they were evaluated along a curve, d_μ takes derivatives of functions *as if* they were evaluated along a section of $\mathcal{B}$.

This notation is very useful, still just a notation for us. We shall use it also by an abuse of language on local functions, e.g. along coordinate functions. For example, in a chart, one has a local function $y^i : b \mapsto y^i(b)$ which gives the value of the coordinate y^i of the point b. Then check that $d_\mu y^i = y_\mu^i$.

Using total derivatives, one can write transition maps on $J^k B$ in compact form

$$\begin{cases} y'^i_\mu = \bar{J}^\rho_\mu d_\rho Y^i(x,y) \\ y'^i_{\mu\nu} = \bar{J}^\rho_\mu d_\rho \left(\bar{J}^\sigma_\nu d_\sigma Y^i(x,y) \right) \end{cases} \tag{18.4.37}$$

and so on by iteration.

Jet functor

Let us consider two bundles $\mathcal{B} = (B, M, \pi, F)$ and $\mathcal{B}' = (B', M', \pi', F')$ and a strong fibered morphism $(\Phi, \varphi) : \mathcal{B} \to \mathcal{B}'$ (i.e. with φ being a diffeomorphism).

Given a section $\sigma : M \to B$, we can define $\sigma' := \Phi \circ \sigma \circ \varphi^{-1} : M' \to B'$ which is automatically a section.

We have to check that $\pi' \circ \sigma' = \mathrm{id}_{M'}$. In fact

$$\pi' \circ \Phi \circ \sigma \circ \varphi^{-1} = \varphi \circ \pi \circ \sigma \circ \varphi^{-1} = \varphi \circ \mathrm{id}_M \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \mathrm{id}_{M'} \tag{18.4.38}$$

Of course, the prescription to define σ' holds also for local sections.

Then we can define a fibered morphism $J^k \Phi : J^k B \to J^k B'$ defined as

$$J^k \Phi : J^k B \to J^k B' : j^k_x \sigma \mapsto j^k_{\varphi(x)} \sigma' \tag{18.4.39}$$

which is called the *k-order jet prolongation* of the map Φ.

The morphism $J^k \Phi$ makes commutative the diagram shown on the side, which of course extends to the whole infinite chain of jet prolongations.

If we consider the identity fibered morphism $(\mathrm{id}_B, \mathrm{id}_M)$ then $\sigma' = \sigma$ and $J^k \mathrm{id}_B = \mathrm{id}_{J^k B}$.

If we consider two strong fibered morphisms $(\Phi, \varphi) : \mathcal{B} \to \mathcal{B}'$ and $(\Psi, \psi) : \mathcal{B}' \to \mathcal{B}''$ one has

$$\sigma'' = (\Psi \circ \Phi) \circ \sigma \circ (\psi \circ \varphi)^{-1} = \Psi \circ (\Phi \circ \sigma \circ \varphi^{-1}) \circ \psi^{-1} = \Psi \circ \sigma' \circ \psi^{-1} \tag{18.4.40}$$

and

$$J^k(\Psi \circ \Phi) : J^k B \to J^k B'' : j^k_x \sigma \mapsto j^k_{\psi \circ \varphi(x)} \sigma'' = j^k_{\psi(x')} \Psi \circ \sigma' \circ \psi^{-1} = J^k \Psi(j^k_{x'} \sigma') = J^k \Psi \circ J^k \Phi(j^k_x \sigma) \tag{18.4.41}$$

Then $J^k(\Psi \circ \Phi) = J^k \Psi \circ J^k \Phi$.

Accordingly, $J^k(\cdot)$ is a covariant functor from the category $\mathfrak{Fib}_M$ in itself.

$$
\begin{array}{ccc}
\cdots & & \cdots \\
\downarrow & & \downarrow \\
J^k B & \xrightarrow{J^k \Phi} & J^k B' \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
J^2 B & \xrightarrow{J^2 \Phi} & J^2 B' \\
{\scriptstyle \pi^2_1} \downarrow & & \downarrow {\scriptstyle (\pi')^2_1} \\
J^1 B & \xrightarrow{J^1 \Phi} & J^1 B' \\
{\scriptstyle \pi^1_0} \downarrow & & \downarrow {\scriptstyle (\pi')^1_0} \\
B & \xrightarrow{\Phi} & B' \\
{\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle \pi'} \\
M & \xrightarrow{\varphi} & M'
\end{array}
$$

Differential equations

Jet bundles are finite dimensional manifolds which account for derivatives of sections up to some (finite) order k. Then, trivially, a differential equation for sections can be translated into an algebraic relation in the bundle which identifies a sub-manifold $D \subset J^k B$.

Consider a manifold $M = \mathbb{R} \times S_2$ with a metric $g = -dt^2 + d\theta^2 + \sin^2(\theta)d\phi^2$. The equation of Laplace for a function is

$$\Box f = \delta df = *d * (\partial_\mu f \, dx^\mu) = *d(\sqrt{g}\partial_\mu f g^{\mu\alpha} \, d\sigma_\alpha) = *\partial_\alpha(\sqrt{g}\partial_\mu f g^{\mu\alpha}) \, d\sigma = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\partial_\mu f g^{\mu\alpha}) = 0 \tag{18.4.42}$$

which, in the specific case ($\sqrt{g} = \sin(\theta)$), becomes

$$\partial_t^2 f = \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta f + \partial_\theta^2 f + \frac{1}{\sin^2(\theta)}\partial_\phi^2 f \tag{18.4.43}$$

Let now consider the bundle $\mathcal{F} = (M \times \mathbb{R}, M, p_1, \mathbb{R})$ of scalar functions on M. One can check that there is a one-to-one correspondence between functions of M and sections of $\mathcal{F}$ (due to the fact that the bundle is trivial, and transition maps $f(x') = f(x)$ agree with glueing conditions for scalar functions).
Fibered coordinates (x^μ, y) on $\mathcal{F}$ induces natural coordinates $(x^\mu, y, y_\mu, y_{\mu\nu})$ on $J^2\mathcal{F}$ and the Laplace differential equation reads as

$$y_{00} = \frac{\cos(\theta)}{\sin(\theta)}y_1 + y_{11} + \frac{1}{\sin^2(\theta)}y_{22} \tag{18.4.44}$$

which defines a sub-manifold $L \subset J^2\mathcal{F}$ which describes the Laplace equation as a geometric object. May one use other coordinates (e.g. stereographic coordinates on the sphere) the expression of the Laplace equation would be different, still it would be a different parameterisation of the same sub-manifold $L \subset J^2\mathcal{F}$.
We decide to be strict on that. The two different parameterisations for us are the *same equation*. There are many properties of the paremetrization which are not properties of sub-manifold itself. We are choosing to ignore those properties just as we decided to ignore contractions like $\alpha_\mu\beta_\mu$ *just* because they depend on the choice of the basis.
On the contrary, we are deciding to focus on properties which are properties of the sub-manifold.

A sub-manifold $D \subset J^k B$ defines a sub-manifolds in all jets of order $k + h$ greater than k by taking simply $(\pi_k^{k+h})^{-1}(D) \subset J^{k+h}B$. Any differential equation of order k can be seen as an equation of higher order which does not actually depend on derivatives higher than k.

Vice versa, a differential equation $D \subset J^k B$ is said to be of *sharp order k*, if there is no lower $h < k$ such that

$$(\pi_h^k)^{-1}\left(\pi_h^k(D)\right) = D \tag{18.4.45}$$

The notion of sharp order is an example of property which does not depend on coordinates and parameterisations. If a differential equation is of order k, no matter if one changes coordinates on M or coordinates on B (which amounts to redefine the unknown function) the order will still be k.

Since the bundle $\pi_{k-1}^k : J^k B \to J^{k-1}B$ is affine it makes sense to define quasi-linear differential equations.

A differential equation is called *quasi-linear* if it is linear in the higher order derivatives, i.e. in the form

$$A_i^{\mu_1 \dots \mu_k}(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1 \dots \lambda_{k-1}}^k)y_{\mu_1 \dots \mu_k}^i + Q(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1 \dots \lambda_{k-1}}^k) = 0 \tag{18.4.46}$$

Analogously, for a quasi-linear system

$$A_i^{A\mu_1 \dots \mu_k}(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1 \dots \lambda_{k-1}}^k)y_{\mu_1 \dots \mu_k}^i + Q^A(x^\lambda, y^k, y_\lambda^k, \dots, y_{\lambda_1 \dots \lambda_{k-1}}^k) = 0 \tag{18.4.47}$$

This makes sense since we know that higher order derivatives transform as

$$y'^i_{\mu_1\ldots\mu_k} = R^{i\nu_1\ldots\nu_k}_{j\mu_1\ldots\mu_k}(x,y)y^j_{\nu_1\ldots\nu_k} + S^i_{\mu_1\ldots\mu_k}(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}}) \tag{18.4.48}$$

and if a differential equation is quasi-linear in the new coordinates

$$A'^{\mu_1\ldots\mu_k}_i(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})y'^i_{\mu_1\ldots\mu_k} + Q'(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}}) = 0 \tag{18.4.49}$$

in the old ones reads as

$$\begin{aligned}
A'^{\mu_1\ldots\mu_k}_i(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})R^{i\nu_1\ldots\nu_k}_{j\mu_1\ldots\mu_k}(x,y)y^j_{\nu_1\ldots\nu_k}+ \\
+ A'^{\mu_1\ldots\mu_k}_i(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})S^i_{\mu_1\ldots\mu_k}(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}}) + Q'(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}}) = 0
\end{aligned} \tag{18.4.50}$$

That is again quasi-linear, if we define

$$\begin{cases}
A^{\mu_1\ldots\mu_k}_i(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}}) = A'^{\mu_1\ldots\mu_k}_i(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})R^{i\nu_1\ldots\nu_k}_{j\mu_1\ldots\mu_k}(x,y) \\
Q(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}}) = A'^{\mu_1\ldots\mu_k}_i(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})S^i_{\mu_1\ldots\mu_k}(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}}) + Q'(x'^\lambda, y'^k, y'^k_\lambda, \ldots, y'^k_{\lambda_1\ldots\lambda_{k-1}})
\end{cases} \tag{18.4.51}$$

Let us mention that the part not depending on higher derivatives $Q(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}})$ is sometimes called the *tail* of quasi-linear equation.
The higher order part $A^{\mu_1\ldots\mu_k}_i(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_{k-1}})y^i_{\mu_1\ldots\mu_k}$ is called the *principal part*. (Not to be confused with *principal* bundles.)

Let us stress here that being quasi-linear has *nothing* to do with a possible linear structure (or affine structure) of fields. We did not required *anything* about the bundle B, nor to be a vector bundle or an affine bundle. Quasi-linear structure is instead something related to the affine structure of jet prolongations.

This will be particularly relevant when discussing GR. Einstein equations are written for a metric which is not a section of a vector bundle (there is no zero metric). Still Einstein equations are quasi-linear (together with (almost) any variational system of equations).

Let us also remark that we can call *degree* of a quasi-linear equation the higher order derivatives appearing in the coefficients $A^{\mu_1\ldots\mu_k}_i$ of the principal part.

Since in the transformation of derivatives of order k, one has $R^{i\nu_1\ldots\nu_k}_{j\mu_1\ldots\mu_k} = \bar{J}^{\nu_1}_{\mu_1}\ldots\bar{J}^{\nu_k}_{\mu_k}J^i_j$ which just depends on (x,y), in general, one has that having degree $0 \le h < k$ is also preserved by change of coordinates and it is thence an intrinsic property of the equation.
This is also particularly relevant for us since Einstein equations are quasi-linear equations of degree 0.
Notice that requiring that $A^{\mu_1\ldots\mu_k}_i(x)$ would in general depend on the coordinate system (unless the bundle B is a vector bundle so that $J^i_j(x)$ does not depend on y).
For similar reasons, in general, we cannot define quasi-linear equations with constant coefficients. We stress we are not claiming that one cannot have a coordinate system in which $A^{\mu_1\ldots\mu_k}_i(x)$ (or in which $A^{\mu_1\ldots\mu_k}_i$ is constant). Just, for us, that is not a property of the equation, in general, since it is not preserved by changing fibered coordinates on the bundle.

For discussing *linear equations* also one needs to restrict the bundle B to be a vector bundle. There is nothing like a linear equation for fields which live in a bundle which is not linear. An equation which happens to be linear in one coordinate system would not be linear in another. On the contrary, if the bundle B is a vector bundle then transition functions would be in the special form $y'^i = A^i_j(x)y^j$ and one has $J^i_j = A^i_j(x)$ which, unlike in the general case, depends on x only.

Then all prolongations $J^k B$ are affine bundles over any lower jet bundle (not just over the one below). Then it makes sense to define a *linear differential equation* (second order to keep it simple) as one in the form

$$A_i^{\mu\nu}(x)y_{\mu\nu}^i + B_i^\mu(x)y_\mu^i + C_i(x)y^i + D(x) = 0 \tag{18.4.52}$$

since this form is preserved by changes of coordinates allowed on vector bundles.

5. Connections

We already defined connections on manifolds. They are quite simple objects and still their nature is quite obscure. We shall here give a more general and geometric definition of connection (on a bundle in general and on a principal bundle in particular). Then connections on the frame bundle $L(M)$ or, more or less equivalently, on the tangent bundle will specify to the connections on manifolds we already defined. On more general principal bundles, connections are the geometric counterparts of gauge fields. The bundle framework also provides a very beautifully simple setting to discuss in general parallel transport and holonomy which have very deep and fundamental implications in physics.

The bad news is that one should discuss at least five or six different equivalent definitions for connection. Each definition is particularly suited to discuss a class of problems. With some effort, we try to reduce that number of definitions to two.

Definition (18.5.1): *a connection on a bundle $\mathcal{B} = (B, M, \pi, F)$ is a distribution H of subspaces $H_b \subset T_b B$ of rank $m = dim(M)$ such that $H_b \oplus V_b(\pi) = T_b B$.*

A connection on a principal bundle $\mathcal{P} = (P, M, \pi, G)$, beside being a connection, has to preserve the right action on P, i.e.

$$T_p R_g (H_p) = H_{p\cdot g} \tag{18.5.2}$$

(in which case, it is called a principal connection, though we shall never consider connections which are not principal on a principal bundle).

Actually, one should require some sort of regularity with respect to the application point b which prevents the subspace from jumping from one position to another for small displacements of b. The easiest way to obtain that is to require that

$$H(\pi) = \coprod_{b \in B} H_b \tag{18.5.3}$$

is a sub-bundle of TB.

By using a connection H, one can split uniquely any vector $w \in T_b B$ into the sum of a *horizontal* and a *vertical part*, $w = h(w) \oplus v(w)$. That is a consequence of being the sum $H_b \oplus V_b(\pi)$ direct. Notice that even though vertical subspaces are intrinsic on a bundle while horizontal subspace depend on the connection, nevertheless both the horizontal and vertical parts depend on the connection.

Also, given a vector $\xi \in T_x M$ and $b \in \pi^{-1}(x) \subset B$ there is a unique vector $\Xi \in T_b B$ such that $T_b \pi(\Xi) = \xi$ and $\Xi \in H_b$.

Suppose that there are two vectors $\Xi, \Xi' \in T_b B$ such that $T_b \pi(\Xi) = T_b \pi(\Xi') = \xi$ and $\Xi, \Xi' \in H_b$. Then $\Xi - \Xi'$ is still horizontal and $T_b \pi(\Xi - \Xi') = 0$, i.e. $\Xi - \Xi'$ is also vertical. But being the sum between horizontal and vertical vectors direct, that means that the only vector which is horizontal and vertical at the same time is the zero vector. Thus $\Xi = \Xi'$ and the vector Ξ is unique.

The vector Ξ is called the *horizontal lift* of ξ at the point b with respect to the connection H. Knowing the horizontal lift $\omega_b : T_x M \to T_b B$ at any point of any vector is equivalent to know the connection H. In fact, one can define the distribution by setting $H_b = \omega_b(T_x M)$.

The horizontal lift is a family of linear maps $\omega_b : T_x M \to T_b B$ or, equivalently, a map $\omega : B \to T^* M \otimes TB$ such that $T\pi \circ \omega = \mathrm{id}_{TM}$; that map is locally described by

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^i(b) \partial_i \right) \tag{18.5.4}$$

By changing fibered coordinates on B, the coefficients ω_μ^i transform as:

$$dx^\mu \otimes \left(\partial_\mu - \omega_\mu^i(b) \partial_b \right) = dx'^\lambda \otimes \left(\partial'_\lambda - \bar{J}_\lambda^\mu \left(\omega_\mu^i(b) J_i^j - J_\mu^j \right) \partial'_j \right) \tag{18.5.5}$$

where we used

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \Rightarrow \begin{cases} \partial_\mu = J_\mu^\nu \partial'_\nu + J_\mu^i \partial'_i \\ \partial_i = J_i^j \partial'_j \end{cases} \tag{18.5.6}$$

Hence the connection coefficients transform as

$$\omega_\lambda'^j(x', y') = \bar{J}_\lambda^\mu \left(\omega_\mu^i(x, y) J_i^j - J_\mu^j \right) \tag{18.5.7}$$

For a principal connection, we still have to apply the behaviour with respect to the right action $R_g : P \to P : [x, k]_\alpha \mapsto [x, k \cdot g]_\alpha$

$$T_{[x,k]_\alpha} R_g(\partial_\mu) = \partial'_\mu \qquad T_{[x,k]_\alpha} R_g(\partial_a) = R_a^b(g) \partial'_b \tag{18.5.8}$$

Let us then define $\rho_A(x, k) := T_A^a R_a^b(k) \partial_b$ which is a right-invariant pointwise basis for vertical vectors on P

$$T_{[x,k]_\alpha} R_g(\rho_A(x, k)) = T_A^a R_a^b(k) T_{[x,k]_\alpha} R_g(\partial_b) = T_A^a R_a^c(k) R_c^b(g) \partial'_b = T_A^a R_a^b(k \cdot g) \partial'_b = \rho_A(x, k \cdot g) \tag{18.5.9}$$

A principal connection $\omega = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^a(x, k) \partial_a \right)$ have to obey

$$T_{[x,k]_\alpha} R_g \left(dx^\mu \otimes \left(\partial_\mu - \omega_\mu^a(x, k) \partial_a \right) \right) = dx^\mu \otimes \left(\partial'_\mu - \omega_\mu^b(x, k \cdot g) \partial'_b \right) \tag{18.5.10}$$

On the left hand, one has

$$T_{[x,k]_\alpha} R_g \left(dx^\mu \otimes \left(\partial_\mu - \omega_\mu^a(x, k) \partial_a \right) \right) = dx^\mu \otimes \left(\partial'_\mu - \omega_\mu^a(x, k) R_a^b(g) \partial'_b \right) \tag{18.5.11}$$

Thus a principal connection must have coefficients for which

$$\omega_\mu^b(x, k \cdot g) = \omega_\mu^a(x, k) R_a^b(g) \tag{18.5.12}$$

Also in this case, condition (18.5.12) tells us that, if we knew ω_μ^a at a point in a fiber, e.g. at $(x, \mathbb{I})$, then we would know it along the whole fiber. Set $\omega_\mu^a(x) := \omega_\mu^a(x, \mathbb{I})$ and we have

$$\omega_\mu^a(x, k) = \omega_\mu^a(x) R_a^b(k) \tag{18.5.13}$$

Then the most general form for a principal connection is

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^a(x) R_a^b(k) \partial_b \right) = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^a(x) T_a^A T_A^c R_c^b(k) \partial_b \right) = dx^\mu \otimes \left(\partial_\mu - \omega_\mu^A(x) \rho_A \right) \tag{18.5.14}$$

where we set $\omega_\mu^A := T_a^A \omega_\mu^a$.

By changing fibered coordinates on $\mathcal{P}$, the basis of vector fields at (x, k) changes to the basis at $(x, \varphi \cdot k)$ as:

$$\begin{cases} x'^\mu = x'^\mu(x) \\ k' = \varphi(x) \cdot k \end{cases} \Rightarrow \begin{cases} \partial_\mu = J_\mu^\nu \partial_\nu' + \partial_\mu \varphi^a \partial_a' \\ \partial_a = L_a^b(\varphi) \partial_b' \end{cases} \Rightarrow \begin{cases} \partial_\mu = J_\mu^\nu \partial_\nu' + \partial_\mu \varphi^a \bar{R}_a^B(\varphi) \rho_B(x, \varphi \cdot k) \\ \rho_A(x, k) = \mathrm{ad}_A^C(\varphi) \rho_C(x, \varphi \cdot k) \end{cases} \tag{18.5.15}$$

where we set $\bar{R}_a^B(\varphi) := \bar{R}_a^b(\varphi) T_b^B$

Then

$$dx^\mu \otimes \left(\partial_\mu - \omega_\mu^A(x) \rho_A \right) = dx'^\lambda \otimes \left(\partial_\lambda' - \bar{J}_\lambda^\mu \left(\mathrm{ad}_A^C(\varphi) \omega_\mu^A(x) - \partial_\mu \varphi^a \bar{R}_a^C(\varphi) \right) \rho_C' \right) \qquad \Rightarrow \omega_\lambda'^C(x') = \bar{J}_\lambda^\mu \left(\mathrm{ad}_A^C(\varphi) \omega_\mu^A(x) - \bar{R}_a^C(\varphi) \partial_\mu \varphi^a \right) \tag{18.5.16}$$

Once we know the local expression of a connection, one has the splitting of a vector $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i \in T_b B$ as

$$\Xi = \xi^\mu \left(\partial_\mu - \omega_\mu^i(b) \partial_i \right) \oplus (\xi^i + \xi^\mu \omega_\mu^i) \partial_i \tag{18.5.17}$$

and we can set $\xi_V^i := \xi^i + \xi^\mu \omega_\mu^i(b)$ for the component of the vertical part.

Notice that ξ^i transforms as $\xi^i = J_j^i \xi^j + \xi^\mu J_\mu^i$ so that it cannot be put to zero in an intrinsic way. If it is zero in a chart, it is not in another. That is the ultimate reason why we need connections to define horizontal vectors. On the contrary, we have

$$\xi_V'^i := \xi'^i + \xi'^\mu \omega_\mu'^i(b) = J_j^i \xi^j + \textcolor{red}{\xi^\mu J_\mu^i} + \xi^\mu J_j^i \omega_\mu^j - \textcolor{red}{\xi^\mu J_\mu^i} = J_j^i \left(\xi^j + \xi^\mu \omega_\mu^j \right) = J_j^i \xi_V^j \tag{18.5.18}$$

Then if $\xi_V^j = 0$ in one chart, it is zero in any chart.

For a principal connection on P, one has the splitting of a vector $\Xi = \xi^\mu \partial_\mu + \xi^A \rho_A \in T_p P$ as

$$\Xi = \xi^\mu \left(\partial_\mu - \omega_\mu^A(x) \rho_A \right) \oplus (\xi^A + \xi^\mu \omega_\mu^A(x)) \rho_A \tag{18.5.19}$$

and we can set $\xi_{(V)}^A := \xi^A + \xi^\mu \omega_\mu^A$ for the components of the vertical part.

One should know that a connection is also in one-to-one correspondence with general sections of the bundle $\pi_0^1 : J^1 B \to B$ and a splitting of the Whitney sequence of vector bundles

$$0 \to V(B) \to TB \to TM \to 0 \tag{18.5.20}$$

As one should know that a principal connection is also in one-to-one correspondence with general sections of the bundle J^1P/G, or that it can be described in terms of a 1-form over P with values in the Lie algebra $\mathfrak{g}$. We shall try to go along with our ignorance and live just by knowing an intrinsic and a local characterization for connections.

Associated connections

A principal connection H on $\mathcal{P}$ induces a connection in any associated bundle $P \times_\lambda F$.

In fact, at the point $[p, f]_\lambda \in P \times_\lambda F$, we can define the subspace $H_{[p,f]_\lambda} = T_p \Phi_f(H_p)$ along the map $\Phi_f : P \to P \times_\lambda F : p \mapsto [p, f]_\lambda$.

We just have to show the definition is well-posed, i.e. independent of the representative, in fact

$$H_{[p \cdot g^{-1}, g \cdot f]_\lambda} = T_{p \cdot g^{-1}} \Phi_{g \cdot f}(H_{p \cdot g^{-1}}) = T_{p \cdot g^{-1}} \Phi_{g \cdot f} \circ T_p R_{g^{-1}}(H_p) = T_p \left(\Phi_{g \cdot f} \circ R_{g^{-1}} \right)(H_p) = T_p \Phi_f(H_p) = H_{[p,f]_\lambda} \tag{18.5.21}$$

where we used the fact that

$$\Phi_{g \cdot f} \circ R_{g^{-1}} : P \to P \times_\lambda F : p \mapsto p \cdot g^{-1} \mapsto [p \cdot g^{-1}, g \cdot f]_\lambda = [p, f]_\lambda \qquad \Rightarrow \Phi_{g \cdot f} \circ R_{g^{-1}} = \Phi_f \tag{18.5.22}$$

The family of subspaces $H_{[p,f]_\lambda}$ is a connection on $P \times_\lambda F$.

Let us consider a vector $\hat{\Xi} \in T_{[p,f]_\lambda} P \times_\lambda F$ which belongs to $H_{[p,f]_\lambda}$ and $V_{[p,f]_\lambda}$. Then one has for some $\Xi \in H_p$

$$\hat{\Xi} = T_p \Phi_f(\Xi) \tag{18.5.23}$$

But $\hat{\Xi}$ is also vertical, hence $T\pi_\lambda(\hat{\Xi}) = 0$, then $T\pi_\lambda \circ T_p \Phi_f(\Xi) = T_p(\pi_\lambda \circ \Phi_f)(\Xi) = 0$. However, $\pi_\lambda \circ \Phi_f : P \to M : p \mapsto [p, f]_\lambda \to \pi(p)$. Accordingly, $T_p \pi(\Xi) = 0$ and also Ξ is vertical. But if Ξ is vertical and horizontal then it is $\Xi = 0$, and then $\hat{\Xi} = 0$. Hence, $H_{[p,f]_\lambda} \oplus V_{[p,f]_\lambda}(\pi_\lambda)$ is a direct sum.

To show that $H_{[p,f]_\lambda}$ is a connection on $P \times_\lambda F$, we just have to show that a vector $\hat{\Xi} \in T_{[p,f]_\lambda} P \times_\lambda F$ can be expanded as a sum of a vertical and an horizontal vector. Let us start by considering the horizontal vector $h(\hat{\Xi}) := T\Phi_f \circ \omega \circ T\pi_\lambda(\hat{\Xi}) \in H_{[p,f]_\lambda}$ where ω denotes the horizontal lift on P. We just have to show that $\hat{\Xi} - h(\hat{\Xi})$ is vertical. For,

$$T\pi_\lambda(\hat{\Xi} - h(\hat{\Xi})) = T\pi_\lambda(\hat{\Xi}) - T(\pi_\lambda \circ \Phi_f) \circ \omega \circ T\pi_\lambda(\hat{\Xi}) = T\pi_\lambda(\hat{\Xi}) - T\pi \circ \omega \circ T\pi_\lambda(\hat{\Xi}) = T\pi_\lambda(\hat{\Xi}) - T\pi_\lambda(\hat{\Xi}) = 0 \tag{18.5.24}$$

Then let us denote by $v(\hat{\Xi}) := \hat{\Xi} - h(\hat{\Xi}) \in V(\pi_\lambda)$ and one has that any $\hat{\Xi} \in T_{[p,f]_\lambda} P \times_\lambda F$ can be split as

$$\hat{\Xi} = h(\hat{\Xi}) \oplus v(\hat{\Xi}) \tag{18.5.25}$$

The connection $H_{[p,f]_\lambda}$ defined on $P \times_\lambda F$ is called the *associated connection* to the principal connection H on $\mathcal{P}$.

All this is important because it shows that connections can be always induced by principal connections. Remember that, in the physical applications, all the bundles we shall consider will be somehow associated to a principal bundle and this is a safe way to characterise connections.

The general definition above, however, is quite weak when it is time to actually *compute* the associated connection. Thus let us collect some examples of how connections are actually dealt with.

Connections on the frame bundle

We already showed that the frame bundle $L(M)$ is a principal bundle for the group $\mathrm{GL}(m)$ and with fibered coordinates (x^μ, e_a^μ). Transition functions are given in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e_a'^\mu = J_\nu^\mu(x) e_a^\nu \end{cases} \tag{18.5.26}$$

The right action of G on $L(M)$ is given by $R_\alpha : L(M) \to L(M) : e_a \mapsto e_b \alpha_a^b$.

We need a right-invariant pointwise basis of vertical vectors on $L(M)$. Let us just consider the (local) vector fields

$$\rho_\nu^\mu = e_a^\mu \partial_\nu^a \tag{18.5.27}$$

They are vertical and independent. Then let us just check that they are right-invariant.

The tangent map of the right action is

$$\begin{cases} TR_\alpha(\partial_\mu) = \partial_\mu \\ TR_\alpha(\partial_\mu^a) = \alpha_b^a \partial_\mu'^b \end{cases} \tag{18.5.28}$$

Then we have

$$TR_\alpha(\rho_\nu^\mu(x^\lambda, e_c^\lambda)) = e_a^\mu TR_\alpha(\partial_\nu^a) = e_a^\mu \alpha_b^a \partial_\nu'^b = e_a'^\mu \partial_\nu'^b = \rho_\nu^\mu(x'^\lambda, e_c'^\lambda) \tag{18.5.29}$$

Since we have a right-invariant basis of vertical vectors, we know that any principal connection on $L(M)$ can be written as

$$\omega = dx^\mu \otimes \left(\partial_\mu - \Gamma_{\beta\mu}^\alpha(x) \rho_\alpha^\beta \right) \tag{18.5.30}$$

Then at least principal connections are locally described by a set of coefficients $\Gamma_{\beta\mu}^\alpha(x)$ as we defined above. A crucial role is played by transformation rules. Both coefficients of principal connections and connections on a manifold come with a transformation rule. If we want to identify principal connections and connections on a manifold, they must come with the same transformation rule so that local descriptions glue together to become a global identification.

Transformation rules for coefficients of a principal connection have been obtained above in general for an arbitrary connection, in an arbitrary principal bundle. The framework we set up for that is quite complex; it encodes the specific group in a number of matrices $L_A^a, R_A^a, \mathrm{ad}_B^A, \dots$ which depend on the details of the group involved. Instead of specifying the result, it is probably better and simpler to get inspiration from the method and repeat the computation from scratch.

We have to determine how $\Gamma_{\beta\mu}^\alpha$ transform under a change of fibered coordinates (18.5.26) on $L(M)$. The basis of vectors changes as

$$\begin{cases} \partial_\mu = J_\mu^\nu \partial_\nu' + J_{\nu\mu}^\epsilon e_a^\nu \partial_\epsilon'^a \\ \partial_\nu^a = J_\nu^\mu \partial_\mu'^a \end{cases} \Rightarrow \begin{cases} \partial_\mu = J_\mu^\nu \partial_\nu' + J_{\nu\mu}^\epsilon e_a^\nu \partial_\epsilon'^a = J_\mu^\nu \partial_\nu' + J_{\nu\mu}^\epsilon \bar{J}_\lambda^\nu \rho_\epsilon'^\lambda \\ \rho_\nu^\mu = e_a^\mu \partial_\nu^a = J_\nu^\epsilon e_a'^\mu \partial_\epsilon'^a = \bar{J}_\lambda^\mu \rho_\epsilon'^\lambda J_\nu^\epsilon \end{cases} \tag{18.5.31}$$

Then the connection on $L(M)$ transforms as

$$\omega = dx^\mu \otimes \left(\partial_\mu - \Gamma_{\beta\mu}^\alpha \rho_\alpha^\beta \right) = dx'^\lambda \otimes \left(\partial_\lambda' - \left(J_\alpha^\epsilon \Gamma_{\beta\mu}^\alpha \bar{J}_\sigma^\beta \bar{J}_\lambda^\mu - \bar{J}_\lambda^\mu J_{\nu\mu}^\epsilon \bar{J}_\sigma^\nu \right) \rho_\epsilon'^\sigma \right) = dx'^\lambda \otimes \left(\partial_\lambda' - J_\alpha^\epsilon \left(\Gamma_{\beta\mu}^\alpha \bar{J}_\sigma^\beta \bar{J}_\lambda^\mu + \bar{J}_{\sigma\lambda}^\alpha \right) \rho_\epsilon'^\sigma \right) \tag{18.5.32}$$

i.e.

$$\Gamma'^{\epsilon}_{\sigma\lambda}(x') = J^{\epsilon}_{\alpha}\left(\Gamma^{\alpha}_{\beta\mu}(x)\bar{J}^{\beta}_{\sigma}\bar{J}^{\mu}_{\lambda} + \bar{J}^{\alpha}_{\sigma\lambda}\right) \tag{18.5.33}$$

which are exactly the transformation rules we postulated for connections on manifolds.

We can then identify connections on manifolds with principal connections on the frame bundles. This identification is not just a matter of notation or curiosity. There are operations and structures that we defined for principal connections which in view of this identification extends as they are to connections on manifolds.

Connections on the tangent bundle

The tangent bundle $TM = L(M) \times_{\lambda} \mathbb{R}^m$ can be defined as an associated bundle to $L(M)$. A connection $\omega = dx^{\mu} \otimes \left(\partial_{\mu} - \Gamma^{\alpha}_{\beta\mu}\rho^{\beta}_{\alpha}\right)$ on $L(M)$ induces an associated connection $\hat{\omega}$ on TM. The horizontal spaces of the connection ω are

$$H = \text{Span}\left(\partial_{\mu} - \Gamma^{\alpha}_{\beta\mu}\rho^{\beta}_{\alpha}\right) \tag{18.5.34}$$

The map $\Phi_v : L(M) \to TM : e_a \mapsto [e_a, v]_{\lambda}$ is locally given by $\Phi_v : (x^{\mu}, e^{\mu}_a) \mapsto (x^{\mu}, v^{\mu} = e^{\mu}_a v^a)$. Then its tangent map is

$$\begin{cases} T\Phi_v(\partial_{\mu}) = \partial_{\mu} \\ T\Phi_v(\partial^a_{\mu}) = v^a \dot{\partial}_{\mu} \end{cases} \qquad \text{where we set } \dot{\partial}_{\mu} = \frac{\partial}{\partial v^{\mu}} \tag{18.5.35}$$

Then the horizontal subspaces in TM are given by

$$\hat{H} = \text{Span}\left(\partial_{\mu} - \Gamma^{\alpha}_{\beta\mu}v^{\beta}\dot{\partial}_{\alpha}\right) \tag{18.5.36}$$

and the associated connection on the tangent bundle is

$$\hat{\omega} = dx^{\mu} \otimes \left(\partial_{\mu} - \Gamma^{\alpha}_{\beta\mu}v^{\beta}\dot{\partial}_{\alpha}\right) \tag{18.5.37}$$

We can repeat the same thing on the cotangent bundle. The map $\Phi^*_{\alpha} : L(M) \to T^*M : e_a \mapsto [e_a, \alpha]_{\lambda}$ is locally given by $\Phi^*_{\alpha} : (x^{\mu}, e^{\mu}_a) \mapsto (x^{\mu}, \alpha_{\mu} = \alpha_a e^a_{\mu})$, where e^a_{μ} denotes the inverse matrix of e^{μ}_b.

Then its tangent map is

$$\begin{cases} T\Phi^*_{\alpha}(\partial_{\mu}) = \partial_{\mu} \\ T\Phi^*_{\alpha}(\partial^a_{\mu}) = -\alpha_b e^b_{\mu} e^a_{\nu} \partial^{\nu} \end{cases} \qquad \text{where we set } \partial^{\mu} = \frac{\partial}{\partial \alpha_{\mu}} \tag{18.5.38}$$

Then the horizontal subspaces in T^*M are given by

$$\hat{H}^* = \text{Span}\left(\partial_{\mu} + \Gamma^{\alpha}_{\beta\mu}\alpha_{\alpha}\partial^{\beta}\right) \tag{18.5.39}$$

and the associated connection on the tangent bundle is

$$\hat{\omega}^* = dx^{\mu} \otimes \left(\partial_{\mu} + \Gamma^{\alpha}_{\beta\mu}\alpha_{\alpha}\partial^{\beta}\right) \tag{18.5.40}$$

Connections induced on tensor bundles

As usual, once one knows how to deal with tangent vectors and covectors, then the extension to any tensor field is easy. For example, the tensor bundle $T^1_1(M)$ of $(1,1)$-tensors is also associated to $L(M)$ with an action which comes from the transformation rules of $(1,1)$-tensors, namely

$$\lambda : \mathrm{GL}(m) \times T^1_1 \to T^1_1 : (J, t^a_b) \mapsto t'^a_b = J^a_c t^c_d \bar{J}^d_b \tag{18.5.41}$$

so that we have fibered coordinates $(x^\mu, t^\mu_\nu := e^\mu_a t^a_b e^b_\nu)$ on $T^1_1(M)$.

The map $\Phi_t : L(M) \to T^1_1(M) : e_a \mapsto [e_a, t]_\lambda$ is locally given by $\Phi_t : (x^\mu, e^\mu_a) \mapsto (x^\mu, t^\mu_\nu = e^\mu_a t^a_b e^b_\nu)$. Then its tangent map is

$$\begin{cases} T\Phi_t(\partial_\mu) = \partial_\mu \\ T\Phi_t(\partial^a_\mu) = e^a_\alpha t^\alpha_\nu \partial^\nu_\mu - t^\rho_\mu e^a_\nu \partial^\nu_\rho \end{cases} \qquad \text{where we set } \partial^\nu_\mu = \frac{\partial}{\partial t^\mu_\nu} \tag{18.5.42}$$

from which we get

$$T\Phi_t(\rho^\nu_\mu) = t^\nu_\rho \partial^\rho_\mu - t^\rho_\mu \partial^\nu_\rho \tag{18.5.43}$$

Then the horizontal subspaces in $T^1_1(M)$ are given by

$$\hat{H} = \mathrm{Span}\left(\partial_\mu - \Gamma^\alpha_{\beta\mu} t^\beta_\rho \partial^\rho_\alpha + \Gamma^\alpha_{\beta\mu} t^\rho_\alpha \partial^\beta_\rho\right) \tag{18.5.44}$$

and the associated connection on the tangent bundle is

$$\hat{\omega} = dx^\mu \otimes \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} t^\beta_\rho \partial^\rho_\alpha + \Gamma^\alpha_{\beta\mu} t^\rho_\alpha \partial^\beta_\rho\right) \tag{18.5.45}$$

So one sees the usual rule appearing: one term for each index, with a sign for any up index, with the opposite sign for lower indices.

Since we are about it, let us consider a vector density f^μ of weight w. Because of the weight, the map $\Phi_f : L(M) \to \mathcal{D}^1(M)$ is locally given by $\Phi_f : (x^\mu, e^\mu_a) \mapsto (x^\mu, f^\mu = \det^{-w}(e) e^\mu_a f^a)$. Then its tangent map is

$$\begin{cases} T\Phi_f(\partial_\mu) = \partial_\mu \\ T\Phi_f(\partial^a_\mu) = e^a_\alpha f^\alpha \dfrac{\partial}{\partial f^\mu} - w e^a_\mu f^\rho \dfrac{\partial}{\partial f^\rho} \end{cases} \tag{18.5.46}$$

from which we get

$$T\Phi_f(\rho^\nu_\mu) = f^\nu \frac{\partial}{\partial f^\mu} - w \delta^\nu_\mu f^\rho \frac{\partial}{\partial f^\rho} \tag{18.5.47}$$

Then the horizontal subspaces in $\mathcal{D}^1(M)$ are given by

$$\hat{H} = \mathrm{Span}\left(\frac{\partial}{\partial x^\mu} - \Gamma^\alpha_{\beta\mu} f^\beta \frac{\partial}{\partial f^\alpha} + w \Gamma^\alpha_{\alpha\mu} f^\rho \frac{\partial}{\partial f^\rho}\right) \tag{18.5.48}$$

and the associated connection on $\mathcal{D}^1(M)$ is

$$\hat{\omega} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} - \Gamma^\alpha_{\beta\mu} f^\beta \frac{\partial}{\partial f^\alpha} + w\Gamma^\alpha_{\alpha\mu} f^\rho \frac{\partial}{\partial f^\rho} \right) \tag{18.5.49}$$

from which one can see how the weight contributes by its own term.

Here the essential thing to catch is that one can *compute* the induced connection (and, consequently, define the covariant derivative as below) of an object. No need to guess it. Rather on the contrary, the connection (and later the covariant derivative) is linked to the bundle in which objects live; if one needs to add terms to the covariant derivative to make things work, that is a powerful indication that objects have been regarded as sections of the wrong bundle.

We find many instances of this issue. When we introduce charged Klein–Gordon field, we saw that it was *wise* to modify the derivative to

$$\overset{A}{\nabla}_\mu \varphi = \overset{*}{\nabla}_\mu \varphi - iqA_\mu \varphi \tag{18.5.50}$$

to take into account the coupling to the electromagnetic field; see (1.10.102). This is an indication that the field φ is not a scalar field, i.e. it is not a section of the bundle $M \times \mathbb{R}$. If it were, it would transform as a scalar field, i.e. as $\varphi'(x') = \varphi(x)$. While this is done to account for gauge transformations, under which the field transforms as $\varphi' = e^{iq\alpha}\varphi$; see (1.10.101), it precisely tells us that it is not a scalar field. That is why one introduces it as a section of a bundle $(P \times_\lambda \mathbb{C})$ which is associated to a $U(1)$-principal bundle P. If one considers the induced connection on $(P \times_\lambda \mathbb{C})$ induced by the electromagnetic field A_μ which is there regarded exactly as a connection on P, then it would find

$$\hat{\omega} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + iqA_\mu \varphi \frac{\partial}{\partial \varphi} - iqA_\mu \varphi^\dagger \frac{\partial}{\partial \varphi^\dagger} \right) \tag{18.5.51}$$

which precisely accounts for the modified gauge-covariant derivative.

A similar situation happens with Vielbein, which, in fact, are not sections of $L(M)$; compare with (2.7.62) where the covariant derivative gets an extra term from the spin connection which would not be there if they were sections of $L(M)$.

Covariant derivatives of sections

Given a connection ω in a bundle $\mathcal{B} = (B, M, \pi, F)$, we have two ways of lifting base vectors.

One is the horizontal lift $\omega_b(\xi) \in T_b B$, the second is by the map $T_x\sigma : T_x M \to T_{\sigma(x)}B$. Then by choosing $b = \sigma(x)$ we can define

$$\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi) \tag{18.5.52}$$

The vector $T\sigma(\xi) - \omega(\xi) \in T_{\sigma(x)}B$ is vertical.

In fact, we have

$$T\pi \left(T\sigma(\xi) - \omega(\xi) \right) = T(\pi \circ \sigma)(\xi) - \mathrm{id}_{T_x M}(\xi) = \mathrm{id}_{T_x M}(\xi) - \mathrm{id}_{T_x M}(\xi) = 0 \tag{18.5.53}$$

Then $\nabla_\xi \sigma$ is a vertical vector field defined on the image of σ, i.e. a section of the bundle $\pi : V(B) \to M$ (which covers the section σ).

Locally, one has

$$\nabla_\xi \sigma = T\sigma(\xi) - \omega(\xi) = \xi^\mu \left(\partial_\mu \sigma^i(x) + \omega^i_\mu(x, \sigma(x)) \right) \partial_i \tag{18.5.54}$$

This is the most general notion of covariant derivative. It applies to any object which can be seen as a section on a bundle on which a connection is defined. For example, tensor fields are sections of bundles which are associated to $L(M)$. Then a connection on $L(M)$ induces a connection on all tensor bundles, and then the covariant derivatives of tensor fields is defined. Finally, one uses a number of coincidences which happen on tensor bundles to obtained the covariant derivatives as we defined them above.

Let us stress that, by this framework, covariant derivatives are not *defined* as better behaving derivatives (as we did above). They can be *computed* since they all are induced by the connection on $L(M)$.

A section $X : M \to TM$ of the tangent bundle is a vector field locally given by $X : x^\mu \mapsto (x^\mu, X^\mu(x))$. Its covariant derivative is defined as

$$\nabla_\xi X = \xi^\mu \left(\partial_\mu X^\alpha + \Gamma^\alpha_{\beta\mu} X^\beta \right) \dot\partial_\alpha \tag{18.5.55}$$

Now, except for the fact that this is a vertical vector on TM (or, better, a section of $VTM \to M$) instead of being a section of TM, the local expression $\nabla_\mu X^\alpha = \partial_\mu X^\alpha + \Gamma^\alpha_{\beta\mu} X^\beta$ exactly reproduces the definition we gave above.

On VTM, we can define the swap map which maps sections of $VTM \to M$ to sections of TM. On VTM, one has coordinates (x^μ, v^μ, ξ^μ) defining a vertical vector $\Xi = \xi^\mu \dot\partial_\mu$ at the point $(x, v) \in TM$. By changing coordinates on M, one gets transition functions in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ v'^\mu = J^\mu_\nu(x) v^\nu \\ \xi'^\mu = J^\mu_\nu(x) \xi^\nu \end{cases} \tag{18.5.56}$$

[This is obtained simply by noticing that the basis for vertical vectors changes as $\dot\partial_\mu = J^\nu_\mu \dot\partial'_\nu$.] Since v^μ and ξ^μ transform in the same way, one can define a global map which is locally represented by

$$\mathrm{swap} : VTM \to VTM : (x^\mu, v^\mu, \xi^\mu) \mapsto (x^\mu, \xi^\mu, v^\mu) \tag{18.5.57}$$

By denoting with $\tau : VTM \to TM$ the bundle projection, one has that $\tau \circ \mathrm{swap} \circ \nabla_\xi X : M \to TM : x^\mu \to (x^\mu, \xi^\mu \left(\partial_\mu X^\alpha + \Gamma^\alpha_{\beta\mu} X^\beta \right))$, i.e. a vector field

$$\tau \circ \mathrm{swap} \circ \nabla_\xi X := \xi^\mu \left(\partial_\mu X^\alpha + \Gamma^\alpha_{\beta\mu} X^\beta \right) \partial_\alpha \tag{18.5.58}$$

which agrees with the definition given above. Then in current notation, the covariant derivative of a vector field X defines a vector field $\tau \circ \mathrm{swap} \circ \nabla_\xi X$ which is also, by an abuse of notation, also denoted as $\nabla_\xi X$ by understanding swap map and projection.

This trick works on any vector bundle $[p : E \to M]$. We can generalize the swap map to $\mathrm{swap} : VE \to VE$ and then consider $\pi \circ \mathrm{swap}$ to obtain a section of E.

Similar swap maps exist, for example, in all tensor bundles and they behaves as on the tangent bundle. Then we have a fundamental notion of covariant derivative which is defined on sections, but it is not section on the same bundle. When the swap map is there one can transform that general covariant derivative into the usual one so that the covariant derivative of a tensor field is a tensor field of the same rank.

6. Parallel transport and holonomy

Given a connection we can define the parallel transport along a curve. Often this is done on a manifold M, in fact transporting points in the tangent bundle TM, i.e. the parallel transport is usually introduced as a way of using a connection on M to parallely transport tangent vectors along curves. This structure will follow as a special case of a more general construction made on principal connections defined on principal bundles P. This more general definition of parallel transport has direct applications to gauge theories, so that there is really no advantage in restricting to the special case.

Given a bundle $[B \to M]$ with a connection ω and a curve $\gamma : [0,1] \to M$ on the base manifold we can define maps among fibers $\mathcal{P}\omega : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(s))$ which describe parallel transport. In general one does not transport tangent vectors, one transports points in a fiber in other fibers.

In the very special case when the bundle is $[TM \to M]$ points in the fibers are in fact vectors tangent to M along the curve γ.

On principal bundles in particular the group structure kicks in and allows extra structure which is called holonomy. Eventually, one can describe a class of connections (up to principal morphisms on P, equivalently in terms of some class of holonomies. Holonomies provide us with a non-local description of a gauge class of connections which, however, transforms nicely unlike the local description given in terms of the components of the connection,

As a matter of fact, that is the framework to deal with gauge theory on a lattice, non-perturbative QFT, as well as loop quantum gravity,.

Parallel transport of principal connections

Let $\mathcal{P} = (P, M, \pi, G)$ be a principal bundle, H_p be a (principal) connection on it, and $\gamma : [0,1] \to M$ be a curve in the base space M.

A curve $\hat{\gamma} : [0,1] \to P$ is called a *lift* of γ iff $\pi \circ \hat{\gamma} = \gamma$. Of course, for any curve γ, one has infinitely many lift curves $\hat{\gamma}$.

A general curve is locally expressed in a trivialisation as $\hat{\gamma} : [0,1] \to P : s \mapsto (x(s), g(s))$. That is a lift of $\gamma : [0,1] \to M : s \mapsto x = \gamma(s)$ iff $x(s) = \gamma(s)$. The *vertical part of the curve*, namely $g(s)$, is completely unconstrained.

Among all possible lifts of a curve γ, we call *horizontal lift* the curve $\hat{\gamma}$ which has the tangent vector that is horizontal.

The tangent vector to $\hat{\gamma}$ is

$$\dot{\hat{\gamma}}(s) = (\gamma(s), \dot{\gamma}^\mu(s)\partial_\mu + \dot{g}^a(s)\partial_a) \tag{18.6.1}$$

That is horizontal iff

$$\dot{\gamma}^\mu \partial_\mu + \dot{g}^a \partial_a = \dot{\gamma}^\mu \left(\partial_\mu - \omega_\mu^A(\gamma(s))\rho_A\right) \qquad \Rightarrow \dot{g}^b(s) = \dot{\gamma}^\mu(s)\omega_\mu^A(\gamma(s))T_A^a R_a^b(g(s)) \tag{18.6.2}$$

Now since, for a given curve $\gamma(s)$ and a given connection $\omega_\mu^A(x)$, one has that

$$\begin{cases} \dot{g}^b(s) = \dot{\gamma}^\mu(s)\omega_\mu^A(\gamma(s))R_A^b(g(s)) \\ g(0) = k \end{cases} \tag{18.6.3}$$

is a well-posed Cauchy problem, the solution exists and it is unique.

Then any curve γ in M allows a unique horizontal lift $\hat{\gamma}$ based at a point $p = [x, k]_\alpha$.

If $\hat{\gamma}$ is the horizontal lift of a curve γ and $\hat{\gamma}(0) = [x, k]_\alpha$ (in the local trivialisation t_α), let us consider the curve $R_g \circ \hat{\gamma}$. Since the right action is vertical, it is another lift of γ and $R_g \circ \hat{\gamma}(0) = [x, k \cdot g]_\alpha$. Moreover, the tangent vector to the curve $R_g \circ \hat{\gamma}$ is

$$TR_g \circ \omega(\dot{\gamma}(s)) \tag{18.6.4}$$

But $TR_g \circ \omega = \omega$ since the connection is principal. Then $R_g \circ \hat{\gamma}$ is a horizontal lift of γ. In view of uniqueness, it is *the* horizontal lift of γ bases at the point $p \cdot g = [x, k \cdot g]_\alpha$.

In other words, the right action translates vertically horizontal lifts into horizontal lifts.

If the curve γ goes from $\gamma(0) = x_0$ to $\gamma(1) = x_1$, we set a starting point $p \in \pi^{-1}(x_0)$, and denote by $\hat{\gamma}_p$ the only horizontal lift of γ based at p, then we have an isomorphism $\mathcal{P}_{(H,\gamma)} : \pi^{-1}(x_0) \to \pi^{-1}(x_1) : p \mapsto \hat{\gamma}_p(1)$ which is called the *parallel transport* along the curve γ with respect to the connection H.

> The map $\mathcal{P}_{(H,\gamma)} : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is one-to-one.
> One can, in fact, prove that the parallel transport $\mathcal{P}_{(H,\sigma)} : \pi^{-1}(x_1) \to \pi^{-1}(x_0)$ along the curve $\sigma(s) = \gamma(1 - s)$ is the inverse to $\mathcal{P}_{(H,\gamma)}$. For, one just needs to notice that, if we set $\mathcal{P}_{(H,\gamma)}(p) = p'$, $\hat{\sigma}_{p'}(s) = \hat{\gamma}_p(1 - s)$ is the horizontal lift of σ.
> Let us just remark that, unlike the usual setting, here we are not parallely transporting tangent vectors, but we are more generally transporting points in the fibers of a principal bundle.

Moreover, if we know parallel transport of a point p, e.g. $\mathcal{P}_{(H,\gamma)}(p) = p'$, then we know parallel transport of the whole fiber

$$\mathcal{P}_{(H,\gamma)} : \pi^{-1}(x_0) \to \pi^{-1}(x_1) : p \cdot g \mapsto p' \cdot g \tag{18.6.5}$$

On a principal bundle $[P \to M]$ one is very tempted to encode parallel transport by fixing a trivialization induced by a local section $\sigma : U \to \pi^{-1}(U) \subset P$ and noting the element of the group $h_{(H,\gamma)} \in G$ such that $\mathcal{P}_{(H,\gamma)}(\sigma(x_0)) = p' = \sigma(x_1) \cdot \overline{h_{(H,\gamma)}}$.

> The bar denotes the inverse. We choose the inverse for being consistent with notation used for holonomy below.
> Once we know $h_{(H,\gamma)}$, we can use it to uniquely obtain $\mathcal{P}_{(H,\gamma)}$, by using (18.6.5).

Unfortunately, we need to stress that the group element $h_{(H,\gamma)}$ depends awkwardly on the trivialization, so that, although it may be convenient convenient, it is not geometrically defined.

> If we consider a new trivialization associated to a new local section $\sigma'(x) = \sigma(x) \cdot \varphi(x)$ for some local map $\varphi : U \to G$, then we have a new element $h'_{H,\gamma}$ such that
>
> $$\mathcal{P}_{(H,\gamma)}(\sigma'(x_0)) = \mathcal{P}_{(H,\gamma)}(\sigma(x_0) \cdot \varphi(x_0)) = \mathcal{P}_{(H,\gamma)}(\sigma(x_0)) \cdot \varphi(x_0) = \sigma(x_1) \cdot \overline{h_{(H,\gamma)}} \cdot \varphi(x_0) = \sigma'(x_1) \cdot \overline{\varphi(x_1)} \cdot \overline{h_{(H,\gamma)}} \cdot \varphi(x_0) = \sigma'(x_1) \cdot \overline{h'_{(H,\gamma)}} \tag{18.6.6}$$
>
> Hence we have
>
> $$h'_{(H,\gamma)} := \overline{\varphi(x_0)} \cdot h_{(H,\gamma)} \cdot \varphi(x_1) \tag{18.6.7}$$
>
> which is called the *gauge transformations* on parallel transport.

Let us remark that is not even an adjoint representation of the group since, when $x_0 \neq x_1$, $\varphi(x_0)$ and $\varphi(x_1)$ are not even necessarily the same.

Parallel transport of associated connections

Once we know parallel transport of a principal connection, we know the parallel transport along any associated connection on any associated bundle. Let $[P \to M]$ be a principal bundle, H_p a principal connection on P, and $\hat{\gamma}_p : [0,1] \to P$ the horizontal lift of a curve $\gamma : [0,1] \to M$.

Let F be a manifold and $\lambda : G \to \mathrm{Diff}(F)$ a left action of the structure group G on F. Let $P \times_\lambda F$ be the associated bundle and $\hat{H}$ the associated connection induced by H on the associated bundle $P \times_\lambda F$.

> Let us remind that by definition of horizontal lift the tangent vectors $\dot{\hat{\gamma}}(s) \in H_{\hat{\gamma}(s)}$ is horizontal with respect to the principal connection H.
>
> Moreover, the horizontal spaces of the associated connection are defined as $\hat{H}_{[p,f]_\lambda} = T_{[p,f]_\lambda}\Phi_f(H_p)$ where $\Phi_f : P \to P \times_\lambda F : p \mapsto [p,f]_\lambda$.

Let us consider the curve $\gamma_\lambda : [0,1] \to P \times_\lambda F : s \mapsto \Phi_f \circ \hat{\gamma}$ and check it is a horizontal lift of γ on the associated bundle. That is trivial since we have

$$\dot{\gamma}_\lambda(s) = T_{\hat{\gamma}(s)}\Phi_f(\dot{\hat{\gamma}}(s)) \in T_{\hat{\gamma}(s)}\Phi_f(H_{\hat{\gamma}(s)}) = \hat{H}_{[p,f]_\lambda} \tag{18.6.8}$$

This shows how the parallel transport on the principal bundle encodes completely all associated parallel transports on any associated bundles of any associated connection.

Then horizontal lift γ_λ on the associated bundle define parallel transport $\mathcal{P}_{(\hat{H},\gamma)} : \pi_\lambda^{-1}(x) \to \pi_\lambda^{-1}(x') : \hat{\gamma}_\lambda(0) \mapsto \hat{\gamma}_\lambda(1)$ on the associated bundle as well.

A curve in TM is locally expressed as $\hat{\gamma} : s \mapsto (\gamma^\mu(s), v^\mu(s))$ and it is a horizontal lift of the curve γ in M iff

$$\dot{\gamma}^\mu\partial_\mu + \dot{v}^\alpha\dot{\partial}_\alpha = \dot{\gamma}^\mu\left(\partial_\mu - \Gamma^\alpha_{\beta\mu}v^\beta\dot{\partial}_\alpha\right) \qquad \dot{v}^\alpha(s) + \Gamma^\alpha_{\beta\mu}(\gamma(s))v^\beta(s)\dot{\gamma}^\mu(s) = 0 \tag{18.6.9}$$

This agrees with equation which determines parallel transport of vectors $v = v^\alpha\partial_\alpha$ along a curve γ.

Holonomy

Let us stress once and for all, that parallel transport of points in the fibers is by no means an element of the group G, even on principal bundles.

> Unlike on vector (or affine) bundles where the fibers are canonically vector (or affine) spaces, on principal bundles fibers are not groups, the *standard fiber* is.
>
> On vector and affine bundles transition maps are restricted to preserve the algebraic structure on the standard fiber. As a result, any local trivialization induces on fibers a linear (or affine) structure and transition maps preserve such a structure which therefore becomes independent of the local trivialization.
>
> If on principal bundles we restricted transition maps to be group homomorphisms, fibers would have a canonical group structure. By the way, these bundles can be defined, they could be called *bundles of groups*, just they are not principal bundles. In principal bundles transition maps are assumed to act by left translation which are not group homomorphisms, in particular they do not fix the identity of the group. In fact, different local trivializations on a principal bundles define different neutral element on each fiber.

In fact, if $\mathcal{P}_{(H,\gamma)}(p) = p'$, then $p \in \pi^{-1}(x_0)$ and $p' \in \pi^{-1}(x_1)$ live in different fibers. The only way to define an element $h \in G$ from parallel transport along an open curve is using a trivialisation σ and set $\mathcal{P}_{(H,\gamma)}(\sigma(x_0)) = \sigma(x_1) \cdot \overline{h}$.

A different story is when $\gamma = \alpha$ is a closed curve, i.e. a loop. In that case $x_1 = x_0$ and $\hat{\alpha}(0), \hat{\alpha}(1) \in \pi^{-1}(x_0)$ are in the same fiber. The parallel transport along a loop is called *holonomy* and it is denoted by $\mathcal{H}_{(H,\alpha)} : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$.

In case of loops, if $\mathcal{H}_{(H,\alpha)} : p \mapsto p'$, both p and p' belongs to the same fiber $\pi^{-1}(x_0)$, and independently of trivializations, there is (since the right action is transitive on the fiber) a unique (since the right action is free) group element $h(H,\alpha;p) = h$ such that $p' = p \cdot \overline{h(H,\alpha;p)}$.

This group element is independent of the trivialization, it depends, however, on the base point p. If we start from $p \cdot g$, we get to

$$p' \cdot g = p \cdot \overline{h(H,\alpha;p)} \cdot g = (p \cdot g) \cdot \overline{g} \cdot \overline{h(H,\alpha;p)} \cdot g = (p \cdot g) \cdot \overline{g \cdot h(H,\alpha;p) \cdot g} \qquad \Rightarrow h(H,\alpha;p\cdot g) = \overline{g} \cdot h(H,\alpha;p) \cdot g \tag{18.6.10}$$

This is much better than gauge transformation of parallel transport, at least acting by a real adjoint action. For example we can define a conjugate class $h(H, \alpha) = [h(H, \alpha; p)]_{Ad}$, which is independent of the starting point in the fiber. Of course, it still depend on the loop and on the connection.

If we consider an automorphism $\Phi : P \to P$ which projects on a diffeomorphism $\phi : M \to M$. This defines a new loop $\alpha' = \phi \circ \alpha$, and a new connection $H'_{\Phi(p)} = T_p \Phi(H_p)$.

If $\hat{\alpha}_p$ is the horizontal lift of the loop α, based at the point p with respect to the connection H, we can consider the curve $\hat{\alpha}'_{\Phi(p)} = \Phi \circ \hat{\alpha}_p$.

The curve $\hat{\alpha}'_{\Phi(p)}$ is a lift of α'.

We have

$$\pi \circ \hat{\alpha}'_{\Phi(p)} = \pi \circ \Phi \circ \hat{\alpha}_p = \phi \circ \pi \circ \hat{\alpha}_p = \phi \circ \alpha = \alpha' \tag{18.6.11}$$

The curve $\hat{\alpha}'_{\Phi(p)}$ is horizontal with respect to the connection H'.

Let us write the tangent vector

$$(\hat{\alpha}'_{\Phi(p)}(s))^{\cdot} = (\Phi \circ \hat{\alpha}_p(s))^{\cdot} = T_{\hat{\alpha}_p(s)}\Phi \circ (\hat{\alpha}_p(s))^{\cdot} \in T_{\hat{\alpha}_p(s)}\Phi(H_{\alpha_p(s)}) = \Phi(H'_{\Phi \circ \hat{\alpha}_p(s)}) = \Phi(H'_{\alpha'_{\Phi(p)}(s)}) \tag{18.6.12}$$

which is hence horizontal with respect to H'.

Then we have

$$\mathcal{H}_{(H', \alpha')}(\Phi(p)) = \Phi(\mathcal{H}_{(H, \alpha)}(p)) = \Phi(p \cdot \overline{h(H, \alpha; p)}) = \Phi(p) \cdot \overline{h(H, \alpha; p)} \qquad \Rightarrow h(H', \alpha'; \Phi(p)) = h(H, \alpha; p) \tag{18.6.13}$$

In other words, automorphism Φ of P simply drags the loop along the diffeomorphism ϕ, without changing the group element.

Holonomy is the starting point for classification for connections. It allows to define quantities associated to connections which are invariant with respect to gauge transformations and are then associated to (and, in the end, completely classify; see[22]) gauge classes of connections.

In order to illustrate this correspondence we need to give loops a *suitable* group structure, define a group homomorphism between classes loops and conjugate classes in the group G.

These maps, actually a class of these maps, encode the holonomy of (a gauge class of) connections. Finally, one can prove that the correspondence between gauge classes of connections and the class of maps that describe holonomies is in fact one-to-one.

Given a connection, one has one and only one holonomy map and, vice versa, given one holonomy map one has a unique connection to produce it, up to automorphisms.

To obtain this result the key fact is giving the loop a suitable group structure. The resulting correspondence is, by construction, independent of gauge transformations. It will give us a way to describe connection by non-local quantities (the holonomy maps) independent of gauge transformation (i.e. automorphisms on P).

Correspondence theorem

Consider a (smooth, oriented) path γ_0 from x_0 to x_1 and a (smooth, oriented) path γ_1 from x_1 to x_2. The two paths can be *concatenated* as

$$\gamma_0 \ltimes \gamma_1 : [0,1] \to M : s \mapsto \begin{cases} \gamma_0(2s) & \left(0 \le s \le \frac{1}{2}\right) \\ \gamma_1(2s-1) & \left(\frac{1}{2} \le s \le 1\right) \end{cases} \tag{18.6.14}$$

Let us stress the concatenated path $\gamma_0 \ltimes \gamma_1$ is from x_0 to x_2, it is oriented, however, is it is not necessarily smooth at x_1.

For regarding concatenation as an operation among paths we need to define oriented paths smooth everywhere but a finite number of points. Let us denote by $\mathbb{P}(x_0, x_1)$ the set of all oriented paths from x_0 to x_1 which are smooth everywhere but a finite number of points. Then we have a concatenation operation defined as $\ltimes : \mathbb{P}(x_0, x_1) \times \mathbb{P}(x_1, x_2) \to \mathbb{P}(x_0, x_2)$. That is far from being a group operation, yet, but it preserves parallel transport, i.e.

$$\mathcal{P}(\omega, \gamma_0 \ltimes \gamma_1) = \mathcal{P}(\omega, \gamma_1) \circ \mathcal{P}(\omega, \gamma_0) \tag{18.6.15}$$

That is not an internal operation on paths, it is defined only when the second path starts at the end of the first one.

If we restrict to (oriented, piecewise smooth) loops $\mathbb{L}(x_0)$ at a point x_0, the concatenation property is satisfied and we have a binary operation $\ltimes : \mathbb{L}(x_0) \times \mathbb{L}(x_0) \to \mathbb{L}(x_0)$ which preserves the holonomy, i.e. for $\alpha \in \mathbb{L}(x_0)$ we have

$$\mathcal{H}_{(\omega,\alpha)}(p) = p \cdot \overline{h(\omega, \alpha; p)} =: p' \tag{18.6.16}$$

and for $\beta \in \mathbb{L}(x_0)$ we have

$$\mathcal{H}_{(\omega,\beta)}(p') = p' \cdot \overline{h(\omega, \beta; p')} \qquad \Rightarrow \mathcal{H}_{(\omega,\alpha \ltimes \beta)}(p) = p \cdot \overline{h(\omega, \alpha; p)} \cdot \overline{h(\omega, \beta; p')} = p \cdot \overline{h(\omega, \beta; p') \cdot h(\omega, \alpha; p)} \tag{18.6.17}$$

By recalling that $h(\omega, \beta; p \cdot \bar{h}) = h \cdot h(\omega, \beta; p) \cdot \bar{h}$, then we have

$$\mathcal{H}_{(\omega,\alpha \ltimes \beta)}(p) = p \cdot \overline{h(\omega, \alpha; p) \cdot h(\omega, \beta; p)} \qquad \Rightarrow h(\omega, \alpha \ltimes \beta; p) = h(\omega, \alpha; p) \cdot h(\omega, \beta; p) \tag{18.6.18}$$

Unfortunately, $\mathbb{L}(x_0)$ is not a group, both because the candidate to be the neutral element, namely the constant loop $c : [0,1] \to M : s \mapsto x_0$, fails to be a neutral element, and because the candidate to be inverse, namely $\alpha^{-1} : [0,1] \to M : s \mapsto \alpha(1-s)$, fails to be inverse (even because we do not have a well-define neutral element to define the inverse).

The solution is to quotient out enough loops to that the group structure becomes well defined.

> Here we have to point out that there are a number of different options to do that. Ideally, one should quotient out the minimal equivalence relation to obtain the group structure. However, as a matter of fact, we shall need further structures later on. Then here we shall jump ahead and propose the final quotient we need to do, for the sake of being brief.

Definition: *two loops $\alpha, \beta \in \mathbb{L}(x_0)$ are said to be holonomic equivalent iff, for any connection ω and any $p_0 \in \pi^{-1}(x_0)$, one has*

$$h(\omega, \alpha; p_0) = h(\omega, \beta; p_0) \tag{18.6.19}$$

This is an equivalent relation on $\mathbb{L}(x_0)$ and we can define the quotient set $\mathbb{H}(x_0) = \mathbb{L}(x_0)/\sim$. An element in $\mathbb{H}(x_0)$ is called a *hoop*, which is the contraction of *holonomic loop*.

Lemma*: we have the following results*

$$\alpha \rtimes c \sim c \rtimes \alpha \sim \alpha \qquad\qquad \alpha^{-1} \rtimes \alpha \sim \alpha \rtimes \alpha^{-1} \sim c \qquad\qquad (18.6.20)$$

Clearly, we have for example

$$h(\omega, \alpha \ltimes c; p_0) = h(\omega, \alpha; p_0) \cdot h(\omega, c; p_0) = h(\omega, \alpha; p_0) \cdot e = h(\omega, \alpha; p_0) \qquad\qquad (18.6.21)$$

as well as

$$h(\omega, \alpha^{-1} \ltimes \alpha; p_0) = h(\omega, \alpha^{-1}; p_0) \cdot h(\omega, \alpha; p_0) = (h(\omega, \alpha; p_0))^{-1} \cdot h(\omega, \alpha; p_0) = e = h(\omega, c; p_0) \qquad (18.6.22)$$

This lemma allows us to define a group structure on $\mathbb{H}(x_0)$. The group operation is the concatenation, the hoop $e = [c]$ is the neutral element, the hoop $\bar\alpha = \alpha^{-1}$ is the inverse of α. Let us also stress that the holonomy (in view of the inverse we postulated in the definition) is a group homomorphism

$$h(\omega; p_0) : \mathbb{H}(x_0) \to G : \alpha \mapsto h(\omega, \alpha; p_0) \qquad\qquad h(\omega; p_0)(c) = e \qquad h(\omega; p_0)(\alpha \ltimes \beta) = h(\omega; p_0)(\alpha) \cdot h(\omega; p_0)(\beta) \qquad (18.6.23)$$

These maps are independent of the base point. If we have a path $\gamma \in \mathbb{P}(x_0, x_1)$ with a horizontal lift $\hat\gamma$ from p_0 and $p_1 = \mathcal{P}(\omega, \gamma; p_0)$, we can define a map $i_\gamma : \mathbb{H}(x_0) \to \mathbb{H}(x_1) : \alpha \mapsto \bar\gamma \ltimes \alpha \ltimes \gamma$, which turns out to be a group isomorphism. Accordingly, we denote by $\mathbb{H}$ the abstract group representing hoops. Moreover, we have

$$h(\omega; p_1)(\bar\gamma \ltimes \alpha \ltimes \gamma) = h(\omega; p_0)(\alpha) \qquad\qquad h(\omega; p_0 \cdot k) = \bar k \cdot h(\omega; p_0) \cdot k \qquad\qquad (18.6.24)$$

Accordingly, if we modify the holonomy map to map into conjugate classes, i.e. $H(\omega) : \mathbb{H} \to G/\mathrm{Ad} : \alpha \mapsto [h(\omega, p_0)]_{Ad}$, for any $p_0 : \pi(p_0) = x_0$.

Definition*: a map $H : \mathbb{H} \to G$ is called Ad-invariant iff $\forall g \in G : H(\alpha) = \bar g \cdot H(\alpha) \cdot g$. The space of Ad-invariant maps is denoted by $Hom(\mathbb{H}, G)/Ad \simeq Hom(\mathbb{H}, G/Ad))$. Ad-invariant maps are also called holonomy maps.*

Finally, we showed that to any connection ω up to automorphisms of P, i.e. to any gauge class of connections $[\omega]$, we can associate a unique holonomy map $H(\omega) : \mathbb{H} \to G/\mathrm{Ad} \in \mathrm{Hom}(\mathbb{H}, G)/\mathrm{Ad} \simeq \mathrm{Hom}(\mathbb{H}, G/\mathrm{Ad}))$ and vice versa. In other words, the holonomy map completely encodes the connection (as well as the bundle P on which it is defined).

7. The curvature of a principal connection

Unlike what happens with ordinary derivatives, covariant derivatives do not need to commute.

We already saw that happening for the connections on manifolds; see (16.2.46). And that happens for pretty much all tensorial objects.

For a 1-form $\alpha \in \Omega^1(M)$ we have

$$
[\nabla_\mu, \nabla_\nu]\alpha_\rho = \left(\nabla_\mu(d_\nu\alpha_\rho - \Gamma^\lambda_{\rho\nu}\alpha_\lambda)\right) - [\mu\nu] = \left(\boxed{d_{\nu\mu}\alpha_\rho} - d_\mu\Gamma^\lambda_{\rho\nu}\alpha_\lambda - \boxed{\Gamma^\lambda_{\rho\nu}d_\mu\alpha_\lambda} - \boxed{\Gamma^\lambda_{\nu\mu}\nabla_\lambda\alpha_\rho} - \Gamma^\sigma_{\rho\mu}(d_\nu\alpha_\sigma - \Gamma^\lambda_{\sigma\nu}\alpha_\lambda)\right) - [\mu\nu] =
$$
$$
= -\left(d_\mu\Gamma^\lambda_{\rho\nu} - d_\nu\Gamma^\lambda_{\rho\mu} + \Gamma^\sigma_{\rho\mu}\Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\lambda_{\sigma\mu}\right)\alpha_\lambda = -R^\lambda{}_{\rho\mu\nu}\alpha_\lambda
$$

(18.7.1)

For a tensor $t \in T^1_1(M)$ tensor we have:

$$
[\nabla_\mu, \nabla_\nu]t^\sigma_\rho = \left(\nabla_\mu(d_\nu t^\sigma_\rho - \Gamma^\lambda_{\rho\nu}t^\sigma_\lambda + \Gamma^\sigma_{\lambda\nu}t^\lambda_\rho)\right) - [\mu\nu] =
$$
$$
= \left(-d_\mu\Gamma^\lambda_{\rho\nu}t^\sigma_\lambda + d_\mu\Gamma^\sigma_{\lambda\nu}t^\lambda_\rho - \Gamma^\lambda_{\rho\nu}d_\mu t^\sigma_\lambda + \Gamma^\sigma_{\lambda\nu}d_\mu t^\lambda_\rho + \Gamma^\sigma_{\theta\mu}(d_\nu t^\theta_\rho - \Gamma^\lambda_{\rho\nu}t^\theta_\lambda + \Gamma^\theta_{\lambda\nu}t^\lambda_\rho) - \Gamma^\theta_{\rho\mu}(d_\nu t^\sigma_\theta - \Gamma^\lambda_{\theta\nu}t^\sigma_\lambda + \Gamma^\sigma_{\lambda\nu}t^\lambda_\theta) - \boxed{\Gamma^\theta_{\nu\mu}\nabla_\nu t^\sigma_\rho}\right) - [\mu\nu] =
$$
$$
= \left(d_\mu\Gamma^\sigma_{\lambda\nu} - d_\nu\Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\theta\mu}\Gamma^\theta_{\lambda\nu} - \Gamma^\sigma_{\theta\nu}\Gamma^\theta_{\lambda\mu}\right)t^\lambda_\rho - \left(d_\mu\Gamma^\lambda_{\rho\nu} - d_\nu\Gamma^\lambda_{\rho\mu} + \Gamma^\theta_{\rho\mu}\Gamma^\lambda_{\theta\nu} - \Gamma^\theta_{\rho\nu}\Gamma^\lambda_{\theta\mu}\right)t^\sigma_\lambda = R^\sigma{}_{\lambda\mu\nu}t^\lambda_\rho - R^\lambda{}_{\rho\mu\nu}t^\sigma_\lambda
$$

(18.7.2)

Actually, for a general tensor, we have one term for each index with the sign depending on the position. We can also consider a vector density $f \in \mathcal{D}^1$ of weight w (see above (16.2.19)):

$$
[\nabla_\mu, \nabla_\nu]f^\sigma = \left(\nabla_\mu(d_\nu f^\sigma + \Gamma^\sigma_{\lambda\nu}f^\lambda - w\Gamma^\lambda_{\lambda\nu}f^\sigma)\right) - [\mu\nu] =
$$
$$
= \left(d_\mu\Gamma^\sigma_{\lambda\nu}f^\lambda - wd_\mu\Gamma^\lambda_{\lambda\nu}f^\sigma + \Gamma^\sigma_{\lambda\nu}d_\mu f^\lambda - w\Gamma^\lambda_{\lambda\nu}d_\mu f^\sigma + \Gamma^\sigma_{\theta\mu}(d_\nu f^\theta + \Gamma^\theta_{\lambda\nu}f^\lambda - w\Gamma^\lambda_{\lambda\nu}f^\theta) - w\Gamma^\theta_{\theta\mu}(d_\nu f^\sigma + \Gamma^\sigma_{\lambda\nu}f^\lambda - \boxed{w\Gamma^\lambda_{\lambda\nu}f^\sigma)}\right) - [\mu\nu] =
$$
$$
= \left(d_\mu\Gamma^\sigma_{\lambda\nu} - d_\nu\Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\theta\mu}\Gamma^\theta_{\lambda\nu} - \Gamma^\sigma_{\theta\nu}\Gamma^\theta_{\lambda\mu}\right)f^\lambda - w\left(d_\mu\Gamma^\lambda_{\lambda\nu} - d_\nu\Gamma^\lambda_{\lambda\mu}\right)f^\sigma - w\left((\Gamma^\sigma_{\lambda\mu}\Gamma^\theta_{\theta\nu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\theta_{\theta\nu})f^\lambda\right) - [\mu\nu] =
$$
$$
= \left(d_\mu\Gamma^\sigma_{\lambda\nu} - d_\nu\Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\theta\mu}\Gamma^\theta_{\lambda\nu} - \Gamma^\sigma_{\theta\nu}\Gamma^\theta_{\lambda\mu}\right)f^\lambda - w\left(d_\mu\Gamma^\lambda_{\lambda\nu} - d_\nu\Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\lambda\mu}\right)f^\sigma = R^\sigma{}_{\lambda\mu\nu}f^\lambda - wR^\lambda{}_{\lambda\mu\nu}f^\sigma
$$

(18.7.3)

which accounts for the contribution due to the weight w (which, of course, vanishes when one is using a Levi Civita connection).

In all tensorial cases, the non-commutation of covariant derivatives is controlled by the curvature tensor (16.2.47).

We can do pretty much the same thing for a principal connection $\omega = dx^\mu(\partial_\mu - \omega^A_\mu(x)\rho_A)$ on a principal bundle P. The easiest way is to consider the associated bundle $P \times_{TAd} \mathfrak{g}$ obtained from the adjoint representation $TAd : G \times \mathfrak{g} \to \mathfrak{g}$, the sections of which are in one-to-one correspondence with sections of the bundle $(VP, M, \tau, \mathfrak{g})$, i.e. with vertical vector fields $\Xi = \xi^A\rho_A$ (defined on a section) in P; transition functions are

$$
\begin{cases} x'^\mu = x'^\mu(x) \\ \xi'^A = \mathrm{Ad}^A_B(\varphi)\xi^B \end{cases}
$$

(18.7.4)

The map $\Phi_\xi : P \to P \times_\xi \mathfrak{g} : (x,g) \mapsto (x, \mathrm{Ad}^A_B(g)\xi^B)$ defines the tangent map

$$
T\Phi_\xi(\partial_\mu) = \partial_\mu \qquad T\Phi_\xi(\partial_a) = \partial_a\mathrm{Ad}^A_B(e)\xi^B\partial_A \qquad (\Rightarrow T\Phi_\xi(\rho_C) = R^a_C(e)\partial_a\mathrm{Ad}^A_B(e)\xi^B\partial_A = -c^A{}_{BC}\xi^B\partial_A)
$$

(18.7.5)

where we used identity (17.1.43). Thus the induced connection is

$$
\omega = dx^\mu \otimes \left(\partial_\mu - \omega^C_\mu c^A{}_{CB}\xi^B\partial_A\right)
$$

(18.7.6)

and the corresponding covariant derivative reads as

$$
\nabla_\mu\xi^A = d_\mu\xi^A + c^A{}_{BC}\omega^B_\mu\xi^C
$$

(18.7.7)

The covariant derivative $\nabla_\mu \xi^A$ is a section of a bundle associated to $P \times_M L(M)$ and one needs also a connection $\Gamma^\alpha_{\beta\mu}$ on M to define the covariant derivative. Show that one has

$$\nabla_\mu \nabla_\nu \xi^A = d_\mu \nabla_\nu \xi^A + c^A{}_{BC} \omega^B_\mu \nabla_\nu \xi^C - \Gamma^\lambda_{\nu\mu} \nabla_\lambda \xi^A \tag{18.7.8}$$

Now we can compute the commutator

$$
\begin{aligned}
{[\nabla_\mu, \nabla_\nu]} \xi^A &= \left(\boxed{d_{\nu\mu} \xi^A} + c^A{}_{BC} d_\mu \omega^B_\nu \xi^C + c^A{}_{BC} \omega^B_\nu d_\mu \xi^C + c^A{}_{BC} \omega^B_\mu \left(d_\nu \xi^C + c^C{}_{ED} \omega^E_\nu \xi^D \right) \right) - [\mu\nu] = \\
&= c^A{}_{BC} \left(d_\mu \omega^B_\nu - d_\nu \omega^B_\mu + 2 c^B{}_{DE} \omega^D_\mu \omega^E_\nu \right) \xi^C + \left(c^A{}_{DE} c^D{}_{CB} + c^A{}_{DB} c^D{}_{EC} \right) \omega^B_\mu \omega^E_\nu \xi^C = \\
&= c^A{}_{BC} \left(d_\mu \omega^B_\nu - d_\nu \omega^B_\mu + 2 c^B{}_{DE} \omega^D_\mu \omega^E_\nu \right) \xi^C - c^A{}_{BC} \left(c^B{}_{DE} \omega^D_\mu \omega^E_\nu \right) \xi^C = \\
&= c^A{}_{BC} \left(d_\mu \omega^B_\nu - d_\nu \omega^B_\mu + c^B{}_{DE} \omega^D_\mu \omega^E_\nu \right) \xi^C =: c^A{}_{BC} F^B_{\mu\nu} \xi^C
\end{aligned}
\tag{18.7.9}
$$

where we set

$$F^A_{\mu\nu} := d_\mu \omega^A_\nu - d_\nu \omega^A_\mu + c^A{}_{BC} \omega^B_\mu \omega^C_\nu \tag{18.7.10}$$

In view of ad-invariance of structure constants, this object must transform as

$$F'^A_{\mu\nu} = \mathrm{ad}^A_B(\varphi) F^B_{\rho\sigma} \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \tag{18.7.11}$$

Thus we define the *curvature* of the principal connection ω to be the (gauge invariant) 2-form F valued in the Lie algebra $\mathfrak{g}$ defined as

$$F = \tfrac{1}{2} F^A_{\mu\nu} \rho_A \otimes dx^\mu \wedge dx^\nu \tag{18.7.12}$$

Notice that, in general, the coefficients of the curvature are Ad-covariant, not invariant. Thus they are not in general gauge invariant and in a gauge theory they are not observables.

However, there are combinations of them which happens to be gauge invariant. They are called *invariant polynomials*, e.g. the trace (in the Lie algebra) is an Ad-invariant polynomial in matrix groups, as well as the trace of powers. In a general gauge theory, the invariant polynomials are observables, while the curvature itself is not. Of course, in electromagnetism the gauge group $U(1)$ is commutative, the adjoint action is trivial and the curvature is observable, though that is a special property of electromagnetism.

We can show that one has Bianchi identities for the curvature, namely

$$\nabla_{[\lambda} F^A_{\mu\nu]} = 0 \tag{18.7.13}$$

The covariant derivative of the curvature is defined as

$$\nabla_\lambda F^A_{\mu\nu} = d_\lambda F^A_{\mu\nu} + c^A{}_{BC} \omega^B_\lambda F^C_{\mu\nu} - \Gamma^\rho_{\mu\lambda} F^A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} F^A_{\mu\rho} \tag{18.7.14}$$

Thus one has

$$
\begin{aligned}
\nabla_{[\lambda} F^A_{\mu\nu]} =& c^A{}_{BC} d_\lambda \omega^B_\mu \omega^C_\nu - c^A{}_{BC} d_\lambda \omega^B_\nu \omega^C_\mu - c^A{}_{BC} F^B_{\mu\nu} \omega^C_\lambda + c^A{}_{BC} d_\mu \omega^B_\nu \omega^C_\lambda - c^A{}_{BC} d_\mu \omega^B_\lambda \omega^C_\nu - c^A{}_{BC} F^B_{\nu\lambda} \omega^C_\mu + \\
&+ c^A{}_{BC} d_\nu \omega^B_\lambda \omega^C_\mu - c^A{}_{BC} d_\nu \omega^B_\mu \omega^C_\lambda - c^A{}_{BC} F^B_{\lambda\mu} \omega^C_\nu = \\
=& c^A{}_{BC} \left(d_\lambda \omega^B_\mu - d_\mu \omega^B_\lambda - F^B_{\lambda\mu} \right) \omega^C_\nu + c^A{}_{BC} \left(d_\nu \omega^B_\lambda - d_\lambda \omega^B_\nu - F^B_{\nu\lambda} \right) \omega^C_\mu + c^A{}_{BC} \left(d_\mu \omega^B_\nu - d_\nu \omega^B_\mu - F^B_{\mu\nu} \right) \omega^C_\lambda = \\
=& - c^A{}_{BC} c^B{}_{DE} \left(\omega^D_\lambda \omega^E_\mu \omega^C_\nu + \omega^D_\nu \omega^E_\lambda \omega^C_\mu + \omega^D_\mu \omega^E_\nu \omega^C_\lambda \right) = - \left(c^A{}_{BC} c^B{}_{DE} + c^A{}_{BD} c^B{}_{EC} + c^A{}_{BE} c^B{}_{CD} \right) \omega^D_\lambda \omega^E_\mu \omega^C_\nu = 0
\end{aligned}
\tag{18.7.15}
$$

8. Multisymplectic bundles

As for cotangent bundles and their symplectic structures, we have bundles with a canonical multisymplectic structure on them.

Given a bundle $[B \to M]$, we define the *first multisymplectic bundle* $Z(B)$. Points z in $Z(B)$ are m-forms on B such that $Y \lrcorner X \lrcorner z = 0$ for any pair of vertical vectors X, Y on B. That means that the most general form z reads locally as

$$
z = f \, d\sigma + f^\mu_i \, dy^i \wedge d\sigma_\mu = \left(f \delta^\mu_\alpha \, dx^\alpha + f^\mu_i \, dy^i \right) \wedge d\sigma_\mu
\tag{18.8.1}
$$

If we fix coordinates (x^μ, y^i) on B, they induce coordinates $(x^\mu, y^i; f^\mu_i, f)$ on $Z(B)$. By changing coordinates on B by

$$
x'^\mu = x'^\mu(x) \qquad\qquad y'^i = y'^i(x, y)
\tag{18.8.2}
$$

we have $d\sigma' = J d\sigma$, $d\sigma'_\mu = J \bar{J}^\alpha_\mu d\sigma_\alpha$, $dy'^i = J^i_\alpha dx^\alpha + J^i_k dy^k$. Then we have

$$
z' = f' d\sigma' + f'^\mu_i dy'^i \wedge d\sigma'_\mu = J f' d\sigma + J f'^\mu_i \bar{J}^\alpha_\mu \left(J^i_\beta dx^\beta + J^i_k dy^k \right) \wedge d\sigma_\alpha = J \left(f' + \bar{J}^\alpha_\mu f'^\mu_i J^i_\alpha \right) d\sigma + J \bar{J}^\alpha_\mu f'^\mu_i J^i_k \, dy^k \wedge d\sigma_\alpha
\tag{18.8.3}
$$

which induces the transformation laws on $Z(B)$ as

$$
f = J \left(f' + \bar{J}^\alpha_\mu f'^\mu_i J^i_\alpha \right) \qquad\qquad f^\alpha_k = J \bar{J}^\alpha_\mu f'^\mu_i J^i_k
\tag{18.8.4}
$$

That suggests that $Z(\cdot)$ is a (controvariant) functor. If $z' \in Z^1(B')$ and $\Phi : B \to B'$ is a bundle map, then $z = Z(\Phi)(z') \in Z(B)$.

Let us consider a fibered isomorphism $\Phi : B \to B'$ locally given by

$$
x'^\mu = \phi^\mu(x) \qquad\qquad y'^i = \phi^i(x, y)
\tag{18.8.5}
$$

As usual, we define the Jacobians

$$
J^\alpha_\mu = \frac{\partial \phi^\alpha}{\partial x^\mu}(x) \qquad\qquad J^i_\mu = \frac{\partial \phi^i}{\partial x^\mu}(x, y) \qquad\qquad J^i_k = \frac{\partial \phi^i}{\partial y^k}(x, y)
\tag{18.8.6}
$$

We can define a map $Z(\Phi) : Z(B') \to Z(B) : f' d\sigma' + f'^\mu_i dy'^i \wedge d\sigma'_\mu \mapsto f d\sigma + f^\mu_i dy^i \wedge d\sigma_\mu$ given by (18.8.4).

As one can define the canonical Liouville form on cotangent bundles, one can check that the m-form

$$\boldsymbol{\Theta} = f\,d\sigma + f_i^\mu\,dy^i \wedge d\sigma_\mu \tag{18.8.7}$$

is canonical on $Z(B)$ which is called the *canonical multisymplectic potential*. Another, intrinsic, characterization of the multisymplectic potential $\boldsymbol{\Theta}$ can be given in terms of the action on sections.

Let $\sigma : M \to Z(B) : x \mapsto (x, y^i(x), f_i^\mu(x), f(x))$ be a section which projects on a section $\underline{\sigma} : M \to B : x \mapsto (x, y^i(x))$. Then we have $\sigma^*\boldsymbol{\Theta} = \underline{\sigma}^* z_\sigma$, where we set $z_\sigma = f(x)d\sigma + f_i^\mu(x)dy^i \wedge d\sigma_\mu$, i.e. the point $z \in Z(B)$ regarded as an m-form on B. In fact, we have

$$\sigma^*\boldsymbol{\Theta} = \left(f(x) + f_i^\mu(x)d_\mu y^i(x)\right)d\sigma = \underline{\sigma}^* z_\sigma \tag{18.8.8}$$

We can also define $\boldsymbol{\Omega} := d\boldsymbol{\Theta}$ which is a canonical $(m+1)$-form on $Z(B)$ called the *multisymplectic form*. It is trivially a closed form (since it is, in fact, exact) and it is also non-degenerate.

The multisymplectic form $\boldsymbol{\Omega}$ can be regarded as a map $\boldsymbol{\Omega}_z : T_z(Z(B)) \to \Lambda_z^m(Z(B)) : v \mapsto v \lrcorner \boldsymbol{\Omega}$ at any point $z \in Z(B)$. Thus, we say that it is non-degenerate when $v \lrcorner \boldsymbol{\Omega} = 0$ iff $v = 0$. Locally, we have

$$\boldsymbol{\Omega} = df \wedge d\sigma + df_i^\mu \wedge dy^i \wedge d\sigma_\mu \tag{18.8.9}$$

For non-degeneracy, let us consider a vector $v = v^\mu \partial_\mu + v^i \partial_i + v\partial + v_i^\alpha \partial_\alpha^i$ tangent to $Z(B)$ and compute

$$v \lrcorner \boldsymbol{\Omega} = v^\alpha \left(-df \wedge d\sigma_\alpha + df_i^\mu \wedge dy^i \wedge d\sigma_{\mu\alpha}\right) - v^i\,df_i^\mu \wedge d\sigma_\mu + v\,d\sigma + v_i^\mu\,dy^i \wedge d\sigma_\mu \tag{18.8.10}$$

We see that $v \lrcorner \boldsymbol{\Omega} = 0 \iff v = 0$, i.e. $\boldsymbol{\Omega}$ is non-degenerate.

A map $\Phi : Z(B) \to Z(B')$ is called a *multi-symplectomorphism* iff $\Phi^*\boldsymbol{\Omega}' = \boldsymbol{\Omega}$.

Since we can lift flows, we can lift a projectable vector field $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$ to $Z(B)$ as

$$Z(\Xi) = \xi^\mu \partial_\mu + \xi^i \partial_i + \xi_i^\mu \partial_\mu^i + \xi\partial \qquad \begin{cases} \xi := -\partial_\alpha \xi^\alpha f - f_i^\alpha \partial_\alpha \xi^i \\ \xi_i^\mu := \partial_\alpha \xi^\mu f_i^\alpha - f_k^\mu \partial_i \xi^k - \partial_\alpha \xi^\alpha f_i^\mu \end{cases} \tag{18.8.11}$$

Let $X = \xi^\mu \partial_\mu + \xi^i \partial_i + \xi\partial + \xi_i^\mu \partial_\mu^i$ be a vector field on $Z(B)$. It is called *locally Hamiltonian* iff $d(X \lrcorner \boldsymbol{\Omega}) = 0$, or a *(globally) Hamiltonian field* iff there exists an $(m-1)$ form F on $Z(B)$ such that $X \lrcorner \boldsymbol{\Omega} = -dF$; in that case F is called a *Poisson form* and the corresponding Hamiltonian vector field is denoted by X_F.

We can turn a Poisson form F into a functional $F_D[\cdot]$ acting on sections $\sigma : M \to Z(B)$ by setting

$$F_D[\sigma] = \int_D (\sigma)^* F$$

on an $(m-1)$-region $D \subset M$ which we call a *smeared observable*.

On $Z(\mathbb{R} \times Q) = T^*(\mathbb{R} \times Q)$, any 0-form $F(t, q^\lambda, p, p_\lambda)$ is a Poisson form. In fact, in that case,

$$X \lrcorner \boldsymbol{\Omega} = X_p \, dt + X_\lambda \, dq^\lambda - X_t \, dp - X^\lambda \, dp_\lambda = -\partial_t F \, dt - \partial_\alpha F \, dq^\alpha - \partial_p F \, dp - \partial^\lambda F \, dp_\lambda = -dF \tag{18.8.12}$$

Hence, $X_F = \frac{\partial F}{\partial p} \frac{\partial}{\partial t} + \frac{\partial F}{\partial p_\lambda} \frac{\partial}{\partial q^\lambda} - \frac{\partial F}{\partial t} \frac{\partial}{\partial p} - \frac{\partial F}{\partial q^\alpha} \frac{\partial}{\partial p_\alpha}$ is a Hamiltonian field and any F is a Poisson function (or a Poisson 0-form). However, in a more general case, we have $X = X^\mu \partial_\mu + X^i \partial_i + X \partial + X_i^\mu \partial_\mu^i$,

$$X \lrcorner \boldsymbol{\Omega} = X \, d\sigma - X^\mu \, df \wedge d\sigma_\mu + X_i^\mu \, dy^i \wedge d\sigma_\mu - X^i \, df_i^\mu \wedge d\sigma_\mu + X^\alpha \, df_i^\mu \wedge dy^i \wedge d\sigma_{\mu\alpha} \tag{18.8.13}$$

On the other hand, a $(m-1)$-form $F = a^\mu \, d\sigma_\mu + a_i^{\mu\alpha} \, dy^i \wedge d\sigma_{\mu\alpha}$ on $Z(B)$ has a differential

$$dF = \partial_\mu a^\mu \, d\sigma + (\partial_i a^\mu - 2\partial_\alpha a_i^{\mu\alpha}) \, dy^i \wedge d\sigma_\mu + \partial a^\mu \, df \wedge d\sigma_\mu + \partial_\alpha^i a^\mu \, df_i^\alpha \wedge d\sigma_\mu + \partial_k a_i^{\mu\alpha} \, dy^k \wedge dy^i \wedge d\sigma_{\mu\alpha} + \partial a_i^{\mu\alpha} \, df \wedge dy^i \wedge d\sigma_{\mu\alpha} + \partial_\beta^k a_i^{\mu\alpha} \, df_k^\beta \wedge dy^i \wedge d\sigma_{\mu\alpha} \tag{18.8.14}$$

By comparing, we get

$$X = -\partial_\mu a^\mu \qquad X^\mu = \partial a^\mu \qquad X_i^\mu = -\partial_i a^\mu + 2\partial_\alpha a_i^{\mu\alpha} \qquad X^i \delta_\alpha^\mu = \partial_\alpha^i a^\mu \qquad X^{[\alpha} \delta_\beta^{\mu]} \delta_i^k = -\partial_\beta^k a_i^{\mu\alpha} \tag{18.8.15}$$

These can be solved for the vector field components

$$X^i = \tfrac{1}{m} \partial_\mu^i a^\mu \qquad\qquad X^\mu = \tfrac{2}{n(m-1)} \partial_\alpha^i a_i^{\mu\alpha} \tag{18.8.16}$$

Hence, we have a Hamiltonian field in the form

$$X_F = \partial a^\mu \partial_\mu + \tfrac{1}{m} \partial_\mu^i a^\mu \partial_i - \partial_\mu a^\mu \partial + (2\partial_\alpha a_i^{\mu\alpha} - \partial_i a^\mu) \, \partial_\mu^i \tag{18.8.17}$$

However, we also have constraints on F to be a Poisson form, namely

$$\partial_{[k} a_{i]}^{\mu\alpha} = 0 \qquad\qquad \partial a_i^{\mu\alpha} = 0 \qquad\qquad 2\partial_\mu^i a_i^{\mu\alpha} = -n(m-1)\partial a^\mu \tag{18.8.18}$$

The second constraint says that $a_i^{\mu\alpha}$ does not depend on f which is something which is preserved by transformation laws, hence it is somehow intrinsic.

Sometimes a choice is made to simplify the situation. One can start from a projectable vector field $v = v^\mu(x)\partial_\mu + v^i(x, y)\partial_i$ on B and produce an $(m-1)$-form

$$F_v = (v^\mu f + v^i f_i^\mu) \, d\sigma_\mu + v^\mu f_i^\alpha \, dy^i \wedge d\sigma_{\mu\alpha} \tag{18.8.19}$$

which, in fact, satisfies the constraints to be a Poisson form. We call such forms F_v *Poisson currents*. They are linear in the momenta, they contain Noether currents. We know transformation laws for $v'^\mu = J_\alpha^\mu v^\alpha$ and $v'^i = J_k^i v^k + J_\alpha^i v^\alpha$ as well as transformation laws for f and f_i^μ and we can check they fit into transformation laws for F_v to be a global m-form, i.e.

$$\begin{aligned} F_{v'}' &= J\bar{J}_\mu^\nu \left(v'^\mu \left(f' + \bar{J}_\alpha^\beta f_i'^\alpha J_\beta^i \right) + v'^i f_i'^\mu - \bar{J}_\alpha^\beta v'^\mu f_i'^\alpha J_\beta^i \right) d\sigma_\nu + J\bar{J}_\mu^\nu \bar{J}_\alpha^\beta v'^\mu f_i'^\alpha J_k^i \, dy^k \wedge d\sigma_{\nu\beta} = \\ &= \left(v^\nu f + \left(v^k + J_\alpha^k v^\alpha - v^\beta J_\beta^k \right) f_k^\nu \right) d\sigma_\nu + v^\nu f_k^\beta \, dy^k \wedge d\sigma_{\nu\beta} = \left(v^\nu f + v^k f_k^\nu \right) d\sigma_\nu + v^\nu f_k^\beta \, dy^k \wedge d\sigma_{\nu\beta} = F_v \end{aligned} \tag{18.8.20}$$

The corresponding Hamiltonian field is

$$X_F = v^\mu \partial_\mu + v^i \partial_i - \left(\partial_\mu v^\mu f + \partial_\mu v^i f_i^\mu \right) \partial + \left(\partial_\alpha v^\mu f_i^\alpha - \partial_\alpha v^\alpha f_i^\mu - \partial_i v^k f_k^\mu \right) \partial_\mu^i \tag{18.8.21}$$

However, that is not the most general solution of constraints. For example, if we have the bundle basis $\mathbb{R}$, we necessarily have $a_i^{\mu\alpha} = 0$, all constraints are satisfied. Generically, however, a^μ can depend arbitrarily on momenta. We knew that from mechanics.

Let us now consider two Poisson forms F and G and their corresponding Hamiltonian fields X_F and X_G. The commutator $[X_F, X_G]$ is an Hamiltonian field, the corresponding Poisson form is denoted by $\{F, G\}$ and it is called the *Poisson bracket* of F and G. In other words, we have

$$[X_F, X_G] = X_{\{F,G\}} \tag{18.8.22}$$

We have $X_F \lrcorner \boldsymbol{\Omega} = -dF$ and $X_G \lrcorner \boldsymbol{\Omega} = -dG$. The, in general, we have

$$
\begin{aligned}
[X_F, X_G] \lrcorner \boldsymbol{\Omega} &= \pounds_{X_F} X_G \lrcorner \boldsymbol{\Omega} - \pounds_{X_G} X_F \lrcorner \boldsymbol{\Omega} = d(X_F \lrcorner X_G \lrcorner \boldsymbol{\Omega}) + X_F \lrcorner d(X_G \lrcorner \boldsymbol{\Omega}) - d(X_G \lrcorner X_F \lrcorner \boldsymbol{\Omega}) - d(X_G \lrcorner X_F \lrcorner \boldsymbol{\Omega}) = \\
&= - d(X_G \lrcorner X_F \lrcorner \boldsymbol{\Omega} - X_F \lrcorner X_G \lrcorner \boldsymbol{\Omega})
\end{aligned}
\tag{18.8.23}
$$

Then we have that the commutator $[X_F, X_G]$ is a Hamiltonian field and we can set the Poisson bracket to be

$$\{F, G\} = X_G \lrcorner X_F \lrcorner \boldsymbol{\Omega} - X_F \lrcorner X_G \lrcorner \boldsymbol{\Omega} \tag{18.8.24}$$

Accordingly, Hamiltonian fields form a Lie subalgebra of $\mathfrak{X}_\pi(Z(B))$, as well as Poisson forms are a Lie subalgebra with Poisson brackets.

Poisson forms are defined up to closed horizontal $(m-2)$-forms (on B), namely, if we set $F' = F + d\alpha$, then $X_F = X_{F'}$.

Let us consider a Poisson form $F = a^\mu \, d\sigma_\mu + a_i^{\mu\alpha} \, dy^i \wedge d\sigma_{\mu\alpha}$ and a horizontal $(m-2)$-form $\alpha = \alpha^{\mu\nu} \, d\sigma_{\mu\nu}$. so that we can define a new Poisson form $F' = F + d\alpha$.

Let us compute the differential $d\alpha$, that is

$$d\alpha = 2\partial_\nu \alpha^{\mu\nu} d\sigma_\mu + \partial_i \alpha^{\mu\nu} dy^i \wedge d\sigma_{\mu\nu} \tag{18.8.25}$$

which corresponds to the new Poisson form

$$F' = a'^\mu d\sigma_\mu + a_i'^{\mu\alpha} dy^i \wedge d\sigma_{\mu\alpha} = (a^\mu + 2\partial_\nu \alpha^{\mu\nu}) \, d\sigma_\mu + (a_i^{\mu\alpha} + \partial_i \alpha^{\mu\alpha}) \, dy^i \wedge d\sigma_{\mu\alpha} \tag{18.8.26}$$

However, we can check that both F and F' define the same Hamiltonian field, namely

$$X_{F'} = \partial a'^\mu \partial_\mu + \tfrac{1}{m} \partial_\mu^i a'^\mu \partial_i - \partial_\mu a'^\mu \partial + \left(2\partial_\alpha a_i'^{\mu\alpha} - \partial_i a'^\mu\right) \partial_\mu^i = \partial a^\mu \partial_\mu + \tfrac{1}{m} \partial_\mu^i a^\mu \partial_i - (\partial_\mu a^\mu + 2\partial_{\mu\nu} \alpha^{\mu\nu}) \partial + (2\partial_\alpha a_i^{\mu\alpha} + \textcolor{red}{2\partial_{i\alpha} \alpha^{\mu\alpha}} - \partial_i a^\mu - \textcolor{red}{2\partial_{i\nu} \alpha^{\mu\nu}}) \partial_\mu^i = X_F \tag{18.8.27}$$

For the Poisson currents, we can write explicitly the Poisson bracket.

Locally, we have $F = (v^\mu f + v^i f_i^\mu) \, d\sigma_\mu + v^\mu f_i^\alpha \, dy^i \wedge d\sigma_{\mu\alpha}$ and $G = (u^\mu f + u^i f_i^\mu) \, d\sigma_\mu + u^\mu f_i^\alpha \, dy^i \wedge d\sigma_{\mu\alpha}$. The corresponding Hamiltonian fields are

$$X_F = v^\mu \partial_\mu + v^i \partial_i - \left(\partial_\mu v^\mu f + \partial_\mu v^i f_i^\mu\right) \partial + \left(\partial_\alpha v^\mu f_i^\alpha - \partial_\alpha v^\alpha f_i^\mu - \partial_i v^k f_k^\mu\right) \partial_\mu^i \qquad X_G = u^\mu \partial_\mu + u^i \partial_i - \left(\partial_\mu u^\mu f + \partial_\mu u^i f_i^\mu\right) \partial + \left(\partial_\alpha u^\mu f_i^\alpha - \partial_\alpha u^\alpha f_i^\mu - \partial_i u^k f_k^\mu\right) \partial_\mu^i \tag{18.8.28}$$

Their commutator is still the Hamiltonian field for a Poisson current, in fact, we have

$$
\begin{aligned}
[X_F, X_G] &= v^\alpha \partial_\alpha u^\mu \partial_\mu + \left(v^\mu \partial_\mu u^i + v^k \partial_k u^i\right) \partial_i + \\
&- \left(v^\alpha \left(\partial_{\alpha\mu} u^\mu f + \partial_{\alpha\mu} u^i f_i^\mu\right) + v^i \partial_{i\mu} u^k f_k^\mu - \left(\boxed{\textcolor{red}{\partial_\mu v^\mu f}} + \partial_\mu v^i f_i^\mu\right) \partial_\beta u^\beta + \left(\partial_\alpha v^\beta f_i^\alpha - \partial_\alpha v^\alpha f_i^\beta - \partial_i v^k f_k^\beta\right) \partial_\beta u^i\right) \partial + \\
&+ \left(v^\beta \left(\partial_{\alpha\beta} u^\mu f_i^\alpha - \partial_{\alpha\beta} u^\alpha f_i^\mu - \partial_{i\beta} u^k f_k^\mu\right) - v^j \partial_{ij} u^k f_k^\mu + \left(\partial_\alpha v^\sigma f_i^\alpha - \textcolor{red}{\partial_\alpha v^\alpha f_i^\sigma} - \partial_i v^k f_k^\sigma\right) \partial_\sigma u^\mu + \right. \\
&\left. - \left(\partial_\alpha v^\mu f_i^\alpha - \boxed{\textcolor{red}{\partial_\alpha v^\alpha f_i^\mu}} - \partial_i v^k f_k^\mu\right) \partial_\beta u^\beta - \left(\partial_\alpha v^\mu f_n^\alpha - \partial_\alpha v^\alpha f_n^\mu - \partial_n v^k f_k^\mu\right) \partial_i u^n\right) \partial_\mu^i - [uv] = \\
&= v^\alpha \partial_\alpha u^\mu \partial_\mu + \left(v^\mu \partial_\mu u^i + v^k \partial_k u^i\right) \partial_i - \left(\partial_\mu(v^\alpha \partial_\alpha u^\mu) f + \partial_\mu(v^\alpha \partial_\alpha u^i + v^k \partial_k u^i) f_i^\mu\right) \partial + \\
&+ \left(\boxed{\textcolor{red}{\partial_\mu v^\alpha \partial_\alpha u^\mu f}} + \partial_\mu v^\alpha \partial_\alpha u^i f_i^\mu + \partial_\mu v^i \partial_i u^k f_k^\mu + \partial_\mu v^i \partial_\beta u^\beta f_i^\mu - \partial_\mu v^\alpha \partial_\alpha u^i f_i^\mu - \partial_\alpha u^\alpha \partial_\beta v^i f_i^\beta - \partial_i u^k \partial_\beta v^i f_k^\beta\right) \partial + \\
&+ \left(\partial_\alpha(v^\beta \partial_\beta u^\mu) f_i^\alpha - \partial_\alpha(v^\beta \partial_\beta u^\alpha) f_i^\mu - \partial_i(v^\alpha \partial_\alpha u^k + v^j \partial_j u^k) f_k^\mu\right) \partial_\mu^i + \\
&+ \left(-\textcolor{red}{\underline{\partial_\alpha v^\beta \partial_\beta u^\mu f_i^\alpha}} + \boxed{\textcolor{blue}{\partial_\alpha v^\beta \partial_\beta u^\alpha f_i^\mu}} + \textcolor{red}{\underline{\partial_\alpha v^\sigma \partial_\sigma u^\mu f_i^\alpha}}\right) \partial_\mu^i - [uv] = \\
&= [v, u]^\mu \partial_\mu + [v, u]^i \partial_i - \left(\partial_\mu[v, u]^\mu f + \partial_\mu[v, u]^i f_i^\mu\right) \partial + \left(\partial_\alpha[v, u]^\mu f_i^\alpha - \partial_\alpha[v, u]^\alpha f_i^\mu - \partial_i[\xi, \zeta]^k f_k^\mu\right) \partial_\mu^i
\end{aligned}
\tag{18.8.29}
$$

where $[uv]$ stands for *the same terms with u and v exchanged.*

We clearly see that if F is a Poisson current associated to $v = v^\mu \partial_\mu + v^i \partial_i$ and G is a Poisson current associated to $u = u^\mu \partial_\mu + u^i \partial_i$, then $\{F, G\}$ is a Poisson current associated to $[v, u] = [v, u]^\mu \partial_\mu + [v, u]^i \partial_i = (v^\alpha \partial_\alpha u^\mu - [uv]) \partial_\mu + (v^\mu \partial_\mu u^i + v^k \partial_k u^i - [uv]) \partial_i$.

Then the Poisson bracket is

$$\{F, G\} = \left([v, u]^\mu f + [v, u]^i f_i^\mu\right) d\sigma_\mu + [v, u]^\mu f_i^\alpha \, dy^i \wedge d\sigma_{\mu\alpha} \tag{18.8.30}$$

Poisson brackets are bilinear and they obey Jacobi identity. They define a Lie algebra on Poisson forms (modulo exact horizontal forms).

If we want to give them a structure of graded associative algebra as well (for example to have some Leibniz rule) we need to extend Poisson forms to more general orders.

Also the interpretation of locally Hamiltonian fields as in mechanics goes through smoothly. The flow of a vector field X is made of canonical multi-simplectomorphisms iff X is a locally Hamiltonian field.

A vector field X on $Z(B)$ is locally Hamiltonian iff $d(X \lrcorner \boldsymbol{\Omega}) = 0$. Since the multi-symplectic form is closed, we have

$$0 = d(X \lrcorner \boldsymbol{\Omega}) + X \lrcorner d\boldsymbol{\Omega} = \pounds_X \boldsymbol{\Omega} \qquad \Longleftrightarrow \qquad (\Phi_s)^* \boldsymbol{\Omega} = \boldsymbol{\Omega} \tag{18.8.31}$$

where $\Phi_s : Z(B) \to Z(B)$ denotes the flow of X.

The second multisymplectic bundle

We can define other multisymplectic bundles. Let us focus on a different multisymplectic bundle which is used in second order theories.

Beside that, we could define, more generally, multisymplectic *manifolds*. In that case, one defines a multisymplectic manifold (M, Ω) of order $(k + 1)$ where Ω is a closed, non-degenerate ($v \lrcorner \Omega = 0$ iff $v = 0$) $(k + 1)$-form. That recovers symplectic manifolds for $k = 1$.

Restricting to multisymplectic bundles allow us to recognise that the fibration $[Z \to M]$ selects variables which enters $\boldsymbol{\Omega}$ in a different way, through the $d\sigma_\mu$, even though the multisymplectic structure on bundles is stricter. We are ready to forget the bundle structure $[Z \to B]$ by multi-symplectomorphisms, as in mechanics, not the fibration onto M which in field theory has a special meaning (unless one goes for a Finsler formulation).

M.Hohmann, C.Pfeifer, N.Voicu, *Mathematical foundations for field theories on Finsler spacetimes*, Journal of Mathematical Physics 63, 032503 (2022); arXiv:2106.14965 [math-ph]

Given a bundle $[B \to M]$, we define the *second multisymplectic bundle* $Z(J^1B)$. A point $z \in Z(J^1B)$ is an m-form on J^1B such that $Y \lrcorner X \lrcorner z = 0$ for any pair of vertical vectors X, Y on $[J^1B \to M]$. That means the most general form z reads locally as

$$z = f d\sigma + f_i^\mu dy^i \wedge d\sigma_\mu + f_i^{\mu\alpha} dy_\alpha^i \wedge d\sigma_\mu \tag{18.8.32}$$

If we fix coordinates (x^μ, y^i) on B, they induce at first coordinates (x^μ, y^i, y_μ^i) on J^1B, and then coordinates $(x^\mu, y^i, y_\mu^i; f_i^\mu, f_i^{\mu\alpha}, f)$ on $Z(J^1B)$. When we change coordinates on B by

$$x'^\mu = x'^\mu(x) \qquad y'^i = y'^i(x, y) \qquad \Rightarrow \qquad y_\mu'^i = \bar{J}_\mu^\nu \left(J_\nu^i + J_k^i y_\nu^k\right) \tag{18.8.33}$$

we have $d\sigma' = Jd\sigma$, $d\sigma'_\mu = J\bar{J}^\alpha_\mu d\sigma_\alpha$, $dy'^i = J^i_\alpha dx^\alpha + J^i_k dy^k$, as well as

$$dy'^i_\mu = \left(\bar{J}^\alpha_{\mu\nu} J^\nu_\beta \left(J^i_\alpha + J^i_k y^k_\alpha\right) + \bar{J}^\nu_\mu d_\nu J^i_\beta\right) dx^\beta + \bar{J}^\nu_\mu d_\nu J^i_k dy^k + \bar{J}^\alpha_\mu J^i_k dy^k_\alpha \tag{18.8.34}$$

Then we have

$$\begin{aligned}
z' &= f'd\sigma' + f'^\mu_i dy'^i \wedge d\sigma'_\mu + f'^{\mu\alpha}_i dy'^i_\alpha \wedge d\sigma'_\mu = \\
&= J\left(f' + f'^\mu_i \bar{J}^\alpha_\mu J^i_\alpha + f'^{\sigma\mu}_i \left(\bar{J}^\alpha_{\mu\sigma}\left(J^i_\alpha + J^i_k y^k_\alpha\right) + \bar{J}^\nu_\mu d_\nu J^i_\beta \bar{J}^\beta_\sigma\right)\right) d\sigma + J\bar{J}^\rho_\sigma \left(f'^\sigma_i J^i_k + f'^{\sigma\mu}_i \bar{J}^\nu_\mu d_\nu J^i_k\right) dy^k \wedge d\sigma_\rho + J\bar{J}^\rho_\sigma f'^{\sigma\mu}_i \bar{J}^\alpha_\mu J^i_k \, dy^k_\alpha \wedge d\sigma_\rho
\end{aligned} \tag{18.8.35}$$

which induces the transformation laws on $Z(J^1B)$ as

$$f = J\left(f' + f'^\mu_i \bar{J}^\alpha_\mu J^i_\alpha + f'^{\sigma\mu}_i \left(\bar{J}^\alpha_{\mu\sigma}\left(J^i_\alpha + J^i_k y^k_\alpha\right) + \bar{J}^\nu_\mu d_\nu J^i_\beta \bar{J}^\beta_\sigma\right)\right) \qquad f^\rho_k = J\bar{J}^\rho_\sigma \left(f'^\sigma_i J^i_k + f'^{\sigma\mu}_i \bar{J}^\nu_\mu d_\nu J^i_k\right) \qquad f^{\rho\alpha}_k = J\bar{J}^\rho_\sigma f'^{\sigma\mu}_i \bar{J}^\alpha_\mu J^i_k \tag{18.8.36}$$

All the results we proved about globality on $Z(B)$ can be applied functorially to $Z(J^1B)$ as well.

The *canonical multisymplectic potential m*-form

$$\boldsymbol{\Theta} = fd\sigma + f^\mu_i dy^i \wedge d\sigma_\mu + f^{\mu\alpha}_i dy^i_\alpha \wedge d\sigma_\mu \tag{18.8.37}$$

is canonical on $Z(J^1B)$. The *multisymplectic form* is $\boldsymbol{\Omega} := d\boldsymbol{\Theta}$ which is a canonical $(m+1)$-form on $Z(J^1B)$. A map $\Phi : Z(J^1B) \to Z(J^1B)$ is called a *multi-symplectomorphism* iff $\Phi^*\boldsymbol{\Omega} = \boldsymbol{\Omega}$.

Since we can lift flows, we can lift a projectable vector field $\Xi = \xi^\mu\partial_\mu + \xi^i\partial_i$ first to J^1B as

$$J^1\Xi = \xi^\mu\partial_\mu + \xi^i\partial_i + \xi^i_\mu\partial^\mu_i \qquad\qquad \xi^i_\mu := d_\mu\xi^i - d_\mu\xi^\nu y^i_\nu \tag{18.8.38}$$

then to $Z(J^1B)$ as $Z(J^1\Xi) = \xi^\mu\partial_\mu + \xi^i\partial_i + \xi^i_\mu\partial^\mu_i + \xi\partial + \xi^\mu_i\partial^i_\mu + \xi^{\mu\alpha}_i\partial^i_{\mu\alpha}$ where we set

$$\begin{cases}
\xi := -\partial_\alpha\xi^\alpha f - f^\alpha_i\partial_\alpha\xi^i + \partial_{\rho\sigma}\xi^\mu f^{\rho\sigma}_k y^k_\mu - f^{\rho\sigma}_k d_{\rho\sigma}\xi^k \\
\xi^\mu_i := \partial_\alpha\xi^\mu f^\alpha_i - f^\mu_k\partial_i\xi^k - f^{\mu\epsilon}_k\partial_i d_\epsilon\xi^k - \partial_\alpha\xi^\alpha f^\mu_i \\
\xi^{\mu\alpha}_i := \partial_\rho\xi^\mu f^{\rho\alpha}_i + \partial_\rho\xi^\alpha f^{\mu\rho}_i - f^{\mu\alpha}_k\partial_i\xi^k - \partial_\epsilon\xi^\epsilon f^{\mu\alpha}_i
\end{cases} \tag{18.8.39}$$

For a vector field $X = \xi^\mu\partial_\mu + \xi^i\partial_i + \xi^i_\mu\partial^\mu_i + \xi\partial + \xi^\mu_i\partial^i_\mu + \xi^{\mu\alpha}_i\partial^i_{\mu\alpha}$ on $Z(J^1B)$, we have

$$X\lrcorner\boldsymbol{\Omega} = Xd\sigma - X^\alpha df \wedge d\sigma_\alpha + X^\mu_i dy^i \wedge d\sigma_\mu - X^i df^\mu_i \wedge d\sigma_\mu + X^\alpha df^\mu_i \wedge dy^i \wedge d\sigma_{\mu\alpha} + X^{\mu\alpha}_i dy^i_\alpha \wedge d\sigma_\mu - X^i_\alpha df^{\mu\alpha}_i \wedge d\sigma_\mu + X^\alpha df^{\mu\nu}_i \wedge dy^i_\nu \wedge d\sigma_{\mu\alpha} \tag{18.8.40}$$

Let $X = \xi^\mu\partial_\mu + \xi^i\partial_i + \xi\partial + \xi^\mu_i\partial^i_\mu + \xi^{\mu\alpha}_i\partial^i_{\mu\alpha}$ be a vector field on $Z(J^1B)$. It is called a *(globally) Hamiltonian field* iff there exists an $(m-1)$ form F on $Z(J^1B)$ such that $X\lrcorner\boldsymbol{\Omega} = -dF$; in that case F is called a *Poisson form* and the vector field is denoted by X_F.

Let us start from a vector field $v = v^\mu(x)\partial_\mu + v^i(x,y)\partial_i + v^i_\alpha(x,y,dy)\partial^\alpha_i$ on J^1B and produce a *Poisson current*

$$F_v = \left(v^\mu f + v^i f^\mu_i + v^i_\alpha f^{\mu\alpha}_i\right) d\sigma_\mu + v^\mu f^\nu_i \, dy^i \wedge d\sigma_{\mu\nu} + v^\mu f^{\nu\alpha}_i \, dy^i_\alpha \wedge d\sigma_{\mu\nu} \tag{18.8.41}$$

We know transformation laws for $v'^\mu = J^\mu_\alpha v^\alpha$, $v'^i = J^i_k v^k + J^i_\alpha v^\alpha$, and

$$v'^i_\mu = \bar{J}^\nu_\mu J^i_k v^k_\nu + \bar{J}^\nu_\mu d_\nu J^i_k v^k + v^\epsilon \left(\bar{J}^\lambda_\alpha d_\lambda J^i_\epsilon + \bar{J}^\lambda_{\alpha\beta} J^\beta_\epsilon \left(J^i_\lambda + J^i_k y^k_\lambda \right) \right) \tag{18.8.42}$$

as well as transformation laws for f, f^μ_i, and $f^{\mu\alpha}_i$ and we can check they fit into transformation laws for F_v to be a global m-form, i.e.

$$
\begin{aligned}
F'_{v'} =& J\bar{J}^\nu_\mu \left(v'^\mu f' + v'^i f'^\mu_i + v'^i_\alpha f'^{\mu\alpha}_i \right) d\sigma_\nu + J\bar{J}^\nu_\mu \bar{J}^\beta_\alpha v'^\mu f'^\alpha_i \, dy'^i \wedge d\sigma_{\nu\beta} + J\bar{J}^\rho_\mu \bar{J}^\sigma_\nu v'^\mu f'^{\nu\alpha}_i \, dy'^i_\alpha \wedge d\sigma_{\rho\sigma} = \\
=& v^\nu J \left(f' + \bar{J}^\beta_\alpha f'^\alpha_i J^i_\beta + f'^{\epsilon\alpha}_i \left(\bar{J}^\lambda_{\alpha\epsilon} \left(J^i_\lambda + J^i_k y^k_\lambda \right) + \bar{J}^\lambda_\alpha d_\lambda J^i_\sigma \bar{J}^\sigma_\epsilon \right) \right) d\sigma_\nu + \\
&+ \left(v^k J \bar{J}^\nu_\mu \left(f'^\mu_i J^i_k + \bar{J}^\beta_\alpha f'^{\mu\alpha}_i d_\beta J^i_k \right) + v^k_\beta J \bar{J}^\nu_\mu \bar{J}^\beta_\alpha f'^{\mu\alpha}_i J^i_k \right) d\sigma_\nu + \\
&+ v^\rho \left(-J\bar{J}^\nu_\alpha f'^\alpha_i J^i_\rho + J\bar{J}^\nu_\mu f'^{\mu\alpha}_i \bar{J}^\lambda_\alpha d_\lambda J^i_\rho + J\bar{J}^\nu_\mu \bar{J}^\lambda_{\alpha\beta} J^\beta_\rho f'^{\mu\alpha}_i \left(J^i_\lambda + J^i_k y^k_\lambda \right) \right) d\sigma_\nu + \\
&+ v^\rho \left(J\bar{J}^\nu_\alpha f'^\alpha_i J^i_\rho - J\bar{J}^\nu_\epsilon f'^{\epsilon\alpha}_i \bar{J}^\lambda_\alpha d_\lambda J^i_\rho - J\bar{J}^\nu_\epsilon \bar{J}^\lambda_{\alpha\beta} J^\beta_\rho f'^{\epsilon\alpha}_i \left(J^i_\lambda + J^i_k y^k_\lambda \right) \right) d\sigma_\nu + \\
&+ v^\rho J\bar{J}^\sigma_\alpha \left(f'^\alpha_i J^i_k + \bar{J}^\epsilon_\mu f'^{\alpha\mu}_i d_\epsilon J^i_k \right) dy^k \wedge d\sigma_{\rho\sigma} + v^\rho \left(J\bar{J}^\sigma_\nu \bar{J}^\epsilon_\alpha f'^{\nu\alpha}_i J^i_k \right) dy^k_\epsilon \wedge d\sigma_{\rho\sigma} = \\
=& \left(v^\nu f + v^k f^\nu_k + v^k_\beta f^{\nu\beta}_k \right) d\sigma_\nu + v^\rho f^\sigma_k \, dy^k \wedge d\sigma_{\rho\sigma} + v^\rho f^{\sigma\epsilon}_k \, dy^k_\epsilon \wedge d\sigma_{\rho\sigma} = F_v
\end{aligned}
\tag{18.8.43}
$$

The corresponding Hamiltonian field $X_F = X^\mu \partial_\mu + X^i \partial_i + X^i_\alpha \partial^\alpha_i + X\partial + X^\mu_i \partial^i_\mu + X^{\mu\alpha}_i \partial^i_{\mu\alpha}$ can be easily obtained by comparing the differential $-dF_v$, i.e.

$$
\begin{aligned}
-dF_v =& -\left(\partial_\mu v^\mu f + \partial_\mu v^i f^\mu_i + \partial_\mu v^i_\alpha f^{\mu\alpha}_i \right) d\sigma - v^\mu \, df \wedge d\sigma_\mu - v^k \, df^\mu_k \wedge d\sigma_\mu - v^k_\alpha \, df^{\mu\alpha}_k \wedge d\sigma_\mu + \\
&- \left(\partial_k v^j f^\rho_j + \partial_k v^j_\alpha f^{\rho\alpha}_j + \partial_\sigma v^\sigma f^\rho_k - \partial_\rho v^\rho f^\sigma_k \right) dy^k \wedge d\sigma_\rho - \left(\partial^\alpha_k v^j_\beta f^{\rho\beta}_i - \partial_\sigma v^\rho f^{\sigma\alpha}_k + \partial_\rho v^\rho f^{\sigma\alpha}_k \right) dy^k_\alpha \wedge d\sigma_\rho + \\
&+ v^\rho \, df^\sigma_k \wedge dy^k \wedge d\sigma_{\sigma\rho} + v^\rho \, df^{\sigma\alpha}_k \wedge dy^k_\alpha \wedge d\sigma_{\sigma\rho}
\end{aligned}
\tag{18.8.44}
$$

with

$$X_F \lrcorner \boldsymbol{\Omega} = X^\alpha \left(-df \wedge d\sigma_\alpha + df^\mu_i \wedge dy^i \wedge d\sigma_{\mu\alpha} + df^{\mu\nu}_i \wedge dy^i_\nu \wedge d\sigma_{\mu\alpha} \right) - X^i df^\mu_i \wedge d\sigma_\mu - X^i_\alpha df^{\mu\alpha}_i \wedge d\sigma_\mu + X d\sigma + X^\mu_i dy^i \wedge d\sigma_\mu + X^{\mu\alpha}_i dy^i_\alpha \wedge d\sigma_\mu \tag{18.8.45}$$

Then we have

$$X_F = v^\mu \partial_\mu + v^i \partial_i + v^i_\alpha \partial^\alpha_i - \left(\partial_\mu v^\mu f + \partial_\mu v^i f^\mu_i + \partial_\mu v^i_\alpha f^{\mu\alpha}_i \right) \partial - \left(\partial_k v^j f^\rho_j + \partial_k v^j_\alpha f^{\rho\alpha}_j + \partial_\sigma v^\sigma f^\rho_k - \partial_\sigma v^\rho f^\sigma_k \right) \partial^k_\rho - \left(\partial^\alpha_k v^j_\beta f^{\rho\beta}_j - \partial_\sigma v^\rho f^{\sigma\alpha}_k + \partial_\sigma v^\sigma f^{\rho\alpha}_k \right) \partial^k_{\rho\alpha} \tag{18.8.46}$$

We showed above intrinsically that when we consider two Poisson forms F and G and their corresponding Hamiltonian fields X_F and X_G, the commutator $[X_F, X_G]$ is a Hamiltonian field, the corresponding Poisson form is denoted by $\{F, G\} = X_G \lrcorner X_F \lrcorner \boldsymbol{\Omega} - X_F \lrcorner X_G \lrcorner \boldsymbol{\Omega}$ and it is called the *Poisson bracket* of F and G so that we have

$$[X_F, X_G] = X_{\{F,G\}} \tag{18.8.47}$$

The proof above goes through on $Z(J^1 B)$ as well and, accordingly, Hamiltonian fields form a Lie subalgebra of $\mathfrak{X}(Z(J^1 B))$, as well as Poisson forms are a Lie subalgebra with Poisson brackets.

For the Poisson currents, we can write explicitly the Poisson bracket.

The Poisson bracket is

$$\{F, G\} = \left([v, u]^\mu f + [v, u]^i f^\mu_i + [v, u]^i_\alpha f^{\mu\alpha}_i \right) d\sigma_\mu + [v, u]^\mu f^\alpha_i \, dy^i \wedge d\sigma_{\mu\alpha} + [v, u]^\mu f^{\nu\alpha}_i \, dy^i_\alpha \wedge d\sigma_{\mu\nu} \tag{18.8.48}$$

Poisson brackets are bilinear, they define a Lie algebra on Poisson forms (modulo exact horizontal forms).

Coisotropic surfaces

Sempre prendendo da [Cantrun, Ibort, De Leon]...

We could define a *linear multisymplectic space* as a pair (V, Ω) where V is a vector space and Ω is a non-degenerate $(m+1)$-form on V. This is a model for multisymplectic manifolds, that we did not even define. We restrict instead to a model for multisymplectic bundles.

Let us start from an exact sequence of vector spaces

$$0 \to V \to V_B \to V_M \to 0 \tag{18.8.49}$$

That splits and one can consider a basis (e_μ, ϵ_i) adapted to the splitting $V_B = V_M \oplus V$. Let us set $m = \dim(V_M)$ and $k = \dim(V)$ which define the range for Greek and Latin indices.

Then we can consider $V_Z = V_B \oplus Z(V_B)$ where $Z(V_B) \subset A_m(V_B)$ is the space of m-forms z on V_B such that $\forall v_1, v_2 \in V$ one has

$$i(v_1) \lrcorner i(v_2) \lrcorner z = 0 \tag{18.8.50}$$

The forms $z \in Z(V_B)$ are expressed as

$$z = f\, V(e) + f_i^\mu\, \epsilon^i \wedge V_\mu(e) \tag{18.8.51}$$

where we set $V(e) = e^1 \wedge \ldots \wedge e^m$ and $V_\mu(e) = e_\mu \lrcorner V(e)$.

Let us remark that this is a very special case of multisymplectic vector space. We could have easily generalized (18.8.50) by requiring z to be an m-form such that

$$i(v_1) \lrcorner \ldots \lrcorner i(v_{r+1}) \lrcorner z = 0 \tag{18.8.52}$$

which would mean, for any $0 \le r \le m$, having local expressions as

$$z = f\, V(e) + f_i^\mu\, \epsilon^i \wedge V_\mu(e) + \ldots + f_{i_1 \ldots i_r}^{\mu_1 \ldots \mu_r}\, \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_r} \wedge V_{\mu_1 \ldots \mu_r}(e) \tag{18.8.53}$$

We prefer here to get stuck to what we used in Hamiltonian formalism as canonical multisymplectic bundles $Z(B)$, which corresponds to $r = 1$.

Then we have a basis $w^a = (e_\mu, \epsilon_i, \epsilon_\mu^i = \epsilon^i \wedge V_\mu(e), \epsilon = V(e))$ for V_Z which corresponds to Cartesian coordinates (v^μ, v^i, v_i^μ, v). Let us then define the $(m+1)$-form on V_Z

$$\Omega = \epsilon \wedge V(e) + \epsilon_i^\mu \wedge \epsilon^i \wedge V_\mu(e) \tag{18.8.54}$$

This is a multisymplectic form since it is non-degenerate.

Let us consider a vector $w = w^\mu e_\mu + w^i \epsilon_i + w_i^\mu \epsilon_\mu^i + w \epsilon$ and compute

$$w \lrcorner \Omega = w V(e) - w^\mu V_\mu(e) + w_i^\mu \epsilon^i \wedge V_\mu(e) - w^i \epsilon_i^\mu \wedge V_\mu(e) + w^\nu \epsilon_i^\mu \wedge \epsilon^i \wedge V_{\mu\nu}(e) \tag{18.8.55}$$

to show that $w \lrcorner \Omega = 0$ iff $w = 0$.

As it happens in mechanics, we can define a symplectic polar $W^\perp$ of a sub-space $W \subset V_Z$. Unlike in mechanics we have a whole family of possibilities. Let us consider the multi-symplectic space (V_Z, Ω), where we set $V_Z := T_z Z(B)$ and $\Omega = \boldsymbol{\Omega}_z$.

Definition: *let $W \subset V_Z$ be a sub-space, we define the k-multisymplectic polar*

$$W_k^\perp = \{v \in V_Z : v \lrcorner w_1 \lrcorner w_2 \lrcorner \dots w_k \lrcorner \Omega = 0, \forall w_1, w_2, \dots, w_k \in W\} \qquad (18.8.56)$$

for an integer $1 \le k \le n := dim(W)$.

One has

$$W_1^\perp \subset W_2^\perp \dots \subset W_k^\perp \subset \dots \subset W_n^\perp \subset W_{n+1}^\perp := V_Z \qquad (18.8.57)$$

In general, we have

    (i) $\{0\}_k^\perp = V_Z$, $(V_Z)_k^\perp = \{0\}$;

    (ii) $U \subset W \quad \Rightarrow \quad W_k^\perp \subset U_k^\perp$;

    (iii) $(U+W)_k^\perp \subset U_k^\perp \cap W_k^\perp$;

    (iv) $U_{k_1}^\perp \cap W_{k_2}^\perp \subset (U+W)_{k_1+k_2-1}^\perp$, for $k_1 + k_2 - 1 < m$ (consequently, $(U+W)_1^\perp = U_1^\perp \cap W_1^\perp$)

    (v) $U_{k_1}^\perp + W_{k_2}^\perp \subset (U \cap W)_{\max(k_1,k_2)}^\perp$.

Definition: *a subspace $W \subset V_Z$ is called k-isotropic iff $W \subset W_k^\perp$, k-coisotropic iff $W_k^\perp \subset W$, k-Lagrangian iff $W_k^\perp = W$, k-multisymplectic iff $W_k^\perp \cap W = \{0\}$.*

That extends to a sub-manifold $S \subset Z(B)$ by restricting the tangent spaces $T_z S$ to be k-(co)isotropic, k-Lagrangian, or k-multisymplectic.

Let us focus on the subspace, e.g., $U = \mathrm{Span}(\partial_\mu, \partial_\mu^i, \partial)$ for a bundle B on a manifold M of dimension $m = 2$. The multisymplectic $(m+1)$-form on $\Omega = df \wedge d\sigma + df_i^\mu \wedge dy^i \wedge d\sigma_\mu$ on $Z(B)$ and, for $u \in U$, we have

$$u \lrcorner \Omega = u d\sigma - u^\mu df \wedge d\sigma_\mu + u_i^\mu dy^i \wedge d\sigma_\mu + u^\nu df_i^\mu \wedge dy^i \wedge d\sigma_{\mu\nu} \qquad (18.8.58)$$

Hence $v \in V$ is in $U_1^\perp$ iff $\forall u \in U$ we have

$$v \lrcorner u \lrcorner \Omega = \left(v^\mu u - v u_i^\mu + v^i u_i^\mu\right) d\sigma_\mu - v^\nu u^\mu df \wedge d\sigma_{\mu\nu} + \left(v_i^\mu u^\nu - v^\nu u_i^\mu\right) dy^i \wedge d\sigma_{\mu\nu} - v^i u^\nu df_i^\mu \wedge d\sigma_{\mu\nu} = 0 \qquad (18.8.59)$$

from which we have $U_1^\perp = \{0\}$. By restricting $v \in U$, for all $u, v \in U$, we have

$$v \lrcorner u \lrcorner \Omega = (v^\mu u - v u_i^\mu) d\sigma_\mu - v^\nu u^\mu df \wedge d\sigma_{\mu\nu} + (v_i^\mu u^\nu - v^\nu u_i^\mu) dy^i \wedge d\sigma_{\mu\nu} = 0 \qquad (18.8.60)$$

Then, for $w \in V$, we have

$$w \lrcorner v \lrcorner u \lrcorner \Omega = w^\nu (v^\mu u - v u_i^\mu) d\sigma_{\mu\nu} - w v^\nu u^\mu d\sigma_{\mu\nu} + w^i (v_i^\mu u^\nu - v^\nu u_i^\mu) d\sigma_{\mu\nu} = 0 \qquad (18.8.61)$$

which implies $U_2^\perp = \mathrm{Span}(\partial_\mu^i) \subset U$ and it shows U is m-coisotropic.

A sub-manifold $S \subset Z(B)$, which is m-coisotropic, with $m \le dim(S)$, comes with a distribution $(T_S)_m^\perp$ on it which is called the *characteristic distribution*, as a rather direct generalization of the symplectic case.

As in the symplectic case, we expect the characteristic distribution to be involutive, hence integrable.

In fact, let X, Y be two characteristic fields. For X to be a characteristic field, it means that

$$X \lrcorner u_1 \lrcorner \ldots \lrcorner u_m \lrcorner \boldsymbol{\Omega} = 0 \qquad \Longleftrightarrow \qquad (X \lrcorner \boldsymbol{\Omega})|_S = 0 \tag{18.8.62}$$

Let us consider the commutator $[X, Y]$; this is characteristic iff $([X, Y] \lrcorner \boldsymbol{\Omega})|_S = 0$. In fact, by using the formula proven in Appendix A one has

$$([X, Y] \boldsymbol{\Omega})|_S = (\pounds_X (i_Y \boldsymbol{\Omega}) - i_Y \pounds_X \boldsymbol{\Omega})|_S = (i_X d (i_Y \boldsymbol{\Omega}) + d (i_X i_Y \boldsymbol{\Omega}) - i_Y i_X d\boldsymbol{\Omega} - i_Y d(i_X \boldsymbol{\Omega}))|_S = -(i_Y i_X d\boldsymbol{\Omega})_S = 0 \tag{18.8.63}$$

and the commutator is characteristic. Then the characteristic distribution is involutive, then it is integrable.

By Frobenious theorem, the characteristic distribution defines a foliation of S into characteristic sub-manifolds C. Then when $Q = S/C$ is a bundle $[\pi : Q \to M]$, we can define a multisymplectic structure $\boldsymbol{\omega}$ on it.

We have the pull-back $\boldsymbol{\Omega}_S = i^*\boldsymbol{\Omega}$ on S. This is constant along the fibers of the bundle $[S \to Q]$, in fact if one evaluates $\boldsymbol{\Omega}_S$ along a vertical vector v of $[S \to Q]$, that is a vector tangent to the characteristic sub-manifods C and one has $v \lrcorner \boldsymbol{\Omega}_S = 0$. Accordingly, for any two vectors $u_1, u_2 \in TS$ such that $u_2 = u_1 + v$ one has $u_1 \lrcorner \boldsymbol{\Omega}_S = u_2 \lrcorner \boldsymbol{\Omega}_S$.

That means the form $\boldsymbol{\Omega}_S$ is projectable along $\hat{\pi} : S \to Q$ and there exists a unique $(m+1)$-form $\boldsymbol{\omega}$ on Q such that $(\hat{\pi})^*\boldsymbol{\omega} = i^*\boldsymbol{\Omega}$. In view of the naturality of the differential, we have $(\hat{\pi})^*d\boldsymbol{\omega} = i^*d\boldsymbol{\Omega} = 0$.

Bisogna mostrare che $\boldsymbol{\omega}$ is closed and non-degenerate so that $(Q, \boldsymbol{\omega})$ is a multisymplectic bundle.

Hence the multisymplectic bundle $(Q, \boldsymbol{\omega})$ is called the *multisymplectic reduction* of $S \subset Z(B)$.

Darboux theorem

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F. Cantrun, A. Ibort, M. De Leon, *On The Geometry Of Multisymplectic Manifolds*, J. Austral. Math. Soc. (Series A) 66 (1999), 303-330

Narciso Román-Roy, *Some properties of multisymplectic manifolds*, Classical and Quantum Physics. 60 Years Alberto Ibort Fest Geometry, Dynamics, and Control, Springer Proceedings in Physics **229**.
G. Marmo, D. Martn de Diego, M.C. Muoz-Lecanda (eds). Springer (2019) 325-336; arXiv:1807.11774 [math-ph]

One can show that in this models, $V_Z = V_B \oplus Z_B$ splits into two subspaces, V_B which is a m-Lagrangian subspace and $Z_B = Z(V_B)$ which is a 1-Lagrangian subspace. Since we have $Z_B = A_m(V_Z/Z_B)$, then

$$\dim(Z_B) = \dim(A_m(V_Z/Z_B)) \tag{18.8.64}$$

Finally, one can prove that if (V, ω) is a multisymplectic vector space for a non-degenerate $(m+1)$-form ω, and $U \subset V$ is a 1-Lagrangian subspace with $\dim(U) = \dim(A_m(V/U))$ then there exists a complement $V = U \oplus S$ which is m-Lagrangian, and iff V is isomorphic to a canonical model (V_Z, Ω).

That corresponds to show that one can find a canonical basis in V, which extends the Darboux theorem for symplectic vector spaces.

For the symplectic case $m = 1$, one has trivially that $Z_B = A_1(V_Z/Z_B) \simeq V_Z/Z_B$ which implies the dimension condition since $\dim(V_Z) = 2n$ and $\dim(Z_B) = n$ since it is a Lagrangian subspace.

9. Lie derivatives of a section

On bundles, we have the most general definition of Lie derivative (of a section) which then specialises on different bundles (e.g. on tensor bundles) to the Lie derivative defined in differential geometry. We shall here present the most general definition which applies in general to sections of any bundle. To obtain the classical prescriptions of differential geometry, one needs more ingredients, namely to consider natural objects, i.e. sections in a natural bundle; see below next Chapter.

In general, let (B, M, π, F) be a bundle, $\Xi \in \mathfrak{X}(\pi)$ be a projectable vector field which projects onto $\xi \in \mathfrak{X}(M)$ and σ a section of B.

We can define the *Lie derivative* of the section σ with respect to the vector field Ξ to be the vertical vector field defined on σ defined as

$$\pounds_\Xi \sigma = T\sigma(\xi) - \Xi \circ \sigma \tag{18.9.1}$$

It is a vertical vector defined on σ, in fact

$$T\pi \circ \pounds_\Xi \sigma = T\pi \circ T\sigma(\xi) - T\pi \circ \Xi \circ \sigma = \xi - \xi = 0 \tag{18.9.2}$$

and

$$\pounds_\Xi \sigma(x) \in T_{\sigma(x)}B \tag{18.9.3}$$

The Lie derivative is also linear with respect to Ξ.

In fibered coordinates, one has $\sigma : M \to B : x^\mu \mapsto (x^\mu, y^i(x))$ and $\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i$; accordingly, one has

$$\pounds_\Xi \sigma = T\sigma(\xi) - \Xi \circ \sigma = \left(d_\mu y^i(x)\xi^\mu(x) - \xi^i(x, y(x)) \right) \partial_i \tag{18.9.4}$$

Let us thus set

$$\pounds_\Xi y^i := y^i_\mu \xi^\mu(x) - \xi^i(x, y) \tag{18.9.5}$$

so that one has

$$\pounds_\Xi \sigma = \left((\pounds_\Xi y^i) \circ j^1\sigma \right) \partial_i \tag{18.9.6}$$

Let us stress that this Lie derivative seems to be very different from the classical Lie derivative of differential geometry. First of all, here the Lie derivative of a section is not a section in the same bundle but a vertical field on a section (while, of course, in differential geometry the Lie derivative of a tensor field is a tensor field itself). Second, here the Lie derivative is computed along a vector field Ξ on the bundle, while in differential geometry it is computed along a vector field on the base manifold M.

We shall see the relation with differential geometry in next Chapter because we need further structure to do it. However, if in differential geometry the Lie derivative measures how much an object changes when it is dragged along the flow of the vector field ξ (on M), the definition (18.9.1) is called *Lie derivative* exactly because it measures how much the section σ changes when dragged along the flow of Ξ (on B).

The flow of Ξ reads as

$$\begin{cases} x'^\mu = x'^\mu_s(x) \\ y'^i = Y^i_s(x, y) \end{cases} \qquad \Longleftrightarrow \qquad \begin{aligned} \xi^\mu(x) &= \frac{dx'^\mu_s}{ds}\Big|_{s=0} \\ \xi^i(x, y) &= \frac{dY^i_s}{ds}\Big|_{s=0} \end{aligned} \tag{18.9.7}$$

and it defines a 1-parameter flow of sections

$$\sigma_s : M \to B : x'^{\mu} \mapsto (x'^{\mu}, Y_s^i(x, y(x))) \tag{18.9.8}$$

where we set $x'^{\mu} = x'^{\mu}_s(x)$ ($\iff x^{\mu} = x'^{\mu}_{-s}(x')$).

If we consider the curve obtained by fixing x' and leaving s change, we have a vertical curve and its tangent vector (which is, in fact, measuring how much σ_s changes when dragged) is

$$\left(\frac{d}{ds}\sigma_s\right)_{s=0} = \left(\xi^i(x, y(x)) - \partial_{\mu} y^i(x)\xi^{\mu}(x)\right)\partial_i = -\pounds_{\Xi}\sigma(x) \tag{18.9.9}$$

We can also show in general that

$$d_{\mu}\pounds_{\Xi}y^i = \pounds_{J^1\Xi}y^i_{\mu} \tag{18.9.10}$$

i.e. the Lie derivative commutes with total derivatives.

We can just expand

$$d_{\mu}\pounds_{\Xi}y^i = d_{\mu}\left(y^i_{\nu}\xi^{\nu} - \xi^i\right) = y^i_{\nu\mu}\xi^{\nu} + y^i_{\nu}d_{\mu}\xi^{\nu} - d_{\mu}\xi^i = y^i_{\mu\nu}\xi^{\nu} - \xi^i_{\mu} = \pounds_{J^1\Xi}y^i_{\mu} \tag{18.9.11}$$

where we remembered by the lift formula for vector fields to J^1B that $\xi^i_{\mu} = d_{\mu}\xi^i - y^i_{\nu}d_{\mu}\xi^{\nu}$.

This can be written more intrinsically (though somehow more obscurely, if not in view of the coordinate expressions) as

$$J^1\pounds_{\Xi}\sigma = \pounds_{J^1\Xi}j^1\sigma \tag{18.9.12}$$

One also has

$$[\pounds_{\Xi_1}, \pounds_{\Xi_2}]\sigma = \pounds_{[\Xi_1,\Xi_2]}\sigma \tag{18.9.13}$$

When we compute $\pounds_{\Xi_2}\sigma$, and the result is a section in $V(B)$, namely $(x^{\mu}, y^i(x), (\xi_2^{\mu}(x)\partial_{\mu}y^i(x) - \xi_2^i(x, y(x))))$.

The vector field Ξ_1 on B can be lifted to $V(B)$ with coordinates (x^{μ}, y^i, ζ^i), since $V(\cdot)$ is a functor, as

$$V(\Xi_1) = \xi_1^{\mu}(x)\frac{\partial}{\partial x^{\mu}} + \xi_1^i(x, y)\frac{\partial}{\partial y^i} + \partial_k\xi_1^i(x, y)\zeta^k\frac{\partial}{\partial \zeta^i} \tag{18.9.14}$$

This is obtained as usual, by considering the flow Φ_s of Ξ_1, lifting the flow $V(\Phi_s)$ to $V(B)$, and finally defining $V(\Xi_1)$ as the infinitesimal generator of $V(\Phi_s)$.

Hence we can write the Lie derivative

$$\pounds_{\Xi_1} \circ \pounds_{\Xi_2}y^i = \xi_1^{\mu}d_{\mu}(\xi_2^{\nu}y^i_{\nu} - \xi_2^i) - \partial_k\xi_1^i(\xi_2^{\mu}y^k_{\mu} - \xi_2^k) = \xi_1^{\mu}d_{\mu}\xi_2^{\nu}y^i_{\nu} + \xi_1^{\mu}\xi_2^{\nu}y^i_{\mu\nu} - \xi_1^{\mu}d_{\mu}\xi_2^i - \partial_k\xi_1^i\xi_2^{\mu}y^k_{\mu} + \partial_k\xi_1^i\xi_2^k \tag{18.9.15}$$

and the commutator

$$\begin{aligned}[\pounds_{\Xi_1}, \pounds_{\Xi_2}]y^i =& [\xi_1, \xi_2]^{\nu}y^i_{\nu} + \xi_2^{\mu}d_{\mu}\xi_1^i - \xi_1^{\mu}d_{\mu}\xi_2^i + \partial_k\xi_2^i\xi_1^{\nu}y^k_{\nu} - \partial_k\xi_1^i\xi_2^{\nu}y^k_{\nu} + \partial_k\xi_1^i\xi_2^k - \partial_k\xi_2^i\xi_1^k = \\ =& [\xi_1, \xi_2]^{\nu}y^i_{\nu} + \xi_2^{\mu}\partial_{\mu}\xi_1^i + \textcolor{red}{\xi_2^{\nu}\partial_k\xi_1^iy^k_{\nu}} - \xi_1^{\mu}\partial_{\mu}\xi_2^i - \textcolor{blue}{\xi_1^{\nu}\partial_k\xi_2^iy^k_{\nu}} + \textcolor{blue}{\xi_1^{\nu}\partial_k\xi_2^iy^k_{\nu}} - \textcolor{red}{\xi_2^{\nu}\partial_k\xi_1^iy^k_{\nu}} + \partial_k\xi_1^i\xi_2^k - \partial_k\xi_2^i\xi_1^k = \\ =& [\xi_1, \xi_2]^{\nu}y^i_{\nu} + \xi_2^{\mu}\partial_{\mu}\xi_1^i - \xi_1^{\mu}\partial_{\mu}\xi_2^i + \xi_2^k\partial_k\xi_1^i - \xi_1^k\partial_k\xi_2^i \end{aligned} \tag{18.9.16}$$

On the other hand, we have

$$[\Xi_1, \Xi_2] = (\xi_1^{\mu}d_{\mu}\xi_2^{\nu} - \xi_2^{\mu}d_{\mu}\xi_1^{\nu})\partial_{\nu} + (\xi_1^{\mu}\partial_{\mu}\xi_2^k - \xi_2^{\mu}\partial_{\mu}\xi_1^k + \xi_1^i\partial_i\xi_2^k - \xi_2^i\partial_i\xi_1^k)\partial_k \tag{18.9.17}$$

and

$$\pounds_{[\Xi_1, \Xi_2]} y^i = [\xi_1, \xi_2]^\nu y^i_\nu + \xi^\mu_2 \partial_\mu \xi^i_1 - \xi^\mu_1 \partial_\mu \xi^i_2 + \xi^k_2 \partial_k \xi^i_1 - \xi^k_1 \partial_k \xi^i_2 = [\pounds_{\Xi_1}, \pounds_{\Xi_2}] y^i \tag{18.9.18}$$

One could also question the right of such a general operator to be called a *Lie derivative* being so much different from the classical Lie derivatives of differential geometry, with which it shares a lot of formal properties as shown above, though. Beside that, of course, that is mainly an issue about how to select the characteristic properties of Lie derivatives, and essentially an argument about notation, still I believe the real reason why these are Lie derivative *de facto* is the role they play in the definition of symmetries (1.4.3) and Noether theorem (1.4.9) for a general field theory.

10. Composite bundles

Let us introduced a (eventually not so much) more structured version of the category of bundles, called the category of *composite bundles*. The objects of the category are a finite ladder of bundles

$$B_n \xrightarrow{\pi^n_{n-1}} \dots \xrightarrow{\pi^4_3} B_3 \xrightarrow{\pi^3_2} B_2 \xrightarrow{\pi^2_1} B_1 \xrightarrow{\pi^1_0} M \tag{18.10.1}$$

The number n is called the *height* of the composite bundle. Hence, ordinary bundles are composite bundles of height 1.

A composite bundle is denoted by $[B_3 \to B_2 \to B_1 \to M]$. The request of being a *finite* ladder is here for the sake of simplicity, to maintain all manifolds ordinary, finite dimensional, smooth manifolds. We already know a number of composite bundles: from any finite sequence of jet bundles, e.g. $[J^2 B \to J^1 B \to B \to M]$, to any sequence of higher order tangent bundles, e.g. $[T^2 M \to TM \to M]$, or the (possibly higher order) vertical bundles, e.g. $[V^2(B) \to V(B) \to B \to M]$.

Often, we refer to a particular bundle within the composite bundle $[B_n \to \dots \to B_2 \to B_1 \to M]$ so we need a systematic way to denote projections. Let us denote by $\pi^{k+h}_k = \pi^{k+1}_k \circ \dots \circ \pi^{k+h}_{k+h-1} : B_{k+h} \to B_k$, for all $0 \le k \le (k+h) \le n$ the projections of the corresponding bundles.

Each level in the ladder is an ordinary bundle, thus *fibered coordinates* on a composite bundle $[B_2 \to B_1 \to M]$ are *stratified* as we have

$$(x^\mu, y^i, z^a) \xrightarrow{\pi^2_1} (x^\mu, y^i) \xrightarrow{\pi^1_0} x^\mu \tag{18.10.2}$$

When we consider a specific bundle in $[B_3 \to B_2 \to B_1 \to M]$, we split the coordinates by semicolon to divide base and fiber coordinates. For example, $(x^\mu, y^i; z^a, v^I)$ will denote fibered coordinates on the bundle $\pi^3_1 : B_3 \to B_1$.

A *composite morphism* from $[B_2 \to B_1 \to M]$ to $[C_2 \to C_1 \to N]$ is triple of maps (Φ_2, Φ_1, ϕ) for which (Φ_2, Φ_1) and (Φ_1, ϕ) are separately fibered morphisms of each level, namely

$$\pi'^2_1 \circ \Phi_2 = \Phi_1 \circ \pi^2_1 \qquad \pi'^1_0 \circ \Phi_1 = \phi \circ \pi^1_0 \tag{18.10.3}$$

Equivalently, a composite morphism makes the diagram alongside commutative.

If we assume that (Φ_2, Φ_1, ϕ) are fibered morphism at each level then we can prove it is fibered along any rectangle in the diagram. For example, we assume

$$\pi_1'^2 \circ \Phi_2 = \Phi_1 \circ \pi_1^2 \qquad\qquad \pi_0'^1 \circ \Phi_1 = \phi \circ \pi_0^1 \tag{18.10.4}$$

and we have

$$\pi_0'^2 \circ \Phi_2 = \pi_0'^1 \circ \pi_1'^2 \circ \Phi_2 = \pi_0'^1 \circ \Phi_1 \circ \pi_1^2 = \phi \circ \pi_0^1 \circ \pi_1^2 = \phi \circ \pi_0^2 \tag{18.10.5}$$

Both transition maps and composite morphisms are stratified. For example, a composite morphism of $[B_3 \to B_2 \to B_1 \to M]$ is locally expressed as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \\ z'^a = Z^a(x, y, z) \\ v'^I = V^I(x, y, z, v) \end{cases} \tag{18.10.6}$$

Stratification is the only difference between a composite bundle and a single bundle. Let us remark that not all fibered bundle morphisms of the bundle $\pi_1^3 : B_3 \to B_1$ can be considered as composite fibered maps. In fact, the general bundle morphism of $\pi_1^3 : B_3 \to B_1$ is locally represented by

$$\begin{cases} x'^\mu = x'^\mu(x, y) \\ y'^i = y'^i(x, y) \\ z'^a = z'^a(x, y, z, v) \\ v'^I = v'^I(x, y, z, v) \end{cases} \tag{18.10.7}$$

which can be considered as a composite map only if $x'^\mu = x'^\mu(x, y) = x'^\mu(x)$ and $z'^a = z'^a(x, y, z, v) = z'^a(x, y, z)$. Hence the category of composite bundles is a genuine new category, since we are really restricting allowed maps. Equivalently, we can say that composite bundles have in fact a richer structure (hence less morphisms which preserve such structures) with respect to the bundles which they contain.

On the other hand, we can always regard a composite bundle as a special bundle, since transition maps (18.10.6) are in fact a special case of transition maps of the bundle $\pi_0^3 : B_3 \to M$, just stratified.

As we do in the category of bundles, we consider a class of vector fields which preserve the composite structure, i.e. such that the their flow is made of composite maps. The components of these vector fields are stratified as well as

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i + \xi^a(x, y, z)\partial_a + \xi^I(x, y, z, v)\partial_I \tag{18.10.8}$$

and they are called *stratified vector fields*.

Definition: *a composite section of $[B_n \to \ldots \to B_1 \to M]$ is a family of sections $\sigma_k^{k+1} : B_k \to B_{k+1}$, each with $\pi_k^{k+1} \circ \sigma_k^{k+1} = id_{B_k}$ being a section itself.*

Given a composite section $\left(\sigma_k^{k+1}\right)$, we can compose them to have sections $\sigma_k^{k+h} : \sigma_{k+h-1}^{k+h} \circ \ldots \circ \sigma_{k+1}^{k+2} \circ \sigma_k^{k+1} : B_k \to B_{k+h}$ at any level.

Prove that $\sigma_k^{k+h} : B_k \to B_{k+h}$ is a section, i.e. show that $\pi_k^{k+h} \circ \sigma_k^{k+h} = \mathrm{id}_{B_k}$.

Given a composite section $\left(\sigma_k^{k+1}\right)$ on $[B_n \to \ldots \to B_2 \to B_1 \to M]$, a section $\sigma_0^n : M \to B_n$ is called a *top section*. On the other hand, $\pi_k^n \circ \sigma_0^n : M \to B_k$ is equal to the section $\sigma_0^k(x) = \pi_k^n \circ \sigma_0^n(x)$.

Locally, a composite section on $[B_2 \to B_1 \to M]$ is represented as

$$\sigma_0^1 : M \to B_1 : x \mapsto (x, y(x)) \qquad \sigma_1^2 : B_1 \to B_2 : (x, y) \to (x, y, z(x, y)) \tag{18.10.9}$$

while the top section is locally represented by

$$\sigma_0^2 : M \to B_2 : x \mapsto (x, y(x), z(x, y(x))) \tag{18.10.10}$$

Let us stress that giving a composite section $\sigma = (\sigma_1^2, \sigma_0^1)$ on $[B_2 \to B_1 \to M]$ corresponds to give the fields $z^a(x, y)$ on B_1 and the field $y(x)$ on M. We shall see in variational calculus some examples in which the same composite variational principle will give equations for fields on different manifolds.

The set of composite sections on $[B_n \to \ldots \to B_1 \to M]$ is denoted by $\Gamma(\pi_{n-1}^n, \ldots, \pi_1^2, \pi_0^1)$.

This is a flexible framework for variational calculus, more general than the usual one. We shall show it allows to treat perturbation theory for mechanics and field theory in a geometric setting.

Let us discuss some examples of composite bundles.

Composite tangent bundles

In mechanics, forces and accelerations play a distinct role which is captured by a composite bundle

$$[\mathbb{R} \times T^2 M \to \mathbb{R} \times TM \to \mathbb{R} \times M \to \mathbb{R}] \tag{18.10.11}$$

which is relevant to (covariant, homogeneous) mechanics, in which we have sections $\sigma_0^1 : \mathbb{R} \to \mathbb{R} \times M$ which are in one-to-one correspondence with parameterized curves, sections $\sigma_1^2 : \mathbb{R} \times M \to \mathbb{R} \times TM$ which describes an act of motion, namely a field of velocities, as well as $\sigma_2^3 : \mathbb{R} \to \mathbb{R} \times T^2 M$ which describes a force field, which is a field of acceleration up to a mass factor.

The top section $\sigma_0^3 : M \to \mathbb{R} \times T^2 M : t \mapsto (t, x(t), v(t), a(t))$ describes the velocity and the acceleration of a particular motion $x(t)$. In this case, velocity and acceleration are defined only on the motion.

Kinematics is captured by the definition of lift of a motion $\sigma_0^1 : \mathbb{R} \to \mathbb{R} \times M$, which provides us with the definition of velocity, acceleration, ... we use to describe a motion. Given a section $\sigma : \mathbb{R} \to \mathbb{R} \times M : t \mapsto (t, x(t))$ (which is in one-to-one correspondence with motions, namely with parameterized curves $\gamma : \mathbb{R} \to M$), we can lift it to a section

$$\sigma_0^3 : \mathbb{R} \to \mathbb{R} \times T^2 M : t \mapsto (t, x(t), v(t), a(t)) \tag{18.10.12}$$

which accounts for the acceleration and it projects to $\sigma_0^2 : (t, x(t), v(t))$ which accounts for velocity. The sections σ_0^3, σ_0^2, and σ_0^1 accounts for position, velocities and acceleration *along* a motion.

However, it is sometime useful (e.g. in dynamics) to regard force fields which are sections of $[\mathbb{R} \times M \to \mathbb{R} \times TM]$, i.e. fields on space instead of their mere remnant along the motion. In this sense a composite section has more information, e.g. it accounts for the force at a point which is outside the motion trajectory as well.

Vertical vector bundles

We already discussed how to define the bundle $[V(B) \to B]$ of vertical vectors. The bundle of vertical vectors is by its nature a composite bundle $[V(B) \to B \to M]$. A composite section $(\sigma, \Xi) : M \to B \to V(B)$ accounts for a section σ of $[B \to M]$ as well as a vertical vector field Ξ on B, which, as we shall see, turns out to be useful to describe perturbations in variational calculus. It also induces a top section $\Xi \circ \sigma : M \to V(B)$ which describes a vertical vector field defined only on the section σ.

The functor $V(\cdot)$ is a covariant functor, so we can iterate it and define

$$[V(V(B)) \to V(B) \to B \to M] \tag{18.10.13}$$

Here we have a very rich structure. For example, on $V(V(B))$ we have composite coordinates $(x^\mu, y^i, v^i; u^i, w^i)$ to describe a vertical vector $\Xi = u^i \frac{\partial}{\partial y^i} + w^i \frac{\partial}{\partial v^i}$ on $V(B)$. Since $V(V(B))$ is obtained functorially from the bundle B, transition maps on it are generated by transition maps on B and they can easily be obtained in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \end{cases} \Rightarrow \begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ v'^i = J^i_j(x,y)v^j \\ u'^i = J^i_j(x,y)u^j \\ w'^i = J^i_j(x,y)w^j + J^i_{kj}(x,y)v^j u^k \end{cases} \tag{18.10.14}$$

where we set $J^i_j := \frac{\partial Y^i}{\partial y^j}(x,y)$ and $J^i_{kj} := \frac{\partial Y^i}{\partial y^j \partial y^k}(x,y)$ as usual. So, first, we notice that transition maps of $V(V(B))$ are stratified as they should in a composite bundle. (Actually, a bit more than stratified since, for example, transition maps on the coordinate u^i do not depend on v.)

Second, we notice that transition maps of the coordinates v^i and u^i are similar in form. Consequently, the local swap maps

$$: (x^\mu, y^i, v^i, u^i, w^i) \mapsto (x^\mu, y^i, u^i, v^i, w^i) \tag{18.10.15}$$

glue together to define a global and intrinsic swap map

$$\text{swap} : V(V(B)) \to V(V(B)) \tag{18.10.16}$$

Also, one can define a sub-bundle $V^2(B) \subset V(V(B))$ by setting the constraint $v^i = u^i$. The constraint is given in coordinates though it could be intrinsically expressed by the intrinsic swap map.

On the composite bundle $[V^2(B), V(B), B, M]$, we have fibered coordinates (x^μ, y^i, v^i, w^i) and transition maps

$$
\begin{cases}
x'^\mu = x'^\mu(x) \\
y'^i = Y^i(x, y) \\
v'^i = J^i_j(x, y)v^j \\
w'^i = J^i_j(x, y)w^j + J^i_{kj}(x, y)v^j v^k
\end{cases}
\tag{18.10.17}
$$

Notice how the bundle $\pi^2_1 : V^2(B) \to V(B)$ is an affine bundle, though $\pi^2_0 : V^2(B) \to B$, in general, is not. Of course, a similar procedure can be carried out at higher orders.

A composite section of $[V^2(B) \to V(B) \to B \to M]$ corresponds to a section $\sigma : M \to B : x \mapsto (x, y(x))$, a vertical vector field $\Xi = v^i(x, y)\frac{\partial}{\partial y^i}$ on B, namely a section of $[V(B) \to B]$, and a section $Z : V(B) \to V^2(B) \subset V(V(B)) : (x, y, v) \mapsto (x, y, v, w(x, y, v))$ of $[V^2(B) \to V(B)]$ given by $Z = v^i(x, y)\frac{\partial}{\partial y^i} + w^i(x, y, z)\frac{\partial}{\partial v^i}$.

Let us also remark that since $V(\cdot)$ is a functor we can lift a vector field on $[B \to M]$ to $[V(B) \to B]$, and hence from there to $[V(V(B)) \to V(B)]$, and so on.

Let us consider a projectable vector field $\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i$ on B. It defines a fibered flow

$$
\Phi_s : B \to B : (x, y) \mapsto (x'_s(x), y'_s(x, y)) \qquad \frac{dx'_s}{ds}\Big|_{s=0}(x) = \xi^\mu \qquad \frac{dy'_s}{ds}\Big|_{s=0}(x, y) = \xi^i
\tag{18.10.18}
$$

Consider $V(B)$ with coordinates (x^μ, y^i, v^i) and the lift of automorphisms to $V(B)$, which is locally expressed as (part of) the transition maps (18.10.17), so we have

$$
x'^\mu = x'^\mu_s(x) \qquad y'^i = y'^i_s(x, y) \qquad v'^i = \partial_j y'^i_s(x, y)v^j
\tag{18.10.19}
$$

That is a flow just because $V(\cdot)$ is a functor and its infinitesimal generator is defined as the lift of Ξ to $V(B)$

$$
V(\Xi) = \xi^\mu(x)\frac{\partial}{\partial x^\mu} + \xi^i(x, y)\frac{\partial}{\partial y^i} + \partial_j \xi^i(x, y)v^j \frac{\partial}{\partial v^i}
\tag{18.10.20}
$$

which, in fact, is stratified. We can use the same procedure to lift Ξ to $V^2(B)$ with coordinates (x^μ, y^i, v^i, w^i) as

$$
V^2(\Xi) = \xi^\mu(x)\frac{\partial}{\partial x^\mu} + \xi^i(x, y)\frac{\partial}{\partial y^i} + \partial_j \xi^i(x, y)v^j \frac{\partial}{\partial v^i} + (\partial_j \xi^i(x, y)w^j + \partial_{kj}\xi^i(x, y)v^j v^k)\frac{\partial}{\partial w^i}
\tag{18.10.21}
$$

Analogously, a projectable vector field on B induces a stratified vector field on any $V^k(B)$.

This result will be particularly relevant in variational calculus, since there a deformation of field, i.e. a vertical vector field on B, will induce a deformation at any order on $V^k(B)$, where we describe perturbations of order k.

Jet prolongations of composite bundles

If we have a composite bundle $[C \to B \to M]$ with coordinates (x^μ, y^i, z^a), the field z^a depends on both (x, y). Accordingly, when we wish to geometrise PDE for composite sections, we need a space which encodes partial derivatives, i.e. both z_μ^a and z_i^a.

For first derivatives, the construction is simple, it has to account for $(y_\mu^i, z_\mu^a, z_i^a)$. If we consider the first prolongation $J^1(\pi_0^1)$ of the bundle $[B \to M]$, we obtain a bundle $[\pi_B^1 : J^1(\pi_0^1) \to B]$ with coordinates $(x^\mu, y^i; y_\mu^i)$. On the other hand, we consider the first prolongation $J^1(\pi_1^2)$ of the bundle $[C \to B]$, i.e. a bundle $[\pi_C^1 : J^1(\pi_1^2) \to C]$ with coordinates $(x^\mu, y^i, z^a; z_\mu^a, z_i^a)$.

We wish to smash these two spaces together, which could be done by fiber product if the were bundles on the same base manifold. Then we can define

$$\mathbb{P}_1 := J^1(\pi_1^2) \times_C (\pi_1^2)^* \left(J^1(\pi_0^1) \right) \tag{18.10.22}$$

which, in fact, has coordinates $(x^\mu, y^i, z^a; y_\mu^i, z_\mu^a, z_i^a)$. The fiber product comes with a projection $\pi_2^3 : \mathbb{P}_1 \to C$.

For the second jet prolongation, the situation is slightly more complicated since we wish to have $(y_{\mu\nu}^i, z_{\mu\nu}^a, z_{ij}^a)$ but also $z_{i\mu}^a$. We consider the second prolongation $J^2(\pi_0^1)$ of the bundle $[B \to M]$, we obtain a bundle $[\pi_B^2 : J^2(\pi_0^1) \to B]$ with coordinates $(x^\mu, y^i; y_\mu^i, y_{\mu\nu}^i)$, as well as the second prolongation $J^2(\pi_1^2)$ of the bundle $[C \to B]$, i.e. a bundle $[\pi_C^2 : J^1(\pi_1^2) \to C]$ with coordinates $(x^\mu, y^i, z^a; z_\mu^a, z_i^a, z_{\mu\nu}^a, z_{i\mu}^a, z_{ij}^a)$. Again, we can smash these bundle together by

$$\mathbb{P}_2 := J^2(\pi_1^2) \times_C (\pi_1^2)^* \left(J^2(\pi_0^1) \right) \tag{18.10.23}$$

which, in fact, has coordinates $(x^\mu, y^i, z^a; y_\mu^i, z_\mu^a, z_i^a, y_{\mu\nu}^i, z_{\mu\nu}^a, z_{i\mu}^a, z_{ij}^a)$. We can define a projection $\pi_3^4 : \mathbb{P}_2 \to \mathbb{P}_1$. Similarly, at higher orders.

Given a composite section $(\sigma : M \to B, \rho : B \to C)$ we can prolong them to $j^1\sigma : M \to J^1B$ and $j^1\rho : C \to J^1C$, as in the diagram alongside. We cannot define a section from C to $\mathbb{P}_1$ (or from B to $\mathbb{P}_1$). As in the standard case, we can define, however, a section $j^1(\sigma, \rho) : M \to \mathbb{P}_1$ such that $\pi_2^3 \circ j^1(\sigma, \rho) = \rho \circ \sigma$ and $\pi_1^3 \circ j^1(\sigma, \rho) = \sigma$.

One starts from $x \in M$ and mark $\sigma(x) \in B$ and $\rho \circ \sigma(x) \in C$, as well as $j^1\sigma(x) \in J^1B$ and $j^1\rho(\sigma(x)) \in J^1C$.

By construction, we have $\pi_B^1\left(j^1\sigma(x)\right) = \sigma(x) = \pi_1^2(\rho \circ \sigma(x))$. Now check in the definition of pull-back bundle, when one has a point $j^1\sigma(x) \in J^1B$ and $\rho \circ \sigma(x) \in C$ such that $\pi_B^1\left(j^1\sigma(x)\right) = \pi_1^2(\rho \circ \sigma(x))$, that defines a point $(\rho \circ \sigma(x), j^1\sigma(x)) \in (\pi_1^2)^*\left(J^1(\pi_0^1)\right)$. The projection on C is defined as $\pi^* : (\pi_1^2)^*\left(J^1(\pi_0^1)\right) \to C : (\rho \circ \sigma(x), j^1\sigma(x)) \mapsto \rho \circ \sigma(x)$. Analogously, as one can check in the definition of product, when one has a point $(\rho \circ \sigma(x), j^1\sigma(x)) \in (\pi_1^2)^*\left(J^1(\pi_0^1)\right)$ and a point $j^1\rho(\sigma(x)) \in J^1C$ such that $\pi^*(\rho \circ \sigma(x), j^1\sigma(x)) = \rho \circ \sigma(x) = \pi_C^1(j^1\rho(\sigma(x))) \in C$, that defines a point $\left(j^1\rho(\sigma(x)), (\rho \circ \sigma(x), j^1\sigma(x))\right) \in J^1C \times_C (\pi_1^2)^*\left(J^1(\pi_0^1)\right) = \mathbb{P}_1$.

Then we can define a map

$$j^1(\sigma, \rho) : M \to \mathbb{P}_1 : x \mapsto \left(j^1\rho(\sigma(x)), (\rho \circ \sigma(x), j^1\sigma(x))\right) \tag{18.10.24}$$

and check it is a section by computing

$$\pi_2^3\left(j^1\rho(\sigma(x)), (\rho \circ \sigma(x), j^1\sigma(x))\right) = \pi_C^1(j^1\rho(\sigma(x)) = \rho(\sigma(x)) \qquad \pi_1^3\left(j^1\rho(\sigma(x)), (\rho \circ \sigma(x), j^1\sigma(x))\right) = \pi_1^2(\rho(\sigma(x))) = \sigma(x) \tag{18.10.25}$$

and, finally, one has $\pi_0^3\left(j^1\rho(\sigma(x)), (\rho \circ \sigma(x), j^1\sigma(x))\right) = \pi_0^1(\sigma(x)) = x$. In other words, $j^1(\sigma, \rho) : M \to \mathbb{P}_1$ is a section on $\rho \circ \sigma$ and hence on σ. Similarly, one can extend the construction to higher order prolongations.

Locally, if the composite section is expressed as $\sigma : M \to B : x \mapsto (x, y(x))$ and $\rho : B \to C : (x, y) \mapsto (x, y; z(x, y))$ (and, accordingly, $\rho \circ \sigma : M \to C : x \mapsto (x, y(x), z(x, y(x)))$), then the prolongation to $\mathbb{P}_1$ is

$$j^1(\sigma, \rho) : M \to \mathbb{P}_1 : x \mapsto \left(x^\mu, y^i(x), z^a(x, y(x)), \partial_\mu y^i(x), \partial_\mu z^a(x, y(x)), \partial_i z^a(x, y(x))\right) \tag{18.10.26}$$
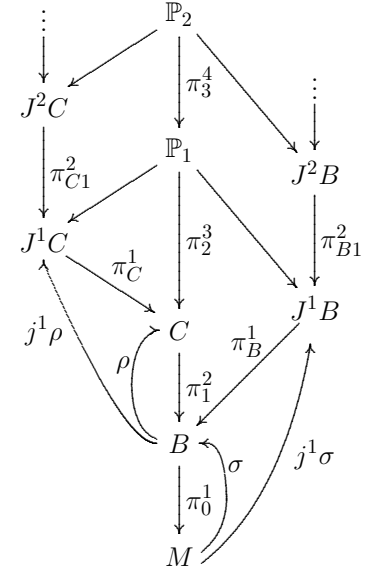
If one tries to define a section : $C \to \mathbb{P}_1$, it should start from $c \in C$ and project to $\pi_1^2(c) = b$ and $\pi_0^1(b) = x$. However, it cannot be guaranteed that $\pi_1^2(c) = b = j^1\sigma(x)$ and, accordingly, it fails to define a point on $(\pi_1^2)^*J^1B$.

Similarly, if one wants to define a section : $B \to \mathbb{P}_1$ the construction starts from $b \in B$ and obtains $\rho(b)$ and $\pi_0^1(b) = x$. Again, $\pi_B^1(j^1\sigma(x)) = \sigma(x)$ fails to be $\sigma(x) = b$ and, accordingly, one cannot define a point on $(\pi_1^2)^*J^1B$.

A vertical vector field on $[C \to B \to M]$ reads as

$$X = X^i(x,y)\frac{\partial}{\partial y^i} + X^a(x,y,z)\frac{\partial}{\partial z^a} \tag{18.10.27}$$

Given a composite section $\rho \circ \sigma : x \mapsto (x, y(x), z(x, y(x)))$, we shall also set $\delta y^k(x) := X^i(x, y(x))$ and $\delta z^a(x) := X^a(x, y(x), z(x, y(x)))$.

Accordingly, the composition $X \circ \rho \circ \sigma$ is locally expressed as $X \circ \rho \circ \sigma : x \mapsto (x, y(x), z(x), \delta y^k(x), \delta z^a(x))$.

References

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Kolar, ...

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Chapter 19. Natural and gauge-natural bundles

1. Introduction

 Here we come to define natural and gauge-natural objects. They are sections of natural and gauge-natural bundles, which, in general, are associated to (higher order) frame bundles and prolongations of principal bundles, respectively.

 Roughly, speaking natural objects are geometric objects defined on a manifold, which carry an action of its diffeomorphisms. As such, they are the most general framework in which one can even state the principle of general covariance.

> Without an action of diffeomorphisms on fields, one does not even know which transformations should be symmetries for the dynamics. The Lagrangian of a system is a horizontal form on some jet prolongation. If one has a transformation on the configuration bundle, then it can be easily prolonged to the jet bundles by the jet functor and required to be a symmetry. Unfortunately, if the configuration bundle is not natural there is no way to lift a base diffeomorphism to a transformation on the configuration bundle.

 Here the relevant issue is being able to lift base diffeomorphisms to the configuration bundle. That is possible if (and only if) the configuration bundle is associated to some (possibly higher order) frame bundle. As a matter of fact, one can lift base diffeomorphisms to the frame bundle. Then the lift is induced to all associated bundles.

 Thus, associated bundles to some frame bundle (i.e. natural bundles) is what is needed to talk about general covariance principle.

 Gauge-natural objects is a (not so) slight extension of what happens when one has a gauge group around. It defines the (most general) objects for which one has a well-defined group of gauge transformations to be required to be symmetries of the dynamics. Gauge transformations are originated by principal automorphisms on some principal structure bundle P. The group $\mathrm{Aut}(P)$ induces gauge transformations on any bundle associated to some prolongation of P and then is prolonged to the jets of the configuration bundle. Its elements can preserve the Lagrangian and be symmetries.

> Roughly speaking, one could say, that gauge transformations *contain* base diffeomorphisms. This would not be accurate. The group $\mathrm{Aut}(P)$ *projects* onto $\mathrm{Diff}(M)$, which is not embedded in it, as shown by the exact sequence
>
> $$0 \to \mathrm{Aut}_V(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M) \to 0 \tag{19.1.1}$$
>
> An embedding of $\mathrm{Diff}(M) \to \mathrm{Aut}(P)$ would be a splitting of this sequence and, if it is defined, as a consequence, one would have $\mathrm{Aut}(P) = \mathrm{Aut}_V(P) \times \mathrm{Diff}(M)$.
> However, not all exact sequences of groups split (as we discussed with spin groups). In particular, this sequence does not split, in general. It does in some particular cases, certainly when P is trivial, see Subsection 2.7.7, but in general the splitting is not global, since it locally depends on the local trivialization.

That is manifest if one tries to define *"horizontal"* automorphisms on P. One would like to consider (in a fixed local trivialization) automorphisms which have "no vertical components", namely

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ g' = g \end{cases} \tag{19.1.2}$$

However, this form is not invariant with respect to changes of local trivializations since, in a new trivialization, one has $g' = \varphi(x) \cdot g$.

Since $\mathrm{Aut}(P)$ *projects* on $\mathrm{Diff}(M)$, it does not *contain* it, requiring that all automorphisms $\mathrm{Aut}(P)$ are symmetries is more general that general covariance. Gauge-natural covariance is *more* general than general covariance in a precise sense. If we consider vertical gauge transformations in $\mathrm{Aut}_V(P)$ these are *contained* in $\mathrm{Aut}(P)$. Then one can require them to be symmetries, before requiring all automorphisms to be symmetries.

Under this partial covariance principle, physical states are identified with equivalence classes of sections with respect to vertical gauge transformations. Precisely, one defines an equivalence relation among sections

$$\sigma \sim \rho \iff \exists \Psi \in \mathrm{Aut}_V(P) \subset \mathrm{Aut}(\mathcal{C}) : \Psi \circ \sigma = \rho \tag{19.1.3}$$

and denote equivalence classes by $[\sigma]$.

These equivalence classes are defined at the level of sections, not at the level of configuration bundle.
By that, we mean that equivalence classes are just that. They cannot be regarded, in general, as sections of some quotient bundle.

Now we can define an action of $\mathrm{Diff}(M)$ on classes. For a diffeomorphism $\phi : M \to M$, we choose an automorphism $\Phi : P \to P$ which projects ϕ and we set

$$\phi([\sigma]) = [\Phi \circ \sigma \circ \phi^{-1}] \tag{19.1.4}$$

This is well defined. If we choose another automorphism $\Phi' : P \to P$ which projects on the same ϕ, then $\Psi = \Phi' \circ \Phi^{-1} : P \to P$ is vertical, i.e. $\Psi \in \mathrm{Aut}_V(P)$ and $\Phi' = \Psi \circ \Phi$. Accordingly, $\Phi' \circ \sigma \circ \phi^{-1} = \Psi \circ \Phi \circ \sigma \circ \phi^{-1}$, so that $\sigma \sim \Phi \circ \sigma \circ \phi^{-1}$, i.e. $[\Phi' \circ \sigma \circ \phi^{-1}] = [\Phi \circ \sigma \circ \phi^{-1}]$.
Then, whatever automorphism one selects over ϕ, the result is unchanged.

Then, one declares the physical state to be represented by equivalence classes of sections which are equivalent modulo $\mathrm{Diff}(M)$, which corresponds to declaring equivalent sections modulo automorphisms $\Phi \in \mathrm{Aut}(P)$, in the beginning.

In other words, using $\mathrm{Aut}(P)$ as symmetries at once, is like imposing vertical gauge symmetries and then base diffeomorphims in two steps. However, in this way, one just works with sections of the configuration bundle.

We shall first consider natural objects in Section 19.2, then gauge-natural objects in Section 19.3. Then we shall collect in Section 19.4 Lie derivatives of some kind of fields, which are used throughout the book.

2. Natural bundles

Let us here collect definitions and properties of natural bundles. Roughly speaking, natural bundles are defined so that eventually one can define an action of $\mathrm{Diff}(M)$ on their sections. That is something one needs to discuss general covariance principle, which, in fact, selects covariant dynamics.

We already discussed some example of natural objects, from tangent and cotangent bundle to the frame bundle. Here we want to be general. We shall start defining the groups which are fibres of the higher order frame bundles, to the higher order frame bundles which are the principal bundles to which natural bundles are associated.

This generality will allow us to discuss natural lifts in general and, eventually, to pin point the difference between natural and gauge-natural bundles.

Group $\mathrm{GL}^k(m)$

Let us start by defining an infinite dimensional group of maps $\alpha : \mathbb{R}^m \to \mathbb{R}^m$, which fix the origin, which are locally invertible at the origin. That means that there exists two neighbourhoods of the origin U and V, that the maps restrict to $\alpha : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^m$ and one has

$$\mathcal{N}_m = \{\alpha : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^m : \alpha(0) = 0, \text{loc. invertible}\} \tag{19.2.1}$$

The map $\alpha : U \to V$ is required to be smooth with a smooth inverse.

That, of course, form a group by composition, unfortunately, an infinite dimensional group. As any map $\alpha : U \to V$ that is a local section of the bundle $(\mathbb{R}^m \times \mathbb{R}^m, \pi, \mathbb{R}^m, \mathbb{R}^m)$, so we can consider jet prolongations of $j_0^k \alpha$.

Accordingly, we have that $j_0^k \alpha$ is described by the numbers $(\alpha_b^a, \alpha_{bc}^a, \ldots, \alpha_{b_1 \ldots b_k}^a)$ in

$$T_0^k(\alpha)(x) = \alpha_b^a x^b + \tfrac{1}{2} \alpha_{bc}^a x^b x^c + \ldots + \tfrac{1}{k!} \alpha_{b_1 \ldots b_k}^a x^{b_1} \ldots x^{b_k} \tag{19.2.2}$$

Let us denote by $\mathrm{GL}^k(m)$ the set of all classes $j_0^k \alpha$ for some $\alpha \in \mathcal{N}_m$.

Since the jet prolongation is a functor, hence it preserves the compositions, it then induces a product on $\mathrm{GL}^k(m)$ which makes it a group. It is finite dimensional, thus, in fact, a Lie group.

At any order k, we can obtain the expression for the product in $\mathrm{GL}^k(m)$. For example, for $k = 2$, it is easy to compute the Taylor expansion for the composition $\alpha \circ \beta \in \mathcal{N}_m$. In fact, we have

$$T_0^2(\alpha \circ \beta)(x) = \alpha_d^a \beta_b^d x^b + \tfrac{1}{2}(\alpha_{de}^a \beta_b^d \beta_c^e + \alpha_d^a \beta_{bc}^d) x^b x^c \tag{19.2.3}$$

That defines a group product on $\mathrm{GL}^2(m)$ as

$$(\alpha_b^a, \alpha_{bc}^a) * (\beta_b^a, \beta_{bc}^a) = (\alpha_d^a \beta_b^d, \alpha_{de}^a \beta_b^d \beta_c^e + \alpha_d^a \beta_{bc}^d) \tag{19.2.4}$$

One can check associativity, which though descends directly from associativity of composition. Moreover, the neutral element is given by $(\delta_b^a, 0)$ and the inverse can be easily obtained as

$$(\alpha_b^a, \alpha_{bc}^a)^{-1} = (\bar{\alpha}_b^a, -\bar{\alpha}_f^a \alpha_{de}^f \bar{\alpha}_b^d \bar{\alpha}_c^e) \tag{19.2.5}$$

Of course, we have an infinite chain of groups with an infinite chain of group monomorphisms

$$\ldots \mathrm{GL}^k(m) \to \mathrm{GL}^{k-1}(m) \to \ldots \to \mathrm{GL}^2(m) \to \mathrm{GL}(m) \tag{19.2.6}$$

where we canonically identified $\mathrm{GL}(m) \simeq \mathrm{GL}^1(m)$.

Higher order frame bundles

Let us now consider maps $e : \mathbb{R}^m \to M$ which are locally diffeomorphisms around the origin. This time we set $e(0) = x \in M$ and we identify maps with the same Taylor polynomial $T_0^k(e)$. The equivalence classes are denoted by $j_0^k e$ and they form a set denoted by $L^k(M)$, which is a manifold since it has coordinates $(e_a^\mu, e_{ab}^\mu, \dots, e_{a_1 \dots a_k}^\mu)$ from the Taylor polynomial

$$T_0^k(e)(x) = x^\mu + e_a^\mu x^a + \tfrac{1}{2} e_{ab}^\mu x^a x^b + \dots + \tfrac{1}{k!} e_{a_1 \dots a_k}^\mu x^{a_1} \dots x^{a_k} \tag{19.2.7}$$

The coordinates on $L^k(M)$ depend on a chart x^μ chosen on M. If we change coordinates $x'^\mu = x'^\mu(x)$ on M, then the coordinates on $L^k(M)$ change as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e_a'^\mu = J_\nu^\mu e_a^\nu \\ e_{ab}'^\mu = J_\alpha^\mu e_{ab}^\alpha + J_{\alpha\beta}^\mu e_a^\alpha e_b^\beta \\ e_{abc}'^\mu = J_\alpha^\mu e_{abc}^\alpha + J_{\alpha\beta}^\mu e_{ab}^\alpha e_c^\beta + J_{\alpha\beta}^\mu e_{ac}^\alpha e_b^\beta + J_{\alpha\beta}^\mu e_{bc}^\alpha e_a^\beta \\ \dots \end{cases} \tag{19.2.8}$$

This is not a group, since maps $e : \mathbb{R}^m \to M$ cannot be composed one with the other.

However, we can define the map $\pi^k : L^k(M) \to M : j_0^k e \mapsto e(0) = x$, which, of course, does not depend on the representative selected in the equivalence class $j_0^k e$.

Notice also that in (19.2.8), the transformation rules for x'^μ just depends on x^μ, the transformation rules for $e_a'^\mu$ just depend on (x^μ, e_a^μ) (and it corresponds to a left product by a matrix $J \in \mathrm{GL}(m)$). Then the transformation rules for $e_{ab}'^\mu$ just depend on $(x^\mu, e_a^\mu, e_{ab}^\mu)$ (and it corresponds to a left product by $(J_\alpha^\mu, J_{\alpha\beta}^\mu)$ in $\mathrm{GL}^2(m)$, as one can see by comparing with (19.2.4)) and so on at higher orders. That shows how the coordinates we chose on $L^k(M)$ are fibered coordinates and we have an infinite chain of bundles

$$\dots L^k(M) \to L^{k-1}(M) \to \dots \to L^2(M) \to L(M) \to M \tag{19.2.9}$$

where we canonically identified the first frame bundle $L^1(M) \simeq L(M)$ with the frame bundle.

We can naturally define a group right action of $\mathrm{GL}^k(m)$ on $L^k(M)$ by

$$j_0^k e * j_0^k \alpha = j_0^k (e \circ \alpha) \tag{19.2.10}$$

which is well defined since $e \circ \alpha : \mathbb{R}^m \to M$ and it is locally invertible at the origin. It is a vertical action, transitive on the fibers, and free.

We have that $e \circ \alpha(0) = e(0)$, hence the right action is vertical.

If we consider two points $j_0^k e_1$ and $j_0^k e_2$ in the same fiber (i.e. $e_1(0) = x = e_2(0)$), then $\epsilon = e_1^{-1} \circ e_2 : \mathbb{R}^m \to \mathbb{R}^m$ fixes the origin and it is locally invertible. Hence, we have $j_0^k \epsilon \in \mathrm{GL}^k(m)$.

We have

$$j_0^k e_2 = j_0^k e_1 * j_0^k \epsilon \tag{19.2.11}$$

and the right action is transitive on the fibers.

Freedom is trickier to prove, since, of course, there are many $\epsilon : \mathbb{R}^m \to \mathbb{R}^m$ such that $j_0^k e_2 = j_0^k e_1 * j_0^k \epsilon$. The point to prove is that they induce a single $j_0^k \epsilon \in \mathrm{GL}^k(m)$. It follows from the local expression of the right action

$$\begin{cases} x'^\mu = x^\mu \\ e'^\mu_a = e^\nu_b \alpha^b_a \\ e'^\mu_{ab} = e^\mu_c \alpha^c_{ab} + e^\mu_{cd} \alpha^c_a \alpha^d_b \\ \dots \end{cases} \tag{19.2.12}$$

For example, at order 2, we have that if $j_0^k e = j_0^k e * j_0^k \alpha$, then we have

$$\begin{cases} e^\mu_a = e^\nu_b \alpha^b_a \\ e^\mu_{ab} = e^\mu_c \alpha^c_{ab} + e^\mu_{cd} \alpha^c_a \alpha^d_b \end{cases} \Rightarrow \begin{cases} \alpha^b_a = \delta^a_b \\ e^\mu_c \alpha^c_{ab} = 0 \quad \Rightarrow \alpha^c_{ab} = 0 \end{cases} \tag{19.2.13}$$

which shows the action is free.

Accordingly, $L^k(M) \to M$ are principal bundles with the group $\mathrm{GL}^k(m)$.

Then, consider a diffeomorphism $\phi : M \to N$. We can define the *natural lift* $L^k(\phi) : L^k(M) \to L^k(N)$ by

$$L^k(\phi) : L^k(M) \to L^k(N) : j_0^k e \mapsto j_0^k(\phi \circ e) \tag{19.2.14}$$

which clearly projects over ϕ and it preserves compositions. Accordingly, $L^k(\cdot)$ is a covariant functor and the natural lift singles out a subgroup of transformations $\mathrm{Diff}(M) \subset \mathrm{Aut}(L^k(M))$.

Natural bundles

A *natural bundle* is a bundle associated to some $L^k(M)$. We already discussed that the tangent bundle, all tensor bundles, as well as tensor densities bundles are associated to $L(M)$. In the next Subsection we shall see that connections on manifolds are sections of a bundle which is associated to $L^2(M)$.

Being associated to $L^k(M)$, a natural bundle $\mathcal{B}$ comes which a $\mathrm{GL}^k(m)$-structure and with a subgroup $\mathrm{Aut}(L^k(M)) \subset \mathrm{Aut}(\mathcal{B})$ of transformations singled out. In view of the natural lift, one also has a smaller subgroup $\mathrm{Diff}(M) \subset \mathrm{Aut}(L^k(M)) \subset \mathrm{Aut}(\mathcal{B})$ singled out.

> We shall see how the difference between natural and gauge-natural theories is precisely that in natural theories the focus is on $\mathrm{Diff}(M) \subset \mathrm{Aut}(\mathcal{B})$ which defines *natural transformations*, while in gauge-natural theories the focus is on $\mathrm{Aut}(L^k(M)) \subset \mathrm{Aut}(\mathcal{B})$ which defines *gauge transformations*.

Once we have natural transformations $\mathrm{Diff}(M) \subset \mathrm{Aut}(\mathcal{B})$ on the natural bundle $\mathcal{B}$, the correspondence is functorial and it associates a natural transformation $\Phi_\lambda : \mathcal{B} \to \mathcal{B}$ to any base diffeomorphism $\phi : M \to M$.

> Let us remark that the whole discussion may seem vague and abstract. However, the correspondence is simply given by the composition of two functors, namely $L^k(M)$ to lift to the frame bundle, and $(\cdot) \times_\lambda F$ to go on the associated bundle.
>
> We described the two functor separately, and we gave example on how to actually compute objects through them. They encode quite a bit of details about the global properties of the base manifold, about the transformation rules of fields (through the group action λ), as well as about the local representation of the fields (through the choice of the standard fiber F in the associated bundle).
>
> Funtors are there exactly to encapsulate details and provide a simple view on in abstraction from details. We shall provide a worked out example in the next Subsection, anyway.

Natural transformations act on a section $\sigma : M \to \mathcal{B}$ as usual, namely

$$\phi_* \sigma = \Phi_\lambda \circ \sigma \circ \phi^{-1} \tag{19.2.15}$$

and they define an action of base diffeomorphisms on fields, which is just what we usually call the transformation rules of fields with respect to base diffeomorphisms.

Being the whole construction functorial also flows are mapped into flows. Accordingly, a base vector field ξ, defines a vector field $\hat{\xi}$ on the frame bundle $L^k(M)$, which induces a vector field $(\hat{\xi})_\lambda$ on the natural bundle. By an abuse of language, sometimes $(\hat{\xi})_\lambda$ is denoted simply by $\hat{\xi}$.

On any bundle $\mathcal{B}$, we can define Lie derivative of a section σ with respect to any projectable vector field Ξ on $\mathcal{B}$. When $\mathcal{B}$ is a natural bundle, however, we can restrict to Lie derivatives with respect to fields $(\hat{\xi})_\lambda$ which are natural lift of base vector fields. By some abuse of notation, we denote these Lie derivatives as

$$\pounds_\xi \sigma := \pounds_{(\hat{\xi})_\lambda} \sigma = T\sigma(\xi) - (\hat{\xi})_\lambda \circ \sigma \tag{19.2.16}$$

and we say they are Lie derivatives along base transformations.

In view of functoriality, we have that

$$[\pounds_\xi, \pounds_\zeta]\sigma = \pounds_{[\xi,\zeta]}\sigma \tag{19.2.17}$$

Let us stress that that is obtained in two steps. First, $[\xi,\zeta]\hat{} = [\hat{\xi}, \hat{\zeta}]$, i.e. the lift is *natural*. Second, $[\pounds_{\Xi_1}, \pounds_{\Xi_2}]\sigma = \pounds_{[\Xi_1,\Xi_2]}\sigma$ which is a general property of Lie derivatives of bundle sections. Let us stress once again also that, if the bundle is not natural, there is no way of defining Lie derivatives with respect to base vector fields and no way of writing down the conditions for which a Lagrangian is covariant with respect to spacetime diffeomorphisms.

Connections as higher order natural objects

Let us here give a simple non-trivial example from scratch, showing as one can regard connections on manifold as natural objects and hence define their Lie derivative (even though they are not tensors).

Now we have a pretty good geometric grasp on what a connection on a manifold actually is. It is a principal connection on the frame bundle $L(M)$. That is a beautiful intrinsic characterisation that readily justifies local representation and transformation rules. However, if it pops out in a physical field theory (as in Palatini formalism) often all one needs to know (and knows) is that there are fields $\Gamma^\alpha_{\beta\mu}(x)$ and that they transforms as

$$\Gamma'^\alpha_{\beta\mu} = J^\alpha_\rho \left(\Gamma^\rho_{\sigma\nu} \bar{J}^\sigma_\beta \bar{J}^\nu_\mu + \bar{J}^\rho_{\beta\mu} \right) \tag{19.2.18}$$

when one acts with a spacetime diffeomorphism.

As a matter of fact, these transformation rules are exactly what ensures that the object locally defined as $\omega = dx^\mu \otimes \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} \rho^\beta_\alpha \right)$ is, in fact, a global object, i.e. it is in that form in any coordinate system on $L(M)$. However, from a field theoretical viewpoint, that information is not what one primarily needs to set up a relativistic field theory, i.e. those fields are sections of a bundle, that they are *general* sections in a bundle, that spacetime diffeomorphisms act functorially on sections, so that Lie derivatives with respect to spacetime vector fields are defined and have good properties.

Luckily enough, that is quite easy to obtain in view of transformation rules (19.2.18), see Subsection 2.1.3. All one needs to do is staring (19.2.18) long enough to guess a Lie group action on a set which captures the local description of fields, i.e. the point of the set needs to be in one-to-one correspondence with pointwise values of fields.

In this case, we have field values expressed as $\Gamma^\alpha_{\beta\mu}$. If we impose the connection to be torsionless that are $d := \frac{1}{2}m^2(m+1)$ numbers, if we allow general connections they are $d := m^3$ numbers. Since transformations (19.2.18) are not linear, we do not select a vector space, so we consider an affine space $\mathbb{A}$ of dimension d modelled on a suitable tensor space $V := T^1_2(\mathbb{R}^m)$ (symmetric in lower indices or not depending on whether we allow torsion).

Since we have Jacobians in transformation rules the group has something to do with $\mathrm{GL}(m)$. However, transformation rules also depend on the Hessian $\bar{J}^\rho_{\beta\nu}$ which is not described in $\mathrm{GL}(m)$. Accordingly, we need to select $\mathrm{GL}^2(m)$ as a group. At this point, we have a left group action $\lambda : \mathrm{GL}^2(m) \times \mathbb{A} \to \mathbb{A}$ defined as

$$\lambda(J^a_b, J^a_{bc})(\Gamma^a_{bm}) = J^a_d \left(\Gamma^d_{ef} \bar{J}^e_b \bar{J}^f_m + \bar{J}^d_{bm} \right) \tag{19.2.19}$$

One can check that it is a left action and that the result is still in $\mathbb{A}$.

Then we can define the associated bundle $\mathrm{Con}(M) := L^2(M) \times_\lambda \mathbb{A}$. It has fibered coordinates $(x^\mu, \Gamma^\alpha_{\beta\nu})$, where we set $\Gamma^\alpha_{\beta\nu} := e^\alpha_a(\Gamma^a_{bc} e^b_\beta e^c_\nu + e^a_{\beta\mu})$ using the inverse $(e^\mu_a, e^\mu_{ab})^{-1} = (e^a_\mu, e^a_{\mu\nu} := -e^a_\alpha e^\alpha_{bc} e^b_\mu e^\nu_c)$. The coordinates $\Gamma^\alpha_{\beta\nu}$ which transform as (19.2.18) under coordinate transformations on $L^2(M)$, in turn, induced by a change of coordinates in M. We automatically have a one-to-one correspondence between sections of the bundle $\mathrm{Con}(M)$ and connections on M.

3. Gauge-natural bundles

Now we know how to deal with natural objects. The good news is that gauge-natural objects are more or less the same, at least if one focuses on the general procedure and forgets the details.

Gauge-natural bundles are associated to a principal bundle P, which is called the *structure bundle*, not to a manifold, they are defined so that, eventually, one can define an action of $\mathrm{Aut}(P)$ on their sections. That allows to deal with gauge symmetries and general covariance at once.

To describe gauge-natural bundles we follow more or less the same path we followed for natural bundles. First, we define groups, in general, then we define gauge-natural prolongations of a principal bundle which are principal bundles themselves though with a bigger group which can accomodate prolongations of gauge group and natural transformations.

Once we do it, then gauge-natural bundles are bundles which are associated to some prolongation of of the structure bundle P. By defining things in this way, we obtain configuration bundles which automatically have the structures needed in field theory. We shall finally comment on the differences between a natural and gauge-natural theory.

Prolongation of a Lie group

Given a Lie group G, we can consider maps $a : \mathbb{R}^m \to G$, declared equivalent maps which the same Taylor polynomial $T^r_0(a)$. Equivalence classes are denoted by $j^s_0 a$ and the quotient space by $J^s_m G = \{j^s_0 a : a : \mathbb{R}^m \to G\}$.

By using the group product on G, one can define a group product on $J_m^s G$ as

$$j_0^s a * j_0^s b = j_0^s(a \cdot b) \tag{19.3.1}$$

where we defined the map $a \cdot b : \mathbb{R}^m \to G : x \mapsto a(x) \cdot b(x)$.

That operation inherits associativity from the associativity in G. The neutral element is from the map $e : \mathbb{R}^m \to G : x \mapsto \mathbb{I}$, the inverse of $j_0^s a$ from the map $a^{-1} : \mathbb{R}^m \to G : x \mapsto a(x)^{-1}$ which uses the group inverse.

The group $J_m^s G$ has coordinates $(g, g_a, g_{ab}, \ldots, g_{a_1 \ldots a_s})$ from the coefficients of the Taylor polynomials

$$T_0^s(a)(x) = g + g_a x^a + \tfrac{1}{2} g_{ab} x^a x^b + \ldots + \tfrac{1}{s!} g_{a_1 \ldots a_s} x^{a_1} \ldots x^{a_s} \tag{19.3.2}$$

which identify equivalence classes $j_0^s a$.

In coordinates, the product of $J_m^s G$ is defined as

$$(g, g_a, g_{ab}, \ldots) * (h, h_a, h_{ab}, \ldots) = (g \cdot h, g_a \cdot h + g \cdot h_a, g_{ab} \cdot h + 2g_{(a} \cdot h_{b)} + g \cdot h_{ab}, \ldots) \tag{19.3.3}$$

When $s \leq k$, we also have a canonical right action of $\mathrm{GL}^k(m)$ on $J_m^s G$, defined by

$$j_0^s a \cdot j_0^k \alpha = j_0^s(a \circ \alpha) \tag{19.3.4}$$

which is well defined every time $s \leq k$ (otherwise the result depends on the representative chosen for $j_0^k \alpha$) since $a \circ \alpha : \mathbb{R}^m \to G$.

In this situation, we can define the semi-direct product

$$W_m^{(s,k)} G := J_m^s G \rtimes \mathrm{GL}^k(m) \tag{19.3.5}$$

An element in this group is a pair $(j_0^s a, j_0^k \alpha) \in W_m^{(s,k)} G$ and the product is defined as

$$(j_0^s a, j_0^k \alpha) * (j_0^s b, j_0^k \beta) = (j_0^s((a \circ \beta) \cdot b), j_0^k(\alpha \circ \beta)) \tag{19.3.6}$$

Let us notice that if $s = 0$ then k can be any integer. However, if $s = 1$, one should have at least $k = 1$ or greater. In a sense, if the gauge group enters at higher order, one needs some frame (at least at the same order) to define prolongations.

The product can be expressed using coordinates on the prolongation group.

For example, let us consider order $(1, 1)$. The product reads as

$$(a, a_a, \alpha_a^b) * (b, b_a, \beta_a^b) = (a \cdot b, a_c \beta_a^c \cdot b + a \cdot b_a, \alpha_c^b \beta_a^c) \tag{19.3.7}$$

The neutral element is $(e, 0, \delta_a^b)$, while the inverse is

$$(a, a_a, \alpha_a^b)^{-1} = (a^{-1}, -a^{-1} \cdot a_c \bar{\alpha}_a^c \cdot a^{-1}, \bar{\alpha}_a^b) \tag{19.3.8}$$

which corresponds to the maps $\bar{a} : \mathbb{R}^m \to G : x \mapsto a(\bar{\alpha}(x))^{-1}$ and $\bar{\alpha} : \mathbb{R}^m \to \mathbb{R}^m$.

Prolongation of a principal bundle

If we start from a structure bundle P, we can define the prolongation

$$W^{(s,k)}P := J^s P \times_M L^k(M) \tag{19.3.9}$$

for any pair of integers (s,k) with $s \le k$, which is called the *prolongation order*.

A point in $W^{(s,k)}P$ is a pair $(j_x^s \sigma, j_0^k e)$. Notice that σ is a local section, not a map $\mathbb{R}^m \to G$ as required by the prolongation group.

On it, we can define a canonical right action of the group $W_m^{(s,k)}G$.

As for the canonical right action, it is defined as

$$(j_x^s \sigma, j_0^k e) * (j_x^s a, j_0^k \alpha) = (j_x^s(\sigma \cdot (a \circ \bar{\alpha} \circ \bar{e})), j_0^k(e \circ \alpha)) \tag{19.3.10}$$

where we have that, in fact, $\sigma \cdot (a \circ \bar{\alpha} \circ \bar{e}) : M \to P$ is again a section of P.

This right action is vertical, transitive on the fibers and free.

The point $(j_x^s \sigma, j_0^k e) \in W^{(s,k)}P$ is applied to $x = e(0)$ and the right action does not change it. Given two points in the same fiber, $(j_x^s \sigma, j_0^k e)$ and $(j_x^s \sigma', j_0^k e')$ one can find one and only one group element $(j_x^s a, j_0^k \alpha)$ such that

$$(j_x^s \sigma', j_0^k e') = (j_x^s \sigma, j_0^k e) * (j_x^s a, j_0^k \alpha) = (j_x^s(\sigma \cdot (a \circ \bar{\alpha} \circ \bar{e})), j_0^k(e \circ \alpha)) \qquad \begin{cases} \sigma' = \sigma \cdot (a \circ \bar{\alpha} \circ \bar{e}) \\ e' = e \circ \alpha \end{cases} \tag{19.3.11}$$

These can be solved for α (since the right action is transitive on the fibres and free on $L^k(M)$) and a (since the right action is transitive on the fibres and free on P).

Hence, $W^{(s,k)}P$ is a principal bundle with a group $W_m^{(s,k)}G$.

Any automorphism $\Phi : P \to P$ which projects onto $\phi : M \to M$ can be prolonged to $W^{(s,k)}(\Phi) : W^{(s,k)}P \to W^{(s,k)}P$ by simply setting

$$W^{(s,k)}(\Phi)(j_x^s \sigma, j_0^k e) = (j_x^s(\Phi \circ \sigma \circ \phi^{-1}), j_0^k(\phi \circ e)) \tag{19.3.12}$$

which is called the *gauge-natural lift* of Φ. That defines a covariant prolongation functor $W^{(s,k)}(\cdot)$ from principal bundles with group G to principal bundles with group $W_m^{(s,k)}G$. That singles out a subgroup $\mathrm{Aut}(P) \subset \mathrm{Aut}(W^{(s,k)}P)$ of transformations on $W^{(s,k)}P$.

Since it is all functorial, that also sends infinitesimal generators of automorphisms (namely, right invariant vector fields) on P, to infinitesimal generators of automorphisms of $W^{(s,k)}P$.

Gauge-natural bundles

Gauge-natural bundles are bundles associated to some $W^{(s,k)}P$. A gauge-natural bundle $\mathcal{B}$, being associated to $W^{(s,k)}P$ comes with a $W_m^{(s,k)}G$-structure and with a subgroup $\mathrm{Aut}(W^{(s,k)}P) \subset \mathrm{Aut}(\mathcal{B})$ of gauge transformations singled out on it. In view of the gauge-natural lift, that has a smaller group $\mathrm{Aut}(P) \subset \mathrm{Aut}(W^{(s,k)}P) \subset \mathrm{Aut}(\mathcal{B})$ in it.

A natural object is associated to the principal bundle $L^k(M)$, so in some sense it is also a $(0,0)$ gauge-natural bundle for the group $\mathrm{GL}^k(m)$. In some sense, any natural bundle can be seen as a special case of a gauge-natural bundle associated to a special principal bundle, namely some frame bundle $L^k(M)$, so one could ask whether one could not avoid natural bundles completely and just work with gauge-natural bundles. This is not the case. Even if natural bundles are (also) objects in the category of gauge-natural bundles, what distinguish them are morphisms.

That is exactly the same thing that happens regarding $\mathbb{R}^2$ as a vector space, or as an affine space, or as a manifold, or as a bundle over $\mathbb{R}$. It is always the same space $\mathbb{R}^2$, though, on it, different transformations are allowed, depending on the category one chooses.

In the case of natural bundles, the two structures are distinguished only by the transformation subgroups singled out on them. As a natural bundle, natural transformations are $\mathrm{Diff}(M) \subset \mathrm{Aut}(L^k(M)) \subset \mathrm{Aut}(\mathcal{B})$. As a gauge-natural bundle, gauge-natural transformations are the whole $\mathrm{Aut}(L^k(M)) \subset \mathrm{Aut}(\mathcal{B})$.

When we shall define the dynamics, the Lagrangian will be required to be covariant with respect to $\mathrm{Diff}(M)$ in the first case, $\mathrm{Aut}(L^k(M))$ in the second case. Hence natural theories can be seen as gauge-natural theories with a smaller transformation group.

In general, however, let us stress once again that a gauge-natural theory on a generic structure bundle P has no natural structure (unless P itself is natural as, in particular, when $G = GL^k(m)$).

Once we have gauge-natural transformations $\mathrm{Aut}(P) \subset \mathrm{Aut}(\mathcal{B})$ on the natural bundle $\mathcal{B}$, the correspondence is functorial and it associates a gauge-natural transformation $\Phi_\lambda : \mathcal{B} \to \mathcal{B}$ to any automorphisms $\Phi : P \to P$ on the structure bundle. Again, the gauge-natural lift to $\mathcal{B}$ is composition of two covariant functors $W^{(s,k)}(\cdot)$ and $(\cdot) \times_\lambda F$ each of which has been described in details.

Gauge-natural transformations then act on sections of $\mathcal{B}$ as

$$\Phi_* \sigma = \Phi_\lambda \circ \sigma \circ \phi^{-1} \tag{19.3.13}$$

and they define an action of automorphisms $\mathrm{Aut}(P)$ of the structure bundle P on fields, which is just what we usually call the transformation rules of fields with respect to gauge transformations.

Being the whole construction functorial, also flows are mapped into flows. Accordingly, a gauge transformation generator Ξ, defines a vector field $\hat{\Xi}$ on the frame bundle $W^{(s,k)}P$, which induces a vector field $(\hat{\Xi})_\lambda$ on the gauge-natural bundle $\mathcal{B}$. By an abuse of language, sometimes $(\hat{\Xi})_\lambda$ is denoted simply by $\hat{\Xi}$ or, even more abusively, by Ξ.

On any bundle $\mathcal{B}$, we can define the Lie derivative of a section σ with respect to any generator Ξ on $\mathcal{B}$. When $\mathcal{B}$ is a gauge-natural bundle, however, we can restrict to Lie derivatives with respect to fields $(\hat{\Xi})_\lambda$ which are gauge-natural lift of gauge generators. By some abuse of notation, we denote these Lie derivatives as

$$\pounds_\Xi \sigma := \pounds_{(\hat{\Xi})_\lambda} \sigma = T\sigma(\xi) - (\hat{\Xi})_\lambda \circ \sigma \tag{19.3.14}$$

where ξ is the base vector field on which Ξ projects, and we say they are Lie derivatives along gauge transformations.

Connections as higher order gauge-natural objects

A principal connection is a field with coefficients ω_μ^A transforming as

$$\omega_\mu'^A = \bar{J}_\nu^\nu \left(\mathrm{ad}_B^A(\varphi)\omega_\nu^B - \bar{R}_a^A(\varphi)\varphi_\nu^a \right) \tag{19.3.15}$$

The coefficients are in $\mathfrak{g} \otimes (\mathbb{R}^m)^*$, which is parameterised by ω_a^A, while the transformation group is $W_m^{(1,1)}G$, which is parameterised by (J_a^b, g^i, g_a^i).

Transformation rules inspire the group action

$$\lambda : W_m^{(1,1)}G \times \mathfrak{g} \otimes (\mathbb{R}^m)^* \to \mathfrak{g} \otimes (\mathbb{R}^m)^* : (J_a^b, g^i, g_a^i, \omega_a^A) \mapsto \bar{J}_a^b \left(\mathrm{ad}_B^A(g)\omega_b^B - \bar{R}_i^A(g)g_b^i \right) \tag{19.3.16}$$

and we define the associate bundle $\mathrm{Con}(P) = W^{(1,1)}P \times_\lambda (\mathfrak{g} \otimes (\mathbb{R}^m)^*)$.

A point in the associated bundle is $[x, e_a^\mu, g^i, g_a^i, \omega_a^A]_\lambda$, which has a canonical representative $[j_x^1 \sigma, \partial_\mu, \omega_\mu^A]$ where σ is the section of P which defines the trivialisation fixed on the structure bundle P and

$$\omega_\mu^A := e_\mu^b \left(\text{ad}_B^A(g) \omega_b^B - \bar{R}_i^A(g) g_b^i \right) \tag{19.3.17}$$

The associated bundle has then coordinates (x^μ, ω_μ^A) which transform under a gauge transformation as in (19.3.15).

There is a one-to-one correspondence between section of the associated bundle $\text{Con}(P)$ and global principal connections on P, hence $\text{Con}(P)$ is called the *bundle of principal connections on P*. That is a gauge-natural bundle of order $(1,1)$ and connections are hence gauge-natural objects.

4. Lie derivatives on natural and gauge-natural bundles

In natural theories, fields are natural objects and the natural lift on the configuration bundle C is expressed as a way of associating a vector field $\hat{\xi} = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i \in \mathfrak{X}(C)$ to a spacetime vector $\xi = \xi^\mu(x)\partial_\mu$. In other words, one can compute the components $\xi^i(x,y)$ out of the components $\xi^\mu(x)$ and their derivatives up to some finite order h.

Again this comes directly from transformation laws of fields.

If one has a tensor field which transforms with Jacobians, the natural lift will be written in terms of the first derivatives of ξ^μ.

Lie derivatives of a metric field

If we have a metric $g_{\mu\nu}$, the transformation laws are

$$g'_{\mu\nu}(x') = \bar{J}_\mu^\alpha g_{\alpha\beta}(x) \bar{J}_\nu^\beta \tag{19.4.1}$$

Let us consider a flow of spacetime transformations $x_s'^\mu(x)$ which is generated by the vector field $\xi = \xi^\mu(x)\partial_\mu$ iff

$$\xi^\mu(x) := \frac{d}{ds}\left(x_s'^\mu\right)\Big|_{s=0}(x) \tag{19.4.2}$$

and lift it to the bundles of metrics to get again a flow

$$x'^\mu = x_s'^\mu(x) \qquad g'_{\mu\nu} = \frac{\partial x_{-s}'^\alpha}{\partial x^\mu} g_{\alpha\beta} \frac{\partial x_{-s}'^\beta}{\partial x^\nu} \tag{19.4.3}$$

That means that fixing $(x^\mu, g_{\mu\nu})$ and letting the parameter s vary, one has a curve $(x'^\mu, g'_{\mu\nu})$ in the configuration bundle $C = \text{Lor}(M)$. Then by taking the derivative of transformation laws with respect to the flow parameter at $s = 0$, one computes the vector tangent to the curve at $(x^\mu, g_{\mu\nu})$

$$\hat{\xi} = \xi^\mu \frac{\partial}{\partial x^\mu} - \left(\partial_\mu \xi^\epsilon g_{\epsilon\nu} + g_{\mu\epsilon}\partial_\nu \xi^\epsilon\right) \frac{\partial}{\partial g_{\mu\nu}} \tag{19.4.4}$$

These vectors form a projectable vector field $\hat{\xi}$ on $\text{Lor}(M)$ which is by construction the natural lift of the spacetime vector field ξ.

:Notation: :Symbols: :AIndex: :Index:

Lie derivatives of a manifold connection

If one has a connection field $\Gamma^\alpha_{\beta\mu}$, transformation laws are

$$\Gamma'^\alpha_{\beta\mu} = J^\alpha_\rho \left(\Gamma^\rho_{\sigma\nu} \bar{J}^\sigma_\beta \bar{J}^\nu_\mu + \bar{J}^\rho_{\beta\mu} \right) \tag{19.4.5}$$

and their derivative with respect to the parameter is

$$\frac{d}{ds}\Gamma'^\alpha_{\beta\mu}\Big|_{s=0} = \partial_\rho \xi^\alpha \Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\sigma\mu}\partial_\beta \xi^\sigma - \Gamma^\alpha_{\beta\nu}\partial_\mu \xi^\nu - \partial_{\beta\mu}\xi^\alpha \tag{19.4.6}$$

A connection transforms with the Hessian of the spacetime transformation, it is a geometric object of order 2, it lives in a bundle which is associated to $L^2(M)$ and the natural lift of ξ to the bundle of connections is

$$\hat{\xi} = \xi^\mu \frac{\partial}{\partial x^\mu} + \left(\partial_\rho \xi^\alpha \Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\sigma\mu}\partial_\beta \xi^\sigma - \Gamma^\alpha_{\beta\nu}\partial_\mu \xi^\nu - \partial_{\beta\mu}\xi^\alpha \right) \frac{\partial}{\partial \Gamma^\alpha_{\beta\mu}} \tag{19.4.7}$$

which in fact depends on the derivatives of ξ^μ up to order 2.

Let us stress that the natural lift is a linear map with respect to the vector field ξ and hence the natural lift to a natural bundle is a *linear combination* of the derivatives of ξ^μ up to some finite order h, as one can check in the two examples above.

The Lie derivative $\pounds_\xi y^i = \xi^\mu y^i_\mu - \xi^i(x,y)$ is a linear combination of ξ^μ and its derivatives up to some finite order h; which order depending on the geometric objects we are using as fields. In other words, we have

$$\pounds_\xi y^i = L^i_\epsilon \xi^\epsilon + L^{i\alpha}_\epsilon \partial_\alpha \xi^\epsilon + \ldots + L^{i\alpha_1\ldots\alpha_s}_\epsilon \partial_{\alpha_1\ldots\alpha_s}\xi^\epsilon \tag{19.4.8}$$

If one has a (torsionless) connection $\tilde{\Gamma}$ on M, we can introduce symmetrised covariant derivatives

$$\pounds_\xi y^i = \tilde{L}^i_\epsilon \xi^\epsilon + \tilde{L}^{i\alpha}_\epsilon \nabla_\alpha \xi^\epsilon + \ldots + \tilde{L}^{i\alpha_1\ldots\alpha_s}_\epsilon \nabla_{\alpha_1\ldots\alpha_s}\xi^\epsilon \tag{19.4.9}$$

The Lie derivative of a metric is

$$\pounds_\xi g_{\mu\nu} = \xi^\epsilon d_\epsilon g_{\mu\nu} + \partial_\mu \xi^\epsilon g_{\epsilon\nu} + g_{\mu\epsilon}\partial_\nu \xi^\epsilon \qquad \Rightarrow \begin{cases} L_{\mu\nu\epsilon} = d_\epsilon g_{\mu\nu} \\ L^\alpha_{\mu\nu\epsilon} = \delta^\alpha_\mu g_{\epsilon\nu} + g_{\mu\epsilon}\delta^\alpha_\nu \end{cases} \tag{19.4.10}$$

If one has a connection $\tilde{\Gamma}$, the expression of the Lie derivative $\pounds_\xi g_{\mu\nu}$ can be recast as

$$\begin{aligned}\pounds_\xi g_{\mu\nu} =&\xi^\epsilon \left(d_\epsilon g_{\mu\nu} - \tilde{\Gamma}^\alpha_{\mu\epsilon} g_{\alpha\nu} - \tilde{\Gamma}^\alpha_{\nu\epsilon} g_{\mu\alpha} \right) + \left(\partial_\mu \xi^\epsilon + \tilde{\Gamma}^\epsilon_{\mu\rho}\xi^\rho \right) g_{\epsilon\nu} + g_{\mu\epsilon}\left(\partial_\nu \xi^\epsilon + \tilde{\Gamma}^\epsilon_{\nu\rho}\xi^\rho \right) = \xi^\epsilon \tilde{\nabla}_\epsilon g_{\mu\nu} + \left(\partial_\mu \xi^\epsilon + \tilde{\Gamma}^\epsilon_{\rho\mu}\xi^\rho + \tilde{T}^\epsilon_{\mu\rho}\xi^\rho \right) g_{\epsilon\nu} + g_{\mu\epsilon}\left(\partial_\nu \xi^\epsilon + \tilde{\Gamma}^\epsilon_{\rho\nu}\xi^\rho + \tilde{T}^\epsilon_{\nu\rho}\xi^\rho \right) = \\ =&\xi^\epsilon \tilde{\nabla}_\epsilon g_{\mu\nu} + \left(\tilde{\nabla}_\mu \xi^\epsilon + \tilde{T}^\epsilon_{\mu\rho}\xi^\rho \right) g_{\epsilon\nu} + g_{\mu\epsilon}\left(\tilde{\nabla}_\nu \xi^\epsilon + \tilde{T}^\epsilon_{\nu\rho}\xi^\rho \right)\end{aligned} \tag{19.4.11}$$

where $\tilde{T}^\epsilon_{\mu\nu}$ denotes the torsion of the connection $\tilde{\Gamma}$. If the connection $\tilde{\Gamma}$ is torsionless, the Lie derivative reduces to

$$\pounds_\xi g_{\mu\nu} = \xi^\epsilon \tilde{\nabla}_\epsilon g_{\mu\nu} + \tilde{\nabla}_\mu \xi^\epsilon g_{\epsilon\nu} + g_{\mu\epsilon}\tilde{\nabla}_\nu \xi^\epsilon \tag{19.4.12}$$

If $\tilde{\Gamma} = \{g\}$ is the Levi Civita connection of the metric g, then the Lie derivative further reduces to

$$\pounds_\xi g_{\mu\nu} = \overset{g}{\nabla}_\mu \xi^\epsilon g_{\epsilon\nu} + g_{\mu\epsilon}\overset{g}{\nabla}_\nu \xi^\epsilon \qquad \Rightarrow \begin{cases} \tilde{L}_{\mu\nu\epsilon} = 0 \\ \tilde{L}^\alpha_{\mu\nu\epsilon} = \delta^\alpha_\mu g_{\epsilon\nu} + g_{\mu\epsilon}\delta^\alpha_\nu \end{cases} \tag{19.4.13}$$

:Index: :AIndex: :Symbols: :Notation:

Let us stress that, in general, the Lie derivative is independent of any connection. The connection is used to have expressions in which each term separately has tensorial character, while, for example in (19.4.10), the whole Lie derivative is tensorial, but single terms are not. Hence, one is free to use any connection is convenient in the context. In some sense, the covariant expressions are just *convenient expressions* and in fact, while single terms depend on the connection, the whole expression of $\pounds_\xi g_{\mu\nu}$ does not.

The same can be said for the Lie derivative of a connection Γ. The bundle $\mathrm{Con}(M)$ comes to us with an action of base diffeomorphisms on sections which is given by the active version of (19.2.18). The Lie derivative of a connection Γ is obtained by simply taking the infinitesimal version of transformation rules, just pretending one has a 1-parameter family of diffeomorphisms and differentiating with respect to the family parameter s at $s=0$. One has the vector field $\hat{\xi} = \xi^\mu \partial_\mu + \xi^\alpha_{\beta\mu} \partial^{\beta\mu}_\alpha$ on $\mathrm{Con}(M)$ defined by

$$\xi^\alpha_{\beta\mu} = \frac{d\Gamma'^\alpha_{\beta\mu}}{ds}\bigg|_{s=0} = \partial_\rho \xi^\alpha \Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\sigma\mu}\partial_\beta \xi^\sigma - \Gamma^\alpha_{\beta\nu}\partial_\mu \xi^\nu - \partial_{\beta\mu}\xi^\alpha \tag{19.4.14}$$

Then the Lie derivative is defined as

$$\pounds_\xi \Gamma = \pounds_\xi \Gamma^\alpha_{\beta\mu}\partial^{\beta\mu}_\alpha = \left(\xi^\epsilon d_\epsilon \Gamma^\alpha_{\beta\mu} - \xi^\alpha_{\beta\mu}\right)\partial^{\beta\mu}_\alpha = \left(\xi^\epsilon d_\epsilon \Gamma^\alpha_{\beta\mu} - \partial_\rho \xi^\alpha \Gamma^\rho_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\partial_\beta \xi^\sigma + \Gamma^\alpha_{\beta\nu}\partial_\mu \xi^\nu + \partial_{\beta\mu}\xi^\alpha\right)\partial^{\beta\mu}_\alpha \tag{19.4.15}$$

In view of the theory, that is already a global vertical vector field (defined on the section Γ). However, we can rearrange terms so that each term is covariant and we can better trace transformation rules and globality.

We have a connection Γ and we can recast derivatives as covariant derivatives with respect to that connection and obtain

$$
\begin{aligned}
\pounds_\xi \Gamma^\alpha_{\beta\mu} =& \xi^\epsilon d_\epsilon \Gamma^\alpha_{\beta\mu} - (\nabla_\rho \xi^\alpha - \Gamma^\alpha_{\sigma\rho}\xi^\sigma)\Gamma^\rho_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}(\nabla_\beta \xi^\sigma - \Gamma^\sigma_{\rho\beta}\xi^\rho) + \Gamma^\alpha_{\beta\nu}(\nabla_\mu \xi^\nu - \Gamma^\nu_{\sigma\mu}\xi^\sigma) + d_\beta(\nabla_\mu \xi^\alpha - \Gamma^\alpha_{\rho\mu}\xi^\rho) = \\
=& \xi^\epsilon d_\epsilon \Gamma^\alpha_{\beta\mu} - (\nabla_\rho \xi^\alpha - \Gamma^\alpha_{\sigma\rho}\xi^\sigma)\Gamma^\rho_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}(\underline{\nabla_\beta \xi^\sigma} - \Gamma^\sigma_{\epsilon\beta}\xi^\epsilon) + \Gamma^\alpha_{\beta\nu}(\nabla_\mu \xi^\nu - \Gamma^\nu_{\epsilon\mu}\xi^\epsilon) + d_\beta \nabla_\mu \xi^\alpha - d_\beta \Gamma^\alpha_{\rho\mu}\xi^\rho - \Gamma^\alpha_{\rho\mu}(\underline{\nabla_\beta \xi^\rho} - \Gamma^\rho_{\sigma\beta}\xi^\sigma) = \\
=& \left(d_\epsilon \Gamma^\alpha_{\beta\mu} - d_\beta \Gamma^\alpha_{\epsilon\mu} + {\color{red}\Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\epsilon\beta}} - \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\epsilon\beta} + \Gamma^\alpha_{\epsilon\rho}\Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\beta\nu}\Gamma^\nu_{\epsilon\mu}\right)\xi^\epsilon - \underline{\nabla_\rho \xi^\alpha \Gamma^\rho_{\beta\mu}} + \underline{\Gamma^\alpha_{\beta\nu}\nabla_\mu \xi^\nu} + \nabla_\beta \nabla_\mu \xi^\alpha - \underline{\Gamma^\alpha_{\rho\beta}\nabla_\mu \xi^\rho} + \Gamma^\sigma_{\mu\beta}\nabla_\sigma \xi^\alpha = \\
=& \left(d_\epsilon T^\alpha_{\beta\mu} + d_\epsilon \Gamma^\alpha_{\mu\beta} - d_\beta T^\alpha_{\epsilon\mu} - d_\beta \Gamma^\alpha_{\mu\epsilon} + \Gamma^\alpha_{\epsilon\rho}T^\rho_{\beta\mu} + \Gamma^\alpha_{\epsilon\rho}\Gamma^\rho_{\mu\beta} - \Gamma^\alpha_{\beta\nu}T^\nu_{\epsilon\mu} - \Gamma^\alpha_{\beta\nu}\Gamma^\nu_{\mu\epsilon}\right)\xi^\epsilon - \nabla_\rho \xi^\alpha T^\rho_{\beta\mu} + T^\alpha_{\beta\rho}\nabla_\mu \xi^\rho + \nabla_\beta \nabla_\mu \xi^\alpha = \\
=& \left(d_\epsilon T^\alpha_{\beta\mu} + \underline{d_\epsilon \Gamma^\alpha_{\mu\beta}} - d_\beta T^\alpha_{\epsilon\mu} - \underline{d_\beta \Gamma^\alpha_{\mu\epsilon}} + \Gamma^\alpha_{\epsilon\rho}T^\rho_{\beta\mu} + \Gamma^\alpha_{\epsilon\rho}\Gamma^\rho_{\mu\beta} + \underline{\Gamma^\alpha_{\rho\epsilon}\Gamma^\rho_{\mu\beta}} - \Gamma^\alpha_{\beta\nu}T^\nu_{\epsilon\mu} - T^\alpha_{\beta\nu}\Gamma^\nu_{\mu\epsilon} - \underline{\Gamma^\alpha_{\nu\beta}\Gamma^\nu_{\mu\epsilon}}\right)\xi^\epsilon - \nabla_\rho \xi^\alpha T^\rho_{\beta\mu} + T^\alpha_{\beta\rho}\nabla_\mu \xi^\rho + \nabla_\beta \nabla_\mu \xi^\alpha = \\
=& \left(R^\alpha_{\ \mu\epsilon\beta} + \underline{d_\epsilon T^\alpha_{\beta\mu}} - d_\beta T^\alpha_{\epsilon\mu} + T^\alpha_{\epsilon\rho}T^\rho_{\beta\mu} + \underline{\Gamma^\alpha_{\rho\epsilon}T^\rho_{\beta\mu}} + T^\alpha_{\epsilon\rho}\Gamma^\rho_{\mu\beta} + T^\alpha_{\epsilon\rho}\Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\beta\nu}T^\nu_{\epsilon\mu} - \underline{T^\alpha_{\beta\nu}\Gamma^\nu_{\mu\epsilon}} - {\color{red}T^\alpha_{\nu\mu}\Gamma^\nu_{\beta\epsilon}} + {\color{red}T^\alpha_{\nu\mu}\Gamma^\nu_{\beta\epsilon}}\right)\xi^\epsilon - \nabla_\rho \xi^\alpha T^\rho_{\beta\mu} + T^\alpha_{\beta\rho}\nabla_\mu \xi^\rho + \nabla_\beta \nabla_\mu \xi^\alpha = \\
=& \left(R^\alpha_{\ \mu\epsilon\beta} + \nabla_\epsilon T^\alpha_{\beta\mu} - \underline{d_\beta T^\alpha_{\epsilon\mu}} - {\color{red}T^\alpha_{\rho\epsilon}T^\rho_{\beta\mu}} + {\color{red}T^\alpha_{\rho\epsilon}T^\rho_{\mu\beta}} - \underline{T^\alpha_{\rho\epsilon}\Gamma^\rho_{\beta\mu}} + \underline{T^\alpha_{\epsilon\rho}\Gamma^\rho_{\mu\beta}} - \underline{T^\alpha_{\rho\beta}T^\rho_{\mu\epsilon}} - \Gamma^\alpha_{\nu\beta}T^\nu_{\epsilon\mu} - \underline{T^\alpha_{\rho\mu}T^\rho_{\epsilon\beta}} + \underline{T^\alpha_{\nu\mu}\Gamma^\nu_{\epsilon\beta}}\right)\xi^\epsilon - \nabla_\rho \xi^\alpha T^\rho_{\beta\mu} + T^\alpha_{\beta\rho}\nabla_\mu \xi^\rho + \nabla_\beta \nabla_\mu \xi^\alpha = \\
=& \left(R^\alpha_{\ \mu\epsilon\beta} + 2\nabla_{[\epsilon} T^\alpha_{\beta]\mu} - 3T^\alpha_{\rho[\epsilon} T^\rho_{\beta\mu]}\right)\xi^\epsilon - \nabla_\rho \xi^\alpha T^\rho_{\beta\mu} + T^\alpha_{\beta\rho}\nabla_\mu \xi^\rho + \nabla_\beta \nabla_\mu \xi^\alpha
\end{aligned}
\tag{19.4.16}
$$

Notice that now each term is a tensor so that $\pounds_\xi \Gamma^\alpha_{\beta\mu}$ is a tensor in the model space $T^1_2(M)$.

If the connection is torsionless, its Lie derivative further simplify to

$$
\begin{aligned}
\pounds_\xi \Gamma^\alpha_{\beta\mu} =& R^\alpha_{\ \mu\epsilon\beta}\xi^\epsilon + \nabla_\beta \nabla_\mu \xi^\alpha = R^\alpha_{\ \mu\epsilon\beta}\xi^\epsilon + \tfrac{1}{2}\left(\nabla_\beta \nabla_\mu \xi^\alpha + [\nabla_\beta, \nabla_\mu]\xi^\alpha + \nabla_\mu \nabla_\beta \xi^\alpha\right) = (R^\alpha_{\ \mu\epsilon\beta} + \tfrac{1}{2}R^\alpha_{\ \epsilon\beta\mu})\xi^\epsilon + \nabla_{\beta\mu}\xi^\alpha = \\
=& \tfrac{1}{2}(R^\alpha_{\ \mu\epsilon\beta} + R^\alpha_{\ \mu\epsilon\beta} + R^\alpha_{\ \epsilon\beta\mu})\xi^\epsilon + \nabla_{\beta\mu}\xi^\alpha = -\tfrac{1}{2}(R^\alpha_{\ \mu\beta\epsilon} + R^\alpha_{\ \beta\mu\epsilon})\xi^\epsilon + \nabla_{\beta\mu}\xi^\alpha = -R^\alpha_{\ (\mu\beta)\epsilon}\xi^\epsilon + \nabla_{\beta\mu}\xi^\alpha
\end{aligned}
\tag{19.4.17}
$$

which makes also symmetry manifest, and one has

$$\tilde{L}^\alpha_{\beta\mu\epsilon} = -R^\alpha_{\ (\beta\mu)\epsilon} \qquad\qquad \tilde{L}^{\alpha\rho}_{\beta\mu\epsilon} = 0 \qquad\qquad \tilde{L}^{\alpha\rho\sigma}_{\beta\mu\epsilon} = \delta^\alpha_\epsilon \delta^\rho_{(\beta}\delta^\sigma_{\mu)} \tag{19.4.18}$$

Admittedly, the computation is a bit tedious. However, it is not a computation for the field theory. The Lie derivative is a property of connections on manifolds. One should find an handy expression once and for all to be used in any theory which has a connection as a fundamental field.

Here the important issues are:

first, that the Lie derivative of any natural field can be written as a linear combination of symmetrised covariant derivatives of the generators ξ^μ;

second, that the expression for the Lie derivative is obtained from transformation laws as shown in the examples above.

Lie derivatives of a principal connection

In gauge-natural theories, infinitesimal gauge transformations are generated by right-invariant vector fields on the structure bundle P, i.e. vector fields in the form

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^A_{(V)}(x)\rho_A \tag{19.4.19}$$

where ρ_A is a right-invariant pointwise basis of vertical vector on P. The index μ labels spacetime tensor, while the index A labels the Lie algebra $\mathfrak{g}$ of the gauge group G of P. The Lie derivative $\pounds_\Xi y^i$ is a linear combination of $(\xi^\mu, \xi^A_{(V)})$ and their derivatives up to some finite order.

The principal connection ω^A_μ transforms as

$$\omega'^A_\mu = \bar{J}^\nu_\mu \left(\mathrm{ad}^A_B(\varphi)\omega^B_\nu - \bar{R}^A_a(\varphi)\varphi^a_\nu \right) \tag{19.4.20}$$

If we fix an infinitesimal gauge transformation $\Xi = \xi^\mu(x)\partial_\mu + \xi^A(x)\rho_A$ on the structure bundle P, that induces on the associated bundle $(\Xi)_\lambda = \xi^\mu(x)\partial_\mu + \xi^A_\mu(x,\omega)\partial^\mu_A$. The component $\xi^A_\mu(x,\omega)$ is obtained by simply differentiate the action (19.4.20) of the gauge transformation which respect to the parameter of the flow, i.e.

$$\xi^A_\mu = -\partial_\mu\xi^\nu\omega^A_\nu + \left(\xi^i\partial_i\mathrm{ad}^A_B(e)\omega^B_\mu - \bar{T}^A_i\partial_\mu\xi^i \right) = -\partial_\mu\xi^\nu\omega^A_\nu + \left(\xi^i\bar{T}^C_i c^A{}_{CB}\omega^B_\mu - \bar{T}^A_i\partial_\mu\xi^i \right) \tag{19.4.21}$$

where we used property (17.1.40).

Then, the Lie derivative of a principal connection is expressed as:

$$\pounds_\Xi\omega = \pounds_\Xi\omega^A_\mu\partial^\mu_A = (\xi^\epsilon d_\epsilon\omega^A_\mu + \partial_\mu\xi^\nu\omega^A_\nu - \xi^i\bar{T}^C_i c^A{}_{CB}\omega^B_\mu + \bar{T}^A_i\partial_\mu\xi^i)\partial^\mu_A \tag{19.4.22}$$

That is a bit of a mess and the fact it transforms as a global vertical vector is not manifest at all. Thus we can write it better as

$$\begin{aligned}\pounds_\Xi\omega^A_\mu =& \xi^\epsilon d_\epsilon\omega^A_\mu + \partial_\mu\xi^\nu\omega^A_\nu - \xi^C c^A{}_{CB}\omega^B_\mu + \partial_\mu\xi^A = \xi^\epsilon d_\epsilon\omega^A_\mu + \partial_\mu\xi^\nu\omega^A_\nu - \xi^C c^A{}_{CB}\omega^B_\mu - \partial_\mu(\xi^A_{(V)} + \xi^\epsilon\omega^A_\epsilon) = \\ =& \xi^\epsilon d_\epsilon\omega^A_\mu + \partial_\mu\xi^\nu\omega^A_\nu - \xi^C c^A{}_{CB}\omega^B_\mu + \nabla_\mu\xi^A_{(V)} - c^A{}_{BC}\omega^B_\mu\xi^C_{(V)} - \partial_\mu\xi^\epsilon\omega^A_\epsilon - \xi^\epsilon d_\mu\omega^A_\epsilon = \\ =& \xi^\epsilon d_\epsilon\omega^A_\mu - \xi^C c^A{}_{CB}\omega^B_\mu + \nabla_\mu\xi^A_{(V)} - c^A{}_{BC}\omega^B_\mu\xi^C - c^A{}_{BC}\omega^B_\mu\omega^C_\epsilon\xi^\epsilon - \xi^\epsilon d_\mu\omega^A_\epsilon = \\ =& \xi^\epsilon \left(d_\epsilon\omega^A_\mu - d_\mu\omega^A_\epsilon + c^A{}_{BC}\omega^B_\epsilon\omega^C_\mu \right) + \nabla_\mu\xi^A_{(V)} = \xi^\epsilon F^A_{\epsilon\mu} + \nabla_\mu\xi^A_{(V)}\end{aligned} \tag{19.4.23}$$

where we set $\xi^C = \bar{T}^C_i\xi^i$, $\xi^C_{(V)} = \xi^C + \xi^\mu\omega^C_\mu$ for the vertical part of Ξ with respect to ω, We also used the expression for the covariant derivative of a right invariant vertical vector field on P that was computed in (18.7.7), as well as the expression for the curvature $F^A_{\mu\nu}$.

Specializing to the electromagnetic field, the gauge group is $G = U(1)$ which is one dimensional, hence commutative, we obtain

$$\pounds_\Xi\omega_\mu = \xi^\epsilon F_{\epsilon\mu} + \overset{*}{\nabla}_\mu\zeta_V \tag{19.4.24}$$

References

add

Analytic Index

References

[1] T.Khun, *The structure of scientific revolutions*

[2] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Prentice-Hall (1963)

[3] L. Fatibene and M. Francaviglia, *Natural and Gauge Natural Formalism for Classical Field Theories: A Geometric Perspective including Spinors and Gauge Theories*, Kluwer, Nov 30, 2003

[4] Anderson bocomplexes

[5] M.J.Gotay, J.Isemberg, J.E.Marsden, R. Montgomery, J. Śniatycki, P.B. Yasskin, *Momentum Map and Classical Relativistic Fields*, (preprint); see also physics/9801019

[6] Ferraris Hamiltonian formalism

[7] R.Abraham, J.E.Marsden, *Foundations of Mechanics: 2nd* ,

[8] G.Sardanashvily, *Noether's Theorems: Applications in Mechanics and Field Theory*, (Atlantis Studies in Variational Geometry) 1st ed. 2016 Edition

[9] J.M. Souriau, *Structure of Dynamical Systems: A Symplectic View of Physics*, (Progress in Mathematics) (1997-09-23)1703

[10] S.M.Carroll, *Spacetime and Geometry: An Introduction to General Relativity*,

[11] C.W.Misner, K.S.Thorne, J.A.Wheeler, *Gravitation*, 3 ott. 2017

[12] J.Baez, J.P.Muniain, *GAUGE FIELDS, KNOTS AND GRAVITY*, (Series on Knots and Everything)Oct 24, 1994

[13] Kolar book

[14] Janiska, Utiyama

[15] Norton, Hole agument

[16] E.Kretschmann, *Über den physikalischen Sinn der Relativitätspostulate, A.Einsteins neue und seine ursprüngliche Relativitätstheorie*, Annalen der Physik, **53**, (1917), 575614

[17] D.Giulini, *Some remarks on the notions of general covariance and background independence*, arXiv:gr-qc/0603087

[18] Articolo EPS il solito

[19] Wheeler's succinct summary of Einstein's theory of general relativity, in Geons, Black Holes, and Quantum Foam, p. 235.

[20] Weyl, unification theory

[21] Ferraris, unification theory

[22] E. Provenzi Quaderno didattico del dipartimento

To be done

To do

Review everything

Add Cauchy problem

Add LQG

Add perturbation theory

Done

2017/09

Log

2017/09/26 – Version 1.0.0 released

2017/10/26 – Added Section on $d\sigma_{\mu...}$

2019/06 – Composite bundles, Linearized equation, Second Variation, stability

2023/09 – Reviewed Part IV

20xx/xx/xx – Version 2.0.0 released

Branches

Currently, we have been asked a branch for Part IV.

It will be given branching number 12 and it is intended to be quite a comprehensive mathematical background for physicist interested in both General Relativity and theoretical physics. Although it will not cover directly physical applications, it will contain physical motivation and examples for introduction of geometric tools in physics. It will be divided into two parts, one for undergrad and one for grad students.

The branch title is *Differential Geometry for Physicists*, by S. Capozziello, L. Fatibene

Branching rules

Each branch is labelled by a natural number.

If you are currently managing a branch 504, you can trace the whole dependency history from other branches. This is done by decomposing the branch number in prime factors: $504 = 2^3 \cdot 3^2 \cdot 7$. Now looking at the powers, you see that there is one factor to the first power, namely 7. By reducing each power by one, you obtain $2^2 \cdot 3 = 12$ which is the branch from which you spin off. Of course, branch 12 was branching off branch 2, which is the root. You also notice that your lowest power prime, 7, is not the first available prime after the one used in your parent branch 12. That means that someone branched off 12 before you and (s)he has got the branch number $360 = 2^3 \cdot 3^2 \cdot 5$. That is your sister version. You cannot know if you have younger brothers without asking the manager of your parent branch.

If someone asks you for a new branch off yours, just add one to each power of your branch number $(2^4 \cdot 3^3 \cdot 7^2)$ and multiply by the first available prime after 7. Thus, for example, the branch numbers you should issue are, in order:

$232848 = 2^4 \cdot 3^3 \cdot 7^2 \cdot 11$

$275184 = 2^4 \cdot 3^3 \cdot 7^2 \cdot 13$

$359856 = 2^4 \cdot 3^3 \cdot 7^2 \cdot 17$

$\ldots$

Humans who cannot understand these branching rules are not allowed to manage a branch, even though I believe they are nice persons.

In your branch, you rule, you can do what you want. You can use any material from parent branch (the branch number is considered a reference to your ancestors), cut what you do not like, add what you want. You are only bound to apply branching rules above.

Each branch should be pubblically accessible in a repository. It is considered polite to provide a link to the depository where your parent branch is stored, as well as connect to your sons' branches.

Warning: this branch will be closed and frozen on June 1st, 2042.